Mathematical notes for *xTensor* and *xCoba* (Not for circulation)

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Notes on the mathematical ideas behind the structure of the *xAct* packages *xTensor* and *xCoba* for abstract and component tensor computations, respectively, for *Mathematica*. We focus on the interplay among connections, metrics and frames of vector fields. We also analyze the geometrical meaning and use of Christoffel tensors, volume forms and densities. Conventions and notations are fixed.

Some implementation notes are also given, describing how some of the mathematical ideas are encoded in the system.

CONTENTS

I. INTRODUCTION

A computer implementation of any branch of mathematics requires the use of a notation which is simultaneously general enough to describe a broad class of problems, and precise enough to avoid ambiguities. Unfortunately, several different notations and points of view have been suggested in the area of Differential Geometry, what makes the choice of such a notation for this area a difficult task.

The notation in *xTensor* [1] is modelled after the geometric notation used nowadays in Classical General Relativity, making heavy use of the *abstract indices* formalism introduced by Penrose [3] in 1969. As nicely explained in Ref. [5], this formalism combines the clarity and ellegance of the index-free geometric approach used by mathematicians with the computational efficiency of the component approach used by physicists, retaining the best of both worlds. In brief, it allows us to think in terms of geometric ideas, without loosing the computational power of indices.

A companion package, called *xCoba* [2], extends the system to work with charts and frames, using the formalism of *marked indices* [7]. We have made a large effort to construct *xCoba* using only the abstract tools supplied by *xTensor*, what allows full compatibility between abstract and components computations in a computer package, something never achieved before. This is mostly based on the explicit marking of chart- or frame-dependent geometrical objects (like parallel derivatives and densities), a rather non-standard idea.

Most of the notations are inspired in three references: Wald's book [4], Ashtekar et al. article on gauge fields [5] and Penrose & Rindler's books [6], in this order of importance, all curiously published in 1982–1984. The arbitrary choices of index positions (like the Riemann $R_{abc}{}^{d}$ or the Christoffel $\Gamma^{a}{}_{ab}$) are taken to be that of Wald, and are rigidly hardwired into the system. Sign conventions are implemented through global variables (which can be configured by the user), and take default values +1 as in Wald. As references on Differential Geometry I will generally cite [7] and [8].

All computations are currently assumed to refer to local properties of the geometric objects involved; no topological information is stored or handled by the current version of *xTensor*, though that could change in the future.

II. MANIFOLDS

Manifolds are assumed to be C^{∞} differentiable manifolds. There is no concept of open set. Each chart defined on a manifold is assumed to fully cover it, and no assumption is made on the properties of the boundaries of the charts defined on the manifolds.

Manifolds in *xTensor* carry very little information. Essentially they are required to play three different roles: as field dependencies of tensor fields, as base of vector bundles, and as submanifolds of other manifolds. The latter is implemented through the concept of (Cartesian) product manifold, though currently only a single product of two submanifolds is supported in *xTensor*.

A manifold is always given a constant dimension, which will also be the dimension of (the fibers of) its tangent vector bundle.

Currently all manifolds are real.

Implementation notes:

- In order to define a manifold we have to supply: name, dimension, and abstract indices for the tangent vbundle.

- Currently we allow only for a single level of inclusion of manifolds and submanifolds. That is, it is not possible to define a submanifold of a submanifold of a manifold.

III. VECTOR BUNDLES

xTensor allows the definition of vector bundles (abbreviated often to "vbundle") which are not the tangent vbundle of a manifold (*gauge* or *inner* vbundles). A vbundle B has a base manifold M and a fiber F (always a vector space), and in the following any reference to the vbundle will be actually understood as a reference to its fiber. The base manifold of the tangent vbundle TM is M.

A vbundle can be real (the number field of the vector space is R and all tensors are self-conjugate) or complex (the field is C and tensors have their conjugates in the conjugate vbundle). A real vbundle can be complexified: it is extended to work with the field C and tensors on the vbundle have as conjugates other tensors on the same vbundle. Currently manifolds are always real, and therefore their tangent vbundles are also real. Following *Mathematica*, in *xTensor* every real vbundle admits a complexification, in the sense that it is always possible to manipulate complex numbers or define complex tensors. By default all conjugate structures are denoted with the character † (this character is stored in the global variable `$DaggerCharacter`).

The concept of "directional field" v on a tangent vbundle (used in the Lie derivative along v, the directional derivative along v, or in the Lie bracket of v and w, etc.) is not present in gauge vbundles.

A vbundle can be constructed as direct sum of previously defined (sub)vbundles. In particular, the tangent vbundle of a product manifold of several submanifolds is the direct sum of the tangent vbundles of the submanifolds.

Implementation notes:

- By default, the name of the tangent bundle of the manifold `M3` is `TangentM3`.
- The conjugate of a vbundle `Spin` is, by default, `Spin†`. If the abstract indices of the former are `a, b, c, ...` then the abstract indices of the latter will be `a†, b†, c†, ...`
- In order to define a (gauge) vbundle we have to supply: name, dimension, base manifold, abstract indices and the "ultraindex" (now only for the `Dir` notation). Note that the dimension of the vbundle means the dimension of the fiber.

IV. TENSORS

A. Generic tensor

In *xTensor* all tensors are fields. A tensor has index-*slots* (each one associated to a vbundle) and *dependencies* (which can be either manifolds or parameters). A tensor is a field on its manifold dependencies and a function of its parameters.

There is no concept of "constant tensor" without additional structure. For example, we can say that a tensor has zero Lie derivative along a given vector field, or that it is parallel with respect to a given connection, but we need to specify the vector field or the connection in advance, respectively. Therefore, a tensor is assumed to be a field on all the base manifolds of the vbundles of its indices. Only scalars can be defined to be constants, but in such a case it is better to define a constant symbol.

Assume the vbundle $\mathcal{B}$ is real and has lowercase latin indices `a, b, c, ...`, and assume the vbundle $\mathcal{C}$ is complex with uppercase latin indices `A, B, C, ...`. Therefore, another vbundle $\mathcal{C}\dagger$ has been defined together with $\mathcal{C}$, having indices `A†, B†, C†, ...` Then a tensor T^a can be defined to be real, so that $(T^a)^\dagger = T^a$, or complex, so that $(T^a)^\dagger = T\dagger^a$, with $T\dagger$ being a different tensor with another slot on $\mathcal{B}$ (we say that the vbundle $\mathcal{B}$ has been complexified). A tensor S^A must be necessarily complex, with $(S^A)^\dagger = S\dagger^{A\dagger}$. The tensor $S\dagger$ is a tensor with slot on the vbundle $\mathcal{C}\dagger$. A tensor X^{aB} is also necessarily complex, with $(X^{aB})^\dagger = X\dagger^{aB\dagger}$. Note the difference between complex conjugation (implemented by †) and Hermitian conjugation (denoted with H), which involves both complex conjugation and a swap of indices: $(W^{AB\dagger})^\dagger = W\dagger^{A\dagger B} = W_H{}^{BA\dagger}$. A tensor is Hermitian if $(W^{AA\dagger})^\dagger = W\dagger^{A\dagger A} = W^{AA\dagger}$, or in other words $W_H = W$. In *xTensor* we do not follow the covention of moving all dagged indices after the non-dagged indices; this is to allow greater flexibility on index positioning (for example grouping pairs $AA\dagger$) and to avoid confusion with the slot-symmetry properties.

Implementation notes:

- The common practice in *Mathematica* is to store the dependencies of an object as arguments of that object, but storing the manifold M dependence of any tensor or its other properties, as in $T[M, \mathrm{props}]^{ab}$, would be painful. The oposite practice is the use of type declarations, favoured in *xTensor*, which also imitates the definitions of the type "let T^{ab} be a tensor on M with properties props", stated just once before the tensor is used. This decision forces us to make extensive use of up-values in *Mathematica*, and confronts us with a severe limitation: up-values in Mathematica are only allowed at level 1, but no deeper.

B. The identity tensor

The identity tensor on a vector bundle is represented as $\delta^a{}_b$. It always has staggered indices. The identity tensor on the dual vbundle would be $\Delta_a{}^b$, which produces identical results as $\delta^b{}_a$, and hence we do not use it. The tensor $\delta_a{}^b$ is not defined in principle, but exists once a metric is defined, and inherits the symmetry properties of that metric. That is, in the presence of a symmetric metric we have $\delta_a{}^b = \delta^b{}_a$, but if the metric is antisymmetric then $\delta_a{}^b = -\delta^b{}_a$.

Implementation notes:

- The δ tensor is exceptional because it has the same name `delta` on all vbundles.
- A representation δ_a^b is not possible in *xTensor* and would be ambiguous when working with an antisymmetric (spinor) metric.
- Unfortunately, for historical reasons related to spinors, we have chosen $\delta_a{}^b$ as the basic delta index configuration in *xTensor*. However, in this notes we always use $\delta^a{}_b$, which is the standard, and agrees with the convention of multiplying vectors from the left.
- The trace of the tensor δ equals the dimension of the vbundle of its indices, and this is computed at evaluation time using the function `DimOfVBundle`. Therefore it is safe to change the dimension of a vector bundle, but it is recommended to change the dimension of the base manifold accordingly.
- The δ tensor is the recommended index-replacement operator (one of the most important operations in the system); these are automatic:

$$\delta^a{}_b\, v^b = v^a, \qquad \omega_b\, \delta^b{}_a = \omega_a. \tag{1}$$

  If both indices can be contracted then the second index of δ will be contracted first:

$$\delta^a{}_b v_a{}^b = v_a{}^a, \qquad \delta_b{}^a v_a{}^b = v_b{}^b. \tag{2}$$

- If a δ tensor is found with two indices having the same character it will be inmediately changed to the first-metric of the vbundle of its indices, throwing an error message if no metric has been defined on it. If a δ tensor is found with a basis-index it will be inmediately changed to a `Basis` object with the same indices.
- All types of derivatives (covariant, Lie, parametric) give zero on the δ tensor.

C. The generalized δ tensor

The tensor $\delta^{a_1\ldots a_k}{}_{b_1\ldots b_k}$ is antisymmetric in the first k indices and separately antisymmetric in the last k indices. The contraction of a $2k$ indices δ tensor gives a $2k-2$ indices δ tensor:

$$\delta^{aa_2\ldots a_k}{}_{ab_2\ldots b_k} = (d-k+1)\delta^{a_2\ldots a_k}{}_{b_2\ldots b_k}, \tag{3}$$

where d is the dimension of the vbundle of the indices of δ. Iteration of that formula gives finally

$$\delta^{a_1\ldots a_k}{}_{a_1\ldots a_k} = \frac{d!}{(d-k)!}. \tag{4}$$

The generalized δ tensor can be expanded as

$$\delta^{a_1\ldots a_k}{}_{b_1\ldots b_k} = \begin{vmatrix} \delta^{a_1}{}_{b_1} & \ldots & \delta^{a_1}{}_{b_k} \\ \vdots & & \vdots \\ \delta^{a_k}{}_{b_1} & \ldots & \delta^{a_k}{}_{b_k} \end{vmatrix}. \tag{5}$$

Note that in dimension d it is possible to have $1 \leq k \leq d$. We could define $\delta \equiv 1$ for $k = 0$, but I don't think we need it.

Implementation notes:

- The generalized δ is represented as `Gdelta`. It is a different tensor because I don't want to overload `delta` with too many definitions, and `Gdelta` is seldom used but `delta` is used very frequently. The tensor `Gdelta` with two indices is automatically converted into `delta`.
- `Gdelta` is defined as antisymmetric in the first half of its indices, and antisymmetric in the second half of indices. If the number of indices is found to be odd and error message is thrown. That symmetry group is computed on real time from the number of indices (this is a trick to use a variable-number-of-indices tensor in *xTensor*).

D. Almost complex structure

A tensor $J^a{}_b$ obeying $J^a{}_b\, J^b{}_c = -\delta^a{}_c$ is called an "almost complex structure" on its vbundle. It is a trace-free tensor. A manifold with an almost complex structure is an "almost complex manifold", and must be even-dimensional.

E. Directions

From the algebraic point of view a tensor is a mapping from a product of vector spaces (and/or their duals) into the reals. Given the tensor T_{ab} and the vectors v^a, w^b we can think of T as a multilinear form acting on those vectors as $T(v,w) \equiv T_{ab} v^a w^b$. In *xTensor* this point of view is represented through the use of *directional indices*: we use the head `Dir` as a new type of index, such that the previous case would be represented as `T[ Dir[v[z]], Dir[w[z]] ]`. Note that we still need to include an abstract index `z` because `v` and `w` are vectors. However that index plays no role at all. To avoid problems with canonicalizations and to force the user to realize that this is not a normal index, *xTensor* requires the use of a special index, called the *ultraindex*, defined together with its corresponding vbundle.

F. Orthogonality to a vector field

V. DECOMPOSITION OF VBUNDLES

This follows closely unpublished notes by Guillaume Faye on the geometry of arbitary d-dimensional foliations of an n-dimensional spacetime.

This section contains only the algebraic (non-differential) manipulations, and hence we do not need the underlaying structure given by the foliation.

A. Projectors and projections

Given an n-dimensional vector space V we decompose it as a direct sum of a d-dimensional vector space V_1 and a $(n-d)$-dimensional vector space V_2. We shall use greek indices on V, latin indices i, j, k on V_1 and latin indices a, b, c on V_2. We choose the projectors:

$$X_i{}^\mu \;\text{ on } V_1, \qquad Y^i{}_\mu \;\text{ on } V_1^*, \qquad t_a{}^\mu \;\text{ on } V_2, \qquad \Theta^a{}_\mu \;\text{ on } V_2^*. \tag{6}$$

Note that the n-dimensional index is always in the second slot. They obey the following duality properties:

$$X_i{}^\mu\, Y^j{}_\mu = \delta_i{}^j, \qquad X_i{}^\mu\, \Theta^a{}_\mu = 0, \qquad Y^i{}_\mu\, t_a{}^\mu = 0, \qquad t_a{}^\mu\, \Theta^b{}_\mu = \delta_a{}^b. \tag{7}$$

The identity can be decomposed as

$$\delta_\mu{}^\nu = P_\mu{}^\nu + Q_\mu{}^\nu \tag{8}$$

with the mutually orthogonal projectors

$$P_\mu{}^\nu = Y^i{}_\mu X_i{}^\nu, \qquad Q_\mu{}^\nu = \Theta^a{}_\mu t_a{}^\nu. \tag{9}$$

This decomposition is fully independent of the existence of a metric or not on V.

Given that set of projectors we now define the projections of arbitrary objects. A tensor $T^{\mu\nu}{}_\sigma$ for example will be decomposed in 8 pieces which require uniquely defined names. One of those pieces would be

$$t^{ia}{}_j = T^{\mu\nu}{}_\sigma \, Y^i{}_\mu \, \Theta^a{}_\nu \, X_j{}^\sigma. \tag{10}$$

Conversely, projected objects can be lifted to n-dimensional objects:

$$t^{\mu\nu}{}_\lambda = t^{ia}{}_j \, X_i{}^\mu \, t_a{}^\nu \, Y^j{}_\lambda. \tag{11}$$

In particular we have $P_\mu{}^\nu = X_\mu{}^\nu = Y^\nu{}_\mu$, and similarly for the complementary vector space.

Implementation notes:

- We need to distinguish two kinds of projectors: those onto vbundles, and those onto their duals. Therefore we introduce two symbols `UProj` and `DProj`, respectively. We then have the following notations for the objects just defined:

$$X_i{}^\mu \to \mathtt{UProj[-i,mu]}, \qquad Y^i{}_\mu \to \mathtt{DProj[i,-mu]}, \qquad t_a{}^\mu \to \mathtt{UProj[-a,mu]}, \qquad \Theta^a{}_\mu \to \mathtt{DProj[a,-mu]}. \tag{12}$$

  There is no need to specify the vectors spaces because they are identified by the indices in the first slot of the expressions. The character of the last index cannot be used to distinguish between the U and D projectors because that can be changed with a metric.

- The names of the projected parts of a tensor will be constructed with the function `Proj`. For example the example given above would be identified as `Proj[T,`V_1`,`V_2`,`$-V_1$`][i,a,-j]`. The user can define `Proj[T,`V_1`,`V_2`,`$-V_1$`] = t`, or else the computer will generate a uniquely defined name (and define the tensor if it does not exist before).

B. Decompose a metric

Let us suppose now that there is the metric $g_{\mu\nu}$ in the vector space V. It is projected into

$$\gamma_{ij} = g_{\mu\nu} \, X_i{}^\mu \, X_j{}^\nu, \qquad \beta_{aj} = \beta_{ja} = g_{\mu\nu} \, t_a{}^\mu \, X_j{}^\nu, \qquad \xi_{ab} = g_{\mu\nu} \, t_a{}^\mu \, t_b{}^\nu. \tag{13}$$

The tensors γ_{ij} and ξ_{ab} define metrics on the vector spaces V_1 and V_2, respectively. Their inverses $(\gamma^{-1})^{ij}$ and $(\xi^{-1})^{ab}$ are not, however, the corresponding projections of the inverse metric $g^{\mu\nu}$. For example, if we denote with a bar the projections of the inverse metric:

$$\bar{\gamma}^{ij} = g^{\mu\nu} \, Y^i{}_\mu \, Y^j{}_\nu, \qquad \bar{\beta}^{aj} = \bar{\beta}^{ja} = g^{\mu\nu} \, \Theta^a{}_\mu \, Y^j{}_\nu, \qquad \bar{\xi}^{ab} = g^{\mu\nu} \, \Theta^a{}_\mu \, \Theta^b{}_\nu, \tag{14}$$

then we have

$$\gamma_{ij} \, \bar{\gamma}^{jk} = \delta_i{}^k - \beta_{ai} \, \bar{\beta}^{ak}, \qquad \xi_{ab} \, \bar{\xi}^{bc} = \delta_a{}^c - \beta_{ai} \, \bar{\beta}^{ci}. \tag{15}$$

It is not possible to relate the proyectors X and Y using these metrics, as it is clear in the relations

$$\bar{\gamma}^{ij} \, X_j{}^\mu = g^{\mu\nu} \, Y^i{}_\nu - \bar{\beta}^{ia} \, t_a{}^\mu, \qquad g_{\mu\nu} \, X_i{}^\nu = \gamma_{ij} Y^j{}_\mu + \beta_{ai} \, \Theta^a{}_\mu. \tag{16}$$

C. Orthogonal decompositions

The presence of a metric introduces further vector spaces. The vectors $\Theta^{a\mu} \equiv g^{\mu\nu} \, \Theta^a{}_\nu$ obey

$$g_{\mu\nu} \, \Theta^{a\mu} \, X_i{}^\nu = 0, \tag{17}$$

and hence span the orthogonal space $V_1{}^{ort}$, which does not coincide with V_2 in general, though naturally shares with it the latin indices a, b, c. In this new space it is more natural to use the metric

$$g_{\mu\nu} \, \Theta^{a\mu} \, \Theta^{b\nu} = \bar{\xi}^{ab} \tag{18}$$

(rather, its inverse), than using ξ_{ab}. We further generalize the treatment by introducing a linear transformation in this space giving an orthonormal frame:

$$n^{a\mu} = \alpha^a{}_b \, \Theta^{b\mu}, \qquad g_{\mu\nu} \, n^{a\mu} \, t_b{}^\nu = \alpha^a{}_b. \tag{19}$$

with

$$g_{\mu\nu} \, n^{a\mu} \, n^{b\nu} = \eta^{ab}. \tag{20}$$

From now on we shall move latin indices a, b, c up and down with the metric η_{ab} and its inverse η^{ab}. As long as no derivative is taken, the formalism can be easily adapted to the original (non-orthonormal) situation by taking $\alpha^a{}_b = \delta^a{}_b$ so that $\eta^{ab} = \bar{\xi}^{ab}$. Latin indices i, j, k will be moved with the metric γ_{ij} and its inverse γ^{ij}. In particular, we can now define the associated objects

$$X^i{}_\mu = \gamma^{ij} \, g_{\mu\nu} \, X_j{}^\nu, \qquad n^a{}_\mu = g_{\mu\nu} \, n^{a\nu}, \qquad n_a{}^\mu = \eta_{ab} \, n^{b\mu}, \tag{21}$$

obeying

$$X^i{}_\mu \, X_j{}^\mu = \delta^i{}_j, \qquad n^a{}_\mu \, n_b{}^\mu = \delta^a{}_b. \tag{22}$$

Note the projections:

$$X^i{}_\mu = Y^i{}_\mu + \beta_a{}^i \, \Theta^a{}_\mu, \qquad t_a{}^\mu = \alpha^b{}_a \, n_b{}^\mu + \beta_a{}^i \, X_i{}^\mu. \tag{23}$$

Using the metric, we have introduced two asymmetries between V_1 and V_2. The first is the fact that we decompose V into the sum of V_1 and V_1^{ort} and not V_1 and V_2. The second is that V_1 will be treated using a general γ_{ij} while V_1^{ort} will use an orthonormal metric η_{ab}. That is, the main objects on V_1 are now the metric γ_{ij} (and its inverse γ^{ij}) and the vectors $X_i{}^\mu$, while the main objects in V_1^{ort} are the metric η_{ab}, the vectors $n^{a\mu}$ and the matrix $\alpha^a{}_b$.

Final decompositions:

$$g_{\mu\nu} = \gamma_{ij} \, X^i{}_\mu \, X^j{}_\nu + \eta_{ab} \, n^a{}_\mu \, n^b{}_\nu, \tag{24}$$

$$g^{\mu\nu} = \gamma^{ij} \, X_i{}^\mu \, X_j{}^\nu + \eta^{ab} \, n_a{}^\mu \, n_b{}^\nu, \tag{25}$$

$$\delta_\mu{}^\nu = X^i{}_\mu \, X_i{}^\nu + n^a{}_\mu \, n_a{}^\nu. \tag{26}$$

Projected objects can be lifted to n-dimensional objects like

$$\gamma_{\mu\nu} = \gamma_{ij} \, X^i{}_\mu \, X^j{}_\nu, \qquad \beta_a{}^\mu = \beta_a{}^i \, X_i{}^\mu. \tag{27}$$

VI. COVARIANT DERIVATIVES

A. Definitions and conventions

A *covariant derivative* or *connection* ∇_a acting on the vbundle V with base manifold M is defined in *xTensor* as given by Wald:

1. Linearity:

$$\nabla_c(\alpha A^{a_1\ldots a_k}{}_{b_1\ldots b_l} + \beta B^{a_1\ldots a_k}{}_{b_1\ldots b_l}) = \alpha \nabla_c A^{a_1\ldots a_k}{}_{b_1\ldots b_l} + \beta \nabla_c B^{a_1\ldots a_k}{}_{b_1\ldots b_l}, \tag{28}$$

   for constants (on V) α and β.

2. Leibnitz rule:

$$\nabla_e[A^{a_1\ldots a_k}{}_{b_1\ldots b_l} B^{c_1\ldots c_{k'}}{}_{d_1\ldots d_{l'}}] = [\nabla_e A^{a_1\ldots a_k}{}_{b_1\ldots b_l}] B^{c_1\ldots c_{k'}}{}_{d_1\ldots d_{l'}} + A^{a_1\ldots a_k}{}_{b_1\ldots b_l} [\nabla_e B^{c_1\ldots c_{k'}}{}_{d_1\ldots d_{l'}}]. \tag{29}$$

3. Commutativity with contraction:

$$\nabla_d(A^{a_1\ldots c\ldots a_k}{}_{b_1\ldots c\ldots b_l}) = \nabla_d A^{a_1\ldots c\ldots a_k}{}_{b_1\ldots c\ldots b_l} \tag{30}$$

4. Consistency with the notion of tangent vectors as directional derivatives on scalar fields:

$$t(f) = t^a \nabla_a f. \tag{31}$$

Any covariant derivative ∇_a has two associated tensors, its *torsion* $T^c{}_{ab}$ and *curvature* or *Riemann* $R_{abc}{}^d$ tensors, defined by

$$\nabla_a \nabla_b v^d - \nabla_b \nabla_a v^d \equiv -s_R R_{abc}{}^d v^c - s_T T^c{}_{ab} \nabla_c v^d, \tag{32}$$

or equivalently,

$$\nabla_a \nabla_b w_c - \nabla_b \nabla_a w_c \equiv s_R R_{abc}{}^d w_d - s_T T^d{}_{ab} \nabla_d w_c. \tag{33}$$

The signs s_T (`$TorsionSign`) and s_R (`$RiemannSign`) are taken to be $+1$ in Wald's book. Both torsion and Riemann are, by construction, antisymmetric in the pair a, b. The *Ricci* tensor is defined as

$$R_{abc}{}^b = s_r R_{ac}, \tag{34}$$

with the sign s_r (`$RicciSign`) taking again the default value $+1$, as in Wald's book.

A covariant derivative with zero curvature is called *flat*. A covariant derivative with zero torsion is called *torsionless* or *symmetric*. A flat and torsionless covariant derivative is called *ordinary* (name taken from Wald's book).

B. Vectors and derivatives

In modern Differential Geometry vectors are defined as derivations, or directional derivatives. In *xTensor* this equivalence is represented by the use of directions: for a vector v^a and a scalar f the index-free notations $v(f)$ or $\partial_v f$ can be represented as `CD[ Dir[v[z]] ][ f[] ]`, for any derivative `CD`. In this way the vector v^a is represented either as a vector `v[a]` or as a derivation `Dv = CD[ Dir[v[z]] ]`, which can act on scalars `Dv[ f[] ]`. The fourth axiom above requires that the operator `Dv` must be independent of the chosen derivative. We could make *xTensor* to change to a special derivative when acting on scalars, but that would generate problems when the scalars are replaced by tensor products, and hence we do not do it. Use the command `ChangeCovD`.

C. Christoffel tensors

Given two covariant derivatives ∇_a and $\tilde{\nabla}_a$ it is possible to describe the difference in action between them using a *Christoffel* tensor $\Gamma^c{}_{ab}$. In *xTensor* we shall always make explicit which two derivatives we are referring to: $\Gamma[\tilde{\nabla}, \nabla]^c{}_{ab}$. Basic definition formulas:

$$\tilde{\nabla}_a v^c = \nabla_a v^c + \Gamma[\tilde{\nabla}, \nabla]^c{}_{ab} v^b, \tag{35}$$

$$\tilde{\nabla}_a \omega_b = \nabla_a \omega_b - \Gamma[\tilde{\nabla}, \nabla]^c{}_{ab} \omega_c. \tag{36}$$

There is no sign-convention variable here, but note that the "derivative index" a always comes in the central position. Clearly Christoffels are antisymmetric in the derivative arguments:

$$\Gamma[\tilde{\nabla}, \nabla]^c{}_{ab} = -\Gamma[\nabla, \tilde{\nabla}]^c{}_{ab} \tag{37}$$

and obey

$$\Gamma[\nabla_1, \nabla_2] + \Gamma[\nabla_2, \nabla_3] = \Gamma[\nabla_1, \nabla_3]. \tag{38}$$

Under a change of covariant derivative the torsion and Riemann tensors change as given by (define temporarily $\Gamma \equiv \Gamma[\tilde{\nabla}, \nabla]$)

$$s_T(\tilde{T}^c{}_{ab} - T^c{}_{ab}) = \Gamma^c{}_{ab} - \Gamma^c{}_{ba}, \tag{39}$$

$$s_R(\tilde{R}_{abc}{}^d - R_{abc}{}^d) = -\nabla_a \Gamma^d{}_{bc} + \nabla_b \Gamma^d{}_{ac} - \Gamma^d{}_{ae}\Gamma^e{}_{bc} + \Gamma^d{}_{be}\Gamma^e{}_{ac} - s_T T^e{}_{ab}\Gamma^d{}_{ec}. \tag{40}$$

If we assume that ∇_a has zero torsion and zero curvature (it is an ordinary derivative) then we get the usual formulas "defining" torsion and curvature for the derivative $\tilde{\nabla}_a$. Both previous formulas are antisymmetric under exchange of the derivatives, though this is not explicit in the latter one.

As pointed out by Penrose and Rindler, if we take $\Gamma^c{}_{ab} = -\frac{1}{2} s_T T^c{}_{ab}$ then automatically $\tilde{T}^c{}_{ab} = 0$, which allows to construct a torsionless derivative $\tilde{\nabla}_a$ starting from any other derivative ∇_a. The curvature, however, is not kept constant in the process.

Another simple case is a Christoffel of the form $\Gamma^c{}_{ab} = (\nabla_a f)\delta^c{}_b$, for a scalar field f, which does not change the curvature tensor, but changes the torsion. It corresponds to the relation $\tilde{\nabla}_a T = e^{-f}\nabla_a(e^f T)$ for any tensor T.

Implementation notes:

- $\Gamma[\nabla^{(1)}, \nabla^{(2)}]$ is represented in *xTensor* as `Christoffelnabla1nabla2`, where `nabla1` anb `nabla2` are sorted alphabetically, unless one of them is `PD`, which is sorted last and removed.

D. The Bianchi identities

For any derivative ∇_a with curvature $R_{abc}{}^d$ and torsion $T^a{}_{bc}$ we have the first identity

$$s_R R_{[abc]}{}^d + s_T \nabla_{[a} T^d{}_{bc]} + T^d{}_{e[a} T^e{}_{bc]} = 0, \tag{41}$$

and the second identity

$$\nabla_{[a} R_{bc]d}{}^e - s_T T^f{}_{[ab} R_{c]fd}{}^e = 0. \tag{42}$$

E. Connections and dependencies

A connection is always defined on one manifold M and it acts on all tensor expressions with a dependency on that manifold. On tensors with no dependence on M it gives zero. The command `CheckZeroDerivative` implements this issue.

A completely different question is whether connections can depend on parameters (currently this is not allowed in *xTensor*). I think this must be possible. The Christoffel tensor between two connections will depend on the union of parameters on which those two connections depend.

F. Lie bracket and Lie derivative

The Lie bracket (or *commutator*) of any two vector fields v^z and w^z can be given in terms of any covariant derivative ∇_a as follows:

$$[v^z, w^z]^a = v^b \nabla_b w^a - w^b \nabla_b v^a - s_T T^a{}_{bc} v^b w^c \equiv v^b \nabla_b w^a - w^b X_b{}^a, \tag{43}$$

where $T^a{}_{bc}$ is the torsion tensor of ∇_a, and we have defined

$$X_b{}^a \equiv \nabla_b v^a - s_T T^a{}_{bc} v^c. \tag{44}$$

From this result we can express the Lie derivative of an arbitrary tensor along v^z as

$$\mathcal{L}_v A^{a_1 \ldots a_k}{}_{b_1 \ldots b_l} = v^c \nabla_c A^{a_1 \ldots a_k}{}_{b_1 \ldots b_l} - \sum_{i=1}^{k} A^{a_1 \ldots c \ldots a_k}{}_{b_1 \ldots b_l} X_c{}^{a_i} + \sum_{j=1}^{l} A^{a_1 \ldots a_k}{}_{b_1 \ldots c \ldots b_l} X_{b_j}{}^c. \tag{45}$$

G. The Nijenhuis tensor of a linear operator

Given a (1,1) tensor $L^a{}_b$ we construct its Nijenhuis tensor as

$$(N_L)^a{}_{bc} X^b Y^c = -L^a{}_b L^b{}_c [X^a, Y^a]^c + L^a{}_b([L^a{}_c X^c, Y^a]^b + [X^a, L^a{}_c Y^c]^b) - [L^a{}_c X^c, L^a{}_c Y^c]^a. \tag{46}$$

Using indices and a torsion free connection ∇_a:

$$(N_L)^a{}_{cd} = L^a{}_b \nabla_c L^b{}_d - L^a{}_b \nabla_d L^b{}_c + L^b{}_d \nabla_b L^a{}_c - L^b{}_c \nabla_b L^a{}_d. \tag{47}$$

It is antisymmetric in its two lower indices. Formula (47) is independent of the choice of connection as long as it is torsion free.

The Nijenhuis tensor of an almost complex structure J vanishes if and only if the structure is integrable. It is then called a "complex structure", without the "almost".

H. Connections and almost complex structures

This section follows closely article [9].

A connection ∇_a on an almost complex manifold (M, J) is called "almost complex" if it preserves J, that is, iff $\nabla_a J^b{}_c = 0$. More generally, the connection is called "compatible" with J iff $\nabla_a J^a{}_b = 0$.

Given an arbitrary connection ∇_a in an almost complex manifold (M, J) we can construct an almost complex connection $\overset{J}{\nabla}_a$ using the Christoffel tensor (called G following [9], though the lower two indices are reversed with respect to that reference, to follow our own convention)

$$G^a{}_{bc} \equiv \Gamma[\overset{J}{\nabla}, \nabla]^a{}_{bc} = \frac{1}{2}(\nabla_b J^a{}_d) J^d{}_c. \tag{48}$$

We have that $G^a{}_{ba} = 0$ and $G^a{}_{bc} J^c{}_a = 0$. The curvature and torsion of $\overset{J}{\nabla}_a$ differ from those of ∇_a in general.

Define the hermitian G_+ and antihermitian G_- parts of G (or of any other (1,2) tensor) as

$$(G_\pm)^a{}_{bc} := \frac{1}{2}\left(G^a{}_{bc} \pm G^a{}_{de} J^d{}_b J^e{}_c\right). \tag{49}$$

The property $G^a{}_{bd} J^d{}_c = -J^a{}_d G^d{}_{bc}$ implies

$$(G_\pm)^a{}_{dc}\, J^d{}_b = \pm J^a{}_d (G_\pm)^d{}_{bc}, \tag{50}$$

$$(G_\pm)^a{}_{bd}\, J^d{}_c = -J^a{}_d (G_\pm)^d{}_{bc}. \tag{51}$$

Adding any constant multiple of G_+ with the two last indices transposed to $\overset{J}{\nabla}$ still produces an almost complex connection $\overset{t}{\nabla}$:

$$\overset{t}{\nabla}_a\, J^b{}_c = \overset{J}{\nabla}_a\, J^b{}_c + t(G_+)^c{}_{da} J^d{}_c - t(G_+)^d{}_{ca} J^b{}_d = 0. \tag{52}$$

Then the hermitian part of the torsion of $\overset{t}{\nabla}$ can be made to vanish:

$$s_T(T[\overset{t}{\nabla}]_+)^a{}_{bc} = s_T(T[\overset{J}{\nabla}]_+)^a{}_{bc} + t\left[(G_+)^a{}_{cb} - (G_+)^a{}_{bc}\right] = s_T(T[\nabla]_+)^a{}_{bc} + (1-t)\left[(G_+)^a{}_{bc} - (G_+)^a{}_{cb}\right] \tag{53}$$

if we start with a torsion-free ∇_a and choose $t = 1$. When we do that we get a privileged almost complex connection $\overset{KN}{\nabla}_a$ whose torsion is antihermitian and proportional to the Nijenhuis tensor of $J^a{}_b$ (which is independent of the original torsionless ∇_a):

$$s_T T[\overset{KN}{\nabla}]^a{}_{bc} = (G_-)^a{}_{bc} - (G_-)^a{}_{cb} = -\frac{1}{4}(N_J)^a{}_{bc}. \tag{54}$$

QUESTION: Is this connection $\overset{KN}{\nabla}_a$ uniquely defined by $J^a{}_b$? We have seen that its torsion is uniquely defined, but what about its curvature?

I. Integrability of projected vector spaces

Take a field of projectors, with the notations of subsection V A. The condition for the projectors $X_i{}^\mu$ to define a submanifold (perhaps a leave of a foliation) is given by the Frobenious theorem:

$$\left[X_i{}^\lambda, X_j{}^\lambda\right]^\mu = C^k{}_{ij}\, X_k{}^\mu, \tag{55}$$

for some structure constants $C^k{}_{ij}$, which will be zero if and only if the $X_i{}^\lambda$ define a coordinated frame. Because the information in the projectors $X_i{}^\mu$ and $Y^i{}_\mu$ is independent, there is no implication on the integrability of the complementary vector spaces. The same observation allows us to multiply that equation by $Q_\mu{}^\nu\, Y^j{}_{[\alpha} Y^k{}_{\beta]}$ without any loss of generality. Expanding the bracket with any symmetric connection ∇_a and exchanging that derivative once we easily arrive at the equivalent condition for the lifted projector:

$$P_{[\alpha}{}^\lambda\, P_{\beta]}{}^\mu \nabla_\lambda P_\mu{}^\nu = 0, \tag{56}$$

which is independent of the choice of $Y^i{}_\mu$. This form of the Frobenious condition ensures integrability, and is independent of any choice of frame, and hence does not involve structure constants.

Let us now introduce ∂_μ, the unique (up to rigid rotations) flat derivative obeying

$$\partial_\lambda X_i{}^\mu = 0, \qquad \partial_\lambda t_a{}^\mu = 0, \qquad \partial_\lambda Y^i{}_\mu = 0, \qquad \partial_\lambda \Theta^a{}_\mu = 0. \tag{57}$$

In particular, it also gives zero on the projectors Q and P. The different projected blocks of the torsion T of this derivative describe the relative properties of the elements of the frame: the diagonal blocks describe the structure constants:

$$C^k{}_{ij} = -s_T\, T^\lambda{}_{\mu\nu}\, Y^k{}_\lambda X_i{}^\mu X_j{}^\nu. \tag{58}$$

The non-diagonal blocks contain the integrability conditions. For example integrability of the projected space of X is

$$T^\lambda{}_{\mu\nu}\, \Theta^a{}_\lambda X_i{}^\mu X_j{}^\nu = 0. \tag{59}$$

Note that for an arbitrary nonintegrable collection of frames all blocks of the torsion, even those with indices on three different subspaces, can be different from zero.

It is also possible to interpret the condition using Christoffel symbols. Define, for any torsionless connection ∇_λ,

$$\nabla_\lambda X_i{}^\mu = \Gamma^j{}_{\lambda i} X_j{}^\mu + \Gamma^a{}_{\lambda i} t_a{}^\mu, \tag{60}$$
$$\nabla_\lambda t_a{}^\mu = \Gamma^b{}_{\lambda a} t_b{}^\mu + \Gamma^i{}_{\lambda a} X_i{}^\mu, \tag{61}$$
$$\nabla_\lambda Y^i{}_\mu = -\Gamma^i{}_{\lambda j} Y^j{}_\mu - \Gamma^i{}_{\lambda a} \Theta^a{}_\mu, \tag{62}$$
$$\nabla_\lambda \Theta^a{}_\mu = -\Gamma^a{}_{\lambda b} \Theta^b{}_\mu - \Gamma^a{}_{\lambda i} Y^i{}_\mu. \tag{63}$$

If we define

$$\Gamma^k{}_{ij} \equiv X_i{}^\lambda \Gamma^k{}_{\lambda j}, \qquad \Gamma^a{}_{ij} \equiv X_i{}^\lambda \Gamma^a{}_{\lambda j}, \tag{64}$$

then we have that the integrability condition is simply

$$\Gamma^a{}_{[ij]} = 0, \tag{65}$$

and the structure constants are given by

$$\Gamma^k{}_{[ij]} = C^k{}_{ij}. \tag{66}$$

VII. METRICS

A. Metric field and its inverse

A metric is a 2-covariant symmetric tensor field g_{ab}.

It is customary in GR to denote the inverse of a metric g_{ab} as g^{bc}, such that $g_{ab}g^{bc} = \delta_a{}^b$. It is not possible to use this notation consistently when there are several metrics, and at the same time raise and lower indices with one (or several) of those metrics. Hence, here we shall denote the inverse of g_{ab} as $\overline{g}^{ab}$, and use indices always in their proper positions.

For any derivation D (a covariant derivative, a parametric derivative, a Lie derivative, etc) we have for any metric g_{ab}

$$D[\overline{g}^{ab}] = -\overline{g}^{ac}\overline{g}^{bd} D[g_{cd}], \tag{67}$$

which is an important rule to canonicalize tensor expressions.

B. Isometries

Two flat metric fields $\hat{g}$ and $\tilde{g}$ on a manifold are isometric: if their respective ordinary connections are $\hat{\partial}_a$ and $\tilde{\partial}_a$, then there is a linear mapping $L_a{}^b$, obeying the integrability condition $\hat{\partial}_{[c} L_{a]}{}^b = 0$ such that $\tilde{g}_{ab} = L_a{}^c\, L_b{}^d\, \hat{g}_{cd}$. Then those connections are related by the Christoffel tensor obeying

$$\tilde{\partial}_c L_a{}^b = \Gamma[\tilde{\partial}, \hat{\partial}]^b{}_{cd} L_a{}^d, \qquad \hat{\partial}_c L_a{}^b = \Gamma[\tilde{\partial}, \hat{\partial}]^d{}_{ca} L_d{}^b. \tag{68}$$

The integrability condition expresses the fact that $L_a{}^b$ is always a gradient:

$$L_a{}^b = \hat{\partial}_a \left(L_c{}^b \tilde{x}^c \right), \tag{69}$$

where $\tilde{x}^c$ is the "position vector" associated to the Cartesian coordinate chart of the metric $\tilde{g}$, defined later in subsection IX B.

C. Metrics and covariant derivatives

Given a metric field g_{ab}, we say that a connection ∇_a is *associated* or *compatible* or *parallel* to g iff $\nabla_a g_{bc} = 0$. For a nondegenerate metric field g_{ab} there is a unique connection without torsion $\overset{g}{\nabla}_a$, called the Levi-Civita connection of that metric, which in general has curvature. When the connection $\overset{g}{\nabla}_a$ is flat (it is then ordinary) we say that the metric is flat. It can be obtained from any other derivative ∇_a using the Christoffel tensor

$$\Gamma[\overset{g}{\nabla}, \nabla]^c{}_{ab} = \frac{1}{2} \overline{g}^{cd} \left(\nabla_a g_{bd} + \nabla_b g_{ad} - \nabla_d g_{ab} \right) + \frac{s_T}{2} \overline{g}^{cd} \left(\underline{T}_{abd} + \underline{T}_{bad} - \underline{T}_{dab} \right), \tag{70}$$

where T is the torsion tensor of ∇_a, and its first index has been lowered with g:

$$\underline{T}_{abc} \equiv g_{ad} T^d{}_{bc}. \tag{71}$$

The combination $\underline{T}_{abd} + \underline{T}_{bad} - \underline{T}_{dab}$ is sometimes called the *contortion* tensor (of ∇ with respect to g). It is convenient to introduce the ChristoffelDown tensor

$$\underline{\Gamma}_{abc} \equiv g_{ad} \Gamma^d{}_{bc}. \tag{72}$$

When working with a single metric we can call both tensors Γ, but when we deal with several metrics the difference is important. Note the nice reverse formula, valid for all connections ∇, and not involving inverse metrics:

$$\nabla_a g_{bc} = \underline{\Gamma}[\overset{g}{\nabla}, \nabla]_{bac} + \underline{\Gamma}[\overset{g}{\nabla}, \nabla]_{cab}. \tag{73}$$

For connections associated to g the Christoffel $\underline{\Gamma}[\overset{g}{\nabla}, \nabla]$ is antisymmetric in its first and last slots.

We can reinterpret formula (70) using the following results. Given an arbitrary rank-3 tensor Q_{abc} we construct

$$\underline{\Gamma}_{cab} = \frac{1}{2} (Q_{abc} + Q_{bac} - Q_{cab}). \tag{74}$$

The relation is always reversible:

$$Q_{abc} = -\underline{\Gamma}_{abc} + \underline{\Gamma}_{acb} + \underline{\Gamma}_{bac} + \underline{\Gamma}_{cab}. \tag{75}$$

Raising one index with a metric we can construct a Christoffel tensor from Q_{abc}:

$$\Gamma[\hat{\nabla}, \nabla]^c{}_{ab} = \overline{g}^{cd} \underline{\Gamma}_{dab}. \tag{76}$$

Then the change of torsion is given by

$$s_T T[\hat{\nabla}]^c{}_{ab} = s_T T[\nabla]^c{}_{ab} - \overline{g}^{cd} Q_{d[ab]}, \tag{77}$$

and the derivatives of that metric are related by

$$\hat{\nabla}_c g_{ab} = \nabla_c g_{ab} - Q_{c(ab)}. \tag{78}$$

Both changes are controlled by the symmetry properties of the last two slots of the tensor Q_{abc}, which in turn controls the symmetry properties of $\underline{\Gamma}_{abc}$.

Formula (70) can be read as a sum of two Christoffel tensors relating $\overset{g}{\nabla}$ and ∇ through an intermediate connection $\hat{\nabla}$ which is still associated to g but now has torsion T. That is, the first term (with $Q_{cab} = \nabla_c g_{ab}$) inforces metric compatibility $\hat{\nabla}_c g_{ab} = 0$:

$$\Gamma[\hat{\nabla}, \nabla]^c{}_{ab} = \frac{1}{2} \overline{g}^{cd} \left(\nabla_a g_{bd} + \nabla_b g_{ad} - \nabla_d g_{ab} \right), \tag{79}$$

which is symmetric in a, b and hence both ∇ and $\hat{\nabla}$ have the same torsion tensor. Then the second term (with $Q_{cab} = \underline{T}_{cab}$) removes the torsion from $\hat{\nabla}$, giving $\overset{g}{\nabla}$:

$$\Gamma[\overset{g}{\nabla}, \hat{\nabla}]^c{}_{ab} = \frac{s_T}{2} \overline{g}^{cd} \left(\underline{T}_{abd} + \underline{T}_{bad} - \underline{T}_{dab} \right), \tag{80}$$

which is antisymmetric in the first and last slots, and hence mantains metric compatibility. That intermediate connection has curvature given by the sum of the curvature of $\overset{g}{\nabla}_a$ and the curvature coming from the "torsion" Christoffel.

D. General decomposition of a Christoffel tensor

Given any two covariant derivatives ∇ and $\tilde{\nabla}$, with respective torsion tensors T and $\tilde{T}$, we can decompose the Christoffel tensor relating them using a metric g_{ab}:

$$\begin{aligned} \Gamma[\tilde{\nabla}, \nabla]^c{}_{ab} &= \frac{1}{2} \overline{g}^{cd} \left(\nabla_a g_{bd} + \nabla_b g_{ad} - \nabla_d g_{ab} \right) + \frac{s_T}{2} \overline{g}^{cd} \left(\underline{T}_{abd} + \underline{T}_{bad} - \underline{T}_{dab} \right) && (81) \\ &- \frac{1}{2} \overline{g}^{cd} \left(\tilde{\nabla}_a g_{bd} + \tilde{\nabla}_b g_{ad} - \tilde{\nabla}_d g_{ab} \right) - \frac{s_T}{2} \overline{g}^{cd} \left(\underline{\tilde{T}}_{abd} + \underline{\tilde{T}}_{bad} - \underline{\tilde{T}}_{dab} \right) \\ &= \Gamma[\hat{\nabla}, \nabla]^c{}_{ab} - \Gamma[\hat{\nabla}, \tilde{\nabla}]^c{}_{ab} \qquad \text{[cf. formula (70)]} && (82) \\ &= \Gamma[\tilde{\nabla}, \hat{\nabla}]^c{}_{ab} + \Gamma[\hat{\nabla}, \nabla]^c{}_{ab}, && (83) \end{aligned}$$

which is a simple composition of two different Christoffel tensors using the Levi-Civita connection of g_{ab} as intermediate step.

The decomposition (81) of a Christoffel tensor in terms of a metric is very unnatural when the metric is related to neither of the two derivatives. The most common case of use of this type of formula is when a) the derivative $\tilde{\nabla}_a$ is associated to the metric (and so $\tilde{\nabla}_a g_{bc} = 0$ and $\tilde{T}_{abc} = 0$), and b) the other derivative ∇_a is the parallel derivative of some basis of vector fields in terms of which we can write the metric g_{ab}. If the basis of vector fields is coordinated then $T_{abc} = 0$, and we get the classical formula.

E. Curvature for a metric-compatible connection

When a connection ∇ (with or without torsion) is associated to a metric g, such that $\nabla_a g_{bc} = 0$ then the Riemann tensor of ∇ gains new symmetries. Define the *RiemannDown* tensor:

$$\underline{R}_{abcd} \equiv R_{abc}{}^e g_{ed}. \tag{84}$$

The latter is antisymmetric in both pairs of indices, but the pairs cannot be exchanged if the connection has torsion. Only for a Levi-Civita connection the RiemannDown tensor has its full symmetries. When there is just one metric, it is customary to denote Riemann and RiemannDown with the same name, but when there are several metrics in the problem we need to be careful with the difference.

From the Riemann tensor we can always construct the Ricci tensor, which is symmetric if the connection is of Levi-Civita type. See equation (34).

From the Ricci tensor we can construct the Ricci scalar:

$$R \equiv \overline{g}^{ab} R_{ab}, \tag{85}$$

and then the Einstein tensor, which is symmetric for a torsionless connection,

$$G_{ab} \equiv R_{ab} - \frac{1}{2} g_{ab} R. \tag{86}$$

We also have the Weyl tensor, always antisymmetric in both pairs, and with exchangeable pairs if the connection is symmetric,

$$W_{abcd} \equiv \underline{R}_{abcd} - \frac{1}{d-2} \left(R_{ac} g_{bd} - R_{ad} g_{bc} + R_{bd} g_{ac} - R_{bc} g_{ad} \right) + \frac{1}{(d-2)(d-1)} \left(g_{ac} g_{bd} - g_{ad} g_{bc} \right) R. \tag{87}$$

If the connection is Levi-Civita of g then it is also possible to construct the Einstein and Weyl tensors. QUESTION: Is this generalizable to the case with torsion?

For a metric-compatible connection we have the contracted form of the second Bianchi identity (42)

$$\overline{g}^{bc}\nabla_c G_{ab} = s_T \overline{g}^{bc} R_{db} T^d{}_{ac} - \frac{s_r\, s_T}{2} \overline{g}^{bc} R_{adb}{}^e T^d{}_{ce}. \tag{88}$$

F. Weyl or conformal connections (not implemented in *xTensor*)

Given a metric field there is a unique torsionless connection associated to it, the Levi-Civita connection. It is interesting to relax the association, but keeping the torsionless condition in the following way:

$$\hat{\nabla}_c\, g_{ab} + A_c\, g_{ab} = 0, \tag{89}$$

for an arbitrary vector field A_c. Such torsionless connection $\hat{\nabla}_c$ can be obtained from any other connection ∇_c by using the Christoffel tensor given in (70) adding this term on the rhs:

$$\frac{1}{2}\overline{g}^{cd}(g_{da}A_b + g_{db}A_a - g_{ab}A_d). \tag{90}$$

These are the so-called *Weyl connections*. I don't know whether every torsionless connection is a Weyl connection for some pair (g_{ab}, A_c).

Equation (89) implies

$$(\hat{\nabla}_d\hat{\nabla}_c - \hat{\nabla}_c\hat{\nabla}_d)g_{ab} = -s_R(\hat{R}_{cda}{}^e g_{eb} + \hat{R}_{cdb}{}^e g_{ea}) = F_{cd}g_{ab}, \tag{91}$$

where we have defined $F_{cd} \equiv \hat{\nabla}_c A_d - \hat{\nabla}_d A_c$. It is important to note that given $\hat{\nabla}_a$ the solution to (89) is unique only up to conformal transformations: the pairs (g_{ab}, A_c) and (g'_{ab}, A'_c) give the same torsionless derivative if they are related by the "gauge" transformation

$$g'_{ab} = e^{\chi} g_{ab}, \qquad A'_c = A_c - \nabla_c\chi, \qquad F'_{ab} = F_{ab}. \tag{92}$$

Equation (91) shows that the condition $F_{ab} = 0$, which invariantly characterizes the Levi-Civita connections among the torsionless connections, is an algebraic property of the Riemann tensor of $\hat{\nabla}_a$, because $(\delta_c{}^c)F_{ab} = -2s_R\hat{R}_{abc}{}^c$. Using the first Bianchi identity we see that the Ricci tensor is not symmetric for general Weyl connections:

$$s_r s_R(\hat{R}_{ab} - \hat{R}_{ba}) = -\frac{\delta_c{}^c}{2}F_{ab}. \tag{93}$$

Note: there is still the question of whether a general connection can be uniquely described by a triplet $(T^a{}_{bc}, g_{ab}, F_{ab})$. The idea would be transforming the connection to a Weyl connection using its torsion.

G. Hermitian geometry

Again, this section follows closely reference [9].

The triple (M, J, g) is called "almost Hermitian" (AH) iff the metric g is orthogonal with respect to J:

$$g_{ab}J^a{}_c J^b{}_d = g_{cd}. \tag{94}$$

Or lowering the indices:

$$J_{ab} = -J_{ba}. \tag{95}$$

We call J_{ab} the Kähler 2-form ω. Note that given any Riemannian metric g_{ab} its Hermitian part

$$(g_+)_{ab} = \frac{1}{2}(g_{ab} + g_{cd}J^c{}_a J^d{}_b) \tag{96}$$

defines an AH structure (M, J, g_+).

In the rest of this section we will assume that (M, J, g) is AH. The Levi-Civita connection $\overset{g}{\nabla}$ of g is not necessarily almost complex. However, we can use the method given in section VI H to construct a connection $\overset{J}{\nabla}$, which is almost complex and still associated to g, but now has torsion. It is called the "canonical Hermitian connnection". Adding a term $t(G_+)^a{}_{cb}$ breaks compatibility with the metric and in fact $\overset{J}{\nabla}$ is the only almost complex connection compatible with the metric.

We have the following properties:

$$G_{abc} = \frac{1}{2}(\overset{g}{\nabla}_b J_{ad})J^d{}_c = -G_{cba}, \tag{97}$$

$$J_{ab}G^b{}_{cd} = -J_{db}G^b{}_{ca}, \tag{98}$$

$$G_{dce}J^d{}_aJ^e{}_b = -G_{abc}. \tag{99}$$

The structure (M, J, g) is called:

- "Hermitian" if $N_J = 0$. Using (54) and (97) we see that this is the case if and only if $(G_-)^a{}_{bc} = 0$.
- "semi-Kähler" or "co-symplectic" if $\delta\omega = 0$ or $\overset{g}{\nabla}_a J^a{}_b = 0$.
- "nearly-Kähler" if $\overset{g}{\nabla}_{(a} J^b{}_{c)} = 0$ or $G^a{}_{(bc)} = 0$.
- "almost-Kähler" or "symplectic" if $d\omega = 0$ or $J_{[a|b}G^b{}_{|cd]} = 0$.
- "Kähler" if $d\omega = 0$ and $N_J = 0$. This is equivalent to $\overset{g}{\nabla}_a J^b{}_c = 0$ or to $G^a{}_{bc} = 0$.

H. Induced metrics

Because they have important applications in General Relativity, *xTensor* has a full collection of commands and predefined structures to work with induced metrics.

Given a non-degenerate metric g_{ab} with Levi-Civita connection ∇_a, and a vector field n^a with constant nonzero modulus $k \equiv g_{ab}n^an^b$ (usually taken as ± 1), we define the projector onto the subspaces orthogonal to n^a (with respect to the metric g_{ab}) as

$$h^a{}_b \equiv \delta^a{}_b - k^{-1}\, n^a g_{bc} n^c. \tag{100}$$

The tensor $h_{ab} \equiv g_{ac}h^c{}_b$ is a metric on the vector spaces of vectors orthogonal to n, independently of whether they are integrable or not. Note that the tensor $h^{ab} \equiv g^{bc}h^a{}_c$ is not the inverse of h_{ab}, and hence we do not have the equivalent of formula (67).

In general the projected subspaces at different points will not be integrable, i.e., they will not form the tangent bundle of a submanifold (a hypersurface). Integrability holds iff the vector n^a obeys the Frobenius condition

$$\text{Frob}: \qquad n_{[a}\nabla_b n_{c]} = 0, \tag{101}$$

for any torsionless connection ∇_b. Then h_{ab} is a metric field on that submanifold. A hypersurface-orthogonal vector is always proportional to the gradient of a scalar field τ:

$$\text{Frob}: \qquad n_a = k\,\alpha\nabla_a\tau. \tag{102}$$

The *time* scalar field τ is defined up to rescalings of the form $\tau \to f(\tau)$, as well as the *lapse* scalar field α, which simply ensures that the modulus of n^a is the constant k. In particular we have

$$\text{Frob}: \qquad \nabla_{[a}n_{b]} = -n_{[a}\nabla_{b]}\log\alpha. \tag{103}$$

Three types of structures must be defined to work with induced metrics: projections, derivatives and induced tensors, and we start with the latter:

1. Induced tensors

For a general pair (g_{ab}, n^c) and its associated projector (100) we define the *extrinsic curvature* tensor K_{ab} and the *acceleration* vector A_a as

$$s_K\, K_{ab} \equiv h_a{}^c h_b{}^d \nabla_c n_d, \qquad s_A\, A_a \equiv k^{-1} n^b \nabla_b n_a, \tag{104}$$

with the signs s_K and s_A respectively encoded as `$ExtrinsicKSign` and `$AccelerationSign` in *xTensor*, such that

$$\nabla_a n_b = s_K\, K_{ab} + s_A\, n_a A_b. \tag{105}$$

In the timelike case ($k < 0$) Smarr & York, Shapiro & Teukolsky, or Choptuik take $s_K = -1$, while Wald, or Hawking & Ellis choose $s_K = 1$. It is conventional to take s_A as the sign of k. If the vector n^a obeys the Frobenius condition then

$$\text{Frob}: \qquad K_{ab} = K_{ba} \qquad \text{and} \qquad s_A A_a = -h_a{}^b\, \nabla_b \log\alpha. \tag{106}$$

Actually the Frobenius condition is equivalent to the symmetry condition on K_{ab}, which in turn implies this form for A_a. Under the rescaling $\tau \to f(\tau)$ mentioned before the derivative $\nabla_a \log\alpha$ develops a term proportional to $f''(\tau) n_a$, which is killed by the projector $h_a{}^b$, so that A_a is insensitive to the choice of scalar τ.

Associated to the metric g_{ab} we can also define the antisymmetric tensor $\epsilon_{a_1 \ldots a_d}$ (with d the dimension of the manifold) and its induced counterpart (note that we contract the first index of ϵ)

$$e_{a_2 \ldots a_d} \equiv n^b\, \epsilon_{b a_2 \ldots a_d}, \tag{107}$$

with inverse

$$k\, \epsilon_{a_1 \ldots a_d} = d\, n_{[a_1}\, e_{a_2 \ldots a_d]} = n_{a_1}\, e_{a_2 \ldots a_d} + n_{a_2}\, e_{a_3 \ldots a_1} + \ldots + n_{a_d}\, e_{a_1 \ldots a_{d-1}}. \tag{108}$$

2. Projections

We now define a projection operation P_h acting on all free indices of a given expression:

$$P_h \left[T^{a_1 \ldots a_k}{}_{b_1 \ldots b_l}\right] \equiv h^{a_1}{}_{c_1} \ldots h^{a_k}{}_{c_k} h_{b_1}{}^{d_1} \ldots h_{b_l}{}^{d_l} T^{c_1 \ldots c_k}{}_{d_1 \ldots d_l}. \tag{109}$$

We say that a tensor is orthogonal to n^a iff $P_h[T] = T$. In particular K_{ab}, A_a and $e_{a_2 \ldots a_d}$ are orthogonal to n^a. In terms of this operator we define the *induced derivative*:

$$D_c T^{a_1 \ldots a_k}{}_{b_1 \ldots b_l} \equiv P_h \left[\nabla_c T^{a_1 \ldots a_k}{}_{b_1 \ldots b_l}\right], \tag{110}$$

for any orthogonal to n^a tensor T, which is the Levi-Civita connection of the metric h_{ab}:

$$D_a h_{bc} = 0. \tag{111}$$

For any orthogonal to n^a tensor T we have the decomposition

$$\begin{aligned} \nabla_c T^{a_1 \ldots a_k}{}_{b_1 \ldots b_l} = {} & D_c T^{a_1 \ldots a_k}{}_{b_1 \ldots b_l} + k^{-1} n_c n^d \nabla_d T^{a_1 \ldots a_k}{}_{b_1 \ldots b_l} \\ & - s_K k^{-1} \sum_{i=1}^{k} n^{a_i} K_{cd} T^{a_1 \ldots d \ldots a_k}{}_{b_1 \ldots b_l} - s_K k^{-1} \sum_{j=1}^{l} n_{b_j} K_c{}^d T^{a_1 \ldots a_k}{}_{b_1 \ldots d \ldots b_l}. \end{aligned} \tag{112}$$

Note that both upper and lower indices produce K-contributions with the same sign. Usually the directional derivative of T along n is further converted into a Lie derivative along n. It is also interesting to note that there are no acceleration vectors in this decomposition.

Formula (110) only defines a derivation on tensors that are orthogonal to n. That is, D_a is not a derivative on tensors with components along n. In particular, it fails to obey the Leibnitz rule: suppose D_a obeyed the Leibnitz rule on the product of two arbitrary tensors v^a and w^b. Then we would have

$$D_a(v^b w^c) = v^b D_a w^c + w^c D_a v^b. \tag{113}$$

The LHS is orthogonal to n by definition, and so must be the RHS. That means that both v and w are orthogonal to n. However, as a formal definition, we can use it on any type of object, having for instance (I thank Cyril for pointing this out):

$$D_a n_b \equiv K_{ab}, \qquad D_a(n_{b_1}...n_{b_k}) \equiv 0, \quad \text{for } k \geq 2. \tag{114}$$

It is obvious that D_a does not obey the Leibnitz rule here.

If the vector n^a does not obey the Frobenius condition then we have, for any scalar field f,

$$D_{[a}D_{b]}f = k^{-1}s_K\, K_{[ab]}n^c\nabla_c f, \tag{115}$$

which means that D_a has a torsion tensor

$$s_T T^c{}_{ab} = -k^{-1}s_K n^c K_{[ab]}. \tag{116}$$

The induced derivative D_a has its own Riemann tensor $\mathcal{R}_{abcd}$, which can be related to the Riemann tensor of ∇_a through the Gauss-Codazzi relations (the Frobenius condition on n^a is essential here, and there does not seem to be an obvious generalization):

$$\text{Frob}: \qquad s_R\left(P_h\left[R_{abcd}\right] - \mathcal{R}_{abcd}\right) = k^{-1}(K_{cb}K_{da} - K_{db}K_{ca}), \tag{117}$$

$$\text{Frob}: \qquad s_R\, P_h\left[R_{abcd}\, n^a\right] = s_K(D_d K_{cb} - D_c K_{db}), \tag{118}$$

$$\text{Frob}: \qquad s_R\, P_h\left[R_{abcd}\, n^b n^d\right] = -s_K\mathcal{L}_n K_{ac} + K_{ab}K_c{}^b - k\,\alpha^{-1}D_a D_c\alpha. \tag{119}$$

Note that we can reconstruct the full Riemann R_{abcd} from those three objects and the vector field n^a. We can also project the Ricci tensor:

$$\text{Frob}: \qquad s_R s_r\left(P_h\left[R_{ab}\right] - \mathcal{R}_{ab}\right) = -\alpha^{-1}D_a D_b\alpha + k^{-1}\left(-s_K\mathcal{L}_n K_{ab} + 2K_{ac}K_b{}^c - KK_{ab}\right), \tag{120}$$

$$\text{Frob}: \qquad s_R s_r\, P_h\left[R_{ab}n^b\right] = s_K\left(D^b K_{ab} - D_a K\right), \tag{121}$$

$$\text{Frob}: \qquad s_R s_r\, R_{ab}n^a n^b = -s_K\mathcal{L}_n K - K_{ab}K^{ab} - k\,\alpha^{-1}D^a D_a\alpha, \tag{122}$$

and hence the Ricci scalar

$$\text{Frob}: \qquad s_R s_r(R - \mathcal{R}) = -2\alpha^{-1}D^a D_a\alpha - k^{-1}\left(2s_K\mathcal{L}_n K + K_{ab}K^{ab} + K^2\right), \tag{123}$$

or the Einstein tensor:

$$\text{Frob}: \qquad s_R s_r\left(P_h\left[G_{ab}\right] - \mathcal{G}_{ab}\right) = , \tag{124}$$

$$\text{Frob}: \qquad s_R s_r\, P_h\left[G_{ab}n^b\right] = s_K\left(D^b K_{ab} - D_a K\right), \tag{125}$$

$$\text{Frob}: \qquad s_R s_r\, G_{ab}n^a n^b = \frac{1}{2}\left(-k s_R s_r\mathcal{R} - K_{ab}K^{ab} + K^2\right), \tag{126}$$

3. "Time" derivatives

A second important derivative in this problem is the Lie derivative along the vector n^a. We have the following basic equations:

$$\mathcal{L}_n n_a = s_A\, k\, A_a, \qquad \mathcal{L}_n n^a = 0, \tag{127}$$

$$\mathcal{L}_n g_{ab} = \nabla_a n_b + \nabla_b n_a, \qquad \mathcal{L}_n g^{ab} = -\nabla^a n^b - \nabla^b n^a, \tag{128}$$

$$\mathcal{L}_n h_{ab} = s_K\left(K_{ab} + K_{ba}\right), \qquad \mathcal{L}_n h^{ab} = -\nabla^a n^b - \nabla^b n^a, \tag{129}$$

$$\mathcal{L}_n h_a{}^b = -s_A\, A_a n^b, \qquad \mathcal{L}_n h^a{}_b = -s_A\, n^a A_b. \tag{130}$$

Then we have:

$$\mathcal{L}_n\epsilon_{a_1\ldots a_d} = s_K\, K^b{}_b\,\epsilon_{a_1\ldots a_d}, \qquad \mathcal{L}_n\epsilon^{a_1\ldots a_d} = -s_K\, K^b{}_b\,\epsilon^{a_1\ldots a_d}, \tag{131}$$

$$\mathcal{L}_n e_{a_2\ldots a_d} = s_K\, K^b{}_b\, e_{a_2\ldots a_d}, \qquad \mathcal{L}_n e^{a_2\ldots a_d} = -s_K\, K^b{}_b\, e^{a_2\ldots a_d} + s_A\, k\, A_b\,\epsilon^{ba_2\ldots a_d}. \tag{132}$$

Two important results for covariant tensors must be mentioned at this point:

i) If $T_{a_1\ldots a_l}$ is a covariant orthogonal to n^a tensor then $\mathcal{L}_n T_{a_1\ldots a_l}$ is also a covariant orthogonal to n^a tensor.

ii) If f is an arbitrary scalar field then $\mathcal{L}_{fn} T_{a_1 \ldots a_l} = f \mathcal{L}_n T_{a_1 \ldots a_l}$.

Unfortunately this result is not valid for orthogonal to n^a tensors with contravariant indices [see for example Eq. (130)], but it can be generalized for a hypersurface-orthogonal vector n^a, once written in the form (102). In this case we can define the new vector field

$$\text{Frob}: \qquad N^a \equiv \alpha\, n^a = (k\alpha^2) \nabla^a \tau \tag{133}$$

which obeys the important properties

$$\text{Frob}: \qquad \mathcal{L}_N\, n_a = n_a\, \mathcal{L}_n\, \alpha, \qquad \mathcal{L}_N\, n^a = -n^a\, \mathcal{L}_n\, \alpha, \qquad \Rightarrow \qquad \mathcal{L}_N\, h^a{}_b = 0, \tag{134}$$

such that the property i) above is valid for all orthogonal to n^a tensors, not only covariant tensors, replacing n by N in the Lie derivative. The vector N^a becomes the natural vector for "time" differentiation because of the duality condition $N^a \nabla_a \tau = 1$. We shall call N^a the *orthogonal time vector*. However, we can have more general time vector fields of the form

$$\text{Frob}: \qquad t^a = N^a + \beta^a, \qquad \text{with} \qquad \beta^a n_a = 0, \tag{135}$$

also obeying $t^a \nabla_a \tau = 1$. The vector β^a will be called the *shift vector*.

It is customary to decompose the metric using a coordinate system $(x^\mu) = (x^0, x^i)$ adapted to the foliation, in the sense that $x^0 \equiv \tau$, $(\partial_0)^a \equiv t^a$ and $(\partial_i)^a n_a = 0$, $t^a \nabla_a x^i = 0$, such that

$$t_a = \left(k\alpha^2 + \beta^k \beta_k, \beta_j \right), \qquad t^a = (1, 0)\,, \tag{136}$$

$$\beta_a = \left(\beta^k \beta_k, \beta_j \right), \qquad \beta^a = \left(0, \beta^i \right), \tag{137}$$

$$N_a = \left(k\alpha^2, 0 \right), \qquad N^a = \left(1, -\beta^i \right), \tag{138}$$

and hence

$$g_{ab} = \begin{pmatrix} k\alpha^2 + \beta^k \beta_k & \beta_j \\ \beta_i & g_{ij} \end{pmatrix}, \qquad g^{ab} = \frac{1}{k\alpha^2} \begin{pmatrix} 1 & -\beta^j \\ -\beta^i & k\alpha^2 g^{ij} + \beta^i \beta^j \end{pmatrix} \tag{139}$$

$$h_{ab} = \begin{pmatrix} \beta^k \beta_k & \beta_j \\ \beta_i & g_{ij} \end{pmatrix}, \qquad h^{ab} = \begin{pmatrix} 0 & 0 \\ 0 & g^{ij} \end{pmatrix} \tag{140}$$

We always find the geometric combination $k\alpha^2 = g^{ab} N_a N_b = (g^{ab} \nabla_a \tau \nabla_b \tau)^{-1}$.

I. Trace-free decomposition of a symmetric tensor

Given any symmetric tensor $S_{a_1 \ldots a_l}$ over a vector space of dimension d with a metric g_{ab}, we construct its trace-free part as

$$[S_{a_1 \ldots a_l}]^{TF} \equiv \sum_{m=0}^{[l/2]} A^{(m)}_{l,d}\; g_{(a_1 a_2} \ldots g_{a_{2m-1} a_{2m}} S_{a_{2m+1} \ldots a_l)}{}^{b_1}{}_{b_1 \ldots}{}^{b_m}{}_{b_m} \tag{141}$$

with $[l/2]$ the integer part of $l/2$. The coefficients of the expansion are determined by the trace-free condition, and are given by

$$A^{(m)}_{l,d} = (-1)^m \frac{l!}{2^m m! (l-2m)!} \frac{\Gamma[l + d/2 - 1 - m]}{2^m \Gamma[l + d/2 - 1]}. \tag{142}$$

Both arguments of Γ are always positive for nontrivial cases ($d \geq 1$ and $l \geq 1$). It is convenient to declare scalars and vectors as traceless

$$[S]^{TF} = S, \qquad [S_a]^{TF} = S_a, \qquad \text{because} \qquad A^{(0)}_{l,d} = 1. \tag{143}$$

We can also decompose the original tensor in a sum of products of metrics and traceless tensors:

$$S_{a_1...a_l} = \sum_{m=0}^{[l/2]} g_{(a_1a_2}...g_{a_{2m-1}a_{2m}} F^{(l-2m)}{}_{a_{2m+1}...a_l)}, \tag{144}$$

with the tensor $F^{(k)}$ being traceless in its k indices. The relation is simply

$$F^{(l-2m)}{}_{a_1...a_{l-2m}} = B^{(m)}_{l,d} \; \left[S_{a_1...a_{l-2m}}{}^{b_1}{}_{b_1}...{}^{b_m}{}_{b_m} \right]^{TF}, \tag{145}$$

with

$$B^{(m)}_{l,d} = \frac{l!}{2^m m!(l-2m)!} \frac{\Gamma[l+d/2-2m]}{2^m \Gamma[l+d/2-m]}. \tag{146}$$

Note on the combinatorial factors: the canonicalization of $g_{(a_1a_2}...g_{a_{2m-1}a_{2m}} T_{a_{2m+1}...a_l)}$ for a symmetric tensor T produces many identical terms, due to the symmetries of g and T. There are $(l-2m)!$ permutations of the indices of T, $m!$ permutations of the metric factors and 2^m permutations of the indices in each metric factor. Therefore the number of actual different terms for a given m is

$$\frac{l!}{2^m m!(l-2m)!} = \begin{pmatrix} l \\ 2m \end{pmatrix} (2m-1)!!. \tag{147}$$

This factor can be absorbed changing the index symmetrization $(a_1...a_l)$ by *canonical symmetrization*, usually denoted with $\{a_1...a_l\}$.

VIII. BASES

A. Definitions

In *xCoba* we call *basis* on a given vbundle a set of vector fields with index on that vbundle. The number of elements in the set must equal the dimension of the vbundle, and the elements will be identified by (positive, negative or 0) integer numbers (called *cnumbers*). An element of the basis B is represented as `Basis[a, {2, -B}]`, with output form $e^a{}_{\hat{\mathbf{2}}}$, where the boldface denotes a basis-index and the hat is understood to represent the basis B. In *xCoba* we associate a color to each basis and the basis-indices of a basis are colored accordingly. The elements of the dual basis are denoted by `Basis[-a, {3, B}]`, with output $e_a{}^{\hat{\mathbf{3}}}$.

The duality condition is expressed as

$$e^a{}_{\hat{\mathbf{b}}} \, e_a{}^{\hat{\mathbf{c}}} = e_{\hat{\mathbf{b}}}{}^{\hat{\mathbf{c}}}, \tag{148}$$

the Kronecker delta of the cnumbers $\hat{\mathbf{b}}$ and $\hat{\mathbf{c}}$. The basis condition is given by

$$e^a{}_{\hat{\mathbf{b}}} \, e_c{}^{\hat{\mathbf{b}}} = \delta^a{}_c. \tag{149}$$

Important comment: The objects `delta` and `Basis` are formally identical, and actually most authors use the same symbol δ for both of them. Those two objects have many associated definitions and we have separated them to avoid too much overloading of the first. It is also conceptually simpler to think of basis vectors as different from the identity tensor. They change automagically into each other when required.

B. Components of tensor fields

Given a tensor field v^a on some manifold and a basis B of vector fields $e^a{}_{\hat{\mathbf{b}}}$, with duals $e_a{}^{\hat{\mathbf{b}}}$, we define the components of v as

$$v^{\hat{\mathbf{b}}} \equiv v^a e_a{}^{\hat{\mathbf{b}}}. \tag{150}$$

We use the so called *marked index* notation, by which the name of the basis used to generate the components is stored in the index. That is, the index has been marked with the basis. Any tensorial slot in which we can have an abstract index can also contain a basis index. There is the question of whether basis indices can be considered to be abstract indices as well (associated to a basis of a vbundle, instead of the vbundle itself), and the answer is probably positive, but I think it is convenient to split the information of vbundles and bases. This also helps in stressing the distinction between abstract indices and basis indices. The opposite notation would stress the difference between coordinate indices and basis indices, which is a different issue.

So, what is really the difference between abstract indices and basis indices? To me, the main point is that v^a is a vector but $v^{\hat{\mathbf{a}}}$ is a scalar, obtained by contraction of the vector v^a and the covector $e_b{}^{\hat{\mathbf{a}}}$. Abstract indices identify the tensor character of an object, and cannot take numerical values. Basis indices can take numerical values and do not contribute to the tensor character of an object.

C. Components of derivatives

There is a very important notational problem when dealing simultaneously with bases and derivatives: is $\nabla_a v_{\hat{\mathbf{3}}}$ the derivative of a component $[\nabla_a(v_b e^b{}_{\hat{\mathbf{3}}})]$ or the component of a derivative $[e^b{}_{\hat{\mathbf{3}}}\nabla_a v_b]$? Both expressions are different when $\nabla_a e^b{}_{\hat{\mathbf{3}}} \neq 0$. Most answers would favor the latter interpretation, unless ∇_a is the "partial derivative", where most of them would favor the former. This inconsistency is unacceptable from the computer-algebra point of view.

Most authors consider $\nabla_a v_b$ to be the notation for a single tensor, and not an operator ∇_a acting on a vector field v_b. That is consistent as long as any other additional structures acting on indices (mainly bases and metrics) are compatible ("parallel") with respect to the derivative ∇_a. For example, asumme that the metric g_{ab} is not associated to the derivative. If $\nabla_a v_b$ is the notation for a tensor, then it would be natural to replace $g^{bc}\nabla_a v_b$ by $\nabla_a v^c$, but this is considered to be incorrect. Of course, this is precisely why most people separates the "partial derivative" case, because it is not the Levi-Civita connection of the metric. But it is clearly wrong to consider this as an exceptional case; it is the other way round: the Levi-Civita connection is the exception because it is the only connection associated to the metric. I believe it is absurd to manipulate simultaneously two incompatible notations: one for the general case, and the other one for a single exceptional case. In *xTensor* we only use the general notation, which is safer, based on the key idea of treating connections as operators. The same analysis can be done for the use of bases in *xCoba*, arriving at the same conclusion.

D. Frame derivatives

Given a basis `B`, it is possible to define directional derivatives along the elements of the basis. The derivation would be, for example, `CD[ Dir[ Basis[{1, -B}, z] ] ]`, which we abbreviate to `CD[{1, -B}]`. In *xTensor* we always have the automatic rewrite rule `Dir[ Basis[{1,-B}, z] ] -> {1,-B}` and its inverse `Basis[{1,B}, Dir[v[z]]] -> v[{1,B}]`.

E. Operations with basis indices

We need to find a complete (hopefully minimal) set of operations allowing us to perform any desired computation involving basis indices.

We first have the issue of contracting and separating elements of the basis of vectors. Examples:

`ContractBasis`: $\quad e^a{}_{\hat{\mathbf{b}}} v^{\hat{\mathbf{b}}} \;\to\; v^a \quad$ or $\quad e^a{}_{\hat{\mathbf{b}}} v_a \;\to\; v_{\hat{\mathbf{b}}}$.

`SeparateBasis`: $\quad v^a \;\to\; e^a{}_{\hat{\mathbf{b}}} v^{\hat{\mathbf{b}}} \quad$ or $\quad v^{\hat{\mathbf{a}}} \;\to\; e^{\hat{\mathbf{a}}}{}_{\bar{\mathbf{b}}} v^{\bar{\mathbf{b}}}$.

These operations essentially relate abstract indices and basis indices. Separation of basis allows us to change a dummy pair by a dummy pair of any other type. This cannot be safely implemented as index replacement because for example next expression can be properly expanded (equality) but the dummy pair cannot be simply replaced (unequality).

$$v^b\,\nabla_a v_b = \left(v^{\hat{\mathbf{c}}}\, e^b{}_{\hat{\mathbf{c}}}\right)\,\nabla_a\left[e_b{}^{\hat{\mathbf{d}}} v_{\hat{\mathbf{d}}}\right] \neq v^{\hat{\mathbf{b}}}\,\nabla_a v_{\hat{\mathbf{b}}}, \tag{151}$$

Then we need operations expanding the basis indices into their correspoding cnumber ranges. Again we need two commands:

`TraceBasisDummy`: $\quad v^{\hat{\mathbf{a}}} w_{\hat{\mathbf{a}}} \;\to\; v^{\hat{\mathbf{1}}} w_{\hat{\mathbf{1}}} + v^{\hat{\mathbf{2}}} w_{\hat{\mathbf{2}}}$.

`TableOfComponents`: $\quad v^{\hat{\mathbf{a}}} w_{\hat{\mathbf{b}}} \;\to\; \{\{v^{\hat{\mathbf{1}}} w_{\hat{\mathbf{1}}}, v^{\hat{\mathbf{1}}} w_{\hat{\mathbf{2}}}\}, \{v^{\hat{\mathbf{2}}} w_{\hat{\mathbf{1}}}, v^{\hat{\mathbf{2}}} w_{\hat{\mathbf{2}}}\}\}$

These functions relate basis indices and component indices, both when the basis indices are dummies or free indices. Tracing can be safely done by index replacing, and listing components of expressions with free basis indices, but listing components with free abstract indices must also be done by multiplying by basis vectors. It is possible to get a single component by using the function `Component`.

F. Parallelism with respect to a connection

Given any connection ∇_a we say that a vector field v^b is parallel with respect to that connection iff $\nabla_a v^b = 0$. It is known that (see Bishop & Goldberg 1968, pp. 236–237) there exists a basis of parallel vector fields $e^b{}_{\bar{\mathbf{c}}}$ with respect to a given connection $\bar{\partial}_a$ if and only if that connection has zero curvature (though torsion is still allowed). For those parallel vectors w.r.t. $\bar{\partial}_a$ we have then

$$\bar{\partial}_a e^b{}_{\bar{\mathbf{c}}} = 0 \quad \Rightarrow \quad s_T \bar{T}^a{}_{\bar{\mathbf{d}}\bar{\mathbf{f}}} = -\left[e^s{}_{\bar{\mathbf{d}}}, e^s{}_{\bar{\mathbf{f}}}\right]^a, \tag{152}$$

which means that the basis will be coordinated if and only if the torsion of the connection $\bar{\partial}_a$ vanishes. The Jacobi identity implies for a flat connection

$$\bar{\partial}_{[a} \bar{T}^d{}_{bc]} + s_T \bar{T}^d{}_{e[a} \bar{T}^e{}_{bc]} = 0, \tag{153}$$

(the first Bianchi identity).

Therefore, given a basis of vector fields it is always possible to construct its (flat) parallel derivative, defined up to a multiplicative constant, which will be torsionless iff the vectors are in involution. Conversely, given a flat connection it is possible to construct its parallel basis, defined up to rigid rotations, which will be coordinated iff it is also torsionless. We conclude that there is an identification between flat connections and bases of vector fields, and in particular between ordinary derivatives and coordinated bases, up to rigid linear transformations. These rigid linear transformations are removed in practice by specifying some geometric structures in terms of others through explicit formulas.

Suppose we have two different bases denoted with bar and hat: $e^a{}_{\bar{\mathbf{b}}}$ and $e^a{}_{\hat{\mathbf{b}}}$, related by

$$e^a{}_{\bar{\mathbf{b}}} = e^a{}_{\hat{\mathbf{c}}}\, e^{\hat{\mathbf{c}}}{}_{\bar{\mathbf{b}}}, \tag{154}$$

where $e^{\hat{\mathbf{c}}}{}_{\bar{\mathbf{b}}}$ is the matrix of change of basis. The respective parallel derivatives $\bar{\partial}_a$ and $\hat{\partial}_a$ obey

$$\bar{\partial}_a e^b{}_{\bar{\mathbf{c}}} = 0, \qquad \hat{\partial}_a e^b{}_{\hat{\mathbf{c}}} = 0, \tag{155}$$

and we can then change from one to the other as

$$\bar{\partial}_a e^b{}_{\hat{\mathbf{c}}} = \Gamma[\bar{\partial}, \hat{\partial}]^b{}_{ad} e^d{}_{\hat{\mathbf{c}}} = \Gamma[\bar{\partial}, \hat{\partial}]^b{}_{a\hat{\mathbf{c}}}. \tag{156}$$

Equivalently,

$$\bar{\partial}_a e_b{}^{\hat{\mathbf{c}}} = -\Gamma[\bar{\partial}, \hat{\partial}]^d{}_{ab} e_d{}^{\hat{\mathbf{c}}} = -\Gamma[\bar{\partial}, \hat{\partial}]^{\hat{\mathbf{c}}}{}_{ab}. \tag{157}$$

Combining both we see that

$$\nabla_b\, e^{\hat{\mathbf{a}}}{}_{\bar{\mathbf{c}}} = \Gamma[\hat{\partial}, \bar{\partial}]^{\hat{\mathbf{a}}}{}_{b\bar{\mathbf{c}}}, \tag{158}$$

with any derivative ∇_a (because the change of basis matrix is formed by scalar fields). This expression shows clearly what is the freedom we have among the set of parallel derivatives: they are all related by Christoffel tensors which are gradients of arbitrary matrices of scalar fields. Under a change of derivative the associated change in torsion is [cf. equation (39)]:

$$s_T(\hat{T}^{\hat{\mathbf{a}}}{}_{bc} - \bar{T}^{\hat{\mathbf{a}}}{}_{bc}) = \bar{\partial}_b\, e^{\hat{\mathbf{a}}}{}_c - \bar{\partial}_c\, e^{\hat{\mathbf{a}}}{}_b. \tag{159}$$

If we have $e^{\hat{\mathbf{a}}}{}_c = \nabla_c\, x^{\hat{\mathbf{a}}}$ for some scalar fields $x^{\hat{\mathbf{a}}}$ (so that the basis is coordinated) then the previous formula consistently implies that $\hat{T}^a{}_{bc} = 0$, but says nothing about the torsion of $\bar{\partial}_a$, which is still free.

G. Metrics and Cartesian bases

Given a metric g_{ab} and a general basis $\hat{B}$ of vector fields $e^c{}_{\hat{\mathbf{d}}}$ we can construct the metric components

$$g_{\hat{\mathbf{a}}\hat{\mathbf{b}}} = g_{cd}\, e^c{}_{\hat{\mathbf{a}}}\, e^d{}_{\hat{\mathbf{b}}}, \tag{160}$$

which are scalars fields, and hence can be given numerical values: the basis can be defined to be orthogonal, normal or orthonormal with respect to that metric in the obvious way. We will say the pair $\{g, \hat{B}\}$ is *Cartesian* if the components of g_{ab} in the basis $\hat{B}$ are constant scalars.

Given a metric there are many Cartesian bases for it, related by arbitrary (point dependent) rotation fields and/or a rigid (constant) dilatation. Each basis has its own parallel derivative, and therefore a Cartesian pair $\{g, \hat{B}\}$ has two preferred derivatives in general: the Levi-Civita connection of g_{ab} has curvature but not torsion; the parallel derivative $\hat{\partial}_a$ of $\hat{B}$ has torsion but not curvature; both are associated to the metric:

$$\nabla_a g_{bc} = 0, \qquad \hat{\partial}_a g_{bc} = 0. \tag{161}$$

The latter is clear from the expression $g_{ab} = g_{\hat{\mathbf{c}}\hat{\mathbf{d}}} e^{\hat{\mathbf{c}}}{}_a e^{\hat{\mathbf{d}}}{}_b$. There are many derivatives giving zero on a metric, but most of them have both curvature and torsion. The condition of having no torsion selects uniquely the Levi-Civita connection ∇_a; the condition of having no curvature selects the parallel derivative $\hat{\partial}_a$ of one of the Cartesian bases.

Using equation (70) we see that the Christoffel between both derivatives is fully given by the torsion tensor $\hat{T}$ of the parallel derivative $\hat{\partial}_a$:

$$\Gamma[\nabla, \hat{\partial}]_{cab} = \frac{s_T}{2}(\hat{T}_{abc} + \hat{T}_{bac} - \hat{T}_{cab}), \qquad \Gamma[\nabla, \hat{\partial}]_{cab} - \Gamma[\nabla, \hat{\partial}]_{cba} = -s_T \hat{T}_{cab}, \tag{162}$$

where we have used the metric g_{ab} to lower all upper indices. Note that this Christoffel tensor is antisymmetric in its first and third indices (to keep the metric compatibility condition), but has no symmetry in the second and third.

It is impossible in general to get a torsionless parallel derivative $\hat{\partial}_a$ by change of Cartesian basis, unless the metric is flat. If the metric is flat then we can take $\hat{\partial}_a = \nabla_a$, which becomes an ordinary derivative. Its parallel basis is automatically Cartesian and coordinated, and defines a “Cartesian” chart.

A cautionary word: given any basis of vector fields $e^a{}_{\hat{\mathbf{b}}}$ and its parallel flat derivative $\hat{\partial}_c$ we can construct a metric $g_{ab} = g_{\hat{\mathbf{c}}\hat{\mathbf{d}}}\, e_a{}^{\hat{\mathbf{c}}}\, e_b{}^{\hat{\mathbf{d}}}$ where we declare the numbers $g_{\hat{\mathbf{c}}\hat{\mathbf{d}}}$ to be constants in such a way that $\hat{\partial}_c g_{ab} = 0$. The derivative will have torsion in general, but we know it is possible to construct the Levi-Civita connection of the metric, without torsion. Whether the metric is flat or not depends on the curvature of the Levi-Civita, in spite of $\hat{\partial}_a$ being flat and associated to the metric.

Note that from the point of view of General Relativity, it might seem confusing to talk about two different flat metrics. This is because in GR we typically start with a metric, expressed in some chart or frame, and we extract both the identity of the metric and the identity of the frame from it, and hence two Cartesian metrics are considered to be the same. However, if we give a non-constant relation between both frames, then it is clear that the metrics are different. For instance, using standard notation, we might ask whether these two metrics are locally the same:

$$ds_1^2 = dx^2 + dy^2, \qquad\qquad ds_2^2 = dr^2 + d\theta^2. \tag{163}$$

We can say that r, θ are just a renaming of x, y. But if we state that $x = r\cos\theta$ and $y = r\sin\theta$ then

$$ds_1^2 = dr^2 + r^2 d\theta^2, \qquad\qquad ds_2^2 = dr^2 + d\theta^2 \tag{164}$$

and both metrics are clearly different flat metrics, though of course isometric (there is a coordinate change taking one to the other).

H. The `eta` tensors and densities

1. Definitions

Given any basis $\hat{B}$ of n vectors (or vector fields), we can always construct the totally antisymmetric parts of the product of all vectors and all covectors in the basis:

$$\tilde{\hat{\eta}}^{a_1 \ldots a_n} \equiv n!\, e^{[a_1}{}_{\hat{\mathbf{1}}} \ldots e^{a_n]}{}_{\hat{\mathbf{n}}}, \qquad\qquad \underset{\sim}{\hat{\eta}}_{a_1 \ldots a_n} \equiv n!\, e_{[a_1}{}^{\hat{\mathbf{1}}} \ldots e_{a_n]}{}^{\hat{\mathbf{n}}}. \tag{165}$$

The hat represents association to the basis $\hat{B}$; the boldface represents basis-indices or component-indices; the tildes will be explained below, and at this point they only serve the purpose of distinguishing between the two η tensors. Each basis has a different pair of η tensors, but they are all proportional, with "Jacobians" as proportionality factors:

$$\tilde{\hat{\eta}}^{a_1 \ldots a_n} = \det(e_{\hat{\mathbf{a}}}{}^{\bar{\mathbf{b}}})\, \tilde{\bar{\eta}}^{a_1 \ldots a_n}, \qquad \underset{\sim}{\hat{\eta}}_{a_1 \ldots a_n} = \det(e_{\bar{\mathbf{a}}}{}^{\hat{\mathbf{b}}})\, \underset{\sim}{\bar{\eta}}_{a_1 \ldots a_n}, \tag{166}$$

for some other $\bar{B}$ basis. We shall denote the Jacobians as

$$J(\hat{B}, \bar{B}) \equiv \det(e_{\hat{\mathbf{a}}}{}^{\bar{\mathbf{b}}}), \qquad J(\bar{B}, \hat{B}) \equiv \det(e_{\bar{\mathbf{a}}}{}^{\hat{\mathbf{b}}}) = \frac{1}{J(\hat{B}, \bar{B})}. \tag{167}$$

2. *Relations and numerical values*

Both η tensors for a given basis are unrelated unless we introduce a metric (as we will see below), but obey

$$\tilde{\hat{\eta}}^{a_1 \ldots a_n}\, \underset{\sim}{\hat{\eta}}_{b_1 \ldots b_n} = \delta^{a_1 \ldots a_n}{}_{b_1 \ldots b_n}, \tag{168}$$

and in particular this implies the normalization

$$\tilde{\hat{\eta}}^{a_1 \ldots a_n}\, \underset{\sim}{\hat{\eta}}_{a_1 \ldots a_n} = n!. \tag{169}$$

When we compute their components in the associated basis we have

$$\tilde{\hat{\eta}}^{\hat{\mathbf{1}} \ldots \hat{\mathbf{n}}} = 1, \qquad \underset{\sim}{\hat{\eta}}_{\hat{\mathbf{1}} \ldots \hat{\mathbf{n}}} = 1, \tag{170}$$

and in general

$$\tilde{\hat{\eta}}^{\hat{\mathbf{a}}_1 \ldots \hat{\mathbf{a}}_n} = \underset{\sim}{\hat{\eta}}_{\hat{\mathbf{a}}_1 \ldots \hat{\mathbf{a}}_n} = \mathrm{signature}(\mathbf{a}_1, \ldots, \mathbf{a}_n). \tag{171}$$

This numerical coincidence only happens in the basis $\hat{B}$ associated to the η tensors. Of course, the signature numerical function does not define a tensor whatsoever in a general vbundle, though it is sometimes called a "relative tensor".

3. *Determinants*

The η tensors allow us to define the determinant of a tensor in a given basis $\hat{B}$:

$$\det{}_{\hat{B}}(T_{a \ldots k}{}^{l \ldots p}) \equiv \frac{1}{n!}\, \tilde{\hat{\eta}}^{a_1 \ldots a_n} \ldots \tilde{\hat{\eta}}^{k_1 \ldots k_n}\, \underset{\sim}{\hat{\eta}}_{l_1 \ldots l_n} \ldots \underset{\sim}{\hat{\eta}}_{p_1 \ldots p_n} T_{a_1 \ldots k_1}{}^{l_1 \ldots p_1} \ldots T_{a_n \ldots k_n}{}^{l_n \ldots p_n}. \tag{172}$$

The number of η tensors is the number of indices of T and the number of T tensors is the number of indices of η. Exchanging two of the T tensors cannot change the result, but this is equivalent to an exchange of a pair of indices per η tensor. Therefore, if T has an odd number of indices then its determinant is zero. The dividing factor $n!$ is introduced to ensure that the determinant is a linear combination of components of T all with coefficient 1.

From this point of view the fact that the determinant of a tensor depends on a basis comes from the dependence of the η tensors on the basis. If we convert the contracted abstract indices into sums in the basis $\hat{B}$ then the η tensors become signature functions and the dependence on the basis is transferred to the components of the tensors (the usual interpretation of the determinant of a tensor). Under certain circumstances, for example when there are equal numbers of upper indices and lower indices in T, the determinant of the tensor does not depend on the basis because the η tensors can be paired into generalized deltas.

4. *The eta tensors and a metric*

Let us now suppose that we introduce a *regular* metric g_{ab}, with any signature. Its determinant in the basis $\hat{B}$ is

$$\hat{g} \equiv \det{}_{\hat{B}}(g_{ab}) \equiv \frac{1}{n!}\, \tilde{\hat{\eta}}^{a_1 \ldots a_n}\, \tilde{\hat{\eta}}^{b_1 \ldots b_n} g_{a_1 b_1} \ldots g_{a_n b_n} = \frac{1}{n!}\, \tilde{\hat{\eta}}^{a_1 \ldots a_n}\, \tilde{\hat{\eta}}_{a_1 \ldots a_n}. \tag{173}$$

Of course, $\det_{\hat{B}}(g^{ab}) = 1/\hat{g}$. Comparing with (169) we see that both eta tensors become proportional to each other (after moving all indices of one of them with the metric):

$$\tilde{\hat{\eta}}_{a_1\ldots a_n} = \hat{g}\, \underset{\sim}{\hat{\eta}}_{a_1\ldots a_n}, \qquad \underset{\sim}{\hat{\eta}}^{a_1\ldots a_n} = \frac{1}{\hat{g}}\, \tilde{\hat{\eta}}^{a_1\ldots a_n}. \tag{174}$$

Under a change of basis from $\hat{B}$ to $\bar{B}$ the determinant of the metric changes as follows:

$$\hat{g} = \det_{\hat{B}}(g_{ab}) = \det(g_{\hat{\mathbf{a}}\hat{\mathbf{b}}}) = \det(e_{\hat{\mathbf{a}}}{}^{\bar{\mathbf{c}}}\, e_{\hat{\mathbf{b}}}{}^{\bar{\mathbf{d}}}\, g_{\bar{\mathbf{c}}\bar{\mathbf{d}}}) = \det(e_{\hat{\mathbf{a}}}{}^{\bar{\mathbf{c}}})^2\, \bar{g}. \tag{175}$$

For further generality, let us suppose that the change of basis could be complex, and define $\hat{\sigma}_g$ to be a phase such that $\hat{\sigma}_g \hat{g}$ is *positive*. We conclude that the following object is an almost basis-independent tensor:

$$\epsilon_{a_1\ldots a_n} \equiv \sqrt{\hat{\sigma}_g \hat{g}}\; \underset{\sim}{\hat{\eta}}_{a_1\ldots a_n} = \frac{\hat{\sigma}_g}{\sqrt{\hat{\sigma}_g \hat{g}}} \tilde{\hat{\eta}}_{a_1\ldots a_n}. \tag{176}$$

We say "almost" because there is still an orientation problem. It comes from the fact that the square root in the previous expression cannot detect a change in the orientation of the basis, or in other words, next formula only contains the absolute value of the Jacobian:

$$\sqrt{\hat{\sigma}_g \hat{g}} = \sqrt{\bar{\sigma}_g \bar{g}}\, \left|\det(e_{\hat{\mathbf{a}}}{}^{\bar{\mathbf{c}}})\right| . \tag{177}$$

Let us perform a polar decomposition of the (possibly complex) Jacobian:

$$\det(e_{\hat{\mathbf{a}}}{}^{\bar{\mathbf{c}}}) \equiv \varphi\, J, \qquad J > 0, \quad |\varphi| = 1. \tag{178}$$

Then we have the transformation formulas

$$\hat{g} = \varphi^2 J^2 \bar{g}, \qquad \hat{\sigma}_g = \varphi^{-2} \bar{\sigma}_g, \qquad \hat{\epsilon}_{a_1\ldots a_n} = \varphi^{-1} \bar{\epsilon}_{a_1\ldots a_n}. \tag{179}$$

We see that under an orientation-reversal change of basis ($\varphi = -1$) the determinant $\hat{g}$ and the phase σ_g do not change sign, but the ϵ tensor does change sign. It is hence a "pseudo-tensor", and not a true tensor.

Implementation notes:

- It is not clear how to implement this orientation problem in *xTensor* and *xCoba*, or how does it relate to the choice of conventions for the η and ϵ objects. In principle we could have respective global variables `$etaSign` and `$epsilonSign` to control the signs of $\tilde{\hat{\eta}}_{\hat{\mathbf{1}}\ldots\hat{\mathbf{n}}}$ and $\hat{\epsilon}_{\hat{\mathbf{1}}\ldots\hat{\mathbf{n}}}$. (Perhaps the former should actually always be declared as +1, given the formula (170).) But then it seems we might also need something like a sign `epsilonToetaSign[metric, basis]` depending on our choices of `metric` and `basis`. The problem is even more complicated by the fact that we should worry about phases, and not just signs, because changes of bases can be complex.

Old: The epsilon tensor is associated only to the metric and defined uniquely by it except for a global constant (as shown below). The sign of that constant (the orientation) can be fixed by convention: in *xCoba* the sign of $\epsilon_{\hat{\mathbf{1}}\ldots\hat{\mathbf{n}}}$ is `$epsilonSign`, which defaults to +1.

5. Densities

We have found several tensors whose very definition depends on a basis in a special way: under a change of that basis the tensor changes with a power of the Jacobian of the transformation. These tensors are called "densities" and that power is referred to as the "weight" of the density. For example, the transformation laws (166) identify $\tilde{\hat{\eta}}$ as a density of weigth +1 and $\underset{\sim}{\hat{\eta}}$ as a density of weight -1. The determinant of the metric $\hat{g}$ is a density of weight +2 [cf. (175)], and the weight of a general determinant (172) is given by the number of upper indices minus the number of lower indices of T.

It is usually said that densities are not tensors because they have a different transformation law. I think this is misleading because they are true tensors, though depending on a basis. They transform as tensors if you do not change the basis, but they transform with Jacobians when you change the basis dependence. The idea is completely parallel to that of Christoffel transformation.

The concept of weight as an integer is too limited. This is clear for instance if you try to compute the weight of a Jacobian $J(\hat{B}, \bar{B})$. It is a density of weight $+1$ when we change the basis $\hat{B}$ and is a density of weight -1 when we change the basis $\bar{B}$. The natural choice is assigning weight $\hat{B} - \bar{B}$ to this Jacobian. In general a weight will be a formal linear combination of the names of the bases, with integer coefficients, such that now we cover the most general case. In the reduced case of working with a single basis B the weight nB can be understood as the traditional weight n.

Following Ashtekar, we denote the weight of a density with tildes. That shows that $\tilde{\hat{\eta}}$ has weight $+\hat{B}$ and that $\underset{\sim}{\hat{\eta}}$ has weight $-\hat{B}$. The determinant $\hat{g}$ of the metric should be denoted as $\tilde{\tilde{\hat{g}}}$ and Jacobians should have one overtilde and one undertilde. In *xTensor* we follow this convention and actually color the tildes according to the bases they are associated to.

6. *Derivatives*

For any derivation D (it could be a covariant derivative ∇_c or a Lie derivativ $\mathcal{L}_v$ or a parametric derivative ∂_t) we have

$$D\underset{\sim}{\hat{\eta}}_{a_1 \dots a_n} = (e^a{}_{\hat{\mathbf{b}}} De_a{}^{\hat{\mathbf{b}}})\, \underset{\sim}{\hat{\eta}}_{a_1 \dots a_n}, \qquad D\tilde{\hat{\eta}}^{a_1 \dots a_n} = (e_a{}^{\hat{\mathbf{b}}} De^a{}_{\hat{\mathbf{b}}})\, \tilde{\hat{\eta}}^{a_1 \dots a_n}. \tag{180}$$

This formula allows us to compute derivatives of densities. Note first that from (168) we can get the useful identity

$$\tilde{\hat{\eta}}^{aa_2 \dots a_n}\, \tilde{\hat{\eta}}_{ba_2 \dots a_n} = (n-1)!\, \hat{g}\, \delta^a{}_b, \tag{181}$$

which implies

$$\hat{\partial}_a \hat{g} = \hat{g}\; g^{bc} \hat{\partial}_a g_{bc} = 2\hat{g}\; \Gamma[\nabla, \hat{\partial}]^b{}_{ab}, \tag{182}$$

where $\hat{\partial}$ is the parallel derivative of the basis $\hat{B}$ and ∇_a is the Levi-Civita connection of the metric g_{ab}. Actually, because $\hat{g}$ is a scalar (a true scalar!), any covariant derivative acting on $\hat{g}$ gives this result, with a Christoffel always relating the derivatives ∇_a and $\hat{\partial}_a$, because the metric g_{ab} and the basis $\hat{B}$ are the two geometrical structures defining the scalar $\hat{g}$. (Compare this with the absurd "definition" $\hat{g}_{;\rho} = 0$ given by MTW in pages 501, 502.) On the other hand, from the definition of $\tilde{\hat{\eta}}$ above we have

$$\hat{\partial}_a \underset{\sim}{\hat{\eta}}_{a_1 \dots a_n} = 0, \qquad \nabla_a \underset{\sim}{\hat{\eta}}_{a_1 \dots a_n} = -\Gamma[\nabla, \hat{\partial}]^b{}_{ab}\, \underset{\sim}{\hat{\eta}}_{a_1 \dots a_n}. \tag{183}$$

Therefore we conclude that

$$\nabla_a\, \epsilon_{a_1 \dots a_n} = 0, \tag{184}$$

which explains the mentioned association of ϵ to the metric. Note that

$$\nabla_a \epsilon_{\hat{\mathbf{a}}_1 \dots \hat{\mathbf{a}}_n} = \left[\nabla_a \sqrt{|\hat{g}|}\right]\; \underset{\sim}{\hat{\eta}}_{\hat{\mathbf{a}}_1 \dots \hat{\mathbf{a}}_n} = \Gamma[\nabla, \hat{\partial}]^b{}_{ab}\; \epsilon_{\hat{\mathbf{a}}_1 \dots \hat{\mathbf{a}}_n}. \tag{185}$$

Compare again with the result $\epsilon_{0123;\rho} = 0$ of MTW. What they really mean is the component 0123 of $\epsilon_{abcd;\rho} = 0$.

Let us revisit again the results of MTW. Suppose we have a true tensor field T and we convert it into a density by multiplication by a power of the determinant of the metric in basis $\hat{B}$. Then we apply an arbitrary covariant derivative D_a:

$$D_a\left[(\hat{g}^{1/2})^n\, T\right] = (\hat{g}^{1/2})^n \left[D_a T + n\Gamma[\nabla, \hat{\partial}]^b{}_{ab} T\right] \tag{186}$$

Because most people perceive a Christoffel tensor as a non-tensorial object, they need to rearrange this result to get a "true tensor" formula, and what they do is modifying the covariant derivative D_a on densities. This is always done only for $D = \nabla$ and so I don't really know how the extension can be done in general, but I guess the extension would depend both in a choice of metric and in a choice of basis. Let us reduce ourselves to the case $D = \nabla$, so that the extension only depends on the basis $\hat{B}$. We define the new covariant derivative $\hat{\nabla}_a$ such that

$$\hat{\nabla}_a \mathcal{T} \equiv \nabla_a \mathcal{T} - n\left[\nabla_a \log(\hat{g}^{1/2})\right] \mathcal{T} = \nabla_a \mathcal{T} - n\Gamma[\nabla, \hat{\partial}]^b{}_{ab} \mathcal{T}, \tag{187}$$

where $\mathcal{T}$ is a tensor density of weight n in the basis $\hat{B}$. We shall say that $\hat{\nabla}_a$ is the 'extension of the Levi-Civita connection ∇_a along the basis $\hat{B}$'. This derivative has the nice property

$$\hat{\nabla}_a \left[(\hat{g}^{1/2})^n T\right] = (\hat{g}^{1/2})^n \hat{\nabla}_a T = (\hat{g}^{1/2})^n \nabla_a T, \tag{188}$$

or in other words,

$$\hat{\nabla}_a \hat{g} = 0. \tag{189}$$

However, given the determinant $\hat{g}$ of the metric g_{ab} in the basis $\hat{B}$ this only works if we are extending the Levi-Civita connection of g_{ab} and we extend it along the basis $\hat{B}$. What happens for example if you want to apply this derivative to a Jacobian, which is a density with respect to two different bases? What happens if you are using several covariant derivatives at the same time? This is a very restrictive trick, totally against the spirit of *xTensor*, and so it will not be implemented by default. It will be an option `WeightedWithBasis` in `DefCovD`, only accepted when defining a Levi-Civita connection.

There is the question of whether the connection (187) is a true connection or not. Equation (188) shows that it is a true connection on the family of densities defined as $\mathcal{T} = (\hat{g}^{1/2})^n T$ for a basis-independent tensor T and any constant n (which includes all basis-independent tensors as the case $n = 0$). I think in no other case $\hat{\nabla}$ is a good derivation. Finally, note that this "densitization" trick can be applied to any derivation D and any scalar f: define the new derivation $D^{(f)}\mathcal{T} \equiv D\mathcal{T} - n(D\log f)\mathcal{T}$ on objects $\mathcal{T} = f^n T$; it automatically obeys $D^{(f)} f = 0$.

As an example, let us compute the derivative of a Jacobian. Again, for any derivation D and starting from

$$J(\hat{B}, \bar{B}) = \det(e_{\hat{\mathbf{a}}}{}^{\bar{\mathbf{b}}}) = \frac{1}{n!} \tilde{\hat{\eta}}^{a_1 \ldots a_n} \underset{\sim}{\bar{\eta}}_{a_1 \ldots a_n}, \tag{190}$$

we get

$$DJ(\hat{B}, \bar{B}) = \left(e_a{}^{\hat{\mathbf{b}}} De^a{}_{\hat{\mathbf{b}}} + e^a{}_{\bar{\mathbf{b}}} De_a{}^{\bar{\mathbf{b}}} \right) J(\hat{B}, \bar{B}). \tag{191}$$

In particular, for any covariant derivative D_a (which could be ∇_a, but not $\hat{\nabla}_a$) we have

$$D_a J(\hat{B}, \bar{B}) = -\Gamma[\hat{\partial}, \bar{\partial}]^b{}_{ab}\, J(\hat{B}, \bar{B}), \tag{192}$$

with no presence of D_a on the right hand side, because the Jacobian is a scalar! An interesting observation from this formula: if the Jacobian is a constant (a generic rotation field and/or a constant dilatation) then the Christoffel tensor is traceless.

Implementation notes:

- The object $\tilde{\eta}$ is represented as `etaUp` and $\underset{\sim}{\eta}$ as `etaDown`, adding then the name of the basis they are associated to.
- The Jacobian must be implemented imitating Christoffels. The Jacobians depend on two bases, in much the same way as Christoffels depend on two connections. Now `Jacobian[basisA, basisB] = 1 / Jacobian[basisB, basisA]`.

IX. CHARTS

A. Definitions

Given a chart with coordinate scalar fields $x^{\hat{\mathbf{a}}}$ we can construct an associated coordinated (i.e. torsionless) basis of vector fields by duality from the basis of 1-forms

$$e_a{}^{\hat{\mathbf{b}}} = \nabla_a x^{\hat{\mathbf{b}}}, \tag{193}$$

for any connection ∇_a. There is of course an ordinary derivative $\hat{\partial}_a$ associated to that basis, such that

$$\hat{\partial}_a \hat{\partial}_b x^{\hat{\mathbf{c}}} = 0. \tag{194}$$

Each chart has an ordinary derivative $\hat\partial_a$, which can be considered to be the "partial derivative" with respect to the chart (altough charts whose bases are related by rigid linear transformations share the same ordinary derivative). The usual interpretation of partial derivative with respect to a coordinate field is finally encoded in

$$\frac{\partial}{\partial x^{\hat{\mathbf{a}}}} \equiv e^b{}_{\hat{\mathbf{a}}} \hat\partial_b = \hat\partial_{\hat{\mathbf{a}}}. \tag{195}$$

The notation $\hat\partial_{\hat{\mathbf{a}}}$ contains the chart in two different places, what might seem redundant, but is not. The marked subindex $\hat{\mathbf{a}}$ indicates the coordinate with respect to which we are differentiating; the superindex in $\hat\partial$ indicates the chart we use to differentiate (we need to "keep other coordinates constant"). In that sense the "d-by-dx" notation is slightly ambiguous and sometimes we find notations like $\frac{\partial}{\partial x}|_{y,z}$ to alleviate that. On the other hand, as we said, the object $\hat\partial_a$ is insentitive to rigid linear transformation of the associated basis, but not the projected object $\hat\partial_{\hat{\mathbf{a}}}$.

Implementation notes:

- We check that the numbers of cnumbers and scalars supplied coincide with the dimension of the manifold on which we define the chart.
- In *xTensor* there is no generic "d-by-dx" operator. Use the `D` (`Derivative`) operator of *Mathematica*, but only on scalar functions. It will never work on tensor fields.
- The basis associated to a chart has the same name as the chart.
- The ordinary derivative of a chart is named in *xTensor* as the "partial derivative" or PD of the chart and it is denoted as `PDchart`. Note that it is the parallel derivative of the basis associated to the chart, and hence there is no name incompatibility (actually I find this double meaning of `PD` really funny).

B. Position vector?

The notation $x^{\hat{\mathbf{a}}}$ suggests that the coordinate scalar fields are components of a vector field x^a. This is only partially true: there exists such a vector, but there is a different vector for each chart, so it is actually $\hat x^a$. We can define the following (chart-associated) vector field

$$\hat x^a = x^{\hat{\mathbf{b}}} e^a{}_{\hat{\mathbf{b}}}, \tag{196}$$

whose components in the basis of the chart are precisely the original coordinate fields:

$$x^{\hat{\mathbf{b}}} = \hat x^a e_a{}^{\hat{\mathbf{b}}}. \tag{197}$$

Therefore we have $\hat x^{\hat{\mathbf{a}}} = x^{\hat{\mathbf{a}}}$. What is false indeed is that this vector has the interpretation of a vector position. That will only happen if the basis is Cartesian with respect to a flat metric.

For the ordinary derivative $\hat\partial_a$ associated to the coordinate basis of the chart we have

$$\hat\partial_a \, \hat x^b = \delta_a{}^b. \tag{198}$$

For any other derivative ∇_a we have

$$\nabla_a \, \hat x^b = \delta_a{}^b + x^{\hat{\mathbf{c}}} \nabla_a e^b{}_{\hat{\mathbf{c}}} = \delta_a{}^b + \Gamma[\nabla, \hat\partial]^b{}_{ac} \hat x^c, \tag{199}$$

as expected for a vector field.

We stress again that each chart has its own "position vector field". Certainly there is not a vector field x^a, but there always are vector fields $\hat x^a$. The precise statement is that there is a vector field $\hat x^a$ for each ordinary derivative $\hat\partial_a$, defining each other through the relation (198).

Implementation notes:

- The default form for the "position vector" of a chart is `Xchart`. Its output is a colored `x`.
- There are several ways in which we can refer to the scalar fields of a chart. They must all be converted into some standard form: we choose convertion to the tensor names of the scalars, like `r[]` or `theta[]`. There is a generic command `Coordinate`, storing those names: `Coordinate[{1,polar}]:= r` or `Coordinate[{2,polar}]:=theta`. (Note that `Coordinate` is not a tensor, and plays the role of x.) Finally, the components of the position vector of the chart are also converted into those names: `Xpolar[{1,polar}]:= r[]` or `Xpolar[{2,polar}]:=theta[]`, where the convertion is only performed when polar labels match. (Note also that `Xpolar` is a tensor.)

C. Functions of coordinates

Most computations are done using a chart. The process has usually two steps: first, compute the components of all tensors using the basis associated to a chart, converting all tensors into collections of scalar fields. Second, express those scalar fields as functions of the coordinate fields. It is extremely important to understand that the scalar field F on the d-dimensional manifold M is conceptually different from the resulting function $\tilde{F}$ of the d coordinate fields. (For instance it makes no sense to ask about the number of arguments of F.) This is called "the first fundamental confusion of calculus" by Woodhouse. In *xTensor* we distiguish scalar fields from scalar functions (though a scalar function of some scalar fields is a scalar field).

In particular, we need to handle coordinate changes, in which a coordinate field is a function of some other coordinate fields.

X. EXTERIOR CALCULUS

A. Differential forms

Differential forms are antisymmetric covariant tensor fields, in exterior products of the cotangent bundle of a differentiable manifold M. As far as I know the construction cannot be made for vector bundles other than the cotangent bundle of M.

As all indices play an equivalent role, it is simple to use index-free notation for them, but that is not currently possible in *xTensor*. A p-form is simply any antisymmetric rank p covariant tensor, with all its indices:

$$\omega_{a_1 \ldots a_p} = \omega_{[a_1 \ldots a_p]}. \tag{200}$$

Scalar fields can be considered 0-forms.

The basic operation is the *exterior product*. CONVENTIONS. NOTATIONS.

A manifold M has an intrinsic notion of exterior differentiation, without the need of introducing a connection. Or in other words, we can construct an exterior derivative associated to any connection ∇_a, acting on any p-form $\omega_{b_1 \ldots b_p}$, following the definition

$$(d^{\nabla}\omega)_{ab_1 \ldots b_p} \equiv f_{p+1} \nabla_{[a}\omega_{b_1 \ldots b_p]}, \tag{201}$$

where f_p is a factor that depends on the convention chosen. Some authors take $f_p = 1$, but here we shall always take $f_p = p!$, and so the corresponding factorial factor coming from the antisymmetrization is cancelled. Note also the sign-convention of putting the derivative index at the beginning.

The curvature of the connection does not affect the result, only its torsion. In general we have

$$(d^{\nabla}\omega)_{ab_1 \ldots b_p} = (d\omega)_{ab_1 \ldots b_p} - p\, f_{p+1}\, s_T\, T[\nabla]^c{}_{[ab_1}\omega_{|c|b_2 \ldots b_p]}. \tag{202}$$

That justifies in particular why $d\omega$ can be computed with any ordinary derivative, and that is what we mean when we write $d\omega$.

B. Vector-valued differential forms

Given any other vector bundle E (which may be itself the tensor product of other bundles) over the same manifold M, we can construct *vector-valued differential forms*, say $\omega^A{}_{a_1 \ldots a_p}$. The space of such fields is called $\Omega^p(M, E)$. Now there is no intrinsic notion of exterior differentiation, and we need to select a connection on E:

$$\nabla : \Omega^0(M, E) \to \Omega^1(M, E), \tag{203}$$

which extends uniquely to the *exterior covariant derivative*

$$d^{\nabla} : \Omega^p(M, E) \to \Omega^{p+1}(M, E). \tag{204}$$

This d^{∇} is completely characterized by linearity:

$$d^{\nabla}(\omega \wedge \eta) = (d^{\nabla}\omega) \wedge \eta + (-1)^p \omega \wedge d\eta, \tag{205}$$

where ω is a E-valued p-form and η is an ordinary q-form.

An important subcase is that of a bundle E in which the fiber space is a Lie algebra.

C. Cartan "moving frames"

The name "moving frames" is inherited from the original application of the idea: the Frenet-Serret frame at the position of a moving particle along a curve. The key idea is that of choosing a vector frame per point and then describe its covariant derivatives in terms of the frame itself. If the frame is orthonormalized then those derivatives describe how the frame "rotates" when moving from point to point, and this motivates the names "Ricci rotation coefficients". This section describes these ideas using xTensor terminology.

Given a frame $e_{\bar{\mathbf{a}}}{}^b$ and its dual $e^{\bar{\mathbf{a}}}{}_b$ we define the following "Ricci rotation coefficients" of a connection ∇_a in that frame:

$$\omega^{\bar{\mathbf{a}}}{}_{b\bar{\mathbf{c}}} \equiv e^{\bar{\mathbf{a}}}{}_d \, \nabla_b e_{\bar{\mathbf{c}}}{}^d = \Gamma[\nabla, \bar{\partial}]^{\bar{\mathbf{a}}}{}_{b\bar{\mathbf{c}}}. \tag{206}$$

(In essence, this is just the Christoffel tensor relating the connection ∇_a with the parallel derivative of the frame.) Now we introduce the exterior derivative of the coframe, and compute it using any derivative ∂_a:

$$(d^\partial e^{\bar{\mathbf{c}}})_{ab} \equiv \partial_a e^{\bar{\mathbf{c}}}{}_b - \partial_b e^{\bar{\mathbf{c}}}{}_a = \Gamma[\partial, \bar{\partial}]^{\bar{\mathbf{c}}}{}_{ba} - \Gamma[\partial, \bar{\partial}]^{\bar{\mathbf{c}}}{}_{ab} = s_T (T[\bar{\partial}]^{\bar{\mathbf{c}}}{}_{ab} - T[\partial]^{\bar{\mathbf{c}}}{}_{ab}). \tag{207}$$

(This gives a quick way of computing the structure constants of the frame starting from an ordinary derivative ∂_a.) We define as well the following exterior product (this is just a complicated notation to get the antisymmetric part of the Christoffel above):

$$\left[\omega^{\bar{\mathbf{c}}}{}_{.\bar{\mathbf{d}}} \wedge e^{\bar{\mathbf{d}}}\right]_{ab} \equiv \omega^{\bar{\mathbf{c}}}{}_{a\bar{\mathbf{d}}} e^{\bar{\mathbf{d}}}{}_b - \omega^{\bar{\mathbf{c}}}{}_{b\bar{\mathbf{d}}} e^{\bar{\mathbf{d}}}{}_a = s_T (T[\nabla]^{\bar{\mathbf{c}}}{}_{ab} - T[\bar{\partial}]^{\bar{\mathbf{c}}}{}_{ab}). \tag{208}$$

Hence we arrive at the first structure equation, giving the torsion 2-forms of the derivative ∇_a in terms of exterior calculus with a frame (this equation is usually given for an ordinary ∂_a and hence the last torsion term disappears):

$$\left[d^\partial e^{\bar{\mathbf{c}}} + \omega^{\bar{\mathbf{c}}}{}_{.\bar{\mathbf{d}}} \wedge e^{\bar{\mathbf{d}}}\right]_{ab} = s_T (T[\nabla]^{\bar{\mathbf{c}}}{}_{ab} - T[\partial]^{\bar{\mathbf{c}}}{}_{ab}). \tag{209}$$

Interestingly, this last equation can be given almost entirely in terms of abstract indices:

$$\left[(d^\partial e^{\bar{\mathbf{d}}}) e_{\bar{\mathbf{d}}}{}^c + \omega^c{}_{.d} \wedge e^d\right]_{ab} = s_T (T[\nabla]^c{}_{ab} - T[\partial]^c{}_{ab}). \tag{210}$$

This raises the questions of what is the meaning of the abstract object e^d and whether $(d^\partial e^{\bar{\mathbf{d}}}) e_{\bar{\mathbf{d}}}{}^c$ can be replaced by de^c, even if only as a notational device. In my opinion the e^d is just a representation of the δ tensor as a vector-valued 1-form. It is needed there simply to produce an antisymmetrization in this mixed notation. The basis is still contained in ω, but is not present in the RHS of the equation, and hence it must be cancelled by the differential term. We conclude that the differential term *does* depend on the basis, and hence the notation de^c would be misleading.

Again, define

$$(d^\partial \omega^{\bar{\mathbf{c}}}{}_{.\bar{\mathbf{d}}})_{ab} \equiv \partial_a \omega^{\bar{\mathbf{c}}}{}_{b\bar{\mathbf{d}}} - \partial_b \omega^{\bar{\mathbf{c}}}{}_{a\bar{\mathbf{d}}}, \tag{211}$$

and

$$\left[\omega^{\bar{\mathbf{c}}}{}_{.\bar{\mathbf{e}}} \wedge \omega^{\bar{\mathbf{e}}}{}_{.\bar{\mathbf{d}}}\right]_{ab} \equiv \omega^{\bar{\mathbf{c}}}{}_{a\bar{\mathbf{e}}} \omega^{\bar{\mathbf{e}}}{}_{b\bar{\mathbf{d}}} - \omega^{\bar{\mathbf{c}}}{}_{b\bar{\mathbf{e}}} \omega^{\bar{\mathbf{e}}}{}_{a\bar{\mathbf{d}}}. \tag{212}$$

Finally, adding both terms and doing some algebra, we arrive at the second structure equation, giving the curvature 2-forms of the derivative ∇_a in terms of exterior calculus with a frame (again, this is usually presented for a torsionless ∂_a):

$$\left[d^\partial \omega^{\bar{\mathbf{c}}}{}_{.\bar{\mathbf{d}}} + \omega^{\bar{\mathbf{c}}}{}_{.\bar{\mathbf{e}}} \wedge \omega^{\bar{\mathbf{e}}}{}_{.\bar{\mathbf{d}}}\right]_{ab} = -s_R R[\nabla]_{ab\bar{\mathbf{d}}}{}^{\bar{\mathbf{c}}} - s_T T[\partial]^e{}_{ab} \omega^{\bar{\mathbf{c}}}{}_{e\bar{\mathbf{d}}}. \tag{213}$$

D. Exterior covariant derivative

Exterior differentiation is only defined on covariant slots. To extend its action to contravariant slots we follow the following route. From the theory of connections on fiber bundles, we introduce the concept of "covariant exterior derivative" D associated to the connection ∇_a and a given frame :

$$D^\nabla \equiv d + \omega \wedge, \tag{214}$$

where ω are the connection 1-forms associated to the derivative ∇_a and the frame. This derivative enters naturally in the Cartan structure equations, in terms of which we can write

$$(D^\nabla e^{\bar{\mathbf{c}}})_{ab} = s_T T[\nabla]^{\bar{\mathbf{c}}}{}_{ab}, \tag{215}$$

$$(D^\nabla \omega^{\bar{\mathbf{c}}}{}_{.\bar{\mathbf{d}}})_{ab} = -s_R R[\nabla]_{ab\bar{\mathbf{d}}}{}^{\bar{\mathbf{c}}}. \tag{216}$$

It has the property

$$(D^\nabla)^2 \phi = \Omega \wedge \phi, \tag{217}$$

on any form ϕ, where Ω are the curvature 2-forms of ∇_a.

With more generality, we can define, for arbitrary connections ∂_a and ∇_a,

$$D^{\partial,\nabla} \equiv d^\partial + \omega\wedge, \tag{218}$$

To understand what it means, let us compute:

$$(D^{\partial,\nabla} v^{\bar{\mathbf{c}}})_{ab} = \left[d^\partial v^{\bar{\mathbf{c}}} + \omega^{\bar{\mathbf{c}}}{}_{.\bar{\mathbf{d}}} \wedge v^{\bar{\mathbf{d}}} \right]_{ab} = (\nabla_{[a} v^d{}_{b]}) e^{\bar{\mathbf{c}}}{}_d + s_T (T[\nabla]^d{}_{ab} - T[\partial]^d{}_{ab}) v^{\bar{\mathbf{c}}}{}_d. \tag{219}$$

(For $v = \delta$ we get back the first structure equation.) We see that $D^{\partial,\nabla}$ acts like ∇, apart from torsion terms, on "vector-valued" forms for which the vector part is given in the appropriate basis.

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