Weyl Invariance in the gravitational sector

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A José Ramón y Trini

The pursuit of knowledge is hopeless and eternal...
Prof. Hubert J. Farnsworth, Futurama (2011)

The following papers, some of them unrelated to the content of this document, where published by the candidate while this thesis was being developed

- E. Alvarez and M. Herrero-Valea, “Unimodular gravity with external sources,” JCAP **1301**, 014 (2013) [arXiv:1209.6223 [hep-th]].
- E. Alvarez and M. Herrero-Valea, “No Conformal Anomaly in Unimodular Gravity,” Phys. Rev. D **87**, 084054 (2013) [arXiv:1301.5130 [hep-th]].
- E. Alvarez, M. Herrero-Valea and C. P. Martin, “Conformal and non Conformal Dilaton Gravity,” JHEP **1410**, 115 (2014) [arXiv:1404.0806 [hep-th]].
- E. Álvarez, S. González-Martin and M. Herrero-Valea, “Some Cosmological Consequences of Weyl Invariance,” JCAP **1503**, no. 03, 035 (2015) [arXiv:1501.07819 [hep-th]].
- E. Álvarez, S. González-Martín, M. Herrero-Valea and C. P. Martín, “Quantum Corrections to Unimodular Gravity,” JHEP **1508**, 078 (2015) [arXiv:1505.01995 [hep-th]].
- E. Álvarez, S. González-Martín, M. Herrero-Valea and C. P. Martín, “Unimodular Gravity Redux,” Phys. Rev. D **92**, no. 6, 061502 (2015) [arXiv:1505.00022 [hep-th]].
- A. O. Barvinsky, D. Blas, M. Herrero-Valea, S. M. Sibiryakov and C. F. Steinwachs, “Renormalization of Hořava gravity,” Phys. Rev. D **93**, no. 6, 064022 (2016) [arXiv:1512.02250 [hep-th]].
- M. Herrero-Valea, “Anomalous equivalence of cosmological frames,” arXiv:1602.06962 [hep-th].

Resumen Es sabido desde hace décadas que aquellas teorías de campos que poseen una invariancia de escala se comportan mejor en el ultravioleta. En este trabajo exploramos las consecuencias que implica esta simetría en presencia de gravedad dinámica, donde viene descrita en terminos de invariancia de Weyl, rescaleos locales de la métrica y de los campos que aparecen en el lagrangiano. Tras describir extensivamente algunas técnicas de uso común en el tratamiento de Gravedad Cuántica como una teoría de campos, tratamos de responder dos cuestiones principales. Primero, estudiamos la supuesta equivalencia entre el referencial de Einstein y el de Jordan en una teoría que denominamos Gravedad Dilatónica Conforme, en la que descubrimos que la equivalencia está rota por la presencia de una anomalía Weyl. Pese a que existe una regularización que resuelve este problema, no está claro que pueda ser implementada en un marco teórico consistente.

Más tarde, centramos nuestra atención en Gravedad Unimodular, donde la invariancia de Weyl se presenta como una forma de relajar el problema de la constante cosmológica, que pasa a estar definida en términos de una constante de integración cuyo valor está determinado por las condiciones iniciales del sistema. Gracias a la construcción de un sector de fantasmas adecuado, cuantizamos la teoría y calculamos las correcciones a un lazo a la acción efectiva. Nuestro resultado sugiere que la protección de la constante cosmológica se preserva en la presencia de correcciones cuánticas supuesto que no econtremos una anomalía Weyl.

Abstract It is been known since long ago that field theories enjoying scale invariance have an improved behaviour in the Ultraviolet. Here we explore the consequences of this symmetry in the presence of dynamical gravitation, where it is described in terms of Weyl invariance, local rescalings of the metric and the fields involved in the lagrangian. After thoroughly introducing some techniques of common use in the field theory approach to Quantum Gravity, we address two main questions. First, we study the equivalence premise between the Einstein and Jordan frame in a theory we dub Conformal Dilaton Gravity, where we find that equivalence is broken in the presence of a Weyl anomaly. Although a regularization that solves this issue exists, it is not clear that it can be implemented in a consistent framework.

Later, we move our attention to Unimodular Gravity, where Weyl invariance appears as a way to relax the problem of the cosmological constant, which is here given in terms of an integration constant, its value determined solely by initial conditions. By constructing a gauge fixing and ghost sector, we quantize the theory and we compute its one loop effective action. Our results suggest that the protection of the cosmological constant is preserved by the quantization process provided no anomaly arises in the Weyl symmetry.

Contents

Introduction

Sir Terry Pratchet wrote once[1] that *"Gravity is a habit that is hard to shake off"* and I cannot think of a better phrase to start this thesis. Although it was not the original intention of the author (he wrote them in a comedy book) these few words wisely represent what I understand as the common feeling of the theoretical physics community in the beginning of the XXI century.

The XX century was undoubtedly an epoch of great advances in Science and, in particular, it gave birth to the pillars of modern physics. On one hand, the General Theory of Relativity came to complete the old Theory of Universal Gravitation of Newton, upgrading the newtonian ghostly force to the concept of a curved space-time that was able to describe classical gravity with a huge precision. On the other hand, the first experiments with subatomic particles gave rise to a program that ended with the constitution of Quantum Field Theory, perhaps the most grandiose human construct ever made. However, these two pillars cannot be used together to hold the same building. General Relativity, when put into the QFT framework, happens to be non-renormalizable. The theory develops, through renormalization, an infinite number of couplings that must be determine by doing an equally infinite number of experiments, thus rendering the framework useless as a real physical theory.

The unpleasant part of this issue is that if the problem with gravity were not there, the Standard Model of Particle Physics, or some minimal extension of it, could be though as a complete theory of Nature, since it is renormalizable, asymptotically free and describes with great precision almost all the phenomena we observe. However, gravity exists and the requirement to describe it together with the other interactions of Nature poses what it is probably the biggest problem in the history of physics. A problem that, as Sir Pratchet described with his words, is *hard to shake off* and that has driven many of the most important advances in physics in the last century.

[1] Small Gods, Terry Pratchet

The seek for a theory of Quantum Gravity has produced some of the most unexpected discoveries in the history of science and almost everything has been tried to construct this ultimate theory. From assuming that perhaps the metric has to be quantized in a special framework, which leads to the asymptotic safety program or to Loop Quantum Gravity; to the substitution of the UV degrees of freedom of the theory, in a similar manner as what happens in QCD, which gave birth to String Theory; stopping by the modification of the symmetry group of the theory, whose main exponent is the theory known as *Hořava-Lifshitz gravity.* Many many things have been tried in the decades since Dirac, Feynman and DeWitt initiated the program of the quantization of the gravitational field and although some of the tries gave interesting insights, specially the AdS/CFT correspondence found in String Theory; we have learnt very few about our initial goal.

Nowadays, it is widely accepted that General Relativity cannot be a complete theory but instead it has to be though as an effective field theory, valid only up to a certain UV scale which in this case is the Planck scale, about 10^{19} GeV in natural units[2] However, this is still problematic, because the presence of the cosmological constant, the extra coupling that allows to account for the expansion of the Universe, breaks the standard lore of effective field theories. While we should expect all higher order operators, those which renormalize the theory, to be suppressed by powers of the UV cut-off, this is only true when $\Lambda = 0$ and the mere presence of the cosmological constant complicates the matter.

In this thesis we introduce a common hint towards the solution of both problems here discussed in the form of Weyl invariance, a local realization of the flat space concept of scale invariance. When interpreted as an internal symmetry in a Quantum Field Theory, it leads to the banishing of both the cosmological constant and the renormalization scale that accompanies quantum corrections. It is then natural to hope that a consistent theory of gravity with Weyl invariance will be free of the problems here referred and this is the main motivation for the work here summarized.

This thesis is organized as follows. In part I we will introduce some common techniques for Quantum Field Theory in the presence of gravitation: the background field method and the Schwinger-DeWitt technique, together with its generalization as introduced by Barvinsky and Vilkovisky.

[2]We define natural units by taking $c = \hbar = 1$.

The first chapters, though, will be devoted to formally introduce the problem of the non-renormalizability of Einstein-Hilbert action and Weyl invariance. Later in part II we will introduce a Weyl invariant model of gravity, which we dub *Conformal Dilaton Gravity*, and we will use it to study the UV structure of Weyl invariant theories, arriving to interesting conclusions about the role of regularizations in the presence of Weyl invariance. Finally, in III we will present *Unimodular Gravity* as a Weyl invariant theory of gravity which solves the cosmological constant problem, half of it at least, at all orders in the loop expansion. This is achieved by relaxing the role of Λ from a physical coupling in the action to an integration constant at the cost of having more complicated interactions. We will construct the path integral formulation for Unimodular Gravity and conclude that the theory is a valid effective field theory.

Part I

Preliminaries

1

Einstein Gravity, the Cosmological Constant and Effective Field Theories

General Relativity (GR)[1] is constructed from a modern point of view under the assumption of Diffeomorphism (Diff) invariance and Ockam razor[2]. That is, we assume that the space-time is a generically curved manifold $\mathcal{M}$ equipped with a metric $g_{\mu\nu}$ whose dynamics are governed by the simplest action invariant under invertible coordinate transformations

$$y^\mu = \Lambda^\mu{}_\nu x^\nu, \qquad \det(\Lambda) \neq 0 \tag{1.1}$$

These act infinitesimally on the metric by Lie dragging its components along a direction ξ^μ

$$g_{\mu\nu} \to \pounds_\xi g_{\mu\nu} \tag{1.2}$$

and demanding the action to be made of local invariants under these transformations fixes it to be a scalar polynomial of the Riemann tensor integrated with the right measure

$$S = \int d^4x \sqrt{|g|}\; P\left(R_{\mu\nu\alpha}{}^\beta, g_{\rho\sigma}\right) \tag{1.3}$$

where all the indices are contracted with the metric . The Einstein-Hilbert action of General Relativity then corresponds to the simplest term in this polynomial (we ignore here the constant term which will produce a cosmological constant to be care about later), the Ricci scalar R

$$S_{EH} = -M_p^2 \int d^4x \sqrt{|g|}\; R \tag{1.4}$$

where a dimensionful coupling M_p^2 must be introduced in order to keep the action dimensionless in natural units. M_p is the Planck mass and when the Einstein-Hilbert action is coupled to matter, it sets the scale at which gravitational interactions are relevant to describe the dynamics of the system. Its value is fixed by solar system physics to be $M_p = (16\pi G)^{-1/2} \sim 10^{19}$ GeV.

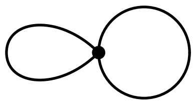

Figure 1.1: An arbitrary diagram with 2 loops, 2 propagators and one vertex. Its superficial degree of divergence is $D = 6$.

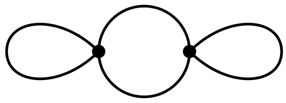

Figure 1.2: Same diagram as before but with an extra loop attached. It has now $D = 8$.

This is important because it is precisely the fact that this coupling is dimensionful what drives the non-renormalizability of the theory when its quantization, along the same lines as with Quantum Electrodynamics, is tried. Since $[M_p] = 1$ in mass dimensions and the propagator of the little perturbations about flat space $h_{\mu\nu} = g_{\mu\nu} - \eta_{\mu\nu}$ scales as $\frac{1}{p^2}$, where p^μ is the relativistic four-momentum, the superficial degree of divergence[1] of an arbitrary Feynman diagram with L loops, V vertices and P propagators, like the ones in figures 1.1 and 1.2, scales as

$$D = 4L + 2(V - P) \tag{1.5}$$

When this is supplemented with the topological identity $L = 1+P-V$, which states no more than the fact that from $P + 1$ original integrals over momenta we are removing V by momentum conservation in the vertices; we are left with the simplified formula

$$D = 2 + 2L \tag{1.6}$$

Therefore, the divergence of a diagram increases with the number of loops, yielding at the end of the day an infinite number of divergences which must be absorbed with an infinite numbers of counter-terms. We conclude that the theory is non-renormalizable.

Therefore, General Relativity, although it describes with great accuracy the solar system and large scale physics, requires a UV completion

[1]This is defined as the scaling power of the diagram under the rescaling $p^\mu \to bp^\mu$.

when quantum physics enter into game and can only be trusted as an *effective field theory* (EFT). In general, at a loop order L we will generate local counter-terms proportional to a power $L+1$ of the Riemann tensor, which are precisely the higher order terms contained in (1.3) that we did not take into account. These terms, by dimensionality, will be suppressed by higher powers of the Planck mass

$$S = -M_p^2 \int d^4x \sqrt{|g|} \left(R + \frac{1}{M_p^2} O\left(R^2\right) + \frac{1}{M_p^4} O\left(R^3\right) + ... \right) \tag{1.7}$$

so that for low energies compared to that scale

$$E << M_p \sim 10^{19} \text{ GeV} \tag{1.8}$$

they can be neglected, leading to the interpretation of the theory as an EFT. When the energy is comparable to this value, higher order corrections must be taken into account and low energy predictions must be modified accordingly. This produces, for instance, a correction to the Newton potential between two static sources that could lead to measurable effects[3]. Although in the particular case of the action (1.4) the one-loop corrections happen to vanish[4], this is not the case anymore at two-loops [5, 6] or in the presence of matter.

The Einstein-Hilbert action is enough to take into account for physical settings in which gravity is weak or, more generally, the curvature of spacetime is low. There, Minkowsky flat spacetime $\eta_{\mu\nu}$ is a solution of the vacuum Einstein equations coming from (1.4)

$$R_{\mu\nu} - \frac{1}{2} g_{\mu\nu} R = 0 \longrightarrow R_{\mu\nu} = 0 \tag{1.9}$$

and one can construct the usual perturbation theory about flat space as a quantum field theory of a spin 2 particle excitation[7]. However, this is not enough to describe, for instance, the physics of large scale cosmology. In 1998 it was confirmed[8, 9] that the average curvature of our Universe is not zero but instead it takes a extremely little and non-vanishing value. This implies that the corresponding metric description of the spacetime manifold is not a solution of Einstein equations any more (1.9) and that they must be supplemented with an extra piece, a cosmological constant Λ which amounts to the average curvature of the manifold

$$R_{\mu\nu} - \frac{1}{2} g_{\mu\nu} R + \Lambda g_{\mu\nu} = 0 \longrightarrow R_{\mu\nu} = \Lambda g_{\mu\nu} \tag{1.10}$$

Moreover, and by general field theory arguments, this term would have to be actually there from the beginning since it corresponds to the zeroth term in the polynomial (1.3) that we neglected before and that must correct Einstein-Hilbert action (1.4)

$$S_{EH\Lambda} = -M_p^2 \int d^4x \sqrt{|g|}\ (R - 2\Lambda) \tag{1.11}$$

In any case, if our Universe happened to be flat, there would be still the open problem to explain why Λ was vanishing. Intead, we face the problem of explaining a very little but non-vanishing value.

However, this is not the only problem introduced by the cosmological constant. The real issue is that its mere presence compromises the EFT picture of General Relativity. The basic assumption of an EFT, as explained before in this section, is that the action and different physical observables can be expanded as a power series in the inverse of a mass parameter which constrains the applicability regime of the theory. By introducing the cosmological constant Λ we are now introducing an infra-red parameter[2] that will jeopardize this assumption. In the presence of Λ, (quantum) corrections to the Einstein-Hilbert action (1.11) can depend both on M_p and Λ and even on a combination of both. As an example, the one-loop on-shell effective action corresponding to the action (1.11) was computed in [10] and read

$$\Gamma_{L=1} = -\frac{M_p^2}{16\pi^2} \log\left(\frac{\mu^2}{M^2}\right) \int d^4x \sqrt{|g|}\ \frac{\Lambda^2}{M_p^2} \tag{1.12}$$

which is clearly not given only in terms of the UV scale M_p. Here μ is the regularization ambiguity and the correction has been computed in the $\overline{MS}$ scheme[11], therefore keeping only the divergence. M denotes the renormalization scale.

Moreover, the origin of Λ as a dynamical coupling compromises the EFT picture even in the absence of gravitational quantum corrections if we include matter in the game. In order to see this, let us introduce a scalar field minimally coupled to gravity

$$S = S_{EH\Lambda} + \int d^4x\ \sqrt{|g|} \left(\frac{1}{2}\nabla_\mu\phi\nabla^\mu\phi - V(\phi)\right) \tag{1.13}$$

[2] It is the most relevant operator in the limit in which the four-momentum vanishes $p^\mu \to 0$ because it does not contain any derivative.

Figure 1.3: Some of the matter contributions to Λ at two and three-loops level in a non-abelian gauge theory.

Here we assume the potential $V(\phi)$ to have a non-trivial minimum $V(\bar{\phi})$ so that the vacuum state of the theory will be at $\phi = \bar{\phi}$. Expanding $V(\phi)$ about the minimum

$$V(\phi) = V(\bar{\phi}) + \frac{1}{2}\left.\frac{d^2V(\phi)}{d\phi^2}\right|_{\phi=\bar{\phi}} (\phi - \bar{\phi})^2 + ... \tag{1.14}$$

we find that the first term on this expansion (the *zero mode*), being a constant value, has exactly the form of a cosmological constant in the gravitational action, so that we can redefine

$$2M_p^2\Lambda' = 2M_p^2\Lambda + V(\bar{\phi}) \tag{1.15}$$

and get a contribution from the matter dynamics to the cosmological constant.

In general, any constant contribution to the action of a Quantum Field Theory, that we normally would dismiss as a non-dynamical contribution, will have a relevance when gravity is turned on, by shifting the value of the cosmological constant. This is what happens when we take into account the Higgs vacuum expectation value[12] of $\langle h \rangle = 246$ GeV or the strong coupling scale in Quantum Chromodynamics $\Lambda_{QCD} = 220$ MeV.

Finally, if we still believe in the EFT approach to gravity, then we must consider any other theory, in particular the Standard Model of Particle Physics, as an effective field theory with validity only up to the gravitational scale M_p, where gravity must start to be relevant. That means that when integrating quantum loops containing matter contributions to the cosmological constant, as seen in figure 1.3, we must cut-off the momentum integral in M_p and thus we expect the value of the cosmological constant induced by these to be close to $M_p^4 \sim 10^{76}$ GeV^4, which is very far from the measured value of $\Lambda \sim 10^{-47}$ GeV^4. This implies that when following Wilsonian ideas of renormalization, we must fine-tune the bare value contained in (1.11) with a huge precision in order to recover experimental results

$$\Lambda + (\text{matter loop corrections}) \sim 10^{-47} \text{ GeV}^4 \tag{1.16}$$

implying a *fine-tuning problem* which is even worse than the *hierarchy problem*[13] for the Higgs particle mass. We call this the *active cosmological constant problem.*

Summarizing, even if the non-completeness of General Relativity in the ultra-violet leads us to consider it as an effective field theory, valid up to a scale of $M_p \sim 10^{19}$ GeV, the presence of a non-vanishing but very little cosmological constant jeopardizes this picture and requires the introduction of additional physics in order to make things work properly.

2

Starring: Weyl Invariance

The simplest way to solve the *active cosmological constant problem* is to protect the value of Λ by introducing a new symmetry in the theory. One option for this is supersymmetry (SUSY)[14], consisting of extending Poincaré invariance by using fermionic generators which span a graded algebra. Its effect is to double the degrees of freedom of a given theory by producing new field partners with a spin difference of $\frac{1}{2}$ with respect to the original fields. Contributions to Λ coming from bosons and fermions carry different signs and thus unbroken SUSY implies that they cancel identically since there are the same number of degrees of freedom of each kind. However, we know that SUSY, if it exists, must be broken in our Universe and recent LHC data[15] constrain the scale of breaking to be above 1 TeV, what repercutes in the requirement of certain amount of fine-tuning (of the TeV order, because now momentum integrals run up to the breaking scale) thus compromising the solution given by SUSY.

Here we will consider instead *Weyl Invariance*[16], a local realization of the concept of scale invariance for Quantum Field Theories defined in flat space. The latter is defined by assigning to the space-time coordinates and to the fields a *scaling dimension* λ_ϕ, where ϕ is a generic field, equal to its *length dimension* under rescalings of the form

$$x^\mu \to \Omega, \qquad \phi \to \Omega^{\lambda_\phi}\phi \tag{2.1}$$

with a constant factor Ω.

The relevance of scale invariance is linked to renormalizability for theories in flat space. When an action is invariant under the scale transformations (2.1), it means, since scaling and length dimension are equivalents, that it does not contain any dimensionful coupling, which is precisely the requirement for power counting renormalizability. Indeed, the condition to break perturbatively the renormalizability of the theory, which is to run the coupling either to a Landau pole or to a region in which it grows monotonically to be greater than one; will produce a scale, breaking effectively scale invariance. Only if this symmetry is perturbatively restored, by falling into a gaussian fixed point, for instance, in which the beta function will vanish and therefore no scale is generated; we can say that the theory is really renormalizable. Moreover, the hint that it forbids dimensionful couplings signal that it may be of interest in order to solve the *active cosmological constant problem*. However, we could be spoiled if there is an anomaly, because global symmetries can be anomalous without harm since they are not required to vanish ghost degrees of freedom from the theory. It is then more interesting to study the case of *local* scale invariance, when we allow the transformation parameter to be a function of the space-time coordinate. This leads to *conformal field theories*, which are of particular interest in two dimensions, where the transformation group becomes infinite dimensional[17].

In the presence of gravity, the concept of coordinate is diluted, the metric is a dynamical field and a scaling dimension must be assigned to it. We do this by using diffeomorphism invariance and requiring that the proper distance between any two space-time points scales as $ds'^2 = \Omega^{-2} ds^2$. This implies

$$ds^2 = g_{\mu\nu} dx^\mu dx^\nu \longrightarrow g'_{\mu\nu} = \Omega^2 g_{\mu\nu} \tag{2.2}$$

What we are doing here is to translate the scaling dimension from the coordinates to the metric. Thus, properly talking, we are *defining* a new symmetry which can be identified with scale invariance in the flat space case but which will be in general different. We call this symmetry thus introduced *rigid Weyl invariance*, the rigid surname implying that it involves a transformation which is equal for any space-time point and for any observer. However, as we said before for scale invariance, global symmetries of this kind are generically broken by the renormalization dependence in quantum corrections[1] and thus it most interesting to consider the case of the local version of the symmetry, which we call from now just *Weyl invariance*.

[1] A recent and interesting phenomenological example of this can be found in [18].

2.1. Weyl Invariance, Noether Current and Ward Identities

We will focus the rest of this work in Weyl invariant theories, defined as Quantum Field Theories whose action functional satisfies the following assumptions

- The action $S(g_{\mu\nu}, \phi_i)$ is a functional of the metric and a collection of other fields which we generically denote as ϕ_i.
- The action $S(g_{\mu\nu}, \phi_i)$ is invariant under Weyl transformations of the form

$$g_{\mu\nu} \to \Omega(x)^2 g_{\mu\nu}, \qquad \phi_i \to \Omega(x)^{\lambda_{\phi_i}} \phi_i \tag{2.3}$$

  with no transformation of coordinates whatsoever.

Indeed, and as we advanced before, the introduction of Weyl invariance forbids the presence of a cosmological constant in the action. The reason is that the coupling

$$S_\Lambda = 2M_p^2 \int d^4x \sqrt{|g|}\, \Lambda \tag{2.4}$$

is not Weyl invariant. This leads to consider a wide range of modified theories of gravity[19] in order to try to explain the acceleration of the Universe in the absence of this term. In part III of this work we will consider *Unimodular Gravity* as one of those.

Moreover, as with scale symmetry in flat space, the vanishing of dimensionful constants from the action could give a hint on how to improve the renormalizability of Weyl invariant theories of gravity. We will examine this issue in Part II, conjecturing that for purely Weyl invariant backgrounds, a finite theory might be achieved.

As with any other symmetry, the presence of Weyl invariance implies the existence of a constraint over a physical quantity of the theory. Under an infinitesimal transformation $\Omega = 1 + \omega + O(\omega^2)$, a Weyl invariant action behaves as[2]

$$\delta S(g_{\mu\nu}, \phi_i) = \delta \left[\int d^4x\, \mathcal{L} \right] = \int d^4x \left(2g_{\mu\nu} \frac{\delta \mathcal{L}}{\delta g_{\mu\nu}} + \sum_i \lambda_{\phi_i} \phi_i \frac{\delta \mathcal{L}}{\delta \phi_i} \right) \omega \tag{2.5}$$

[2]Note that we are defining the lagrangian density including the factor of $\sqrt{|g|}$.

So that for an arbitrary variation ω, Weyl invariance implies that

$$2g_{\mu\nu}\frac{\delta\mathcal{L}}{\delta g_{\mu\nu}} + \sum_i \lambda_{\phi_i}\phi_i\frac{\delta\mathcal{L}}{\delta\phi_i} = 0 \tag{2.6}$$

If gravity is not dynamical, that is, if we consider a theory of the fields ϕ_i living in a fixed background geometry and whose dynamics do not backreact onto it, this statement would imply the known fact that for local scale invariant theories, the trace of the energy momentum tensor

$$\sqrt{|g|}\; g_{\mu\nu}T^{\mu\nu} = 2g_{\mu\nu}\frac{\delta\mathcal{L}}{\delta g_{\mu\nu}} \tag{2.7}$$

vanishes on the mass-shell[3]. However, for theories with dynamical gravity, this is not true anymore. What Weyl invariance states instead is that the ponderated trace (2.6) of the equations of motion is vanishing. This is a subtle but important difference because, since now all the terms in the constraint are dynamical and proportional to the equations of motion for the propagating fields in the action, this does not give any new information for on-shell quantities. This gives the hint that probably one is not allowed to decouple gravitational dynamics from the rest of fields in the theory when Weyl invariance is introduced. At the end of the day, if the theory is indeed Weyl invariant, there is no physical scale at all that would stablish the separation between the dynamical and non-dynamical regimens. Only when the symmetry is broken by the introduction of a scale, one can stablish this difference.

The case of a quantum theory is even more interesting. Let us start by introducing the path integral formulation for the action $S(g_{\mu\nu}, \phi_i)$

$$\mathcal{Z}[J^{\mu\nu}, J_i] = \int \mathcal{D}g_{\mu\nu}\left(\prod_i \mathcal{D}\phi_i\right) e^{iS(g_{\mu\nu},\phi_i)+\int d^4x\sqrt{|g|}\,\left(g_{\mu\nu}J^{\mu\nu}+\sum_i J_i\phi_i\right)} \tag{2.8}$$

with currents $J_{\mu\nu}$ and J_i which couple to the fields.

[3]In flat-space theories invariant under *conformal transformations* $SO(d,2)$, where d is the space-time dimension, this appears as a consequence of the current conservation [20] by Noether theorem $\partial_\mu D^\mu = T^\mu_\mu = 0$.

After an infinitesimal Weyl transformation, and assuming that the integration measure is invariant, we have

$$0=\delta\mathcal{Z}=\int\mathcal{D}g_{\mu\nu}\left(\prod_i\mathcal{D}\phi_i\right)e^{iS(g_{\mu\nu},\phi_i)+\int d^4x\sqrt{|g|}\,\left(g_{\mu\nu}J^{\mu\nu}+\sum_i J_i\phi_i\right)}\times \tag{2.9}$$
$$\times\left\{\int d^4x\sqrt{|g|}\,\left(2g_{\mu\nu}\frac{\delta\mathcal{L}}{\delta g_{\mu\nu}}+\sum_i\lambda_{\phi_i}\phi_i\frac{\delta\mathcal{L}}{\delta\phi_i}+2g_{\mu\nu}J^{\mu\nu}+\sum_i\lambda_{\phi_i}\phi_iJ_i\right)\omega\right\}$$

When the sources vanish, this implies that the ponderated trace (2.6) of the equations of motion must vanish not only clasically as a Noether identity but also its expectation value between any pair of states connected by the path integral must do so. The vacuum expectation value is a particular case of this

$$\left\langle\ 0\ \middle|\ 2g_{\mu\nu}\frac{\delta\mathcal{L}}{\delta g_{\mu\nu}}+\sum_i\lambda_{\phi_i}\phi_i\frac{\delta\mathcal{L}}{\delta\phi_i}\ \middle|\ 0\ \right\rangle=0 \tag{2.10}$$

While deriving this, we have considered, as usual, that the integration measure in the path integral is invariant under Weyl transformations. However, this does not need to be true and in general it will not. Whenever this happens, we will find that the Ward identity (2.10) is not satisfied, obtaining a non-vanishing right hand side. In this situation, we will say that we have found a *Weyl (conformal) anomaly.*

2.2. Weyl anomalies Weyl anomalies were first discovered by Duff and Capper [21] in 1973 by evaluating the flat space limit of quantum corrections to the gravitational propagator given by loops of massless fields. They found that the tracelesness condition imposed by the classical Weyl invariance was lost due to finite parts when quantum corrections entered into game. Being explicit, for some clasically Weyl invariant systems, it may happen that the Ward identity (2.10) is substituted by

$$\left\langle\ 0\ \middle|\ 2g_{\mu\nu}\frac{\delta\mathcal{L}}{\delta g_{\mu\nu}}+\sum_i\lambda_{\phi_i}\phi_i\frac{\delta\mathcal{L}}{\delta\phi_i}\ \middle|\ 0\ \right\rangle=\sum_j\mathcal{C}_j\mathcal{O}_j \tag{2.11}$$

where $\mathcal{C}_j$ are constant coefficients and $\mathcal{O}_j$ Weyl covariant operators with the right scaling dimension[4]. The Weyl anomaly breaks Weyl invariance but its structure is Weyl invariant by itself.

[4]We say that an operator is Weyl covariant if it transforms homogeneously under Weyl transformations

$$\mathcal{O}^{'}=\Omega^{\lambda_{\mathcal{O}}}\mathcal{O} \tag{2.12}$$

The operators entering into the rhs of the anomalous Ward identity depend on the spacetime dimension n. For example, for $n = 2$ at the one-loop level there is a single one

$$\mathcal{O}_{n=2} = \sqrt{|g|}\; R \tag{2.13}$$

while for $n = 4$ the possible choices grow

$$\mathcal{O}_1 = \sqrt{|g|}\; C_{\mu\nu\rho\sigma}C^{\mu\nu\rho\sigma}, \qquad \mathcal{O}_2 = \sqrt{|g|}\; E_4 \qquad \mathcal{O}_3 = \sqrt{|g|}\; \Box R \tag{2.14}$$

where $C_{\mu\nu\rho\sigma}$ and E_4 are the Weyl tensor and the Euler density in four dimensions (Gauss-Bonnet term), given by

$$C_{\mu\nu\rho\sigma} = R_{\mu\nu\rho\sigma} - g_{\mu[\rho}R_{\sigma]\nu} + g_{\nu[\rho}R_{\sigma]\mu} + \frac{R}{3}g_{\mu[\rho}g_{\sigma]\nu} \tag{2.15}$$

$$E_4 = R^2 - 4R_{\mu\nu}R^{\mu\nu} + R_{\mu\nu\rho\sigma}R^{\mu\nu\rho\sigma} \tag{2.16}$$

with [...] meaning complete antysimmetrization of the indices contained inside the brackets[5].

The operator $\mathcal{O}_{n=2}$ happens to be a total derivative[6] and the same happens for $\mathcal{O}_3$. $\mathcal{O}_1$ and $\mathcal{O}_2$ however remain and the important point about them is that they cannot be cancelled by the addition of any *local* counterterm in the original action[23]. Therefore, a local renormalization process cannot restore Weyl invariance at an arbitrary loop order. This is the real statement of the existence of an anomaly.

For a four-dimensional theory, the anomalous Ward identity will generically take the form

$$\left\langle\, 0 \;\middle|\; 2g_{\mu\nu}\frac{\delta\mathcal{L}}{\delta g_{\mu\nu}} + \sum_i \lambda_{\phi_i}\phi_i\frac{\delta\mathcal{L}}{\delta\phi_i} \;\middle|\; 0 \,\right\rangle = \mathcal{C}_1 C_{\mu\nu\rho\sigma}C^{\mu\nu\rho\sigma} + \mathcal{C}_2 E_4 + \mathcal{C}_3\Box R \tag{2.17}$$

The value of the coefficients $\mathcal{C}_i$ must be computed case by case by looking to, for instance, two-point functions or, more generally, to the quantum effective action. In the next chapter we will introduce the Schwinger-DeWitt

[5]In the presence of a gauge field A_μ, also the square of the field strength $Tr\,(F_{\mu\nu}F^{\mu\nu})$ is an admissible operator.

[6]This is a consequence of the *Theorema Egregium* by Gauss[22]. It can be proven by noting that for $n = 2$ any manifold admits a foliation in spacelike surfaces. Computing $\sqrt{|g|}\; R$ explicitly by means of this foliation shows that it is a total derivative.

technique, which will become a powerful tool in computing this. Here just let us note that Weyl anomalies are intimately related to the appearance of the renormalization scale for quantum corrections, as we would have expected from the fact that any scale breaks the symmetry. For a generic Weyl invariant theory, the one-loop divergence of the quantum effective action of these theories takes the form

$$\Gamma_{L=1} \sim \frac{1}{n-4} \int d^n x \sqrt{|g|} \; (\mathcal{C}_1 C_{\mu\nu\rho\sigma} C^{\mu\nu\rho\sigma} + \mathcal{C}_2 E_4) \tag{2.18}$$

when evaluated in dimensional regularization, thus the pole in $n = 4$. If we perform a Weyl transformation of this, we find that that the operators transform with a scaling dimension $\lambda_{\mathcal{O}} = -4$, while $\sqrt{|g|}$ has dimension n. The total integrand then transforms as Ω^{n-4}. Therefore, under an infinitesimal transformation we have

$$\delta\Gamma_{L=1} \sim \int d^n x \sqrt{|g|} \; (\mathcal{C}_1 C_{\mu\nu\rho\sigma} C^{\mu\nu\rho\sigma} + \mathcal{C}_2 E_4)\, \omega \tag{2.19}$$

The pole has been cancelled by the $n - 4$ factor coming from the transformation, leaving an *evanescent operator*, a finite part that remains after taking the limit $n = 4$ and breaks Weyl invariance. However, if the theory was finite, that is, there is no renormalization dependence or, in our language, no pole dependence, there is no possibility of an anomaly arising through this mechanism. Therefore, one can envision that a truly Weyl invariant theory should be finite in the ultra-violet regime[24].

2.3. Regularizations and the fate of the anomaly

Up to here we have thoroughly remarked that Weyl anomalies cannot be removed by local counterterms, but they can be removed by non-local ones. This is linked to the fact that an anomaly can arise also if we regularize the theory in such a way that our regularization process breaks explicitly the symmetry. Then, the Ward identity related to such symmetry is not guaranteed for UV divergent quantities and an anomaly can generically arise. In our case, the quest for a regularization that preserves Weyl invariance and thus ensures the absence of anomalies, is only possible if we choose to introduce non-local counterterms.

However, there is a simple way to understand how this works[25, 26, 27]. In particular, let us exemplify it with the case of a single scalar field ϕ (with scaling dimension λ_ϕ) coupled to the metric by a generic action

$S(g_{\mu\nu}, \phi)$. In this case, there exists a regularization scheme that removes the conformal anomaly at the cost of introducing non-local interactions with the scalar field. For example, for the counterterm (2.18) it amounts to substitute the counterterm by

$$\Gamma_{L=1} \sim \frac{1}{n-4} \int d^n x \sqrt{|g|}\, \phi^{-\frac{n-4}{\lambda_\phi}} \left(\mathcal{C}_1 C_{\mu\nu\rho\sigma} C^{\mu\nu\rho\sigma} + \mathcal{C}_2 E_4 \right) \tag{2.20}$$

Note that this counterterm reduces to the previous one (2.18) when the limit $n \to 4$ is taken but now the dimension dependent coupling with ϕ makes it Weyl invariant in any dimension and thus it cancels the anomaly at the cost of introducing this new factor that will produce a non-polynomial counterterm when expanded about $n = 4$. We will come back to this issue in part II.

If we were working on a theory containing only the metric, or the metric together with fields which cannot be accommodated as compensating factors in the regularization scheme, one can always unveil the conformal factor of the metric by a change of variables to the Jordan frame[7] and use it for the purpose here introduced.

Finally, we must stress that there is an extra way of solving the anomaly issue *without introducing any non-local counterterm.* The statement of the anomaly being impossible to be removed by local counterterms is only true if one wants to preserve *diffeomorphism invariance*, since it is this symmetry what forces the structure of the counterterm to be of the form

$$\Gamma_{L=1} \sim \int d^n x \sqrt{|g|}\, \mathcal{O} \tag{2.22}$$

with some local operator $\mathcal{O}$ of dimension four. If we forget about *Diff* invariance, then we can consider counterterms with arbitrary powers of the determinant of the metric in the integration measure. In particular, we can consider

$$\Gamma_{L=1} \sim \int d^n x\, |g|^{\frac{2}{n}}\, \mathcal{O} \tag{2.23}$$

[7]This is achieved by transforming the metric as

$$g_{\mu\nu} \to g_{\mu\nu} \phi^{\frac{4}{2-n}} \tag{2.21}$$

which reduces to the appropiate counterterm once the dimensional limit is taken and is Weyl invariant in any dimension provided that $\lambda_{\mathcal{O}} = 4$. This will be of relevance for our work in Part III.

3

The Schwinger-DeWitt technique

Up to here we have talked about generalities of gravitational actions, the effective action and Weyl anomalies, but the explicit form of those objects will obviously depend of the theory at hand. A computational technique is thus required to compute the effective action of a given lagrangian. Although diagrammatic techniques are always available, they tend to be very involved in the presence of spin two excitations and non-linear theories. Moreover, the need to include gauge breaking terms in the action jeopardizes the obvious gauge invariance of the computations and only when computing physical quantities it is explicit. In this chapter we present a more suitable alternative by combining the background field method with the Schwinger-DeWitt technique, focusing on the properties of determinants of differential operators in riemannian manifolds.

3.1. The background field method

We are interested in obtaining the effective action of a quantum field theory described by an action functional $S(Q)$ depending on a collection of fields that we generically denote Q among whose we include the metric. In general, the fields Q can carry space-time indices as well as internal bundle indices but we will suppress any label for the discussion here. The quantum dynamics of those fields will be given by the path integral

$$\mathcal{Z}[J] = \int \mathcal{D}Q \; e^{iS(Q)+i\langle Q,J\rangle} \tag{3.1}$$

where J is a source and we have defined the inner product[1]

$$\langle \mathcal{I}, \mathcal{J} \rangle = \int d^4x \sqrt{|g|} \; \mathrm{Tr}\,(\mathcal{I}\mathcal{J}) \tag{3.3}$$

[1] This product also induces a measure in the path integral

$$1 = \int \mathcal{D}Q \; e^{i\langle Q,Q\rangle} \tag{3.2}$$

where the trace runs over all internal and space-time indices.

Using this, we can define some standard quantities. We will introduce the most important ones by following closely the arguments in [28] and [29]. Whenever something is omitted, we refer the reader to these documents and references therein.

We start by defining the Green's functions of the theory as time-ordered n-point functions with coincident argument

$$G^{(n)}(Q) = \langle 0|T\{\underbrace{Q...Q}_{n-times}\}|0\rangle = \int \mathcal{D}Q\ (\underbrace{Q...Q}_{n-times})e^{iS(Q)+i\langle Q,j\rangle} = \left(\frac{1}{i}\frac{\delta}{\delta J}\right)^n \mathcal{Z}[J]|_{J=0} \tag{3.4}$$

These are the disconnected Green functions, containing completely disjointed parts. However, these will not contribute to the S-matrix of the theory and it is useful to get rid of them by defining the generator of only connected Green functions

$$W[J] = -i\log\mathcal{Z}[J] \tag{3.5}$$

Surprisingly, the logarithm implies that, when taking functional derivatives of $W[J]$ instead of $Z[J]$, the terms rearrange in such a way that the disconnected parts are subtracted from the total result. However, this is not the simplest way to write these Green's functions. Their computation can be further simplified by writing them in terms of 1PI graphs, those which cannot be split into two inequivalent functions by cutting a line, that can be stringed together afterwards to recover the full connected answer. The 1PI functions are generated by the so-called *effective action*, defined as

$$\Gamma[\bar{Q}] = W[J] - \langle J, \bar{Q}\rangle \tag{3.6}$$

where the mean field

$$\bar{Q} = \frac{\delta W}{\delta J} \tag{3.7}$$

can be understood as the vacuum expectation value of Q in the presence of the source J. Differentiating $\Gamma[\bar{Q}]$ with respect to the mean field now computes the 1PI functions. In particular, the first derivative of (3.6) gives

$$\frac{\delta\Gamma}{\delta\bar{Q}} = -J \tag{3.8}$$

which can be though as the quantum-mechanical correction to the classical equations of motion.

Computing $\Gamma[\bar{Q}]$ to the desired order in a loop expansion gives all the 1PI functions to that order. Obtaining the S-matrix from it is then just a matter of generating the full connected functions, amputating external propagators, putting all momenta on-shell and adding wave-function factors using the LSZ formula. The effective action is therefore an easy way to access to all the relevant information of a given quantum theory. Now we will show how the background field method is a suitable tool to compute it.

We start by introducing, analogously to (3.1) a partition functional $\tilde{\mathcal{Z}}$ by taking the classical action $S(Q)$ and writing it in terms of a shifted field

$$\tilde{\mathcal{Z}}[J,\bar{\phi}] = \int \mathcal{D}Q \; e^{iS(Q+\bar{\phi})+i\langle J,Q\rangle} \tag{3.9}$$

Here $\tilde{\mathcal{Z}}$ depends both on the conventional source and on the background field $\bar{\phi}$, which can be though as an alternate source. We continue with the analogy and introduce

$$\tilde{W}[J,\bar{\phi}] = -i\log\tilde{\mathcal{Z}}[J,\bar{\phi}] \tag{3.10}$$

and by defining the new mean field

$$\tilde{Q} = \frac{\delta\tilde{W}}{\delta J} \tag{3.11}$$

we arrive to the background field effective action

$$\tilde{\Gamma}[\tilde{Q},\bar{\phi}] = \tilde{W}[J,\bar{\phi}] - \langle J,\tilde{Q}\rangle \tag{3.12}$$

To see the point of these new definitions, let us rewrite (3.9) by shifting the integration variable, finding

$$\tilde{\mathcal{Z}}[J,\bar{\phi}] = \mathcal{Z}[J]e^{-i\langle J,\bar{\phi}\rangle} \tag{3.13}$$

and by taking logarithms

$$\tilde{W}[J,\bar{\phi}] = W[J] - \langle J,\bar{\phi}\rangle \tag{3.14}$$

If we now differentiate both sides with respect of J and keep in mind the definitions (3.7) and (3.11), we find that both mean fields are related in a simple way

$$\tilde{Q} = \bar{Q} - \bar{\phi} \tag{3.15}$$

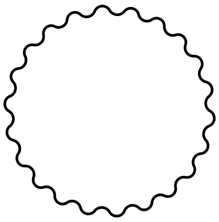

Figure 3.1: The only vacuum diagram at one loop in the background field approach. It is obtained by tracing the two-legs vertex.

and therefore the effective actions are also simply related

$$\tilde{\Gamma}[\tilde{Q},\bar{\phi}] = W[J] - \langle J,\bar{\phi}\rangle - \langle J,\tilde{Q}\rangle + \langle J,\bar{\phi}\rangle = \Gamma[\bar{Q}] = \Gamma[\tilde{Q}+\bar{\phi}] \tag{3.16}$$

This is the main result of the background field procedure. As a special case, we can take $\tilde{Q}=0$, thus identifying the background field with the mean field $\bar{Q}=\bar{\phi}$ and having

$$\tilde{\Gamma}[0,\bar{\phi}] = \Gamma[\bar{\phi}] \tag{3.17}$$

implying that in order to compute the effective action of the theory it is enough to compute an equivalent effective action $\tilde{\Gamma}[0,\bar{\phi}]$.

The background field effective action $\tilde{\Gamma}[\tilde{Q},\bar{\phi}]$ produces all the 1PI Green's functions in the presence of such background field. By setting $\tilde{Q}=0$ we are killing all the dependence on the field $\tilde{Q}$ and thus generating only graphs with no external lines. $\tilde{\Gamma}[0,\bar{\phi}]$ is then the generator of all 1PI vacuum graphs in the presence of $\bar{\phi}$.

This is particularly helpful when facing one-loop computations. In such a case, let us go back to the partition function $\tilde{\mathcal{Z}}[J,\bar{\phi}]$

$$\tilde{\mathcal{Z}}[J,\bar{\phi}] = \int \mathcal{D}Q\; e^{iS(Q+\bar{\phi})+i\langle J,Q\rangle} \tag{3.18}$$

and consider the field Q perturbatively. Since we are now interested in one-loop computations and we only need to care about vacuum diagrams, there is only one single contribution, shown in figure 3.1, which tells us that it is enough to expand the theory up to second order in the field Q. Thus, for the action we have

$$S(Q+\bar{\phi}) = S(\bar{\phi}) + \left\langle \frac{\delta\mathcal{L}}{\delta Q}\bigg|_{Q=\bar{\phi}}, Q\right\rangle + \frac{1}{2}\left\langle Q, \frac{\delta^2\mathcal{L}}{\delta Q}\bigg|_{Q=\bar{\phi}} Q\right\rangle + O(Q^3) \tag{3.19}$$

where we have assumed that $S(Q) = \int d^4x \sqrt{|g|}\, \mathcal{L}(Q)$.

When setting later $\bar{\phi} = \bar{Q}$ and using both equation (3.7) and the fact that at the lowest order $\Gamma \equiv S$, the linear term will vanish. The partition function is then rewritten as

$$\tilde{\mathcal{Z}}[J, \bar{\phi}] = \int [\mathcal{D}Q] e^{iS(\bar{\phi})} e^{i\langle Q, \hat{D}Q\rangle + i\langle J, Q\rangle} \tag{3.20}$$

with the differential operator $\hat{D}$ defined as

$$\hat{D} = \left.\frac{\delta^2 \mathcal{L}}{\delta Q}\right|_{Q=\bar{\phi}} \tag{3.21}$$

Here we are using hats to denote operators while un-hatted quantities are c-numbers.

The first factor is ultra-local and can be absorbed in the overall normalization of the wave-function. In the absence of sources (which are irrelevant for vacuum graphs computations) and assuming that the differential operator $\hat{D}$ has non-vanishing determinant[2], this is simply[3]

$$\tilde{\mathcal{Z}}[0, \bar{\phi}] = \det{}^{-\frac{1}{2}}(\hat{D}) \tag{3.22}$$

and thus we have

$$\tilde{W}[0, \bar{\phi}] = \frac{i}{2} \log \left[\det(\hat{D})\right] \tag{3.23}$$

and, since we are setting $J = \tilde{Q} = 0$, by equations (3.9) and (3.17) we finally arrive to

$$\Gamma[\bar{\phi}] = \tilde{\Gamma}[0, \bar{\phi}] = \tilde{W}[0, \bar{\phi}] = \frac{i}{2} \log \left[\det(\hat{D})\right] \tag{3.24}$$

Therefore, we find that the effective action in which we are interested can be computed as the determinant of some differential operator living in a riemannian manifold. In order to perform this computation and to obtain the divergent pieces and the counterterms required to renormalize the theory we will use the known as Schwinger-DeWitt or Heat Kernel technique.

Finally, let us comment that by using the background field definition of the effective action, gauge invariance is preserved explicitly in the computations[30, 31, 32].

[2]The case of vanishing determinant will imply the existence of a zero mode in the spectrum of the operator and thus of a gauge symmetry. We will consider this situation later in chapter 5.

[3]Of course, this is only true if D is self-adjoint with respect to the inner product $\langle ..., ...\rangle$.

3.2. Zeta function regularization Up to here we have seen that one can obtain the effective action of a given theory by computing the determinant of the self-adjoint operator that governs second order fluctuations about a background configuration [4]

$$\Gamma[\bar{\phi}] = \frac{1}{2} \log \left[\det(\hat{D}) \right] \tag{3.25}$$

The operator $\hat{D}$ lives in a Riemannian manifol equipped with a metric[5] $g_{\mu\nu}$ and a connection ${\Gamma^{\mu}}_{\alpha\beta}$ given by the Levi-Civita condition. Its determinant will be in general divergent and requires the introduction of a regularization scheme. As well as with any other technique involving quantum corrections we must cut-off the momentum space in its upper-limit and differentiate between divergent and finite terms in our result. Only the first ones will contribute to the renormalization of physical couplings and the UV structure of the theory.

We do this by using *zeta function regularization*, first introduced in [33]. We assume that a suitable set of eigenfunctions Q_i of $\hat{D}$ exist such that each one carries an eigenvalue ρ_i. Then, we first define the zeta function $\zeta(s, \epsilon, \hat{D})$ of the operator as

$$\zeta(s, \epsilon, \hat{D}) = \Gamma(s)^{-1} \int_0^\infty dt \; t^{s-1} \langle Q_i | \epsilon \; e^{-t\hat{D}} | Q_i \rangle \tag{3.26}$$

where $\epsilon = \epsilon(x)$ is a local function of the space-time coordinates.

The bracket contained inside this definition is the trace of a functor[6] known as the *Heat Kernel*[34, 35]

$$\hat{K}(t, \epsilon, \hat{D}) = \epsilon \; e^{-t\hat{D}}, \qquad K(t, \epsilon, \hat{D}) = \mathrm{Tr} \left[\hat{K}(t, \epsilon, \hat{D}) \right] = \langle Q_i | \epsilon \; e^{-t\hat{D}} | Q_i \rangle \tag{3.27}$$

where its name comes from the fact that, when $\epsilon = 1$, it solves a heat equation with appropriate initial conditions

$$(\partial_t + \hat{D}) \hat{K}(t, 1, \hat{D}) = 0, \qquad \hat{K}(0, 1, \hat{D}) = \delta^{(4)}(y - x) \tag{3.28}$$

[4] Here we have turned to euclidean signature by Wick rotating the temporal coordinate $x_0 = it$ in order to avoid subtleties with the behavior of the path integral.

[5] In the case of theories with a dynamical metric, i.e., when the set of fields Q contains the metric $g_{\mu\nu}$ itself, the metric defining the manifold will be the background metric $\bar{g}_{\mu\nu}$.

[6] Being sloppy, a functor can be though as a function of an operator.

being this the reason as to why the variable t is sometimes referred to as *proper time*. Thought like this, the Heat Kernel describes the diffusion of a fluid in a space-time with an extra dimension.

More interesting, in particular for us, is the connection of the Heat Kernel with the spectrum of the operator $\hat{D}$. For a given non-vanishing eigenvalue[7] we can always write

$$\log \rho_i = -\int_0^\infty \frac{dt}{t} e^{-t\rho_i} \tag{3.29}$$

This identity is always true up to an infinite constant that can be disregarded for the purposes here[8]. It can be also equally extended to the full spectrum of the operator, allowing us to *define*

$$\log \left[\det(\hat{D})\right] = -\int_0^\infty \frac{dt}{t} \langle Q_i|\epsilon \; e^{-t\hat{D}}|Q_i\rangle \tag{3.30}$$

and therefore, by using (3.24) and (3.27), the effective action takes the form

$$\Gamma[\bar{\phi}] = -\frac{1}{2}\int_0^\infty \frac{dt}{t} K(t, 1, \hat{D}) \tag{3.31}$$

Up to here we did not get rid of the infinities yet but only rewritten the relevant expression. However, in this form ultraviolet infinities are associated to the lower limit of the proper time integral in (3.31). The reason is that, assuming Poincaré invariance, the operator $\hat{D}$ must be given in Fourier space as a polynomial in the four-momenta squared

$$\hat{D} = \sum_i^\beta \hat{\alpha}_i (p^2)^i \tag{3.32}$$

with some operator functions $\hat{\alpha}_i$ and higher order term $(p^2)^i$. Therefore, for the exponential to be dimensionless when $|p| \to \infty$, $t \to 0$ and the UV infinity is related to the vanishing limit of t.

There are two ways of regularizing this infinities in the present formulation. The obvious one is to introduce a cut-off by changing the lower limit of the integral

$$\int_0^\infty \longrightarrow \int_{\Lambda^{-2\beta}}^\infty \tag{3.33}$$

[7]Recall that we are considering that our operator contains no zero modes and therefore no vanishing eigenvalues.

[8]This can be proven by differentiating both sides of the identity.

This is however not Lorentz invariant and, even worse, it explicitly breaks scale invariance and therefore Weyl invariance. Counterterms in this regularization are not constrained to preserve any of these symmetries but complicated Slavnov-Taylor identities[36] instead and cancellations must be checked order by order. We then prefer to use the second option, which consists in shifting the power of t inside the integral, arriving to a regularized effective action

$$\Gamma_{reg}[\bar{\phi}] = -\frac{1}{2}\,\tilde{\mu}^{2s}\int_0^{\infty}\frac{dt}{t^{1-s}}K(t,1,\hat{D}) = -\frac{1}{2}\tilde{\mu}^{2s}\Gamma(s)\zeta(s,1,\hat{D}) \tag{3.34}$$

where the zeta function has appeared explicitly and we have introduced a mass scale $\tilde{\mu}$ in order to preserve dimensions.

In this way, ultraviolet divergences are identified with divergences at $s \to 0$ in (3.34). Those are inherited from the pole of the Gamma function

$$\Gamma(s)\big|_{s\to 0} = \frac{1}{s} - \gamma_E + O(s) \tag{3.35}$$

with γ_E being the Euler-Mascheroni constant. Thus

$$\Gamma_{reg}[\bar{\phi}]\big|_{s\to 0} = -\frac{1}{2}\left(\frac{1}{s} - \gamma_E + \log\left(\tilde{\mu}^2\right)\right)\zeta(0,1,\hat{D}) - \frac{1}{2}\frac{\partial}{\partial s}\,\zeta(s,1,\hat{D})\Big|_{s=0} \tag{3.36}$$

If we choose a minimal subtraction scheme, the pole term will be removed by renormalization and the remaining terms will define the *renormalized effective action*

$$\Gamma_{\mathbf{ren}}[\bar{\phi}] = -\frac{1}{2}\log\left(\mu^2\right)\zeta(0,1,\hat{D}) - \frac{1}{2}\frac{\partial}{\partial s}\,\zeta(s,1,\hat{D})\Big|_{s=0} \tag{3.37}$$

where we have introduced a rescaled parameter by $\mu^2 = e^{-\gamma_E}\tilde{\mu}^2$. This parameter represents in this approach the renormalization ambiguity that must be fixed by a suitable normalization condition.

Through this work, we will in-distinctively use $s \to 0$ or $s = (n-4) \to 0$ (when working in four dimensions) to denote the cutoff of the theory in order to connect sometimes with dimensional regularization and with the discussions about evanescent operators that we did in chapter 2.

Together with (3.30), equation (3.37) yields a *definition* of the determinant for a positive elliptic operator which is commonly used in mathematics

$$\log\left[\det(\hat{D})\right] = -\log\left(\mu^2\right)\zeta(0,1,\hat{D}) - \frac{\partial}{\partial s}\,\zeta(s,1,\hat{D})\Big|_{s=0} \tag{3.38}$$

and that here will represent the core of our computational tools, to be described in the following section.

3.3. The short-time expansion In the last sections we have shown, by using the background field method and zeta function regularization, how to reduce the problem of computing the one-loop effective action of a given field theory in the presence of a background metric to the problem of computing the zeta function of the operator governing second order fluctuations when the proper time s goes to zero

$$\Gamma_{\mathbf{ren}}[\bar{\phi}] = -\frac{1}{2}\log\left(\mu^2\right)\zeta(0,1,\hat{D}) - \frac{1}{2}\frac{\partial}{\partial s}\,\zeta(s,1,\hat{D})\Big|_{s=0} \tag{3.39}$$

The zeta function was defined in (3.26) through the *Heat Kernel*

$$\zeta(s,\epsilon,\hat{D}) = \Gamma(s)^{-1}\int_0^\infty dt\; t^{s-1}K(t,\epsilon,\hat{D}) \tag{3.40}$$

This is a Mellin transform[9] and thus it can be inverted

$$K(t,\epsilon,\hat{D}) = \frac{1}{2\pi i}\oint ds\; t^{-s}\Gamma(s)\zeta(s,\epsilon,\hat{D}) \tag{3.43}$$

and this equation can be used to relate the poles of the Heat Kernel with those of the integrand.

The key point here is that under fairly general assumptions the traced Heat Kernel of a second order operator in a space-time of dimension n with no boundary enjoys a short-time expansion when $t \to 0$ of the form

$$K(t,\epsilon,\hat{D}) = \sum_k t^{\frac{k-n}{2}}a_k(\epsilon,\hat{D}) \tag{3.44}$$

Here the functions $a_k(\epsilon,\hat{D})$ are known as *Heat Kernel coefficients* and they are given as integrals

$$a_k(f,\hat{D}) = \int d^n x\sqrt{|g|}\;\epsilon(x)b_k(x) \tag{3.45}$$

[9] The Mellin transform $\phi(s) = \mathcal{M}[f](s)$ of a function $f(x)$ is defined as

$$\phi(s) = \int_0^\infty dx\; x^{s-1}f(x) \tag{3.41}$$

Its inverse is given by

$$f(x) = \frac{1}{2\pi i}\oint ds\; x^{-s}\phi(s) \tag{3.42}$$

with the contour encircling all the poles in the integrand.

where $b_k(x)$ are *local invariants of the manifold* of scaling dimension k. The coefficients $a_k(f, \hat{D})$ are non-vanishing only for even k. For odd labels, the $b_k(x)$ are total derivatives and thus they vanish in the absence of a boundary. The existence of such a power law is a very non-trivial statement that can be however proven under fairly general assumptions[37].

Let us note however that this expansion is only valid for very particular hypothesis. First, it is only valid for second order operators and, among those, only for *minimal operators*, those whose principal symbol, in Fourier space, is of the form

$$\hat{D}_{ps} = \hat{G}\, p^2 \tag{3.46}$$

where $\hat{G}$ is a matrix valued metric which accounts for the possible index and internal bundle structure of the operator[10]. For minimal operators of arbitrary order $2z$, the technique can be extended by considering a more general ansatz for the short-time expansion[38]

$$K(t, \epsilon, \hat{D}) = \sum_k t^{\frac{k-n}{2z}} a_k(\epsilon, \hat{D}) \tag{3.47}$$

This can be also extended to operators which are non relativistic[39] but we will not consider this case here. In the case of non-minimal operators, one has to relay in aditional techniques, which will be developed later in chapter 4.

By going back to (3.44) we can relate the heat kernel coefficients with the poles of the integrand as

$$a_k(\epsilon, \hat{D}) = \mathrm{Res}_{s=(n-k)/2}\{\Gamma(s)\zeta(s, \epsilon, \hat{D})\} \tag{3.48}$$

and, in particular, we find that

$$\zeta(0, \epsilon, \hat{D}) = a_n(\epsilon, \hat{D}) \tag{3.49}$$

where, remind, n is the space-time dimension.

Therefore, and going back to (3.37) we find that the renormalized effective action of our theory can be finally written[11] as

$$\Gamma_{\mathbf{ren}}[\bar{\phi}] = -\frac{1}{2}\log\left(\frac{\mu^2}{M^2}\right) a_n(\epsilon, \hat{D}) + \textbf{regular terms} \tag{3.50}$$

[10]For instance, $\hat{G} = g_{\mu\nu}\delta_{ab}$ for gauge fields $A^{(a)\mu}$.

[11]Here we have introduced the renormalization scale M.

and so, computing the renormalization dependent piece of the effective action is reduced to computing the relevant heat kernel coefficient in a given dimension. Since they are local invariants, this can be done in an universal way which is blind to the explicit spin or gauge content of the operator D and thus we can give general formulas for them.

There are two main techniques to compute these coefficients. They were originally introduced by Schwinger and DeWitt [34, 40, 41, 42, 43] and their method of obtaining them was based on solving recursively the heat equation (3.28) order by order. This is a very general method that can be used for many different operators that go even beyond the original purpose [44], but it is very involved. For second orders, it is easier to rely in the universality of the coefficients and use the functorial properties introduced by Gilkey[45]. Those give three simple recursions between the different coefficients that can be extensively exploited. A pedagogical review on how to do this can be found in [29].

The two main properties of the Heat Kernel coefficients have been already commented here but let us remark them. Let us start by assuming that we are working with a minimal second order operator, which can be always taken to the form

$$\hat{D} = -\hat{G}\Box - \hat{E} \tag{3.51}$$

with $\hat{E}$ containing no derivatives and $\hat{G}$ taking a role of a metric in field (configuration) space. Then

- Heat Kernel coefficients with odd index vanish, $a_{2j+1}(\epsilon, \hat{D}) = 0$.
- Coefficients $a_{2j}(\epsilon, \hat{D})$ are locally computable in terms of invariants of the manifold.

The first statement is only true since we are working in manifolds without boundary. Otherwise, odd coefficients would be proportional to total derivatives evaluated in the boundary. However, the second statement is more important, since in it relays the effectiveness of the method here presented. It can be summarized by saying that the Heat Kernel coefficients are of the form

$$a_k(\epsilon, \hat{D}) = \int \sqrt{|g|}\mathrm{Tr}\left\{\epsilon(x)\sum_i \mathcal{A}_i\, \hat{\mathcal{U}}_i\right\} \tag{3.52}$$

where $\mathcal{A}_i$ are constants depending only on the space-time dimension and $\hat{\mathcal{U}}_i$ represent all the possible independent invariants of scaling dimension k constructed by using $R_{\mu\nu\rho\sigma}$, $\hat{E}$ and a curvature $\hat{F}_{\mu\nu}$ constructed out of any extra bundle carried by the fields by using Ricci's identity

$$[\nabla_\mu, \nabla_n]\phi = \hat{F}_{\mu\nu} \cdot \phi \tag{3.53}$$

The trace Tr[...] is taken over both all space-time and internal indices (gauge bundles, etc...).

The consequences of these two simple properties are very far reaching. For instance, a simple statement that can be now proven is the universal dependence of $\mathcal{A}_i$ on the space-time dimension n if the operator if second order. The proof comes as follows. Let us consider a space-time manifold $\mathcal{M}$ which is factorizable, so

$$\mathcal{M} = \mathcal{M}_1 \otimes \mathcal{M}_2 \tag{3.54}$$

and introduce the only natural differential operator over this

$$\hat{D} = \hat{D}_1 \otimes \mathbb{I} + \mathbb{I} \otimes \hat{D}_2 \tag{3.55}$$

This means that the bundle indices are also independent, then in the definition (3.27) of the traced Heat Kernel, the traces also factorize and, following the argument, we find for the short-time expansion of the full operator $\hat{D}$

$$a_k(\epsilon_1\epsilon_2, \hat{D}) = \sum_{p+q=k} a_p(\epsilon_1, \hat{D}_1) a_q(\epsilon_2, \hat{D}_2) \tag{3.56}$$

Now, let us assume that our factorizable manifold is, in particular

$$\mathcal{M} = S^1 \otimes \mathcal{M}_2 \tag{3.57}$$

and that the operator is then

$$\hat{D} = \left(-\partial^2_{x_1}\right) \otimes \mathbb{I} + \mathbb{I} \otimes \hat{D}_2 \tag{3.58}$$

where x_1 is the cyclic coordinate over the S^1, $0 < x_1 < 2\pi$. Then[12]

$$\begin{aligned} a_k(\epsilon_2, \hat{D}) &= \int_{S^1 \otimes \mathcal{M}_2} d^n x \sqrt{|g|} \operatorname{Tr}\{\epsilon_1\epsilon_2 \sum_i \mathcal{A}_i(n)\hat{\mathcal{U}}_i\} = \\ &= 2\pi \int_{\mathcal{M}_2} d^{n-1}x \sqrt{|g|} \operatorname{Tr}\{\epsilon(x) \sum_i \mathcal{A}_i(n)\hat{\mathcal{U}}_i\} \end{aligned} \tag{3.59}$$

[12]Note that since S^1 is flat, the invariants $\hat{\mathcal{U}}_i$ can here only depend on the details of $\hat{D}_2$. This justifies integration over the cyclic coordinate.

where we have made explicit the possible space-time dependence of $\mathcal{A}_i(n)$.

On the other hand, we can use (3.56) and the fact that we can compute the spectrum of $\hat{D}_1$. Its eigenvalues are l^2 with $l \in \mathbb{Z}$ and the kernel can be summed by using Poisson summation formula

$$K(t, \epsilon_1, \hat{D}_1) = \sum_{l\in\mathbb{Z}} \epsilon_1 \ \exp(-tl^2) = \epsilon_1 \sqrt{\frac{\pi}{t}} \sum_{l\in\mathbb{Z}} \exp(-\pi^2 l^2/t^2) \sim \\ \sim \epsilon_1 \sqrt{\frac{\pi}{t}} + O(e^{-1/t}) \tag{3.60}$$

Since exponentially suppressed terms do not contribute to the power law, we find that the only non-vanishing coefficient for the short-time expansion of $\hat{D}_1$ is

$$a_0(\epsilon_1, \hat{D}_1) = \sqrt{\pi}\, \epsilon_1 \tag{3.61}$$

and by (3.56)

$$a_k(\epsilon_1\epsilon_2, \hat{D}) = \sqrt{\pi} \int_{\mathcal{M}_2} d^{n-1}x \, \sqrt{|g|} \ \mathrm{Tr}\{\epsilon_1\epsilon_2 \sum_i \mathcal{A}_i(n-1)\hat{\mathcal{U}}_i\} \tag{3.62}$$

So that comparing equations (3.59) and (3.62) we have

$$\mathcal{A}_i(n-1) = \sqrt{4\pi}\ \mathcal{A}_i(n) \tag{3.63}$$

and we find that all the space-time dependence is contained in an overal normalization. Therefore, for a second order operator we can rewrite the ansatz for the coefficients as

$$a_k(\epsilon, \hat{D}) = (4\pi)^{-\frac{n}{2}} \int \sqrt{|g|} \mathrm{Tr}\{\epsilon(x) \sum_i \mathcal{A}_i \hat{\mathcal{U}}_i\} \tag{3.64}$$

where now $\mathcal{A}_i$ are just numerical universal factors.

Now we will introduce three functorial properties which we will relate the different Heat Kernel coefficients and that we will exploit them in order to compute the expansion in a very general way. Of course, let us remark that we are always talking of the expansion for a second order minimal operator of the form (3.51).

The first property is introduced by considering an operator depending of a factor γ. Then, we note that

$$\left.\frac{d}{d\gamma}\right|_{\gamma=0} \mathrm{Tr}\Big\{ \exp\left(e^{-2\gamma\epsilon} t\hat{D}\right) \Big\} = \mathrm{Tr}\Big\{ 2\epsilon t \hat{D} e^{-t\hat{D}} \Big\} = -2t\frac{d}{dt}\mathrm{Tr}\Big\{ \epsilon e^{-t\hat{D}} \Big\} \tag{3.65}$$

If we now expand both sides of this equation by using the short-time expansion ansatz, we arrive to the first functorial property

$$\left.\frac{d}{d\gamma}\right|_{\gamma=0} a_k(1, e^{-2\gamma\epsilon}\hat{D}) = (n-k) = a_k(\epsilon, D) \tag{3.66}$$

This property states, in particular, that when the operator is Weyl invariant (since the lhs of these equation can be regarded as a Weyl transformation), the Heat Kernel critical coefficient which appears in the effective action $a_n(\epsilon, \hat{D})$ must be also invariant.

In the same way, by relying in the universality of the expansion, one can prove in a similar manner that when $\hat{O}$ contains no derivatives

$$\left.\frac{d}{d\gamma}\right|_{\gamma=0} a_k(1, \hat{D} - \gamma\hat{O}) = a_{k-2}(\hat{O}, \hat{D}) \tag{3.67}$$

which restricts the dependence of the coefficients on the potential part $\hat{E}$.

Finally, let us consider and operator such that

$$\hat{D}(\gamma, \beta) = e^{-2\gamma\epsilon}(\hat{D} - \beta\hat{O}) \tag{3.68}$$

and use the first property (3.66) with $n = k$, then

$$0 = \left.\frac{d}{d\gamma}\right|_{\gamma=0} a_k(1, \hat{D}(\gamma, \beta)) \tag{3.69}$$

Now, we vary it also with respect to β and interchange derivation order

$$0 = \left.\frac{d}{d\gamma}\right|_{\beta=0} \left.\frac{d}{d\gamma}\right|_{\gamma=0} a_k(1, \hat{D}(\gamma, \beta)) = \left.\frac{d}{d\gamma}\right|_{\gamma=0} \left.\frac{d}{d\gamma}\right|_{\beta=0} a_k(1, \hat{D}(\gamma, \beta)) \tag{3.70}$$

and by using (3.67) we finally arrive to

$$\left.\frac{d}{d\gamma}\right|_{\gamma=0} a_{n-2}(e^{-2\gamma\epsilon}\hat{O}, e^{-2\gamma\epsilon}\hat{D}) = 0 \tag{3.71}$$

Now, to calculate the Heat Kernel coefficients, we must write an ansatz for them. As we saw, they are going to be given solely in terms of invariants of the appropiate dimension constructed by using $R_{\mu\nu\rho\sigma}$, $\hat{E}$ and $\hat{F}_{\mu\nu}$, multiplied by some constant unknown factors that we must obtain. The number of invariants of every given dimension is finite and this allows us to write for the first three non-vanishing coefficients

$$a_0(\epsilon, \hat{D}) = (4\pi)^{-n/2} \int d^n x \sqrt{|g|}\, \mathrm{Tr}\{\alpha_0 \epsilon\} \tag{3.72}$$

$$a_2(\epsilon, \hat{D}) = \frac{1}{6}(4\pi)^{-n/2} \int d^n x \sqrt{|g|}\, \mathrm{Tr}\{\epsilon(\alpha_1 \hat{E} + \alpha_2 R\hat{\mathbb{I}})\} \tag{3.73}$$

$$\begin{aligned} a_4(\epsilon, \hat{D}) = \frac{1}{360}(4\pi)^{-n/2} \int d^n x \sqrt{|g|}\, \mathrm{Tr}\{\epsilon(\alpha_3 \Box \hat{E} + \alpha_4 R\hat{E} + \alpha_5 \hat{E}^2 + \\ + \hat{\mathbb{I}}(\alpha_6 \Box R + \alpha_7 R^2 + \alpha_8 R_{\mu\nu}R^{\mu\nu} + \alpha_9 R_{\mu\nu\rho\sigma}R^{\mu\nu\rho\sigma}) + \alpha_{10}\hat{F}_{\mu\nu}\hat{F}^{\mu\nu})\} \end{aligned} \tag{3.74}$$

where the numerical factors have been chosen for convenience.

Now the rules of the game have been set up. We know that this expansion is completely universal as long as the operator at hand is second order and minimal. In that case, all the α_i coefficients are purely numerical factors and all the information about the manifold is encoded in the geometric invariants. Thus, we can choose particular examples of space-times and operators in different dimensions to fix these coefficients one by one.

As a first example, the α_0 coefficient follows trivially from the expansion of the scalar laplacian in S^1 in (3.60), reading $\alpha_0 = 1$.

Using this and formula (3.67) with $k = 2$ we find

$$\frac{1}{6} \int d^n x \sqrt{|g|}\, \mathrm{Tr}\{\alpha_1 \hat{O}\} = \int d^n x \sqrt{|g|}\, \mathrm{Tr}\{\hat{O}\} \tag{3.75}$$

and thus $\alpha_1 = 6$.

α_2 is fixed by using the scaling property (3.71) in dimension $n = 4$. There, for the operator to be Weyl invariant, the square root of the metric determinant, the Ricci scalar and the endomorphism $\hat{E}$ must transform as

$$\left.\frac{d}{d\gamma}\right|_{\gamma=0} \sqrt{|g|} = n\epsilon\sqrt{|g|} \tag{3.76}$$

$$\left.\frac{d}{d\gamma}\right|_{\gamma=0} R = -2\epsilon R + 2(n-1)\Box\epsilon \tag{3.77}$$

$$\left.\frac{d}{d\gamma}\right|_{\gamma=0} \hat{E} = -2\epsilon\hat{E} + \frac{1}{2}(n-2)\Box\epsilon\, \hat{\mathbb{I}} \tag{3.78}$$

Thus, by using (3.71) with $n = 4$ and collecting the terms in $\Box\epsilon$ we find straightforward that $\alpha_1 = 6\alpha_2$.

The rest of the coefficients can be computed in a similar fashion. By exploiting the functorial properties (3.66), (3.67) and (3.71) together with the factorization rule (3.56) one can fix all the coefficients but one. That single final value is then obtained by computing explicitly the trace of $e^{-t\hat{D}}$ for a simple operator in flat space-time, by going to Fourier space. We will not go into the full detail of this computation here but we will instead refer the interested reader to section 4 of [29].

After all this work, one finds that the first three non-vanishing Heat Kernel coefficients for a second order minimal operator are

$$a_0(\epsilon,\hat{D}) = \int d^n x\sqrt{|g|}\ \mathrm{Tr}\{\epsilon(x)\} \tag{3.79}$$

$$a_2(\epsilon,\hat{D}) = \frac{1}{6}\int d^n x\sqrt{|g|}\ \mathrm{Tr}\{\epsilon(x)(6\hat{E}+R)\} \tag{3.80}$$

$$\begin{aligned} a_4(\epsilon,\hat{D}) = \frac{1}{360}\int d^n x\sqrt{|g|}\ &\mathrm{Tr}\{\epsilon(x)(60\Box\hat{E}+60R\hat{E}+180\hat{E}^2+12\Box R\hat{\mathbb{I}}+\\ &+\hat{\mathbb{I}}(5R^2-2R_{\mu\nu}R^{\mu\nu}+2R_{\mu\nu\rho\sigma}R^{\mu\nu\rho\sigma})+30\hat{F}_{\mu\nu}\hat{F}^{\mu\nu}\} \end{aligned} \tag{3.81}$$

with the higher terms growing exponentially in complexity and the ones with odd label vanishing.

These coefficients $a_0(\epsilon,D)$ to $a_4(\epsilon,D)$ were computed originally in [40, 46]. $a_6(\epsilon,D)$ was first computed by Gilkey[37]. Higher order coefficients have been also computed for particular and general cases [47, 48, 49, 50].

Therefore, by using (3.37) and (3.79) we find that the divergent part of the renormalized effective action for a quantum field theory in four space-time dimensions and whose second order dynamics is driven by the operator D can be written as

$$\begin{aligned} \Gamma_{\mathbf{ren}}[\bar{\phi}] = -\frac{1}{32\pi^2}\frac{1}{360}\log(\frac{\mu^2}{M^2})\int d^n x\sqrt{|g|}\ &\mathrm{Tr}\{60\Box\hat{E}+60R\hat{E}+180\hat{E}^2+\\ &+\hat{\mathbb{I}}(12\Box R+5R^2-2R_{\mu\nu}R^{\mu\nu}+2R_{\mu\nu\rho\sigma}R^{\mu\nu\rho\sigma})+30\hat{F}_{\mu\nu}\hat{F}^{\mu\nu}\} \end{aligned} \tag{3.82}$$

Here there are two remarks to be done. First, $R_{\mu\nu\alpha\beta}$ is the Riemann tensor of the background field. If we are working with a quantum field theory defined in a curved space-time (without gravity being a quantum field) the terms including them can be disregarded. If however we are considering a theory of Quantum Gravity, those give the renormalization of the operators in the bare action, through formula (3.17). Let us also note that the invariants involved in $a_4(\epsilon, D)$ in (3.79) are exactly the different operators in (2.14), just expressed in a different basis. This is of course not surprising since in chapter 2 we argued that the effective action and the anomaly are intimately related.

3.4. The Weyl Anomaly from the Heat Kernel For completeness of the discussion and although we have presented before a method to obtain the Weyl anomaly as an evanescent operator from the effective action, let us here rederive it by using the zeta function regularization that leads to (3.37).

Since we are interested anomalies in Weyl invariance, we assume that the differential operator $\hat{D}$ is such that it transforms homogeneously under Weyl transformations

$$\hat{D} \to \Omega^{-2}\hat{D} \tag{3.83}$$

so that the classical action will be invariant.

This will induce a transformation on the zeta function (3.26) that can be written as

$$\zeta(s, 1, \hat{D}') = \Gamma^{-1}(s)\int_0^\infty dt\ t^{s-1}\mathrm{Tr}\Big\{e^{-t\Omega^{-2}\hat{D}}\Big\} \tag{3.84}$$

and this can be rewritten, after a trivial change of variable, as

$$\zeta(s, 1, |Dh') = \Gamma^{-1}(s)\int_0^\infty dt\ \Omega^{2s}t^{s-1}\mathrm{Tr}\Big\{e^{-t\hat{D}}\Big\} \tag{3.85}$$

which at the linear level simplifies to

$$\delta\zeta(s, 1, \hat{D}) = 2s\Gamma^{-1}\int_0^\infty dt\ t^{s-1}\omega\mathrm{Tr}\Big\{e^{-t\hat{D}}\Big\} = 2s\zeta(s, \omega, \hat{D}) \tag{3.86}$$

We see that in the variation of the zeta function, the Weyl linearised parameter ω takes the role of the smearing function ϵ.

Now, the rest of the story is simple. We take the expression (3.37) for the renormalized effective action and perform a Weyl transformation at the linear level using (3.86). We obtain

$$\delta\Gamma_{\mathbf{ren}}[\bar{\phi}] = -\zeta(0, \omega, \hat{D}) = -a_n(\omega, \hat{D}) \tag{3.87}$$

where we have used the relation (3.48) between the poles of the zeta function and the Heat Kernel coefficients.

We find exactly what we expected from the discussion about evanescent operators. The Weyl anomaly is a finite effect (it does not depend on the renormalization scale M) which is proportional to the pole term of the effective action in the presence of gravity.

4

Beyond the Schwinger-DeWitt technique

In the previous chapter we have presented the Schwinger-DeWitt technique as a useful tool to compute the functional determinant that gives the one-loop divergences of the quantum effective action in the background field approach. However, our analysis was focused in the case of second oder operators whose principal symbol was minimal. Although this is the most common situation in known physics[1], there might be cases in which non-minimality remains. In those cases, one has to work further and go beyond the Schwinger-DeWitt technique to obtain the extra contributions to the effective action given by the non-minimal terms.

This can be done in different manners. As always, the diagramatic techniques are available but they are very involved for curved space computations. Other option is to use the techniques of [51, 52], both based upon using the Baker-Campbell-Hausdorff lemma to split the Heat Kernel into pieces that perturb the minimal part. However, here we are going to rely in the technique of functional traces developed quite some time ago by Barvinsky and Vilkovisky [53] and known as *Generalized Schwinger-DeWitt technique.*

We start by considering a non-minimal second order operator[2] given by

$$\hat{F} = \hat{G}\Box + \hat{J}^{\mu\nu}\nabla_\mu\nabla_\nu + \hat{D}^{(-)} \tag{4.1}$$

where $\hat{D}^{(-)}$ contains at most one derivative.

The basic point of the method is to assume that $\hat{F}$ can be embedded in a one-parameter family

$$\hat{F}(\tau) = \hat{G}\Box + \tau\hat{J}^{\mu\nu}\nabla_\mu\nabla_\nu + \hat{D}^{(-)} \tag{4.2}$$

[1] Both Yang-Mills theory and General Relativity can be taken to this form by an appropiate choice of gauge fixing.

[2] Here we deal with a second order operator but the technique can be trivially extended to higher order operators.

such that for $\tau = 0$ we get a minimal operator and for $\tau = 1$ we go back to our starting point.

Now, let us apply formula (3.24) to the case of $\hat{F}(\tau)$. By differentiating with respect to τ we find that

$$\frac{d}{d\tau}\bar{\Gamma}[\bar{\phi}](\tau) = \frac{1}{2}\log\left[\det(\hat{F}(\tau))\right] = \frac{1}{2}\frac{d}{d\tau}\mathrm{Tr}\left[\log(\hat{F}(\tau))\right] = -\frac{1}{2}\mathrm{Tr}\left[\frac{d\hat{F}(\tau)}{d\tau}\hat{Q}(\tau)\right] \tag{4.3}$$

where $\hat{Q}(\tau)$ is the Green's function of $\hat{F}(\tau)$ defined as[3]

$$\hat{F}(\tau)\hat{Q}(\tau) = -\hat{\mathbb{I}} \tag{4.4}$$

By integrating now (4.3) between $\tau = 0$ and $\tau = 1$, the quantum effective action of the initial non-minimal operator $\hat{F}$ is obtained in terms of $\hat{Q}(\tau)$

$$\Gamma[\bar{\phi}](1) = \Gamma[\bar{\phi}](0) - \frac{1}{2}\int_0^1 d\tau \mathrm{Tr}\left[\frac{d\hat{F}(\tau)}{d\tau}\hat{Q}(\tau)\right] \tag{4.5}$$

where $\Gamma[\bar{\phi}](0)$ corresponds to the effective action of a minimal operator and can be computed by the Schwinger-DeWitt technique of chapter 3.

All the problem has been moved to the issue of computing the Green's function $\hat{Q}(\tau)$ and the traces inside (4.5). In order to do this, perturbation theory in τ seems to be the obvious way but it is however very inefficient. The expression of $\hat{Q}(\tau)$ will be in general a non-polynomial function in τ and thus its computation requires to sum up all the different graphs to obtain terms of order $O(1)$. A more efficient expansion is obtained by considering an expansion in scaling dimension[4]. The reason is that since we know that $\Gamma[\bar{\phi}]$ must be marginal in the UV, this uniquely fixes the scaling dimension of the one-loop logarithmic divergences to be $\lambda = 4$. Then, we can expand all the quantities in the problem by using this parameter, throwing away those that enjoy $\lambda > 4$ since they will be irrelevant for the one-loop computation. By doing this, it is important to have in mind that covariant derivatives have scaling dimension $\lambda_\nabla = 1$ and that the Riemann tensor has $\lambda_R = 2$. That implies that every commutator of covariant derivatives will produce

[3]Note that this definition introduces an extra minus sign when taking the derivative of the logarithm.

[4]Also referred as to *background dimension* in [53].

terms with increasing scaling dimension and that by reordering derivatives we have a natural power series in λ which is finite up to the desired order. The key idea is then to go first to flat space and introduce a four-momentum p^μ. The principal part of $\hat{F}(\lambda)$, which we will denote by $\hat{D}(\nabla)$ from now, can be inverted there in terms of simple fractions

$$\hat{D}^{-1}(p) = \frac{\hat{K}(p)}{(p^2)^m}, \qquad \hat{D}(p)\hat{K}(p) = \left(p^2\right)^m \hat{\mathbb{I}} \tag{4.6}$$

with some power m which will depend on particular details of the operator.

Going back to curved space, the only difference will be the order of derivatives due to its anticommutative character. However, as we said, this will only produce lower order terms that will increase their background dimensionality and then they will be cut off at some point. Then, we have

$$\hat{D}(\nabla)\hat{K}(\nabla) = \hat{\mathbb{I}}\,\Box^m + \hat{K}_1(\nabla) \tag{4.7}$$

where $\hat{K}_1$ will contain at most $2m-1$ derivatives.

The same happens when we do the trick with the full operator $\hat{F}(\tau)$. By multiplying it by $\hat{K}(\nabla)$ we produce a higher order minimal operator

$$\hat{F}(\tau)\hat{K}(\nabla) = \hat{\mathbb{I}}\,\Box^m + \hat{M} \tag{4.8}$$

with some lower order terms contained in $\hat{M}$.

Now this can be worked out by using usual perturbation theory and write the inverse of $\hat{F}(\tau)$, in the sense of (4.4), as

$$\hat{Q}(\tau) = -\hat{K}(\nabla)\frac{\hat{\mathbb{I}}}{\Box^m}\sum_{p=0}^{4}\left(-\hat{M}\frac{\hat{\mathbb{I}}}{\Box^m}\right)^p + O(\Omega^5) \tag{4.9}$$

where we have cut-off the series at scaling dimension $\lambda = 4$.

Here $\frac{\hat{\mathbb{I}}}{\Box^m}$ in the m-th power of the inverse laplacian in a operator sense and we will clarify how to compute it in a while. For now, let us just commute all of them to the right, rewriting last formula as

$$\hat{Q}(\tau) = -\hat{K}(\nabla)\sum_{p=0}^{4}\hat{M}_p\frac{\hat{\mathbb{I}}}{\Box^{m(p+1)}} + O(\Omega^5) \tag{4.10}$$

where the operators $\hat{M}_p$ are given recursively by the rule

$$\hat{M}_0 = \hat{\mathbb{I}} \tag{4.11}$$

$$\hat{M}_p + 1 = \hat{M}\hat{M}_p + \left[\hat{\mathbb{I}}\Box^m, \hat{M}_p\right] \tag{4.12}$$

Furthermore, we are not going to proof it here, but it can be checked that when the metric accompanying the minimal factor is covariantly conserved

$$\nabla_\mu \hat{G} = 0 \tag{4.13}$$

and there are no linear derivative term in the original operator $\hat{F}$, then both $\hat{M}_3$ and $\hat{M}_4$ are automatically of scaling dimension equal or greater than five and we have just that

$$\hat{M}_1 = \hat{M}, \qquad \hat{M}_2 = M^2 + m[\hat{\mathbb{I}}\Box, \hat{M}]\hat{\mathbb{I}}\Box^{m-1}, \qquad \hat{M}_3 = \hat{M}_4 = O(\Omega^5) \tag{4.14}$$

Expression (4.10) gives then a closed operative way to compute the Green's function $\hat{Q}(\tau)$ as a series of elements ordered by its scaling dimension. Once this is done, the final step is to take care of the different functional traces in (4.5). All these traces have, and that is why we have commuted all the inverse laplacians to the right, the general form

$$\nabla_{\mu_1}\nabla_{\mu_2}...\nabla_{\mu_p}\frac{\hat{\mathbb{I}}}{\Box^m} \tag{4.15}$$

They can be computed by using the Heat Kernel representation for $\Box$. In order to do that, we write

$$\frac{\hat{\mathbb{I}}}{\Box^m} = \frac{1}{(m-1)!}\int_0^\infty ds\ s^{m-1}e^{-s\hat{\mathbb{I}}\Box} = \frac{1}{(m-1)!}\int_0^\infty ds\ s^{m-1}\hat{K}(s,1,\hat{\mathbb{I}}\Box) \tag{4.16}$$

In opposite to the case of a minimal operator now we need the the untraced Heat Kernel in order to apply to them the derivatives in the trace and then take the trace afterwards. But we are lucky, since the untraced Heat Kernel also enjoys a short-time expansion that can be used to extract divergences. In this case, it reads[53]

$$\hat{K}(s,1,\hat{\mathbb{I}}\Box) = \frac{\mathcal{D}^{1/2}(x,x')}{(4\pi)^{\frac{n}{2}}}e^{-\frac{\sigma(x,x')}{2s}}\sum_{k=0}^\infty s^{(k-n)/2}\hat{A}_k(x,x') \tag{4.17}$$

Here $\hat{A}_n(x,x')$ are the corresponding Heat Kernel coefficients, made up of local invariants of the manifold, but in opposite to the $a_n(\epsilon,\hat{D})$ coefficients of chapter 3 now they will be operators. $\sigma(x,x') = \frac{1}{2}\sigma_\mu\sigma^\mu$ is the world-function, representing the geodesic path between x and x' travelled in a proper time s and $\mathcal{D}^{1/2}(x,x)$, known as the Van-Vleck determinant, takes care of the integration measure connecting both points. The trace of these objects, also known as *coincidence lmit*, can be taken with the help of recursive relations. A detailed table of these can be found in section 4 of [53] but they can be also derived from the basic ones, inherited from the Heat equation

$$
\begin{aligned}
&\sigma^\mu \nabla_\mu \hat{A}_0(x,x') = 0, \qquad \hat{A}_0(x,x') = \hat{\mathbb{I}} \\
&\sigma(x,x) = 0, \qquad \nabla_\mu \sigma(x,x') = 0, \qquad \nabla_\mu \nabla_\nu \sigma(x,x') = g_{\mu\nu} \\
&(n+1)\hat{A}_{2(n+1)}(x,x') + \sigma^\mu \nabla_\mu \hat{A}_{2(n+1)}(x,x') = \mathbf{\Delta}^{-1/2} \hat{\mathbb{I}} \Box (\mathbf{\Delta}^{1/2} \hat{A}_{2n}(x,x'))
\end{aligned}
$$

where $\mathcal{D}(x,x') = g^{1/2}(x) g^{1/2}(x') \mathbf{\Delta}$

After taking this representation, one just has to apply all the derivatives in (4.15), plug them in the definition (4.10) of $\hat{Q}(\tau)$ and take finally the trace with the help of the recursive relations. Integration in s will remain, but only those traces satisfying $p - 2n + 4 \leq 4$ will contain divergences, all of them arising from the basic integral

$$
\int_0^\infty \frac{ds}{s^{n/2+k}}, \quad \text{with} \quad k = -1, 0, 1 \tag{4.18}
$$

whose pole part can be extracted by the method of part integration, which gives the result of the integral as a Laurent series.

This generalized Schwinger-DeWitt technique here presented may seem very involved but it provides a closed technique to be used to compute the quantum effective action for a huge variety of different theories, regardless their complicated differential structure. In particular, it will be of great help when we come to the issue of quantizing Unimodular Gravity.

4.1. The functional traces The biggest computational challenge for the technique we have described is to compute the different functional traces that appear in (4.15). Almost all the traces we are going to need in Part III are contained in [53] but we have rederived them for completeness. Here we show them together with the new one which we needed to compute from scratch.

$$
\begin{aligned}
&\nabla_\mu\nabla_\nu\nabla_\alpha\nabla_\beta\frac{\mathbb{I}}{\Box^2} = \\
&\frac{\sqrt{g}}{8(n-4)\pi^2}\Bigg\{\Bigg[\frac{1}{36}\left(R_{\mu\nu}R_{\alpha\beta}+R_{\mu\alpha}R_{\nu\beta}+R_{\mu\beta}R_{\nu\alpha}\right)+\frac{1}{180}\Big(R^\lambda_\mu\left(11R_{\nu\alpha\beta\lambda}-R_{\beta\alpha\nu\lambda}\right)+ \\
&+R^\lambda_\nu\left(11R_{\mu\alpha\beta\lambda}-R_{\beta\alpha\mu\lambda}\right)+R^\lambda_\alpha\left(11R_{\mu\nu\beta\lambda}-R_{\beta\nu\mu\lambda}\right)+R^\lambda_\beta\left(11R_{\mu\nu\alpha\lambda}-R_{\alpha\nu\mu\lambda}\right)\Big)+ \\
&+\frac{1}{90}\left(R_\mu{}^\lambda{}_\nu{}^\sigma\left(R_{\lambda\alpha\sigma\beta}+R_{\lambda\beta\sigma\alpha}\right)+R_\mu{}^\lambda{}_\alpha{}^\sigma\left(R_{\lambda\nu\sigma\beta}+R_{\lambda\beta\sigma\nu}\right)+R_\mu{}^\lambda{}_\beta{}^\sigma\left(R_{\lambda\nu\sigma\alpha}+R_{\lambda\alpha\sigma\nu}\right)\right)+ \\
&+\frac{1}{20}\left(\nabla_\mu\nabla_\nu R_{\alpha\beta}+\nabla_\mu\nabla_\alpha R_{\nu\beta}+\nabla_\mu\nabla_\beta R_{\nu\alpha}+\nabla_n\nabla_a R_{\mu\beta}+\nabla_\nu\nabla_\beta R_{\mu\alpha}+\nabla_a\nabla_\beta R_{\mu\nu}\right)\Bigg]\mathbb{I}+ \\
&+\frac{1}{12}\left[R_{\mu\nu}\hat{F}_{\alpha\beta}+R_{\mu\alpha}\hat{F}_{\nu\beta}+R_{\mu\beta}\hat{F}_{\nu\alpha}+R_{\nu\alpha}\hat{F}_{\mu\beta}+R_{\nu\beta}\hat{F}_{\mu\alpha}+R_{\alpha\beta}\hat{F}_{\mu\nu}\right]+ \\
&+\frac{1}{2}\left[\nabla_\mu\nabla_\nu\hat{F}_{\alpha\beta}+\nabla_\mu\nabla_\alpha\hat{F}_{\nu,b}+\nabla_\nu\nabla_\alpha\hat{F}_{\mu\beta}\right]+\frac{1}{8}\Big[\hat{F}_{\mu\nu}\hat{F}_{\alpha\beta}+\hat{F}_{\alpha\beta}\hat{F}_{\mu\nu}+\hat{F}_{\mu\alpha}\hat{F}_{\nu\beta}+ \\
&+\hat{F}_{\nu\beta}\hat{F}_{\mu\alpha}+\hat{F}_{\mu\beta}\hat{F}_{\nu\alpha}+\hat{F}_{\nu\alpha}\hat{F}_{\mu\beta}\Big]-\frac{1}{12}\Big[\hat{F}_{\mu\lambda}\left(R^\lambda_{\alpha\nu\beta}+R^\lambda_{\beta\nu\alpha}\right)+\hat{F}_{\nu\lambda}\left(R^\lambda_{\alpha\mu\beta}+R^\lambda_{\beta\mu\alpha}\right)+ \\
&+\hat{F}_{\alpha\lambda}\left(R^\lambda_{\nu\mu\beta}+R^\lambda_{\beta\mu\nu}\right)+\hat{F}_{\beta\lambda}\left(R^\lambda_{\mu\nu\alpha}+R^\lambda_{\alpha\nu\mu}\right)\Big]-\frac{1}{2}\Bigg[-\frac{1}{9}\left(R_{\alpha\mu\beta\nu}+R_{\beta\mu\alpha\nu}\right)R\mathbb{I}+ \\
&+g_{\mu\nu}\Bigg[\left(\frac{1}{36}R_{\alpha\beta}R+\frac{1}{90}R^{\lambda\sigma}R_{\lambda\alpha\sigma\beta}+\frac{1}{90}R_{\rho\sigma\lambda\alpha}R^{\rho\sigma\lambda}{}_\beta-\frac{1}{45}R_{\alpha\lambda}R^\lambda_\beta+\frac{1}{60}\Box R_{\alpha\beta}+\frac{1}{20}\nabla_\alpha\nabla_\beta R\right)\mathbb{I}+ \\
&+\frac{1}{12}\left(\hat{F}_{\alpha\lambda}\hat{F}^\lambda_\beta+\hat{F}_{\beta\lambda}\hat{F}^\lambda_\alpha\right)-\frac{1}{12}\left(\nabla_\alpha\nabla^\lambda\hat{F}_{\lambda\beta}+\nabla_\beta\nabla^\lambda\hat{F}_{\lambda\alpha}\right)+\frac{1}{12}R\hat{F}_{\alpha\beta}\Bigg]+ \\
&+g_{\mu\alpha}\Bigg[\left(\frac{1}{36}R_{\nu\beta}R+\frac{1}{90}R^{\lambda\sigma}R_{\lambda\nu\sigma\beta}+\frac{1}{90}R_{\rho\sigma\lambda\nu}R^{\rho\sigma\lambda}{}_\beta-\frac{1}{45}R_{\nu\lambda}R^\lambda_\beta+\frac{1}{60}\Box R_{\nu\beta}+\frac{1}{20}\nabla_\nu\nabla_\beta R\right)\mathbb{I}+ \\
&+\frac{1}{12}\left(\hat{F}_{\nu\lambda}\hat{F}^\lambda_\beta+\hat{F}_{\beta\lambda}\hat{F}^\lambda_\nu\right)-\frac{1}{12}\left(\nabla_\nu\nabla^\lambda\hat{F}_{\lambda\beta}+\nabla_\beta\nabla^\lambda\hat{F}_{\lambda\nu}\right)+\frac{1}{12}R\hat{F}_{\nu\beta}\Bigg]+ \\
&+g_{\mu\beta}\Bigg[\left(\frac{1}{36}R_{\nu\alpha}R+\frac{1}{90}R^{\lambda\sigma}R_{\lambda\nu\sigma\alpha}+\frac{1}{90}R_{\rho\sigma\lambda\nu}R^{\rho\sigma\lambda}{}_\alpha-\frac{1}{45}R_{\nu\lambda}R^\lambda_\alpha+\frac{1}{60}\Box R_{\nu\alpha}+\frac{1}{20}\nabla_\nu\nabla_\alpha R\right)\mathbb{I}+ \\
&+\frac{1}{12}\left(\hat{F}_{\nu\lambda}\hat{F}^\lambda_\alpha+\hat{F}_{\alpha\lambda}\hat{F}^\lambda_\nu\right)-\frac{1}{12}\left(\nabla_\nu\nabla^\lambda\hat{F}_{\lambda\alpha}+\nabla_\alpha\nabla^\lambda\hat{F}_{\lambda\nu}\right)+\frac{1}{12}R\hat{F}_{\nu\alpha}\Bigg]+ \\
&+g_{\nu\alpha}\Bigg[\left(\frac{1}{36}R_{\mu\beta}R+\frac{1}{90}R^{\lambda\sigma}R_{\lambda\mu\sigma\beta}+\frac{1}{90}R_{\rho\sigma\lambda\mu}R^{\rho\sigma\lambda}{}_\beta-\frac{1}{45}R_{\mu\lambda}R^\lambda_\beta+\frac{1}{60}\Box R_{\mu\beta}+\frac{1}{20}\nabla_\mu\nabla_\beta R\right)\mathbb{I}+ \\
&+\frac{1}{12}\left(\hat{F}_{\mu\lambda}\hat{F}^\lambda_\beta+\hat{F}_{\beta\lambda}\hat{F}^\lambda_\mu\right)-\frac{1}{12}\left(\nabla_\mu\nabla^\lambda\hat{F}_{\lambda\beta}+\nabla_\beta\nabla^\lambda\hat{F}_{\lambda\mu}\right)+\frac{1}{12}R\hat{F}_{\mu\beta}\Bigg]+ \\
&+g_{\nu\beta}\Bigg[\left(\frac{1}{36}R_{\mu\alpha}R+\frac{1}{90}R^{\lambda\sigma}R_{\lambda\mu\sigma\alpha}+\frac{1}{90}R_{\rho\sigma\lambda\mu}R^{\rho\sigma\lambda}{}_\alpha-\frac{1}{45}R_{\mu\lambda}R^\lambda_\alpha+\frac{1}{60}\Box R_{\mu\alpha}+\frac{1}{20}\nabla_\mu\nabla_\alpha R\right)\mathbb{I}+ \\
&+\frac{1}{12}\left(\hat{F}_{\mu\lambda}\hat{F}^\lambda_\alpha+\hat{F}_{\alpha\lambda}\hat{F}^\lambda_\mu\right)-\frac{1}{12}\left(\nabla_\mu\nabla^\lambda\hat{F}_{\lambda\alpha}+\nabla_\alpha\nabla^\lambda\hat{F}_{\lambda\mu}\right)+\frac{1}{12}R\hat{F}_{\mu\alpha}\Bigg]+ \\
&+g_{\alpha\beta}\Bigg[\left(\frac{1}{36}R_{\mu\nu}R+\frac{1}{90}R^{\lambda\sigma}R_{\lambda\mu\sigma\nu}+\frac{1}{90}R_{\rho\sigma\lambda\mu}R^{\rho\sigma\lambda}{}_\nu-\frac{1}{45}R_{\mu\lambda}R^\lambda_\nu+\frac{1}{60}\Box R_{\mu\nu}+\frac{1}{20}\nabla_\mu\nabla_\nu R\right)\mathbb{I}+ \\
&+\frac{1}{12}\left(\hat{F}_{\mu\lambda}\hat{F}^\lambda_\nu+\hat{F}_{\nu\lambda}\hat{F}^\lambda_\mu\right)-\frac{1}{12}\left(\nabla_\mu\nabla^\lambda\hat{F}_{\lambda\nu}+\nabla_\nu\nabla^\lambda\hat{F}_{\lambda\mu}\right)+\frac{1}{12}R\hat{F}_{\mu\nu}\Bigg]\Bigg]+ \\
&+\frac{1}{4}\left(g_{\mu\nu}g_{\alpha\beta}+g_{\mu\alpha}g_{\nu\beta}+g_{\mu\beta}g_{\nu\alpha}\right)\Bigg[\Bigg[\frac{1}{180}\left(R_{\lambda\sigma\rho\gamma}R^{\lambda\sigma\rho\gamma}-R_{\lambda\partial}R^{\lambda\sigma}\right)+\frac{1}{30}\Box R-\frac{1}{72}R^2\Bigg]\mathbb{I}+ \\
&+\frac{1}{12}\hat{F}_{\lambda\sigma}\hat{F}^{\lambda\sigma}\Bigg]\Bigg\}
\end{aligned}
\tag{4.19}
$$

$$\nabla_\mu\nabla_\nu\frac{\mathbb{I}}{\Box}=$$
$$\frac{\sqrt{g}}{8(n-4)\pi^2}\frac{1}{2}\left\{\left[-g_{\mu\nu}\left(\frac{1}{180}R_{\alpha\beta\lambda\sigma}R^{\alpha\beta\lambda\sigma}-\frac{1}{180}R_{\alpha\beta}R^{\alpha\beta}+\frac{1}{72}R^2+\frac{1}{30}\Box R\right)\mathbb{I}+\right.\right.$$
$$+\frac{1}{45}R^{\alpha\beta}R_{\alpha\mu\beta\nu}+\frac{1}{45}R_{\alpha\beta\lambda\mu}R^{\alpha\beta\lambda}_{\nu}-\frac{2}{45}R_{\mu\alpha}R^{\alpha}_{\nu}+\frac{1}{18}RR_{\mu\nu}+\frac{1}{30}\Box R_{\mu\nu}+$$
$$\left.+\frac{1}{10}\nabla_\mu\nabla_\nu R\right]-\frac{1}{12}g_{\mu\nu}\hat{F}_{\alpha\beta}\hat{F}^{\alpha\beta}+\frac{1}{6}R\hat{F}_{\mu\nu}+\frac{1}{6}\hat{F}_{\mu\alpha}\hat{F}^{\alpha}_{\nu}+\frac{1}{6}\hat{F}_{\nu\alpha}\hat{F}^{\alpha}_{\mu}-$$
$$\left.-\frac{1}{6}\nabla_\mu\nabla^\alpha\hat{F}_{\alpha\nu}-\frac{1}{6}\nabla_\nu\nabla^\alpha\hat{F}_{\alpha\mu}\right\} \tag{4.20}$$

$$\nabla_\alpha\nabla_\beta\nabla_\mu\nabla_\nu\frac{\mathbb{I}}{\Box^3}=$$
$$-\frac{\sqrt{g}}{8(n-4)\pi^2}\frac{1}{4}\left\{-\frac{2}{6}R_{\mu\beta\nu\alpha}-\frac{2}{6}R_{\nu\beta\mu\alpha}\mathbb{I}+g_{\mu\nu}\left(\frac{1}{6}R_{\alpha\beta}\mathbb{I}+\frac{1}{2}\hat{F}_{\alpha\beta}\right)+\right.$$
$$+g_{\beta\nu}\left(\frac{1}{6}R_{\alpha\mu}\mathbb{I}+\frac{1}{2}\hat{F}_{\alpha\mu}\right)+g_{\alpha\nu}\left(\frac{1}{6}R_{\mu\beta}\mathbb{I}+\frac{1}{2}\hat{F}_{\mu\beta}\right)+g_{\beta\mu}\left(\frac{1}{6}R_{\alpha\nu}\mathbb{I}+\frac{1}{2}\hat{F}_{\alpha\nu}\right)+$$
$$\left.+g_{\alpha\mu}\left(\frac{1}{6}R_{\beta\nu}\mathbb{I}+\frac{1}{2}\hat{F}_{\beta\nu}\right)+g_{\alpha\beta}\left(\frac{1}{6}R_{\mu\nu}\mathbb{I}+\frac{1}{2}\hat{F}_{\mu\nu}\right)-\frac{1}{12}g^{(2)}_{\mu\nu\alpha\beta}\mathbb{I}\right\} \tag{4.21}$$

$$\nabla_\alpha\nabla_\beta\nabla_\mu\nabla_\nu\nabla_\sigma\nabla_\lambda\frac{\mathbb{I}}{\Box^3}=$$
$$-\frac{\sqrt{g}}{8(n-4)\pi^2}\frac{1}{6}\left\{g^{(2)}_{\mu\nu\alpha\beta}\hat{B}_{\sigma\lambda}+g^{(2)}_{\mu\nu\alpha\sigma}\hat{B}_{\beta\lambda}+g^{(2)}_{\mu\nu\beta\sigma}\hat{B}_{\alpha\lambda}+g^{(2)}_{\mu\alpha\beta\sigma}\hat{B}_{\nu\lambda}+\right.$$
$$+g^{(2)}_{\mu\nu\alpha\lambda}\hat{B}_{\beta\sigma}+g^{(2)}_{\mu\nu\beta\lambda}\hat{B}_{\alpha\sigma}+g^{(2)}_{\mu\alpha\beta\lambda}\hat{B}_{\nu\sigma}+g^{(2)}_{\nu\alpha\beta\lambda}\hat{B}_{\mu\sigma}+g^{(2)}_{\mu\nu\sigma\lambda}\hat{B}_{\alpha\beta}+$$
$$+g^{(2)}_{\mu\alpha\sigma\lambda}\hat{B}_{\nu\beta}+g^{(2)}_{\nu\alpha\sigma\lambda}\hat{B}_{\mu\beta}+g^{(2)}_{\mu\beta\sigma\lambda}\hat{B}_{\nu\alpha}+g^{(2)}_{\nu\beta\sigma\lambda}\hat{B}_{\mu\alpha}+g^{(2)}_{\alpha\beta\sigma\lambda}\hat{B}_{\mu\nu}-$$
$$-\frac{1}{12}\left[g_{\sigma\lambda}\left(R_{\beta\nu\alpha\mu}+R_{\alpha\nu\beta\mu}\right)+g_{\beta\lambda}\left(R_{\sigma\nu\alpha\mu}+R_{\alpha\nu\sigma\mu}\right)+g_{\alpha\lambda}\left(R_{\sigma\nu\beta\mu}+R_{\beta\nu\sigma\mu}\right)+\right.$$
$$+g_{\nu\lambda}\left(R_{\sigma\alpha\beta\mu}+R_{\beta\alpha\sigma\mu}\right)+g_{\mu\lambda}\left(R_{\sigma\alpha\beta\nu}+R_{\beta\alpha\sigma\nu}\right)+g_{\beta\sigma}\left(R_{\lambda\nu\alpha\mu}+R_{\alpha\nu\lambda\mu}\right)+$$
$$+g_{\alpha\sigma}\left(R_{\lambda\nu\beta\mu}+R_{\beta\nu\lambda\mu}\right)+g_{\mu\sigma}\left(R_{\lambda\alpha\beta\nu}+R_{\beta\alpha\lambda\nu}\right)+g_{\alpha\beta}\left(R_{\lambda\nu\sigma\mu}+R_{\sigma\nu\lambda\mu}\right)+$$
$$+g_{\nu\beta}\left(R_{\lambda\alpha\sigma\mu}+R_{\sigma\alpha\lambda\mu}\right)+g_{\mu\beta}\left(R_{\lambda\alpha\sigma\nu}+R_{\sigma\alpha\lambda\nu}\right)+g_{\nu\alpha}\left(R_{\lambda\beta\sigma\mu}+R_{\sigma\beta\lambda\mu}\right)+$$
$$\left.\left.+g_{\mu\alpha}\left(R_{\lambda\beta\sigma\nu}+R_{\sigma\beta\lambda\nu}\right)+g_{\mu\nu}\left(R_{\lambda\beta\sigma\alpha}+R_{\sigma\beta\lambda\alpha}\right)+\frac{1}{8}g^{(3)}_{\mu\nu\alpha\beta\sigma\lambda}R\right]\mathbb{I}\right\} \tag{4.22}$$

$$\nabla_\mu\frac{\mathbb{I}}{\Box}=\frac{\sqrt{g}}{8(n-4)\pi^2}\left(\frac{1}{12}\nabla_\mu R\mathbb{I}-\frac{1}{6}\nabla^\nu\hat{F}_{\nu\mu}\right) \tag{4.23}$$

$$\frac{\mathbb{I}}{\Box}=\frac{\sqrt{g}}{8(n-4)\pi^2}\frac{1}{6}R\mathbb{I} \tag{4.24}$$

$$\nabla_\mu \nabla_\nu \frac{\mathbb{I}}{\Box^2} = -\frac{\sqrt{g}}{8(n-4)\pi^2}\left[\frac{1}{6}\left(R_{\mu\nu} - \frac{1}{2}g_{\mu\nu}R\right)\mathbb{I} + \frac{1}{2}\hat{F}_{\mu\nu}\right] \quad (4.25)$$

$$\nabla_{\mu_1}.....\nabla_{\mu_{2n-4}}\frac{\mathbb{I}}{\Box^n} = -\frac{\sqrt{g}}{8(n-4)\pi^2}\frac{g^{(n-2)}_{\mu_1....\mu_{2n-4}}}{2^{n-2}(n-1)!} \quad (4.26)$$

where we have defined

$$\begin{aligned}
&g^{(0)} = 1\\
&g^{(1)}_{\mu\nu} = g_{\mu\nu}\\
&g^{(2)}_{\mu\nu\alpha\beta} = g_{\mu\alpha}g_{\nu\beta} + g_{\mu\beta}g_{\nu\alpha} + g_{\mu\nu}g_{\alpha\beta}\\
&g^{(3)}_{\mu\nu\alpha\beta\sigma\lambda} = g_{\mu\nu}g^{(2)}_{\alpha\beta\sigma\lambda} + g_{\mu\alpha}g^{(2)}_{\nu\beta\sigma\lambda} + g_{\mu\beta}g^{(2)}_{\nu\alpha\sigma\lambda} + g_{\mu\sigma}g^{(2)}_{\nu\alpha\beta\lambda} + g_{\mu\lambda}g^{(2)}_{\nu\alpha\beta\sigma}\\
&g^{(n+1)}_{\mu_1....\mu_{2n+2}} = \sum_{i=2}^{2n+2} g_{\mu_1\mu_i}g^{(n)}_{\mu_2...\mu_{i-1}\mu_{i+1}\mu_{2n+2}}\\
&\hat{B}_{\alpha\beta} = \frac{1}{24}R_{\alpha\beta}\mathbb{I} + \frac{1}{8}\hat{F}_{\alpha\beta} \qquad (4.27)
\end{aligned}$$

4.2. The case of $G(\Box)$ operators

For the computation of Part III we are going to need also the contribution to the effective action of operators which are scalar functions of the laplacian $G(\Box)$. However, this is not a problem since its Heat Kernel coefficient can be easily computed by means of the definition of the zeta function.

Given the asymptotic expansion for the heat-kernel of an operator $\Box$, one can easily write the heat-kernel coefficients of any polynomial $G(\Box)$ in terms of those of $\Box$ [54, 55, 51]. Here we will make this relation by using the elegant derivation in terms of ζ functions present in [55, 54]. As shown in [55], given an operator $\Box$ of class u acting on a bundle in a D dimensional manifold with asymptotic expansion

$$Tr(\epsilon e^{-\tau\Box}) \sim \sum_{k=0}^{\infty} a_k(\epsilon, \Box)\tau^{(k-D)/u} \quad (4.28)$$

the following asymptotic expansion holds

$$Tr(\epsilon\,\Box^m e^{-\tau\Box^n}) \sim \sum_{k=0}^{\infty} a_k(\epsilon, \Box; n, m)\tau^{(k-D-mu)/(un)} \quad (4.29)$$

with

$$a_k(\epsilon,\Box;n,m) = \frac{1}{n}\lim_{\epsilon\to 0}\left[\Gamma\left(\frac{D-k+\epsilon}{u}\right)^{-1}\Gamma\left(\frac{D-k+\epsilon}{un}+\frac{m}{un}\right)\right]a_k(\epsilon,\Box) \tag{4.30}$$

Notice that for $m, n \in \mathbb{N}$, the previous coefficient is always finite. Furthermore, it vanishes for $(k-D)/u \in \mathbb{N}$ unless $(k-D-m)/(un) \in \mathbb{N}$. For a polynomial $G(\Box) = \sum_r^n c_r \Box^r$, one has

$$Tr(\epsilon e^{-\tau G(\Box)}) \sim \sum_k a_k(\epsilon, G(\Box))\tau^{(k-D)/(un)} \tag{4.31}$$

To find the heat-kernel coefficients, one uses (4.29) to compute (one can always take $c_n = 1$)

$$Tr(\epsilon e^{-\tau G(\Box)}) = Tr\left(\epsilon e^{-\tau\Box^n}\sum_m \frac{(G(\Box)-\Box^n)^m(-\tau)^m}{m!}\right) \tag{4.32}$$

Here we have rewritten $G(\Box) = \Box^n + (G(\Box) - \Box^n)$ and expanded the exponential of the latter in a power series. Correspondingly, the final result can be organised as a series

$$Tr(\epsilon e^{-\tau G(\Box)}) = \lim_{z\to 0}\sum_k a_k(\epsilon,\Box)\Gamma\left(\frac{D-k+z}{u}\right)^{-1}\int_0^\infty dq\, q^{(D-k+z)/u-1}e^{-\tau G(q)} \tag{4.33}$$

from where the different terms of fixed τ order in (4.29) can be easily retrieved. This result coincides with the expressions of [51, **?**]. Afterwards, one can use the method of functional traces to compute the extra contribution coming from any non-minimal piece in the operator of interest.

5

Dealing with gauge symmetries

In the previous chapters we introduced techniques, based upon the definition of the zeta function, to compute the one-loop renormalized effective action in the background field formalism. In doing so we assumed that the differential operator $\hat{D}$ that emerges from the second variation of the full action contained no zero modes or at least just a finite number of those. However, this is not the case for many of the most interesting physical theories out there and also of the ones we are going to take care in Parts II and III. In general, physical theories may contain zero modes as a consequence of gauge invariance, the fact that the action is invariant under some local (depending on the spacetime point) symmetry, where the fields ϕ transform under some group algebra. In that case, configurations that are related to the trivial $\phi = 0$ are said to be *pure gauge* and they appear as zero modes since they must be annihilated by the action regarded as an operator. Two well-knows examples of this are Yang-Mills theories, invariant under $SU(N)$ groups; and Einstein-Hilbert theory, which is invariant under diffeomorphisms.

Classically, gauge invariance only tells us that we have to supplement the equations of motion with an extra condition which allows to solve them, like for instance the well-known Lorentz gauge $\partial_\mu A^\mu = 0$ in QED. The issue is however more problematic when talking about a quantum theory. In the path integral approach, where we defined the partition function as

$$\mathcal{Z}[J] = \int [\mathcal{D}\phi] e^{iS(\phi)+i\langle\phi,J\rangle} \tag{5.1}$$

the gauge invariance reflects in the fact that the integration measure $[\mathcal{D}\phi]$ is not well-defined. Its integration domain is not compact and we are integrating more than once over the same physical configuration.

We do not expect here to write a detailed work on gauge symmetries and BRST invariance (to be introduced later) but just to present the latter as a useful tool to fix the gauge and be able to integrate over quantum dynamics with complicated symmetries, killing all the zero modes.

The usual way to come to grips with the problem of the ill-defined integration measure is to fix then the integration regime to a single orbit of the gauge group. This can be regarded as performing a redefinition of the integration variable which will introduce thus the determinant of a jacobian in the integral when expressed in terms of the original variable. Using the fact that a determinant can be written as a path integral suggest to exponentiate this factor and introduce a new set of fields called ghosts that compensate for this phenomena. This is the usual Faddeev-Popov method which is used extensively in quantum field theory and that, although it is very convenient for Yang-Mills theories, it is quite cumbersome when we need to deal with complicated symmetries.

An alternative which is more suitable to suit our needs here is to use the method of BRST (Becchi, Rouet, Stora and Tyutin)[56, 57]. Let us start by assuming that we have an action $S[\phi]$ depending on a set of fields ϕ which is invariant under some local symmetry which acts over the fields infinitesimally as

$$\phi \to \phi + R\omega + O(\omega^2) \tag{5.2}$$

where ω is the infinitesimal parameter of the transformation and R are the infinitesimal generators of the algebra associated to the invariance group.

The idea behind the method is to substitute the gauge invariance of the original theory by a new symmetry (the BRST invariance) which does not constrain the integration measure by itself. This symmetry is generated by an operator $\smallint$ which acts over the field as

$$\smallint \phi = Rc \tag{5.3}$$

where c is a grassman odd field which will take later the role of the ghost. In this way, the BRST operator acts on the physical field as a gauge transformation with an anti-commuting parameter, thus ensuring that the classical action will be still invariant. The BRST operator then has ghost number one and it is also grassman odd.

The transformation of the ghost under the BRST action is defined as the Lie bracket with itself

$$\smallint c = \frac{1}{2}[c, c] \tag{5.4}$$

which is not vanishing thanks to its anticommutativity.

By defining the action of $\int$ in this way, the original unfixed action automatically satisfies $\int S = 0$ due to gauge invariance. Moreover, because of the anticonmuting nature of c, the operator $\int$ is nilpotent over all the fields present here

$$\int^2 = 0 \tag{5.5}$$

The implications of this relation have a direct physical correspondence. Because the lagrangian will have now the continuous symmetry σ, there will be a conserved charge $\mathbf{Q}$ which commutes with the hamiltonian and thus it will divide the Hilbert space of the quantum theory at hand in three parts

- Those states $|\Psi_1\rangle$ which are not annihilated by $\mathbf{Q}$ will not belong to the physical Hilbert state of the system. We call this $\mathcal{H}_1$
- Those states which satisfy $|\Psi_2\rangle = \mathbf{Q}\,|\Psi_1\rangle$, where $|\Psi_1\rangle$ is in $\mathcal{H}_1$ and thus they are anhilated by a second application of $\mathbf{Q}$ due to nilpotency

  $$\mathbf{Q}\,|\Psi_2\rangle = \mathbf{Q}^2\,|\Psi_1\rangle = 0 \tag{5.6}$$

  We call this subspace $\mathcal{H}_2$.
- Those states which are annihilated by $\mathbf{Q}$

  $$\mathbf{Q}\,|\Psi_0\rangle = 0 \tag{5.7}$$

  but they are not in $\mathcal{H}_2$. We call this $\mathcal{H}_0$.

It is easy to see that $\mathcal{H}_2$ contains only states of zero norm and that all the states in $\mathcal{H}_2$ are orthogonal to those in $\mathcal{H}_0$.

Now, because of the way that BRST transformations act onto the fields, we see that non-physical polarizations of ϕ (the gauge modes) are taken to be ghosts by the action of $\int$, which are later annihilated by the second action of $\int$. This means that states containing gauge modes will belong to $\mathcal{H}_1$ while states containing ghosts will be in $\mathcal{H}_2$. States containing physical modes will be then contained in $\mathcal{H}_0$ and, since $\mathcal{H}_2$ and $\mathcal{H}_0$ are orthogonal, the scattering of two physical states will not receive contributions of $\mathcal{H}_2$, ensuring gauge independence of the S matrix.

This is all formal but it can be also exploited to construct a path integral formulation free from the sickness introduced by zero modes of the gauge symmetry. The idea is to introduce a new term in the action S_{gb} which is so that it satisfies two simple conditions

- It is BRST invariant

$$\smallint S_{gb} = 0 \tag{5.8}$$

- It is not gauge invariant

In order to construct such a term it is necessary to introduce two new extra fields b and f with transformations under the action of $\smallint$ obeying

$$\smallint b = f \tag{5.9}$$

$$\smallint f = 0 \tag{5.10}$$

The field b is what we called the anti-ghost and it is here to compensate the ghost number and grassman character of c in order to obtain a hermitian action term. It is thus also grassman odd and has ghost number -1. The other field, f, it is known as a Nakanishi-Lautrup auxiliary field and it is there in order to be able to close the BRST algebra $\smallint^2$ over all the fields.

Now, we can construct S_{gb} satisfying all the conditions above by simply taking it to be the BRST transformation of a local polynomial on b, f and ϕ of ghost number -1 and odd grassman character

$$S_{gb} = \smallint \int d^4x \, \mathbf{X} \tag{5.11}$$

so that it is automatically BRST invariant because of the nilpotency of $\smallint^2$.

By adding this piece to the action, we can now integrate over the physical field, the auxiliary field and the ghosts; since the path integral does not contain an infinite number of zero modes any more. Moreover, by choosing S_{gb} to be BRST invariant, any correlation function of a gauge invariant operator is automatically BRST invariant and, since correlation functions can only depende on the physical field ϕ and for this one, BRST transformations are gauge transformations, it will be also gauge invariant.

As a simple example of how this works, let us arrive to the usual Faddeev-Popov prescription by means of BRST invariance. We start by choosing the $\mathbf{X}$ polynomial to be

$$\mathbf{X} = b\,(F + f) \tag{5.12}$$

where $F = 0$ is the gauge fixing condition for the system.

Acting with the BRST operator, we find

$$\begin{aligned} S_{gb} &= \int d^4x \left\{ f\ F + f^2 - b\ \mathcal{s}F \right\} = \\ &= \int d^4x \left\{ \left(f + \frac{1}{2}F \right)^2 - \frac{1}{4}F^2 - b\ \mathcal{s}F \right\} \end{aligned} \tag{5.13}$$

where in the second step we have completed the square.

Now we can shift the f variable in such a way that

$$f \rightarrow f - \frac{1}{2}F \tag{5.14}$$

This change will have no effect at all in the path integral for f since it is just a local shift. After it, the gauge breaking term reduces to

$$S_{gb} = \int d^4x \left\{ f^2 - \frac{1}{4}F^2 - b\ \mathcal{s}F \right\} \tag{5.15}$$

We find then three terms to be add to the original action. The first one has no effect since it will just introduce an ultralocal factor related to the normalization of f that we can get rid of in connected amplitudes. The second one is what we would call the gauge fixing term in Faddeev-Poppov technique, while the last one is the ghost action, since $\mathcal{s}F$ is equivalent to a gauge transformation of the gauge fixing condition. We then arrive to the complete Faddeev-Popov condition by using the BRST operator method.

More complicated choices for $\mathbf{X}$ can be done, involving local operators acting on f and more bizarre combinations of different auxiliary fields but the logic behind is exactly the same as the one presented here.

Part II

Conformal Dilaton Gravity

6

Dilaton Gravity

As we thoroughly introduced in 2, the notion of scale invariance, upgraded to Weyl invariance in the presence of gravity, is believed to be relevant when studying physics a very short distances. Indeed, it is easy to construct a gravitational theory enjoying this symmetry by using the Weyl tensor. *Conformal (super)gravity* [58] is such a theory which in four space-time dimensions and is given by the lagrangian

$$L \equiv \sqrt{-g}\; C_{\mu\nu\rho\sigma}\; C^{\mu\nu\rho\sigma} \equiv \sqrt{|g|}\; C^2 \tag{6.1}$$

where $C_{\mu\nu\rho\sigma}$ is Weyl's tensor, the tracefree piece of Riemann's tensor. It is explicitily defined in terms of the Riemann tensor as

$$\begin{aligned} C_{\mu\nu\rho\sigma} \equiv R_{\mu\nu\rho\sigma} - \frac{1}{n-2}\left(g_{\mu\rho}R_{\nu\sigma} - g_{\mu\sigma}R_{\nu\rho} - g_{\nu\rho}R_{\mu\sigma} + g_{\nu\sigma}R_{\mu\rho}\right) + \\ + \frac{1}{(n-1)(n-2)} R\left(g_{\mu\rho}g_{\nu\sigma} - g_{\mu\sigma}g_{\nu\rho}\right) \end{aligned} \tag{6.2}$$

so that

$$C^2 \equiv R^2_{\mu\nu\rho\sigma} - \frac{4}{n-2}R^2_{\mu\nu} + \frac{2}{(n-1)\,(n-2)}R^2. \tag{6.3}$$

This theory is Weyl invariant in four dimensions only[1]. There are local invariants in arbitrary dimensions, involving derivatives of the Weyl tensor and the Fefferman-Graham obstruction, whose existence is guaranteed, but which is not known explicitly in general [59].

Conformal (super)gravities (as any other theory quadratic in curvature) are renormalizable and it has been argued that they can even be finite at the quantum level provided they have enough supersymmetry. Nevertheless, there is always some tension, at least at the perturbative level, with unitarity, because the propagator is quartic in the four-momentum, implying

[1] Here n is the space-time dimension. We will keep it arbitrary with an eye on using dimensional regularization later. However, all our results will be evaluated in four space-time dimensions only.

the existence of new poles that lead to either unitarity or causality violations. It is actually not clear in spite of some insightful attempts[60, 61] whether a non-perturbative unitary definition of the theory is possible at all. The topic has however attracted a renovated attention recently, since those theories, when coupled to a scale-invariant version of the Standard Model, seem to reproduce qualitatively well the known values for the coupling constants in nature[18].

It is nevertheless quite easy to construct a much simpler theory in which both dynamical gravitation and Weyl invariance are present and which is free of these problems by the procedure of *group averaging*, that is, perform a Weyl transformation on the Einstein-Hilbert lagrangian and promote the Weyl rescaling factor to the status of a new field.

Under a Weyl rescaling the Einstein-Hilbert lagrangian behaves as

$$\sqrt{|g|}\, R \rightarrow \sqrt{|g|}\left[\Omega^{n-2} R + (n-1)(n-2)\Omega^{n-4}(\nabla\Omega)^2\right] \tag{6.4}$$

where we have neglected a total derivative which yields a boundary term.

We then define a *gravitational scalar* field through

$$\Omega \equiv \frac{1}{M_p}\left(\frac{(n-2)}{4(n-1)}\right)^{\frac{1}{n-2}} \phi^{\frac{2}{n-2}} \tag{6.5}$$

(where the n-dimensional Planck mass is defined as $M_p^{n-2} \equiv \frac{1}{16\pi G_n}$) obtaining the lagrangian of our theory

$$S_{CDG} = \int d^n x \sqrt{|g|}\, \left(-\frac{n-2}{8(n-1)} R\phi^2 - \frac{1}{2} g^{\mu\nu}\nabla_\mu\phi\nabla_\nu\phi\right) \tag{6.6}$$

We end up with nothing more than the well known action for a scalar conformally coupled to gravity, with the difference that now we consider that gravity is dynamical. In this way, the field ϕ takes the role of a *dilaton* and we will dub this theory *Conformal Dilaton Gravity(CDG).*

It seems that the first to consider CDG was Dirac [62] in a very interesting paper in which he related the *large numbers hypothesis* with the old unified theory of Hermann Weyl. Other interesting pioneering works on this theory include[25, 63, 64, 65]. In those works, it was considered as a conformally invariant off-mass shell extension of quantum gravity in the context of the early attempts to understand the physical meaning of the conformal anomaly [21]. In particular and in Duff's words *"Real Weyl invariance has* *anomalies; pseudo-Weyl invariance (i.e. involving a compensator field) does not. This is a regularization-scheme-independent statement."*

This statement, as phrased by Duff, represents our main motivation to study Conformal Dilaton Gravity. As it is presented, this theory is classically equivalent to General Relativity. We have just redefined the fields to go from the *Einstein Frame*, where the kinetic term for the graviton has the standard form R, to the *Jordan Frame*[66], where the coupling $R\phi^2$ appears[2]. However, in the quantum theory this is not so obvious any more. In particular, let us recall our definition (3.24) for the quantum effective action in the background field approach

$$\Gamma[\bar{\phi}] = \frac{1}{2} \log \left[\det(\hat{D}) \right] \tag{6.7}$$

Under a transformation of the fields

$$Q' = Q'(Q) \tag{6.8}$$

it behaves as[67]

$$\Gamma^{'}[\bar{\phi}] = \frac{1}{2} \log \left[\det(\hat{D}') + \left(\frac{\partial Q'}{\partial Q} \right)^2 \frac{\partial^2 Q}{\partial Q'^2} \frac{\partial S}{\partial Q'} \right] \tag{6.9}$$

where the first term is the determinant of the corresponding transformed operator.

This implies that when redefining fields, only quantities which are evaluated on-shell ($\frac{\partial S}{\partial Q'} = 0$), in particular the S-matrix, are equivalent in both formulations, meaning that they are the transformation one of the other. This is a fact known since a long time ago and sometimes referred to as *Kallosh-DeWitt theorem*[68, 69, 40]. In the particular case of transformations between the Einstein and Jordan frame, this is of capital importance, since non-minimally coupled scalar-fields appear commonly in models of inflation and going to the simplest frame is a extensively used technique. One of the main results of this part of this thesis will be to shed new light to this question when Weyl invariance is present.

The above considerations are taken as a motivation to study the non-minimally coupled system gravitational-scalar field in the following sense

$$S = -\int d^n x \sqrt{|g|} \left(\xi R\phi^2 + \frac{1}{2}(\nabla\phi)^2 \right) \tag{6.10}$$

[2] Of course, if we couple matter to the system, the question of which metric we should consider as the physical one arises and equivalence is lost.

with in principle arbitrary non-minimal coupling ξ. Only when $\xi = \xi_c = \frac{n-2}{8(n-1)}$ is the symetry of the theory upgraded to include Weyl invariance. In that case, the scalar field enjoys scaling dimension $\lambda_\phi = (2-n)/2$. The global sign in front of the action is irrelevant as it stands, but it is the correct one to couple to a matter lagrangian containing matter fields.

The vacuum structure of this theory is quite peculiar. There is a global $\mathbb{Z}_2$ symmetry

$$\phi(x) \to -\phi(x) \tag{6.11}$$

which is promoted to an $U(1)$ when the scalar field is complex and ϕ^2 is replaced by $|\phi|^2$. There are then two different phases, depending on whether the background field vanishes or not. Only the vanishing solution is compatible with the $\mathbb{Z}_2$ symmetry. In this *symmetric phase*, quantum perturbations are defined around the symmetric classical solution

$$\bar{\phi}(x) = 0 \tag{6.12}$$

In this case there is no propagator for the gravitational fluctuation, and we do not know how to proceed (although some possible paths will be suggested later). In the *broken phase* we consider a classically non-vanishing solution

$$\bar{\phi}(x) \neq 0 \tag{6.13}$$

that determines the graviton propagator.

Our aim in this part is then to study dilaton gravity both in the both Weyl and non-Weyl invariant regimes. Using a combination of the background field and heat kernel techniques introduced in chapter 3, the one-loop effective action will be first determined for generic value of the coupling constant ξ. This calculation wont be valid at the conformal point, $\xi = \xi_c$, because there the gauge symmetry will be enhanced by Weyl invariance. If Kallosh-DeWitt theorem holds as it is stated before, then the counterterms of CDG and GR would be the transformation one of the other, preserving the group averaging character in the path integral computation. Our objective is to check whether this is the case or not.

7

Non-Conformal Dilaton Gravity

Let us begin by analyzing the non-Weyl invariant action of Dilaton Gravity, that is

$$S = -\int d^n x \sqrt{|g|}\, \left(\xi R \Phi^2 + \frac{1}{2} (\nabla \Phi)^2 \right) \tag{7.1}$$

with free coupling constant ξ. The reason for the notation Φ will be apparent in a moment.

We are going to obtain the one-loop effective action of this theory by means of the Schwinger-DeWitt technique described in Chapter 3. For that reason, let us start by expanding the action around an arbitrary background for both the gravitational and scalar fields

$$\begin{aligned} g_{\mu\nu} &= g_{\mu\nu} + h_{\mu\nu} \\ \Phi &= \bar{\phi} + \phi \end{aligned} \tag{7.2}$$

where we have omitted the bars over the background metric in order to keep the notation clean.

Demanding that the linear terms in the expansion cancel determines the background equations of motion (EM). When the background fields are so restricted, absence of tadpoles in the quantum theory is guaranteed. In four space-time dimensions and with arbitrary parameter ξ they read

$$\begin{aligned} \xi \bar{\phi} R_{\mu\nu} = &\frac{1}{4} g_{\mu\nu} \nabla^2 \bar{\phi} - \left(\frac{1}{2} - 2\xi \right) \frac{\nabla_\mu \bar{\phi} \nabla_\nu \bar{\phi}}{\bar{\phi}} - \left(2\xi - \frac{1}{4} \right) g_{\mu\nu} \frac{(\nabla \bar{\phi})^2}{\bar{\phi}} + \\ &+ 2\xi \nabla_\mu \nabla_\nu \bar{\phi} - 2\xi g_{\mu\nu} \nabla^2 \bar{\phi} \end{aligned} \tag{7.3}$$

$$\bar{\phi} R = \frac{1}{2\xi} \nabla^2 \bar{\phi} \tag{7.4}$$

while the second order terms determine the differential operator over which we have to integrate in order to compute the background field effective action. This is given by

$$S_2 = -\int d^n x \sqrt{|g|}\, (H + F + HF) \tag{7.5}$$

where

$$H = \xi\bar{\phi}^2 \left[-\frac{1}{4} h\nabla^2 h + \frac{1}{4} h^{\alpha\beta}\nabla^2 h_{\alpha\beta} - \frac{1}{2}\nabla_\mu h\nabla_\nu h^{\mu\nu} + \frac{1}{2}\nabla_\mu h^{\mu\alpha}\nabla_\nu h^\nu_\alpha + \right.$$
$$\left. + \frac{1}{8} h^2 R - \frac{1}{4} h_{\mu\nu}h^{\mu\nu} R - -\frac{1}{2} h h^{\alpha\beta} R_{\alpha\beta} + \frac{1}{2} h^{\mu\nu} h^\alpha_\nu R_{\mu\alpha} + \frac{1}{2} R_{\mu\nu\alpha\beta} h^{\mu\alpha} h^{\nu\beta} \right] +$$
$$+ \xi(\nabla_\alpha\bar{\phi}^2) \left(\frac{1}{4} h\nabla^\alpha h - \frac{3}{4} h^{\mu\nu}\nabla^\alpha h_{\mu\nu} + \frac{3}{2} h^{\alpha\beta}\nabla_\mu h^\mu_\beta + \frac{1}{2} h^{\mu\nu}\nabla_\mu h^\alpha_\nu - \right.$$
$$\left. - h^{\alpha\beta}\nabla_\beta h - \frac{1}{2} h\nabla_\beta h^{\alpha\beta} \right) + \frac{1}{2} h^{\mu\alpha} h^\nu_\alpha \nabla_\mu\bar{\phi}\nabla_\nu\bar{\phi} - \frac{1}{4} h h^{\mu\nu}\nabla_\mu\bar{\phi}\nabla_\nu\bar{\phi} +$$
$$- \frac{1}{8} h_{\mu\nu}h^{\mu\nu}\nabla_\alpha\bar{\phi}\nabla^\alpha\bar{\phi} + \frac{1}{16} h^2\nabla_\mu\bar{\phi}\nabla^\mu\bar{\phi}$$

$$F = \frac{1}{2}\nabla^\alpha\phi\nabla_\alpha\phi + \xi R\phi^2$$

$$HF = \xi\bar{\phi}\phi\left(-2h^{\mu\nu}R_{\mu\nu} + hR + 2\nabla_\mu\nabla_\nu h^{\mu\nu} - 2\nabla^2 h\right) - h^{\alpha\beta}\nabla_\alpha\phi\nabla_\beta\bar{\phi} +$$
$$+ \frac{1}{2} h\nabla_\mu\bar{\phi}\nabla^\mu\phi$$

Since gravitational fluctuations are symmetric tensors, $h_{\mu\nu} = h_{\nu\mu}$ only the symmetric part of the quadratic term contributes. We find convenient to define the operators

$$\mathcal{P}^{\alpha\beta}_{\mu\nu\rho\sigma} = \frac{1}{8}\left(g_{\mu\rho}\delta^\alpha_\nu\delta^\beta_\sigma + g_{\mu\sigma}\delta^\alpha_\nu\delta^\beta_\rho + g_{\nu\rho}\delta^\alpha_\mu\delta^\beta_\sigma + g_{\nu\sigma}\delta^\alpha_\mu\delta^\beta_\rho \right) + \frac{1}{8}(\alpha \leftrightarrow \beta) \quad (7.6)$$
$$\mathcal{K}^{\alpha\beta}_{\mu\nu\rho\sigma} = \frac{1}{4}\left(g_{\mu\nu}\delta^\alpha_\rho\delta^\beta_\sigma + g_{\rho\sigma}\delta^\alpha_\mu\delta^\beta_\nu \right) + \frac{1}{4}(\alpha \leftrightarrow \beta)$$

which project over the helicity states

$$h^{\mu\nu}h_{\mu\nu} = h^{\mu\nu}h^{\rho\sigma}\mathcal{P}^{\alpha\beta}_{\mu\nu\rho\sigma}g_{\alpha\beta}$$
$$h^2 = \mathcal{K}^{\alpha\beta}_{\mu\nu\rho\sigma}g_{\alpha\beta}h^{\mu\nu}h^{\rho\sigma} \quad (7.7)$$

and which simplify a lot the quadratic operators. Using them, the second order expanded action can be further reduced to

$$S_2 = -\int d^n x\sqrt{|g|}\ \left[h^{\mu\nu}\hat{H}_{\mu\nu\rho\sigma}h^{\rho\sigma} + \phi(\widehat{HF})_{\mu\nu}h^{\mu\nu} + \phi\hat{F}\phi + \right. \quad (7.8)$$
$$\left. + \xi\bar{\phi}^2\left(-\frac{1}{2}\nabla_\mu h\nabla_\nu h^{\mu\nu} + \frac{1}{2}\nabla_\mu h^{\mu\alpha}\nabla_\nu h^{\nu\alpha} \right) \right]$$

where we have kept apart the non-diagonal contributions to the graviton sector in order to cancel them later with a proper gauge fixing and where the corresponding operators are given by

$$
\begin{aligned}
\hat{H}_{\mu\nu\rho\sigma} = {} & \frac{\xi}{4}(\nabla_\alpha \bar{\phi}^2)\left(\left(\mathcal{K}^{\gamma\omega}_{\mu\nu\rho\sigma} - 3\mathcal{P}^{\gamma\omega}_{\mu\nu\rho\sigma}\right) g_{\gamma\omega} g^{\alpha\beta} + X^{\alpha\beta}_{\mu\nu\rho\sigma}\right)\nabla_\beta + \\
& + \left(\frac{1}{2}\mathcal{P}^{\alpha\beta}_{\mu\nu\rho\sigma} - \frac{1}{4}\mathcal{K}^{\alpha\beta}_{\mu\nu\rho\sigma}\right)\left(\nabla_\alpha\bar{\phi}\nabla_\beta\bar{\phi} - \frac{1}{4}g_{\alpha\beta}(\nabla\bar{\phi})^2\right) + \\
& + \xi\bar{\phi}^2\left[\frac{1}{4}\left(\mathcal{P}^{\alpha\beta}_{\mu\nu\rho\sigma} - \mathcal{K}^{\alpha\beta}_{\mu\nu\rho\sigma}\right) g_{\alpha\beta}\nabla^2 + \frac{1}{2}\left(\mathcal{P}^{\alpha\beta}_{\mu\nu\rho\sigma} - \mathcal{K}^{\alpha\beta}_{\mu\nu\rho\sigma}\right) R_{\alpha\beta} + \right. \\
& \left. + \frac{1}{2}R_{(\mu\rho\nu\sigma)} + \left(\frac{1}{8}\mathcal{K}^{\alpha\beta}_{\mu\nu\rho\sigma} - \frac{1}{4}\mathcal{P}^{\alpha\beta}_{\mu\nu\rho\sigma}\right) g_{\alpha\beta} R\right]
\end{aligned}
\tag{7.9}
$$

with the tensor $X^{\alpha\beta}_{\mu\nu\rho\sigma}$ defined as

$$
\begin{aligned}
X^{\alpha\beta}_{\mu\nu\rho\sigma} = {} & \frac{3}{2}\left(g^\alpha_\mu g^\beta_\rho g_{\nu\sigma} + g^\alpha_\nu g^\beta_\rho g_{\mu\sigma} + g^\alpha_\mu g^\beta_\sigma g_{\nu\rho} + g^\alpha_\nu g^\beta_\sigma g_{\mu\rho}\right) - \\
& - \left(g^\alpha_\rho g^\beta_\sigma g_{\mu\nu} + g^\alpha_\sigma g^\beta_\rho g_{\mu\nu}\right) - 2\left(g^\alpha_\mu g^\beta_\nu g_{\rho\sigma} + g^\alpha_\nu g^\beta_\mu g_{\rho\sigma}\right) + \\
& + \frac{1}{2}\left(g^\alpha_\rho g^\beta_\mu g_{\nu\sigma} + g^\alpha_\rho g^\beta_\nu g_{\mu\sigma} + g^\alpha_\sigma g^\beta_\mu g_{\nu\rho} + g^\alpha_\sigma g^\beta_\nu g_{\mu\rho}\right)
\end{aligned}
\tag{7.10}
$$

and

$$
\begin{aligned}
(\widehat{HF})_{\mu\nu} = {} & \xi\bar{\phi}\left[R g_{\mu\nu} - 2R_{\mu\nu} + 2\nabla_\mu\nabla_\nu - 2g_{\mu\nu}\nabla^2\right] - \frac{1}{2}g_{\mu\nu}\nabla_\alpha\bar{\phi}\nabla^\alpha + \\
& + \nabla_\mu\nabla_\nu\bar{\phi} - \frac{1}{2}g_{\mu\nu}\nabla^2\bar{\phi} + \frac{1}{2}\left(\nabla_\mu\bar{\phi}\nabla_\nu + \nabla_\nu\bar{\phi}\nabla_\mu\right)
\end{aligned}
\tag{7.11}
$$

$$
\hat{F} = -\frac{1}{2}\nabla^2 + \xi R \tag{7.12}
$$

Here the round parenthesis mean complete symmetrization of the indices contained inside.

The kinetic term for the graviton is, however, non-standard. It contains a second power of the background scalar field $\bar{\phi}$ that complicates things. In particular, it makes the metric on the field space $\hat{G}$, as defined in (3.51) to be non-covariantly constant. To solve this, it is useful then to change variables to

$$
k_{\mu\nu} = \bar{\phi} h_{\mu\nu} \tag{7.13}
$$

in order to eliminate all the dependence on $\bar{\phi}$ out of the kinetic term. This only makes sense in the broken phase, since this transformation is ill-defined when $\bar{\phi} = 0$. It will also mean nothing for the path integral, since it will only add an ultralocal factor. In this case, the action is then rewritten as

$$S_2 = -\int d^n x\sqrt{|g|}\,\left[k^{\mu\nu}\hat{K}_{\mu\nu\rho\sigma}k^{\rho\sigma} + \phi(\widehat{KP})_{\mu\nu}k^{\mu\nu} + \phi\hat{P}\phi + \right. \\ \left. +\xi\left(-\frac{1}{2}\nabla_\mu k\nabla_\nu k^{\mu\nu} + \frac{1}{2}\nabla_\mu k^{\mu\alpha}\nabla_\nu k^\nu_\alpha\right)\right] \tag{7.14}$$

where the explicit values of the coefficients are now

$$\begin{aligned}\hat{H}_{\mu\nu\rho\sigma} =& + \left(\frac{1}{2}\mathcal{P}^{\alpha\beta}_{\mu\nu\rho\sigma} - \frac{1}{4}\mathcal{K}^{\alpha\beta}_{\mu\nu\rho\sigma}\right)\left(\frac{\nabla_\alpha\bar{\phi}\nabla_\beta\bar{\phi}}{\bar{\phi}^2} - \frac{1}{4}g_{\alpha\beta}\frac{(\nabla\bar{\phi})^2}{\bar{\phi}^2}\right) + \\ &+ 2\xi\frac{\nabla_\alpha\bar{\phi}}{\bar{\phi}}\left(\left(\frac{1}{2}\mathcal{K}^{\gamma\omega}_{\mu\nu\rho\sigma} - \mathcal{P}^{\gamma\omega}_{\mu\nu\rho\sigma}\right)g_{\gamma\omega}g^{\alpha\beta} + \frac{1}{2}\mathcal{K}_{\mu\nu\rho\sigma} + \frac{1}{4}Y^{\alpha\beta}_{\mu\nu\rho\sigma}\right)\nabla_\beta + \\ &+ \frac{\xi}{2}\left[\frac{1}{2}\left(\mathcal{P}^{\alpha\beta}_{\mu\nu\rho\sigma} - \mathcal{K}^{\alpha\beta}_{\mu\nu\rho\sigma}\right)g_{\alpha\beta}\left(2\frac{(\nabla\bar{\phi})^2}{\bar{\phi}^2} - \frac{\nabla^2\bar{\phi}}{\bar{\phi}}\right) + \right. \\ &\left. + \left(\mathcal{P}^{\alpha\beta}_{\mu\nu\rho\sigma} - \mathcal{K}^{\alpha\beta}_{\mu\nu\rho\sigma} - \left(\mathcal{K}^{\gamma\delta}_{\mu\nu\rho\sigma} - 3\mathcal{P}^{\gamma\delta}_{\mu\nu\rho\sigma}\right)g_{\gamma\delta}g^{\alpha\beta} - X^{\alpha\beta}_{\mu\nu\rho\sigma}\right)\frac{\nabla_\alpha\bar{\phi}\nabla_\beta\bar{\phi}}{\bar{\phi}^2}\right] + \\ &+ \xi\left[\frac{1}{4}\left(\mathcal{P}^{\alpha\beta}_{\mu\nu\rho\sigma} - \mathcal{K}^{\alpha\beta}_{\mu\nu\rho\sigma}\right)g_{\alpha\beta}\nabla^2 + \frac{1}{2}\left(\mathcal{P}^{\alpha\beta}_{\mu\nu\rho\sigma} - \mathcal{K}^{\alpha\beta}_{\mu\nu\rho\sigma}\right)R_{\alpha\beta} + \frac{1}{2}R_{(\mu\rho\nu\sigma)}\right. \\ &\left. + \left(\frac{1}{8}\mathcal{K}^{\alpha\beta}_{\mu\nu\rho\sigma} - \frac{1}{4}\mathcal{P}^{\alpha\beta}_{\mu\nu\rho\sigma}\right)g_{\alpha\beta}R\right]\end{aligned} \tag{7.15}$$

$$\begin{aligned}(\widehat{HF})_{\mu\nu} =& \,\xi\left[Rg_{\mu\nu} - 2R_{\mu\nu} + 2\nabla_\mu\nabla_\nu - 2g_{\mu\nu}\nabla^2\right] + \\ &+ \frac{1}{2}\left(\frac{\nabla_\mu\bar{\phi}}{\bar{\phi}}\delta^\beta_\nu + \frac{\nabla_\nu\bar{\phi}}{\bar{\phi}}\delta^\beta_\mu - g_{\mu\nu}\frac{\nabla^\beta\bar{\phi}}{\bar{\phi}}\right)\nabla_\beta - \\ &- 2\xi\left(\frac{\nabla_\mu\bar{\phi}}{\bar{\phi}}\delta^\beta_\nu + \frac{\nabla_\nu\bar{\phi}}{\bar{\phi}}\delta^\beta_\mu - 2g_{\mu\nu}\frac{\nabla^\beta\bar{\phi}}{\bar{\phi}}\right)\nabla_\beta + \\ &+ 2\xi\left(2\frac{\nabla_\mu\bar{\phi}\nabla_\nu\bar{\phi}}{\bar{\phi}^2} - \frac{\nabla_\mu\nabla_\nu\bar{\phi}}{\bar{\phi}} - 2\frac{(\nabla\bar{\phi})^2}{\bar{\phi}^2}g_{\mu\nu} + \frac{\nabla^2\bar{\phi}}{\bar{\phi}}g_{\mu\nu}\right) + \\ &+ \frac{1}{2}\left(2\frac{\nabla_\mu\nabla_\nu\bar{\phi}}{\bar{\phi}} - 2\frac{\nabla_\mu\bar{\phi}\nabla_\nu\bar{\phi}}{\bar{\phi}^2} + \frac{(\nabla\bar{\phi})^2}{\bar{\phi}^2} - g_{\mu\nu}\frac{\nabla^2\bar{\phi}}{\bar{\phi}}\right)\end{aligned} \tag{7.16}$$

$$\hat{F} = -\frac{1}{2}\nabla^2 + \frac{(n-2)}{8(n-1)}R \tag{7.17}$$

and finally

$$\begin{aligned}Y^{\alpha\beta}_{\mu\nu\rho\sigma} =& \left(g^\alpha_\mu g^\beta_\rho g_{\nu\sigma} + g^\alpha_\nu g^\beta_\rho g_{\mu\sigma} + g^\alpha_\mu g^\beta_\sigma g_{\nu\rho} + g^\alpha_\nu g^\beta_\sigma g_{\mu\rho}\right) - \\ &- \left(g^\alpha_\rho g^\beta_\sigma g_{\mu\nu} + g^\alpha_\sigma g^\beta_\rho g_{\mu\nu}\right) - 2\left(g^\alpha_\mu g^\beta_\nu g_{\rho\sigma} + g^\alpha_\nu g^\beta_\mu g_{\rho\sigma}\right) + \\ &+ \frac{1}{2}\left(g^\alpha_\rho g^\beta_\mu g_{\nu\sigma} + g^\alpha_\rho g^\beta_\nu g_{\mu\sigma} + g^\alpha_\sigma g^\beta_\mu g_{\nu\rho} + g^\alpha_\sigma g^\beta_\nu g_{\mu\rho}\right)\end{aligned} \tag{7.18}$$

At this point, the action is still not suitable to use the Schwinger-DeWitt technique and compute the one-loop divergences of the quantum effective action. The reason is that, as it stands now, the theory contains zero modes inherited from gauge invariance. The original action (7.1) is invariant under the full group of diffeomorphisms, represented at the infinitesimal level by Lie dragging the original fields

$$g'_{\mu\nu} = \pounds_\xi g_{\mu\nu} \tag{7.19}$$
$$\phi' = \pounds_\xi \phi \tag{7.20}$$

along a vector ξ^μ which contains the infinitesimal generators of the connected part of the *Diff* group. Alternatively, and more suitable for the quantization of the theory, we can define a $BRST$ operator as we did in chapter 5 that implements the symmetry

$$\mathbf{s}_D \equiv \pounds_c \tag{7.21}$$

where now c^μ is an odd Grasman field that takes the role of the Faddeev-Popop ghost.

In the background field approach the operator splits in background and quantum counterparts and it is this last part what represents a redundancy in the path integral measure. In particular, this means that the quantum fluctuations $h_{\mu\nu}$ and ϕ can be transformed by

$$\mathbf{s}_D h_{\mu\nu} = \nabla_\mu c_\nu + \nabla_\nu c_\mu + c^\rho \nabla_\rho h_{\mu\nu} + \nabla_\mu c^\rho h_{\rho\nu} + \nabla_\nu c^\rho h_{\rho\mu} \tag{7.22}$$
$$\mathbf{s}_D \phi = c^\lambda \nabla_\lambda (\bar{\phi} + \phi) \tag{7.23}$$

while the background fields are non-sensitive to these transformations. These leave the second order action (7.5) unchanged. For the field $k_{\mu\nu}$ the corresponding transformation is

$$\mathbf{s}_D k_{\mu\nu} = \bar{\phi} \left(\nabla_\mu c_\nu + \nabla_\nu c_\mu\right) + c^\rho \nabla_\rho k_{\mu\nu} + \nabla_\mu c^\rho k_{\rho\nu} + \nabla_\nu c^\rho k_{\rho\mu} - c^\rho k_{\mu\nu} \frac{\nabla_\rho \bar{\phi}}{\bar{\phi}} \tag{7.24}$$

Now, there are two things that we aim to do with the gauge fixing, constructed with the BRST method of chapter 5. We must, of course, cancel the zero modes in order to have a well-defined action that we can integrate over and, second, we want to apply the Schwinger-DeWitt technique of chapter 3 so we will try to cancel any kinetic term which is non-minimal. This can be achieved in different ways, some of them simple modifications of the well-known harmonic or de Donder gauge. It is however actually possible to choose a very general gauge interpolating between two functions

$$\hat{F}_\mu = (1-\gamma)F^1_\mu + \gamma F^2_\mu \tag{7.25}$$

with

$$F^1_\mu = \nabla^\nu k_{\mu\nu} - \frac{1}{2}\nabla_\mu k - 2\nabla_\mu \phi \tag{7.26}$$

$$F^2_\mu = \bar{\phi}\left(\nabla^\nu h_{\mu\nu} - \frac{1}{2}\nabla_\mu h\right) - 2\nabla_\mu \phi \tag{7.27}$$

Although each of the two functions F^1_μ and F^2_μ represent perfectly admissible gauge choices separately, we have decided to consider this more general linear combination of them as above in order to be able to track the dependence on the γ parameter along the computation and explicitly check that it vanishes on-shell, as it should. The full gauge fixing choice is then

$$\hat{F}_\mu = \nabla^\nu k_{\mu\nu} - \frac{1}{2}\nabla_\mu k - 2\nabla_\mu\phi - \gamma k^\nu_\mu \frac{\nabla_\nu\bar{\phi}}{\bar{\phi}} + \gamma\frac{1}{2}k\frac{\nabla_\mu\bar{\phi}}{\bar{\phi}} \tag{7.28}$$

Here the situation is easy enough as to use Faddeev-Popov quantization, instead of the full BRST method of chapter 5. We thus introduce a gauge fixing term of the form

$$S_{diff} = \frac{\xi}{2}\int d^n x\sqrt{|g|}\;\hat{F}_\mu \hat{F}^\mu \tag{7.29}$$

with

$$\begin{aligned}\hat{F}_\mu\hat{F}^\mu =& 2\left(\frac{1}{2}\nabla_\mu k\nabla_\nu k^{\mu\nu} - \frac{1}{2}\nabla_\mu k^{\mu\alpha}\nabla_\nu k^\nu_\alpha\right) + 2\left[-2\phi\nabla_\mu\nabla_\nu k^{\mu\nu} + \phi\nabla^2 k\right] + 4\phi\nabla^2\phi + \\ &+ \frac{1}{4}k\nabla^2 k + \gamma\left[k^{\mu\nu}k^\alpha_\nu\frac{\nabla_\mu\bar{\phi}\nabla_\alpha\bar{\phi}}{\bar{\phi}^2} + \frac{1}{4}k^2\frac{(\nabla\bar{\phi})^2}{\bar{\phi}^2} - kk^{\mu\nu}\frac{\nabla_\mu\bar{\phi}\nabla_\nu\bar{\phi}}{\bar{\phi}^2} + k\frac{\nabla_\mu\bar{\phi}}{\bar{\phi}}\nabla_\nu k^{\mu\nu} - \right.\\ &\left. -2k^{\mu\nu}\frac{\nabla_\nu\bar{\phi}}{\bar{\phi}}\nabla^\alpha k_{\alpha\mu} - \frac{1}{2}k\nabla_\mu k\frac{\nabla^\mu\bar{\phi}}{\bar{\phi}} + k^{\mu\nu}\nabla_\mu k\frac{\nabla_\nu\bar{\phi}}{\bar{\phi}} - 2k\frac{\nabla_\mu\bar{\phi}}{\bar{\phi}}\nabla^\mu\phi + 4k^{\mu\nu}\frac{\nabla_\mu\bar{\phi}}{\bar{\phi}}\nabla_\nu\phi\right]\end{aligned} \tag{7.30}$$

which cancels exactly the non-minimal terms in (7.5).

By adding the gauge-fixing term to the original action we can finally write an operator which is *almost* suitable for the application of the Schwinger-DeWitt technique. The original action with the gauge fixing added then reads

$$S^{full}_2 = -\int d^n x\sqrt{|g|}\left[k^{\mu\nu}\hat{H}_{\mu\nu\rho\sigma}k^{\rho\sigma} + \phi(\widehat{HF})_{\mu\nu}k^{\mu\nu} + \phi\hat{F}\phi\right] \tag{7.31}$$

where the values of the coefficients are again quite contrived

$$\begin{aligned}\hat{H}_{\mu\nu\rho\sigma} =& + 2\xi\frac{\nabla_\alpha\bar{\phi}}{\bar{\phi}}\left(\left(\frac{1}{2}\mathcal{K}^{\gamma\omega}_{\mu\nu\rho\sigma} - \mathcal{P}^{\gamma\omega}_{\mu\nu\rho\sigma}\right)g_{\gamma\omega}g^{\alpha\beta} + \frac{1}{2}\mathcal{K}_{\mu\nu\rho\sigma} + \frac{1}{4}Y^{\alpha\beta}_{\mu\nu\rho\sigma}\right)\nabla_\beta + \\ &+ \frac{\gamma^2\xi}{4}E^{\alpha\beta}_{\mu\nu\rho\sigma}\frac{\nabla_\alpha\bar{\phi}\nabla_\beta\bar{\phi}}{\bar{\phi}^2} + \frac{\xi}{2}\left[\frac{1}{2}\left(\mathcal{P}^{\alpha\beta}_{\mu\nu\rho\sigma} - \mathcal{K}^{\alpha\beta}_{\mu\nu\rho\sigma}\right)g_{\alpha\beta}\left(2\frac{(\nabla\bar{\phi})^2}{\bar{\phi}^2} - \frac{\nabla^2\bar{\phi}}{\bar{\phi}}\right) + \right. \\ &\left. + \left(\mathcal{P}^{\alpha\beta}_{\mu\nu\rho\sigma} - \mathcal{K}^{\alpha\beta}_{\mu\nu\rho\sigma} - \left(\mathcal{K}^{\gamma\delta}_{\mu\nu\rho\sigma} - 3\mathcal{P}^{\gamma\delta}_{\mu\nu\rho\sigma}\right)g_{\gamma\delta}g^{\alpha\beta} - X^{\alpha\beta}_{\mu\nu\rho\sigma}\right)\frac{\nabla_\alpha\bar{\phi}\nabla_\beta\bar{\phi}}{\bar{\phi}^2}\right] + \\ &+ \left(\frac{1}{2}\mathcal{P}^{\alpha\beta}_{\mu\nu\rho\sigma} - \frac{1}{4}\mathcal{K}^{\alpha\beta}_{\mu\nu\rho\sigma}\right)\left(\frac{\nabla_\alpha\bar{\phi}\nabla_\beta\bar{\phi}}{\bar{\phi}^2} - \frac{1}{4}g_{\alpha\beta}\frac{(\nabla\bar{\phi})^2}{\bar{\phi}^2}\right) + \\ &\xi\left[\frac{1}{4}\left(\mathcal{P}^{\alpha\beta}_{\mu\nu\rho\sigma} - \frac{1}{2}\mathcal{K}^{\alpha\beta}_{\mu\nu\rho\sigma}\right)g_{\alpha\beta}\nabla^2 + \frac{1}{2}\left(\mathcal{P}^{\alpha\beta}_{\mu\nu\rho\sigma} - \mathcal{K}^{\alpha\beta}_{\mu\nu\rho\sigma}\right)R_{\alpha\beta} + \frac{1}{2}R_{(\mu\rho\nu\sigma)}\right. \\ &\left. + \left(\frac{1}{8}\mathcal{K}^{\alpha\beta}_{\mu\nu\rho\sigma} - \frac{1}{4}\mathcal{P}^{\alpha\beta}_{\mu\nu\rho\sigma}\right)g_{\alpha\beta}R\right] - \frac{\gamma\xi}{2}E^{\alpha\beta}_{\mu\nu\rho\sigma}\frac{\nabla_\alpha\bar{\phi}}{\bar{\phi}}\nabla_\beta \end{aligned} \tag{7.32}$$

$$\begin{aligned}(\widehat{HF})_{\mu\nu} =& \xi\left[Rg_{\mu\nu} - 2R_{\mu\nu} - g_{\mu\nu}\nabla^2\right] + \\ &+ \left(\gamma\xi + \frac{1}{2}\right)\left(\frac{\nabla_\mu\bar{\phi}}{\bar{\phi}}\delta^\beta_\nu + \frac{\nabla_\nu\bar{\phi}}{\bar{\phi}}\delta^\beta_\mu - g_{\mu\nu}\frac{\nabla^\beta\bar{\phi}}{\bar{\phi}}\right)\nabla_\beta - \\ &- 2\xi\left(\frac{\nabla_\mu\bar{\phi}}{\bar{\phi}}\delta^\beta_\nu + \frac{\nabla_\nu\bar{\phi}}{\bar{\phi}}\delta^\beta_\mu - 2g_{\mu\nu}\frac{\nabla^\beta\bar{\phi}}{\bar{\phi}}\right)\nabla_\beta + \\ &+ 2\xi\left(2\frac{\nabla_\mu\bar{\phi}\nabla_\nu\bar{\phi}}{\bar{\phi}^2} - \frac{\nabla_\mu\nabla_\nu\bar{\phi}}{\bar{\phi}} - 2\frac{(\nabla\bar{\phi})^2}{\bar{\phi}^2}g_{\mu\nu} + \frac{\nabla^2\bar{\phi}}{\bar{\phi}}g_{\mu\nu}\right) + \\ &+ \frac{1}{2}\left(2\frac{\nabla_\mu\nabla_\nu\bar{\phi}}{\bar{\phi}} - 2\frac{\nabla_\mu\bar{\phi}\nabla_\nu\bar{\phi}}{\bar{\phi}^2} + \frac{(\nabla\bar{\phi})^2}{\bar{\phi}^2} - g_{\mu\nu}\frac{\nabla^2\bar{\phi}}{\bar{\phi}}\right)\end{aligned} \tag{7.33}$$

$$\hat{F} = \left(2\xi - \frac{1}{2}\right)\nabla^2 + \xi R \tag{7.34}$$

where we have introduced a new tensor

$$\begin{aligned}E^{\alpha\beta}_{\mu\nu\rho\sigma} =& \frac{1}{2}\left(g_{\mu\nu}\delta^\alpha_\rho\delta^\beta_\sigma + g_{\mu\nu}\delta^\alpha_\sigma\delta^\beta_\rho + g_{\rho\sigma}\delta^\alpha_\mu\delta^\beta_n + g_{\rho\sigma}\delta^\alpha_\nu\delta^\beta_m - g_{\mu\nu}g_{\rho\sigma}g^{\alpha\beta}\right) - \\ &- \frac{1}{2}\left(g_{\mu\rho}\delta^\alpha_\nu\delta^\beta_\sigma + g_{\mu\sigma}\delta^\alpha_\nu\delta^\beta_\rho + g_{\nu\rho}\delta^\alpha_\mu\delta^\beta_\sigma + g_{\nu\sigma}\delta^\alpha_\mu\delta^\beta_\rho\right)\end{aligned} \tag{7.35}$$

In order to apply the Schwinger-DeWitt technique and the short-time expansion of the Heat Kernel contained in chapter 3 we need now to write this as

$$S = \int d^n x\sqrt{|g|}\ \Psi^A\left(-G_{AB}\Box - E_{AB}\right)\Psi^B \tag{7.36}$$

where A, B are generalized indices. In our case, they will run over the different components of the fields in the action. That is, we introduce a

DeWitt superfield

$$\Psi^A = \begin{pmatrix} k^{\mu\nu} \\ \phi \end{pmatrix} \tag{7.37}$$

and the interpretation of the index A is a label for the different polarizations[1]. Therefore, the rank of the index A is $n(n+1)/2+1$.

By a first manipulation and using this, we can write the full expanded gauge fixed action as

$$S_{full} = \int d^n x \sqrt{|g|}\, \Psi^A \left(-G_{AB}\Box + N^{\mu}_{AB}\nabla_\mu + M_{AB}\right)\Psi^B \tag{7.38}$$

where the metric G_{AB} is symmetric and given by

$$G_{AB} = \begin{pmatrix} \frac{\xi}{4}\left(\frac{1}{2}\mathcal{K}^{\alpha\beta}_{\mu\nu\rho\sigma} - \mathcal{P}^{\alpha\beta}_{\mu\nu\rho\sigma}\right) g_{\alpha\beta} & \frac{\xi}{2} g_{\mu\nu} \\ \frac{\xi}{2} g_{\rho\sigma} & \frac{1}{2} - 2\xi \end{pmatrix} \tag{7.39}$$

with inverse

$$G^{AB} = \frac{1}{8\xi(n-1)-(n-2)} \begin{pmatrix} G^{\mu\nu\rho\sigma} & 8g^{\mu\nu} \\ 8g^{\rho\sigma} & -2(n-2) \end{pmatrix} \tag{7.40}$$

$$G^{\mu\nu\rho\sigma} = -\frac{2}{\xi}\left[(8\xi(n-1)-(n-2))(g^{\mu\sigma}g^{\nu\rho} + g^{\mu\rho} + g^{\nu\sigma}) + 2(1-8\xi)g^{\mu\nu}g^{\rho\sigma}\right]$$

defined in such a way that

$$G_{AB}G^{BC} = G^{CB}G_{BA} = \begin{pmatrix} \frac{1}{2}\left(\delta^\rho_\mu\delta^\sigma_\nu + \delta^\rho_\nu\delta^\sigma_\mu\right) & 0 \\ 0 & 1 \end{pmatrix} \tag{7.41}$$

and the matrices N^μ_{AB} and M_{AB} are anti-symmetric and symmetric respectively. They are obtaining by partial integration of the different terms in (7.31) and read

$$N^\beta_{AB} = \begin{pmatrix} N^\beta_{kk} & N^\beta_{k\phi} \\ N^\beta_{\phi k} & N^\beta_{\phi\phi} \end{pmatrix} \tag{7.42}$$

where

$$N^\beta_{kk} = \frac{\xi}{4}\left(Y^{\alpha\beta}_{\mu\nu\rho\sigma} - Y^{\alpha\beta}_{\rho\sigma\mu\nu} - \gamma E^{\alpha\beta}_{\mu\nu\rho\sigma} + \gamma E^{\alpha\beta}_{\rho\sigma\mu\nu}\right)\frac{\nabla_\alpha\bar{\phi}}{\bar{\phi}}$$

$$N^\beta_{\phi\phi} = 0$$

$$N^\beta_{k\phi} = -N^\beta_{\phi k} = \frac{1}{2}\left(\frac{1}{2} - 2\xi\right)\left(\frac{\nabla_\mu\bar{\phi}}{\bar{\phi}}\delta^\beta_\nu + \frac{\nabla_\nu\bar{\phi}}{\bar{\phi}}\delta^\beta_\mu\right) - \frac{1}{2}\left(4\xi - \frac{1}{2}\right) g_{\mu\nu}\frac{\nabla^\beta\bar{\phi}}{\bar{\phi}} - \\ - \frac{\xi\gamma}{2}\left(\frac{\nabla_\mu\bar{\phi}}{\bar{\phi}}\delta^\beta_\nu + \frac{\nabla_\nu\bar{\phi}}{\bar{\phi}}\delta^\beta_\mu - g_{\mu\nu}\frac{\nabla^\beta\bar{\phi}}{\bar{\phi}}\right)$$

[1] Mind that here we are not killing the gauge polarizations yet. Those will be taken care of by the ghost action to be introduced later.

$$M_{AB} = \begin{pmatrix} M_{kk} & M_{k\phi} \\ M_{\phi k} & M_{\phi\phi} \end{pmatrix} \tag{7.43}$$

where the different elements are

$$\begin{aligned}
&M_{kk} = \\
&= \frac{\xi}{2}\Bigg\{ \frac{1}{2}\left(\mathcal{P}^{\alpha\beta}_{\mu\nu\rho\sigma} - \mathcal{K}^{\alpha\beta}_{\mu\nu\rho\sigma}\right) g_{\alpha\beta}\left(2\frac{(\nabla\bar{\phi})^2}{\bar{\phi}^2} - \frac{\nabla^2\bar{\phi}}{\bar{\phi}}\right) + \frac{\nabla_\alpha\bar{\phi}\nabla_\beta\bar{\phi}}{\bar{\phi}^2}\left[\left(\mathcal{P}^{\alpha\beta}_{\mu\nu\rho\sigma} - \mathcal{K}^{\alpha\beta}_{\mu\nu\rho\sigma}\right) + \right. \\
&\left. + \left(3\mathcal{P}^{\gamma\omega}_{\mu\nu\rho\sigma} - \mathcal{K}^{\gamma\omega}_{\mu\nu\rho\sigma}\right) g_{\gamma\omega} g^{\alpha\beta} - X^{\alpha\beta}_{\mu\nu\rho\sigma}\right]\Bigg\} - \xi\nabla_\beta\left(\frac{\nabla_\alpha\bar{\phi}}{\bar{\phi}}\right)\left[\mathcal{G}^{\gamma\omega}_{\mu\nu\rho\sigma} g_{\gamma\omega} g^{\alpha\beta} + \right. \\
&\left. + \frac{1}{2}\mathcal{K}^{\alpha\beta}_{\mu\nu\rho\sigma} + \frac{1}{8}\left(Y^{\alpha\beta}_{\mu\nu\rho\sigma} + Y^{\alpha\beta}_{\rho\sigma\mu\nu}\right)\right] - \frac{1}{2}\mathcal{G}^{\alpha\beta}_{\mu\nu\rho\sigma}\left(\frac{\nabla_\alpha\bar{\phi}\nabla_\beta\bar{\phi}}{\bar{\phi}^2} - \frac{1}{4}g_{\alpha\beta}\frac{(\nabla\bar{\phi})^2}{\bar{\phi}^2}\right) + \\
&+ \xi\left[\frac{1}{2}\left(\mathcal{P}^{\alpha\beta}_{\mu\nu\rho\sigma} - \mathcal{K}^{\alpha\beta}_{\mu\nu\rho\sigma}\right) R_{\alpha\beta} + \frac{1}{4}\mathcal{G}^{\alpha\beta}_{\mu\nu\rho\sigma} g_{\alpha\beta} R + \frac{1}{2}R_{(\mu\rho\nu\sigma)}\right] + \\
&+ \frac{\xi}{8}\left(E^{\alpha\beta}_{\mu\nu\rho\sigma} + E^{\alpha\beta}_{\rho\sigma\mu\nu}\right)\left(\gamma\nabla_\beta\left(\frac{\nabla_\alpha\bar{\phi}}{\bar{\phi}}\right) + \gamma^2\frac{\nabla_\alpha\bar{\phi}\nabla_\beta\bar{\phi}}{\bar{\phi}^2}\right)
\end{aligned} \tag{7.44}$$

$$\begin{aligned}
M_{k\phi} = M_{\phi k} &= \frac{\xi}{2}\left(R g_{\mu\nu} - 2R_{\mu\nu}\right) - \frac{1}{2}\left(\frac{1}{2} - 2\xi\right)\nabla_\mu\left(\frac{\nabla_\mu\bar{\phi}}{\bar{\phi}}\right) + \\
&+ \xi\left(2\frac{\nabla_\mu\bar{\phi}\nabla_\nu\bar{\phi}}{\bar{\phi}^2} - \frac{\nabla_\mu\nabla_\nu\bar{\phi}}{\bar{\phi}} - 2g_{\mu\nu}\frac{\nabla_\beta\bar{\phi}\nabla^\beta\bar{\phi}}{\bar{\phi}^2} + g_{\mu\nu}\frac{\nabla_\beta\nabla^\beta\bar{\phi}}{\bar{\phi}}\right) + \\
&+ \frac{1}{4}\left(2\frac{\nabla_\mu\nabla_\nu\bar{\phi}}{\bar{\phi}} - 2\frac{\nabla_\mu\bar{\phi}\nabla_\nu\bar{\phi}}{\bar{\phi}^2} + g_{\mu\nu}\frac{\nabla_\beta\bar{\phi}\nabla^\beta\bar{\phi}}{\bar{\phi}^2} - g_{\mu\nu}\frac{\nabla_\beta\nabla^\beta\bar{\phi}}{\bar{\phi}}\right) - \\
&- \frac{1}{4}\left(4\xi - \frac{1}{2}\right) g_{\mu\nu}\nabla_\beta\left(\frac{\nabla^\beta\bar{\phi}}{\bar{\phi}}\right) - \frac{\gamma\xi}{2}\left[\frac{1}{2}g_{\mu\nu}\nabla_\beta\left(\frac{\nabla^\beta\bar{\phi}}{\bar{\phi}}\right) - \nabla_\mu\left(\frac{\nabla_\nu\bar{\phi}}{\bar{\phi}}\right)\right]
\end{aligned} \tag{7.45}$$

$$M_{\phi\phi} = \xi R \tag{7.46}$$

where

$$\mathcal{G}^{\alpha\beta}_{\mu\nu\rho\sigma} = \frac{1}{2}\mathcal{K}^{\alpha\beta}_{\mu\nu\rho\sigma} - \mathcal{P}^{\alpha\beta}_{\mu\nu\rho\sigma} \tag{7.47}$$

Finally, to compute the effective action corresponding to Dilaton Gravity out of the Weyl invariant point by means of the Heat Kernel expansion,we define a bundle connection ω_μ and the endomorphism E that will allow us to express S_{full} by getting rid of the linear derivative term as follows

$$\begin{aligned}
S_{full} &= \int d^n x\sqrt{|g|}\;\Psi_A\left(-g^{\mu\nu}[\nabla_\mu\delta^A{}_C + \omega^A_{\mu}{}_C][\nabla_\nu\delta^C{}_B + \omega^C_{\nu}{}_B] - E^A{}_B\right)\Psi^B = \\
&= \int d^n x\sqrt{|g|}\;\Psi^A\left(-G_{AB}\tilde{\Box} - E_{AB}\right)\Psi^B
\end{aligned} \tag{7.48}$$

where we are using the notation $\tilde{\nabla}_\mu = \nabla_\mu \delta^A{}_C + \omega^A_{\mu\,C}$.

The bundle connection is then constructed so that

$$G^{AC}\left(-G_{CB}\nabla^2 + N^\mu_{CB}\nabla_\mu + M_{CB}\right) = -g^{\mu\nu}(\nabla_\mu \delta^A{}_C + \omega^A_{\mu\,C})(\nabla_\nu \delta^C{}_B + \omega^C_{\nu\,B}) \; - \; E^A{}_B, \tag{7.49}$$

which is true if

$$\omega^A_{\mu\,B} = \frac{1}{2} G^{AC}\, N_{\mu\,CB} \tag{7.50}$$

$$E^A{}_B = G^{AC}(-M_{CB} - \omega_{\mu CF}\omega^{\mu F}{}_B) \tag{7.51}$$

A final ingredient we need is the field strength given by Ricci's identity

$$[\tilde{\nabla}_\mu, \tilde{\nabla}_\nu]\Psi^A = F_{\mu\nu}{}^A{}_B \tag{7.52}$$

so that

$$\begin{aligned} F_{\alpha\beta}{}^A{}_B = & \frac{1}{2}\left(R^\mu{}_{\rho\,\alpha\beta}\delta^\nu_\sigma + R^\nu{}_{\rho\,\alpha\beta}\delta^\mu_\sigma + R^\mu{}_{\sigma\,\alpha\beta}\delta^\nu_\rho + R^\nu{}_{\sigma\,\alpha\beta}\delta^\mu_\rho\right)\begin{pmatrix}1 & 0\\ 0 & 0\end{pmatrix} + \\ & + \nabla_\alpha \omega_\beta{}^A{}_B - \nabla_\beta \omega_\alpha{}^A{}_B + \omega_\alpha{}^A{}_C \omega_\beta{}^C{}_B - \omega_\beta{}^A{}_C \omega_\alpha{}^C{}_B \end{aligned} \tag{7.53}$$

Therefore and through these manipulations, we have finally arrived to an operator which can be written in the form

$$\hat{D}_{AB} = -G_{AB}\tilde{\Box} - E_{AB} \tag{7.54}$$

so its contribution to the effective action can be computed by direct application of expression (3.82) as

$$\begin{aligned} \Gamma^{h\phi}_{\mathbf{ren}}[g_{\mu\nu}, \bar{\phi}] = & -\frac{1}{32\pi^2}\frac{1}{360}\log(\frac{\mu^2}{M^2})\int d^n x\sqrt{|g|}\;\mathrm{Tr}\{60\Box E + 60RE + \\ & + 180E^2 + 12\Box R + 5R^2 - 2R_{\mu\nu}R^{\mu\nu} + 2R_{\mu\nu\rho\sigma}R^{\mu\nu\rho\sigma} + \\ & + 30F_{\mu\nu}F^{\mu\nu}\} \end{aligned} \tag{7.55}$$

Finally and as a last ingredient, we have to take care of the ghost fields induced by the gauge choice (7.25)

$$S_{ghost} = \int d^n x\sqrt{|g|}\; g^{\mu\nu}\,\bar{c}_\mu s_D \tilde{F}_\nu, \tag{7.56}$$

where $s_D\tilde{F}_\nu$ denotes the order-one variation of the gauge-fixing function $\tilde{F}_\mu$, as explained in chapter 5. Here $\bar{c}_\mu$ is the anti-ghost field.

A little algebra yields the contribution to S_{ghost} that is quadratic in the quantum fields. This contribution reads

$$S_2^{ghost} = \int d^n x \sqrt{|g|}\, \bar{c}^\rho \left(-g_{\rho\sigma}\Box + N^\mu{}_{\rho\sigma}\bar{\nabla}_\mu + M_{\rho\sigma}\right) c^\sigma, \tag{7.57}$$

where

$$\begin{aligned} N^\mu{}_{\rho\sigma} &= -(1-\gamma)\,\bar{g}_{\rho\sigma}\,\frac{\bar{\nabla}^\mu\bar{\phi}}{\bar{\phi}} + (1+\gamma)\,\frac{\bar{\nabla}_\sigma\bar{\phi}}{\bar{\phi}}\,\delta^\mu{}_\rho + (1-\gamma)\,\frac{\bar{\nabla}_\rho\bar{\phi}}{\bar{\phi}}\,\delta^\mu{}_\sigma \\ M_{\rho\sigma} &= -\bar{R}_{\rho\sigma} + 2\frac{\bar{\nabla}_\rho\bar{\nabla}_\sigma\bar{\phi}}{\bar{\phi}} \end{aligned} \tag{7.58}$$

The contribution to the effective action associated to S_2^{ghost} will be then the corresponding coefficient of the heat kernel expansion of the following operator

$$\hat{D}^{(ghost)} = -(g^{\mu\nu}[\bar{\nabla}_\mu\delta^\rho{}_\lambda + \omega_\mu{}^\rho{}_\lambda][\bar{\nabla}_\nu\delta^\lambda{}_\sigma + \omega_\nu{}^\lambda{}_\sigma] + E_{(g)}^\rho{}_\sigma), \tag{7.59}$$

where we define

$$\omega_{(g)\mu}{}^\rho{}_\lambda = \bar{g}_{\mu\nu}\omega_{(g)}^{\nu\rho}{}_\lambda, \quad \omega_{(g)}^{\nu\rho}{}_\sigma = -\frac{1}{2}\bar{g}^{\rho\lambda}N^\mu{}_{\lambda\sigma} \tag{7.60}$$

$$E_{(g)}^\rho{}_\sigma = -\bar{g}^{\rho\lambda}(M_{\lambda\sigma} + \omega_{\mu\lambda\delta}\omega^{\mu\delta}{}_\sigma + \bar{\nabla}_\mu\omega^\mu{}_{\lambda\sigma}) \tag{7.61}$$

$$F_{(g)\rho\sigma}{}^\mu{}_\nu = \bar{R}^\mu{}_{\nu\rho\sigma} + \bar{\nabla}_\rho\,\omega_\sigma{}^\mu{}_\nu - \bar{\nabla}_\sigma\,\omega_\rho{}^\mu{}_\nu + [\omega_\rho,\omega_\sigma]^\mu{}_\nu \tag{7.62}$$

so that we have

$$\begin{aligned} \Gamma_{\mathbf{ren}}^{ghost}[g_{\mu\nu},\bar{\phi}] = &-\frac{1}{32\pi^2}\frac{1}{360}\log(\frac{\mu^2}{M^2})\int d^n x\sqrt{|g|}\ \mathrm{Tr}\{60\Box E_{(g)} + 60RE_{(g)} + \\ &+ 180E_{(g)}^2 + 12\Box R + 5R^2 - 2R_{\mu\nu}R^{\mu\nu} + 2R_{\mu\nu\rho\sigma}R^{\mu\nu\rho\sigma} + \\ &+ 30F_{(g)\mu\nu}F_{(g)}^{\mu\nu}\} \end{aligned} \tag{7.63}$$

Summarizing, we finally find that the one-loop divergent correction to the quantum effective action in the background field formalism will be given by a combination of expressions (7.55) and (7.63)

$$\Gamma_{\mathbf{ren}}[g_{\mu\nu},\bar{\phi}] = \Gamma_{\mathbf{ren}}^{h\phi}[g_{\mu\nu},\bar{\phi}] - 2\Gamma_{\mathbf{ren}}^{ghost}[g_{\mu\nu},\bar{\phi}] \tag{7.64}$$

where the ghost contribution comes twice because of the presence of both ghost and anti-ghost fields and with a minus sign due to its anti-commuting character. The final result comes after a quite lengthy computation, for which extensive use of Mathematica and the package xAct[70] were used. It reads

$$\Gamma_{\mathbf{ren}}[g_{\mu\nu},\bar{\phi}]=\frac{\log\left(\mu^2\right)}{16\pi^2}\frac{1}{720\xi^2(2-8\xi+4(8\xi-1))^2}\int d^nx\sqrt{|g|}\;\tilde{\mathcal{L}} \tag{7.65}$$

where $\tilde{\mathcal{L}}$ is a collection of counterterms given by

$$\begin{aligned}\tilde{\mathcal{L}}=&(12\xi-1)\Bigg\{P_0(\xi,\gamma)\frac{\nabla^\mu\bar{\phi}\nabla^\nu\bar{\phi}\nabla_\mu\bar{\phi}\nabla_\nu\bar{\phi}}{\bar{\phi}^4}P_1(\xi,\gamma)\frac{\nabla^\alpha\bar{\phi}\nabla^\beta\bar{\phi}\nabla_\alpha\nabla_\beta\bar{\phi}}{\bar{\phi}^2}+\\&+P_2(\xi,\gamma)\frac{\nabla_\mu\nabla_\nu\bar{\phi}\nabla^\mu\nabla^\nu\bar{\phi}}{\bar{\phi}^2}+P_3(\xi,\gamma)\frac{(\nabla\bar{\phi})^2\nabla^2\bar{\phi}}{\bar{\phi}^3}\Bigg\}+P_4(\xi,\gamma)\frac{\nabla^2\bar{\phi}\nabla^2\bar{\phi}}{\bar{\phi}^2}+\\&+P_5(\xi,g)\frac{\nabla^\alpha\nabla^\beta\bar{\phi}\;R_{\alpha\beta}}{\bar{\phi}}+P_6(\xi,\gamma)\frac{\nabla^\alpha\bar{\phi}\nabla^\beta\bar{\phi}\;R_{\alpha\beta}}{\bar{\phi}^2}-P_7(\xi,\gamma)R_{\mu\nu}R^{\mu\nu}+\\&+P_8(\xi,\gamma)\frac{(\nabla\bar{\phi})^2R}{\bar{\phi}^2}+P_9(\xi,\gamma)\frac{\nabla^2\bar{\phi}\;R}{\bar{\phi}}+P_{10}(\xi,\gamma)R^2+P_{11}(\xi,\gamma)R_{\mu\nu\alpha\beta}R^{\mu\nu\alpha\beta}\Bigg\}\end{aligned} \tag{7.66}$$

and the different off-shell polynomials $P_i(\xi,\gamma)$, which depend both on the coupling and on the gauge fixing parameter are

$$\begin{aligned}P_0(\xi,\gamma)=&720(-5+104\xi-728\xi^2+2784\xi^3-18\xi\gamma+72\xi^2\gamma+1536\xi^3\gamma+8\xi\gamma^2-\\&-260\xi^2\gamma^2+2064\xi^3\gamma^2-16\xi^2\gamma^3+216\xi^3\gamma^3)\\P_1(\xi,\gamma)=&-960\xi(-29+450\xi-840\xi^2-15\gamma+88\xi\gamma+912\xi^2\gamma-38\xi\gamma^2+\\&+508\xi^2\gamma^2-8\xi\gamma^3+108\xi^2\gamma^3)\\P_2(\xi,\gamma)=&480\xi\left(1-78\xi+984\xi^2-68\xi\gamma+720\xi^2\gamma-2\xi\gamma^2+28\xi^2\gamma^2\right)\\P_3(\xi,\gamma)=&-480\xi(-2+228\xi-3072\xi^2+9\gamma+64\xi\gamma-1680\xi^2\gamma+16\xi\gamma^2-\\&-368\xi^2\gamma^2-8\xi\gamma^3+108\xi^2\gamma^3)\\P_4(\xi,\gamma)=&-480\xi(-1-48\xi+672\xi^2-3312\xi^3+56\xi\gamma-1248\xi^2\gamma+6912\xi^3\gamma+\\&+2\xi\gamma^2-52\xi^2\gamma^2+336\xi^3\gamma^2)\\P_5(\xi,\gamma)=&-3840\xi^2\,(-1+12\xi)(3-12\xi-\gamma+6\xi\gamma)\\P_6(\xi,\gamma)=&-480\xi(-1+12\xi)\left(-1+42\xi-744\xi^2+52\xi\gamma-528\xi^2\gamma-10\xi\gamma^2+116\xi^2\gamma^2\right)\\P_7(\xi,\gamma)=&-48\xi^2(-1+12\xi)\,(-241+2412\xi)\\P_8(\xi,\gamma)=&-960\xi(-1+12\xi)\left(1-41\xi+432\xi^2-32\xi\gamma+348\xi^2\gamma-6\xi\gamma^2+90\xi^2\gamma^2\right)\\P_9(\xi,\gamma)=&1920\xi^2\left(-11+189\xi-1008\xi^2+\gamma-18\xi\gamma+72\xi^2\gamma\right)\\P_{10}(\xi,\gamma)=&120\xi^2\left(29-576\xi+3168\xi^2\right)\\P_{11}(\xi,\gamma)=&3408\xi^2(-1+12\xi)^2\end{aligned} \tag{7.67}$$

All these expressions are of course off-shell. We have just obtained the functional generator of all 1PI n-point functions for Dilaton Gravity in the non-Weyl invariant phase. If now we want to extract S-matrix information from this result, we must take the background fields to obey the tree level mean field equations, which are just the classical equations of motion. Therefore, we must put all the fields in the last expressions on the mass-shell. In order to do that, apart from the application of the equations of motion (7.3), we must relate the scalar quantities appearing in $\tilde{\mathcal{L}}$ by using integration by parts.

All the different operators that we find can be classified as

$$
\begin{aligned}
G_1 &\equiv \frac{\nabla_\mu\bar{\phi}\nabla_\nu\bar{\phi}\, R^{\mu\nu}}{\bar{\phi}^2} & \qquad A &= \frac{\nabla^2\bar{\phi}\nabla^2\bar{\phi}}{\bar{\phi}^2} \\
G_2 &\equiv \frac{\nabla_\mu\nabla_\nu\bar{\phi}\, R^{\mu\nu}}{\bar{\phi}} & B &= \frac{\nabla^2\bar{\phi}(\nabla\bar{\phi})^2}{\bar{\phi}^3} \\
G_3 &\equiv \frac{\nabla^2\bar{\phi}\, R}{\bar{\phi}} & C &= \frac{(\nabla\bar{\phi})^2(\nabla\bar{\phi})^2}{\bar{\phi}^4} \\
G_4 &\equiv \frac{(\nabla\bar{\phi})^2 R}{\bar{\phi}^2} & D &= \frac{\nabla_\mu\nabla_\nu\bar{\phi}\nabla^\mu\nabla^\nu\bar{\phi}}{\bar{\phi}^2} \\
G_5 &\equiv R_{\mu\nu}R^{\mu\nu} & E &= \frac{\nabla_\mu\bar{\phi}\nabla^\nu\bar{\phi}\nabla^\mu\nabla_\nu\bar{\phi}}{\bar{\phi}^3} \\
G_6 &\equiv R^2 & F &= \frac{\nabla_\mu\bar{\phi}\nabla^2\nabla^\mu\bar{\phi}}{\bar{\phi}^2} \\
G_7 &\equiv R_{\mu\nu\alpha\beta}R^{\mu\nu\alpha\beta} & &
\end{aligned}
$$

and the equations of motion, together with integration by parts, introduce the following equivalences

$$G_3 = \frac{\nabla^2\bar{\phi}\, R}{\bar{\phi}} = \frac{1}{2\xi} A \tag{7.68}$$

$$G_4 = \frac{(\nabla\bar{\phi})^2 R}{\bar{\phi}^2} = \frac{1}{2\xi} B \tag{7.69}$$

$$G_6 = R^2 = \frac{1}{4\xi^2} A \tag{7.70}$$

$$G_1 = \frac{\nabla^\mu\bar{\phi}\nabla^\nu\bar{\phi}\, R_{\mu\nu}}{\bar{\phi}^2} = \left(\frac{1}{4\xi} - 2\right) B - \frac{1}{4\xi} C + 2E \tag{7.71}$$

$$G_2 = \frac{\nabla_\mu\nabla_\nu\bar{\phi}\, R^{\mu\nu}}{\bar{\phi}} = \left(\frac{1}{4\xi} - 2\right)(A+B) + 2D + \left(2 - \frac{1}{2\xi}\right) E \tag{7.72}$$

$$\begin{aligned} G_5 = R_{\mu\nu}R^{\mu\nu} = \left(\frac{1}{4\xi} - 2\right) R \left[\frac{\nabla^2\bar{\phi}}{\bar{\phi}} + \frac{(\nabla\bar{\phi})^2}{\bar{\phi}^2}\right] + 2\frac{\nabla^\mu\nabla^\nu\bar{\phi}\, R_{\mu\nu}}{\bar{\phi}} + \\ + \left(2 - \frac{1}{2\xi}\right) \frac{\nabla^\mu\bar{\phi}\nabla^\nu\bar{\phi}\, R_{\mu\nu}}{\bar{\phi}^2} \end{aligned} \tag{7.73}$$

$$\int d(vol) D = \int d(vol) \left(A - 2B + 2E - \frac{\nabla^\mu\bar{\phi}\nabla^\nu\bar{\phi}\, R_{\mu\nu}}{\bar{\phi}^2}\right) \tag{7.74}$$

$$\int d(vol) E = \int d(vol) \left(\frac{3}{2} C - \frac{1}{2} B\right) \tag{7.75}$$

$$\int d(vol) F = \int d(vol) \left(-D + 2E\right) \tag{7.76}$$

And also, whenever $\xi \neq \frac{1}{12}$ there is an extra relation that we can use and that comes from the fact that the two equations of motion (7.3) for the metric and the scalar field must be compatible. Taking the trace of the first one we have

$$R = \left(\frac{n}{4\xi} + 2 - 2n\right) \frac{\nabla^2\bar{\phi}}{\bar{\phi}} + \left(\frac{n}{4\xi} + 2 - 2n - \frac{1}{2\xi}\right) \frac{(\nabla\bar{\phi})^2}{\bar{\phi}^2} \tag{7.77}$$

so requiring agreement with the scalar equation of motion forces

$$\frac{\nabla^2\bar{\phi}}{\bar{\phi}} + \frac{(\nabla\bar{\phi})^2}{\bar{\phi}^2} = 0 \tag{7.78}$$

which implies

$$A = C = -B \tag{7.79}$$

In the case $\xi = \frac{1}{12}$ this identity is satisfied identically and these last relations cannot be used.

Therefore, and by using these identities, we can work out the on-shell effective action from our result (7.65), finding it to reduce to

$$\Gamma_{\mathbf{ren}}[g_{\mu\nu}, \bar{\phi}]\Big|_{\text{on-shell}} = \frac{1}{16\pi^2} \log\left(\frac{\mu^2}{M^2}\right) \int d^n x \sqrt{|g|} \left\{ \frac{71}{60} C^2 + \frac{1259}{1440} \frac{(1-12\xi^2)}{\xi^2} \frac{(\nabla\phi)^4}{\phi^4} \right\} \tag{7.80}$$

There are various remarks we must do about this result. First, the appearance of a pole when $\xi = 0$ signals the fact that for that concrete value of the coupling, the gravitational fluctuation are not propagating any more, because the coupling to curvature $\xi R\phi^2$ disappears. We come back to a phase rather similar to the symmetric phase $\bar{\phi} = 0$ and our pertubative computations fail. Second, when the value of the coupling takes the conformal value in four dimension $\xi = 1/12$ the only surviving counterterm is the one which is precisely Weyl invariant. However, in that phase we have an extended gauge invariance, Weyl invariance, and the ghost sector will include new corrections that will shift the numerical value in front of C^2. We plan to take care of this in the next chapter.

Finally, both as a check of the robustness of our computation and of the validity of Kallosh-DeWitt theorem[68, 69, 40] we find that all the dependence on the gauge fixing parameter γ disappears on-shell.

8

Conformal Dilaton Gravity

We move now to the analysis of the quantum dynamics of Dilaton Gravity in the Weyl invariant point, with action in arbitrary dimension[1] n

$$S_{CDG} = -\int d^n x \sqrt{|g|} \left\{ \frac{n-2}{8(n-1)} \Phi^2 R + \frac{1}{2} \nabla_\mu \Phi \nabla^\mu \Phi \right\} \tag{8.1}$$

In this case, the result obtained in the last chapter is not valid anymore because the matrix G_{AB} given in (7.39) is not invertible. Its determinant is singular and this signals the presence of a new zero mode inherited from an extra gauge symmetry in the theory. Indeed, with that particular choice of the non-minimal coupling the theory is invariant under Weyl transformations of the form

$$g_{\mu\nu} \to \Omega^2 g_{\mu\nu} \tag{8.2}$$

$$\Phi \to \Omega^{\frac{2-n}{2}} \tag{8.3}$$

so we must enlarge the gauge sector of the theory at hand with a new gauge fixing and new ghost fields to cope with this.

When expanding around a background field

$$g_{\mu\nu} = g_{\mu\nu} + h_{\mu\nu} \tag{8.4}$$

$$\Phi = \bar{\phi} + \phi \tag{8.5}$$

the gauge symmetries translate into infinitesimal symmetries acting on the fluctuations. In the non-Weyl invariant case, Faddeev-Popov quantization was enough totake care of the theory. Here, however, the presence of two independent symmetries requires the introduction of more complicated BRST techniques. Following chapter 5 we the construct two BRST operators $\mathbf{s}_D$

[1]Again, let us remind that we are using arbitrary dimension in order to connect later with dimensional regularization, but all the consequences here derived will be specified in dimension $n = 4$.

and $\mathbf{s}_W$ such that they implement the transformations when acting over the fields

$$\mathbf{s}_D h_{\mu\nu} = \nabla_\mu c_\nu + \nabla_\nu c_\mu + c^\rho \nabla_\rho h_{\mu\nu} + \nabla_\mu c^\rho h_{\rho\nu} + \nabla_\nu c^\rho h_{\rho\mu} \tag{8.6}$$

$$\mathbf{s}_D \phi = c^\lambda \nabla_\lambda (\bar{\phi} + \phi) \tag{8.7}$$

$$\mathbf{s}_W h_{\mu\nu} = 2c(g_{\mu\nu} + h_{\mu\nu}) \tag{8.8}$$

$$\mathbf{s}_W \phi = \frac{2-n}{2} c(\bar{\phi} + \phi) \tag{8.9}$$

where c is the grassman odd ghost associated with Weyl transformations in the same way as c^μ is the one of *Diff.* The action of both operators over them is given by

$$\begin{aligned}
&\mathbf{s}_D c^\mu = B^\mu, && \mathbf{s}_D B^\mu = 0 \\
&\mathbf{s}_W \bar{c}^\mu = 0, && \mathbf{s}_W B^\mu = 0 \\
&\mathbf{s}_W \bar{c} = f, && \mathbf{s}_W f = 0 \\
&\mathbf{s}_D \bar{c} = c^\rho \nabla_\rho \bar{c}, && \mathbf{s}_D f = c^\lambda \nabla_\lambda f
\end{aligned} \tag{8.10}$$

Here B^μ and f are the Nakanishin-Lautrup auxiliary fields needed to close the BRST algebra while $\bar{c}^\mu$ and $\bar{c}$ are the corresponding anti-ghost fields. With the transformation so defined, we have that

$$\mathbf{s}_D^2 = 0, \qquad \mathbf{s}_W^2 = 0, \qquad \{\mathbf{s}_W, \mathbf{s}_D\} = 0 \tag{8.11}$$

and so we can define a full BRST operator $\int = \mathbf{s}_D + \mathbf{s}_W$ in order to implement the gauge fixing and ghost action through the method of chapter 5, by adding to the action (8.1) a term constructed as a local variation

$$S = S_{CDG} + \int (X_D + X_W) \tag{8.12}$$

where we choose

$$X_D = \int d^n x \sqrt{|g|}\, \bar{c}^\mu \left(-\frac{4(n-1)}{(n-2)} B^\mu + \hat{F}_\mu \right) \tag{8.13}$$

$$X_W = \int d^n x \sqrt{|g|}\, (g^{\mu\nu} - h^{\mu\nu}) \nabla_\mu \bar{c} \nabla_\mu (f - \alpha(\bar{\phi} + \phi) \tag{8.14}$$

with $\hat{F}_\mu$ as given in (7.25) for the non-Weyl invariant case. Of all those terms we only need to keep those that are second order in the quantum fields.

Again, as in the non-Weyl invariant case, we choose these functions not only to cancel the gauge modes but also with the requirement of obtaining minimal operators to apply the Schwinger-DeWitt technique of chapter 3.

After applying $\int$ we can integrate out the field B^μ, that appears with no derivatives and thus it does not propagate, leaving us only with the extra ghost and anti-ghost pair for the Diff gauge sector. However, it is not possible to do the same with f since it appears quadratically coupled to ϕ through the gauge fixing[2]. Thus, now the bosonic sector of the theory contains three fields and the corresponding DeWitt superfield that we need to introduce reads

$$\Psi^A = \begin{pmatrix} k^{\mu\nu} \\ \phi \\ f \end{pmatrix} \tag{8.15}$$

This means that now the metric G_{AB} and the matrices M_{AB} and N^μ_{AB} will have new entries corresponding to terms with the new field f. Indeed, the metric reads

$$G_{AB} = \frac{(n-2)}{4(n-1)} \begin{pmatrix} \frac{1}{8}\mathcal{G}^{\alpha\beta}_{\mu\nu\rho\sigma}g_{\alpha\beta} & \frac{1}{4}g_{\mu\nu} & 0 \\ \frac{1}{4}g_{\rho\sigma} & \frac{n}{(n-2)} & -\frac{2\alpha(n-1)}{n-2} \\ 0 & -\frac{2\alpha(n-1)}{n-2} & \frac{4(n-1)}{n-2} \end{pmatrix} \tag{8.16}$$

with $\mathcal{G}^{\alpha\beta}_{\mu\nu\rho\sigma}$ as defined in (7.47). Its inverse now exists in the same sense as in (7.40), and happens to be

$$G^{AB} = \begin{pmatrix} G^{11} & \frac{16}{\alpha^2(n-2)}g^{\rho\sigma} & \frac{8}{\alpha(n-2)}g^{\rho\sigma} \\ \frac{16}{\alpha^2(n-2)}g^{\mu\nu} & -\frac{4}{\alpha^2} & -\frac{2}{\alpha} \\ \frac{8}{\alpha(n-2)}g^{\mu\nu} & -\frac{2}{\alpha} & 0 \end{pmatrix} \tag{8.17}$$

where

$$G^{11} = -\frac{16(n-1)}{n-2}\left[g^{\mu\rho}g^{\nu\sigma} + g^{\nu\rho}g^{\mu\sigma} + \frac{2(2+\alpha^2(1-n))}{\alpha^2(2-3n+n^2)}g^{\mu\nu}g^{\rho\sigma}\right] \tag{8.18}$$

The matrices are extended in such a way that

$$N^\beta_{AB} = \begin{pmatrix} N^\beta_{kk} & N^\beta_{k\phi} & N^\beta_{kf} \\ N^\beta_{\phi k} & N^\beta_{\phi\phi} & N^\beta_{\phi f} \\ N^\beta_{fk} & N^\beta_{f\phi} & N^\beta_{ff} \end{pmatrix} \qquad M_{AB} = \begin{pmatrix} M_{kk} & M_{k\phi} & M_{kf} \\ M_{\phi k} & M_{\phi\phi} & M_{\phi f} \\ M_{fk} & M_{f\phi} & M_{ff} \end{pmatrix} \tag{8.19}$$

[2] Actually, we could integrate it out but then we would obtain a non-local action for the field ϕ that exits minimality.

where the kk, $k\phi$ and $\phi\phi$ elements are the same as in the non-Weyl-invariant case (provided that we substitute the coupling ξ by its Weyl invariant value) and the new elements read

$$
\begin{aligned}
&N^{\beta}_{kf} = -N^{\beta}_{fk} = \frac{\alpha}{4}\frac{\nabla^{\alpha}\bar{\phi}}{\bar{\phi}}\left(g_{\alpha\nu}\delta^{\beta}_{\mu} + g_{\alpha\mu}\delta^{\beta}_{\nu} - g_{\mu\nu}\delta^{\beta}_{\alpha}\right)\\
&N^{\beta}_{ff} = 0\\
&N^{\beta}_{\phi f} = -N^{\beta}_{f\phi} = 0\\
&M_{kf} = M_{fk} = -\frac{\alpha}{8}\left(\nabla_{\mu}\left(\frac{\nabla_{\nu}\bar{\phi}}{\bar{\phi}}\right) + \nabla_{\nu}\left(\frac{\nabla_{\mu}\bar{\phi}}{\bar{\phi}}\right) - g_{\mu\nu}\nabla^{\beta}\left(\frac{\nabla_{\beta}\bar{\phi}}{\bar{\phi}}\right)\right)\\
&M_{\phi f} = M_{f\phi} = 0\\
&M_{ff} = 0
\end{aligned}
$$

The algorithm now is the same as in the non-conformal case, we only need to substitute the matrices and construct a new field strength as

$$
\begin{aligned}
F_{\alpha\beta}{}^{A}{}_{B} =& \frac{1}{2}\left(R^{\mu}{}_{\rho\,\alpha\beta}\delta^{\nu}_{\sigma} + R^{\nu}{}_{\rho\,\alpha\beta}\delta^{\mu}_{\sigma} + R^{\mu}{}_{\sigma\,\alpha\beta}\delta^{\nu}_{\rho} + R^{\nu}{}_{\sigma\,\alpha\beta}\delta^{\mu}_{\rho}\right)\begin{pmatrix}1 & 0 & 0\\ 0 & 0 & 0\\ 0 & 0 & 0\end{pmatrix} +\\
&+ \nabla_{\alpha}\omega_{\beta}{}^{A}{}_{B} - \nabla_{\beta}\omega_{\alpha}{}^{A}{}_{B} + \omega_{\alpha}{}^{A}{}_{C}\omega_{\beta}{}^{C}{}_{B} - \omega_{\beta}{}^{A}{}_{C}\omega_{\alpha}{}^{C}{}_{B} \qquad (8.20)
\end{aligned}
$$

With this, the contribution to the effective action in the background field formalism of the bosonic sector is a trivial extension of the non-Weyl invariant case and the expression looks the same

$$
\begin{aligned}
\Gamma^{h\phi f}_{\mathbf{ren}}[g_{\mu\nu},\bar{\phi}] =& -\frac{1}{32\pi^2}\frac{1}{360}\log\left(\frac{\mu^2}{M^2}\right)\int d^n x\sqrt{|g|}\ \mathrm{Tr}\{60\Box E + 60RE+\\
&+ 180E^2 + 12\Box R + 5R^2 - 2R_{\mu\nu}R^{\mu\nu} + 2R_{\mu\nu\rho\sigma}R^{\mu\nu\rho\sigma}+\\
&+ 30F_{\mu\nu}F^{\mu\nu}\} \qquad (8.21)
\end{aligned}
$$

where we just need to substitute the different objects contained in the expression by the new ones.

With respect to the ghost sector, now we have two kind of ghosts fields, c^{μ} coming from the diffeomophism invariance and c coming from Weyl invariance, together with their corresponding anti-ghost partners. Moreover, the fact that the gauge fixing choice X_D in (8.13) is not Weyl invariant will produce interaction terms between *Diff* anti-ghosts and Weyl ghosts. This implies that both ghost sectors are not decoupled and their path integral cannot be taken separately. thus, we must introduce, as with the bosonic sector, two DeWitt super-ghost fields, defined by

$$\eta^A = \begin{pmatrix} c^\mu \\ c \end{pmatrix}, \qquad \bar{\eta}^A = \begin{pmatrix} \bar{c}^\mu \\ \bar{c} \end{pmatrix} \tag{8.22}$$

so that we can arrange their action to be

$$S_{ghost} = \int d^n x \sqrt{|g|}\, \bar{\eta}^A \left(-G^{(g)}_{AB}\Box + N^{(g)\mu}_{AB}\nabla_\mu + M^{(g)}_{AB} \right) \eta^B \tag{8.23}$$

where

$$G^{(g)}_{AB} = \begin{pmatrix} g_{\mu\nu} & 0 \\ 0 & 1 \end{pmatrix}, \quad N^{(g)\alpha}_{AB} = \begin{pmatrix} N^\alpha_{\mu\nu} & 0 \\ N^\alpha_{W,\mu} & 0 \end{pmatrix}, \quad M^{(g)}_{AB} = \begin{pmatrix} M_{\mu\nu} & M_{\mu,W} \\ M_{W,\mu} & M_{WW} \end{pmatrix}$$

with the different elements being

$$\begin{aligned}
N^\alpha_{\mu\nu} &= -(1-\gamma) g_{\mu\nu} \frac{\nabla^\alpha \phi}{\phi} + (1+\gamma) \frac{\nabla_\nu \phi}{\phi} \delta^\alpha_\mu + (1-\gamma) \frac{\nabla_\mu \phi}{\phi} \delta^\alpha_\nu \\
N^\alpha_{W,\nu} &= \frac{2}{n-2} \frac{\nabla^2 \phi}{\phi} \delta^\alpha_\nu \\
M_{\mu\nu} &= -R_{\mu\nu} + 2 \frac{\nabla_\mu \nabla_\nu \phi}{\phi} \\
M_{\mu,W} &= -\gamma(n-2) \frac{\nabla_\mu \phi}{\phi} \\
M_{W,\nu} &= \frac{2}{n-2} \frac{\nabla_\nu \nabla^2 \phi}{\phi} \\
M_{WW} &= \frac{\nabla^2 \phi}{\phi}
\end{aligned}$$

This allows us to use again the Schwinger-DeWitt technique of chapter 3 to compute the ghost contribution to the quantum effective action as

$$\begin{aligned}
\Gamma^{\eta\bar{\eta}}_{\mathbf{ren}}[g_{\mu\nu}, \bar{\phi}] = &- \frac{1}{32\pi^2} \frac{1}{360} \log(\mu^2) \int d^n x \sqrt{|g|}\, \mathrm{Tr}\{ 60\Box E^{(g)} + 60 R E^{(g)} + \\
&+ 180 \left(E^{(g)} \right)^2 + 12\Box R + 5R^2 - 2R_{\mu\nu}R^{\mu\nu} + 2R_{\mu\nu\rho\sigma}R^{\mu\nu\rho\sigma} + \\
&+ 30 F^{(g)}_{\mu\nu} F^{(g)\mu\nu} \}
\end{aligned} \tag{8.24}$$

where here the endomorphism and bundle connection are defined by

$$\omega^A_{\mu B} = \frac{1}{2} G^{(g)AC} N^{(g)}_{\mu BC} \tag{8.25}$$

$$E^{(g)A}_B = G^{AC} \left(-M^{(g)}_{BC} - \omega_{\mu CD}\omega^{\mu D}_B - \nabla_\mu \omega^\mu_{CB} \right) \tag{8.26}$$

and the field strength reads[3]

$$F^{(g)A}_{\rho\sigma B} = \begin{pmatrix} R^{\mu}_{\nu\rho\sigma} & 0 \\ 0 & 0 \end{pmatrix} + \nabla_\rho \omega^A_{\sigma B} - \nabla_\sigma \omega^A_{\rho B} + [\omega_\rho, \omega_\sigma]^A_B \tag{8.27}$$

Finally, we arrive to a closed expression for the quantum effective action of Dilaton Gravity in the Weyl invariant point, given by a combination of contributions (8.21) and (8.27)

$$\Gamma_{\mathbf{ren}}[g_{\mu\nu}, \bar{\phi}] = \Gamma^{h\phi f}_{\mathbf{ren}}[g_{\mu\nu}, \bar{\phi}] - 2\Gamma^{\eta\bar{\eta}}_{\mathbf{ren}}[g_{\mu\nu}, \bar{\phi}] \tag{8.28}$$

where, again, the ghost contribution enters with a minus two due to the presence of two anti-commuting ghost fields.

When this expression is evaluated explicitely, we find that the one-loop divergent part of the quantum effective action in the background field formalism reads off-shell

$$\Gamma_{\mathbf{ren}}[g_{\mu\nu}, \bar{\phi}] = \frac{1}{16\pi^2} \log(\frac{\mu^2}{M^2}) \int d^n x \sqrt{|g|}\; \hat{\mathcal{L}} \tag{8.29}$$

where $\hat{\mathcal{L}}$ contains a complicated combination of all operators with the right scaling dimension

$$\begin{aligned}\hat{\mathcal{L}} =& Q_1(\alpha\,\gamma)\frac{(\nabla\bar{\phi})^2(\nabla\bar{\phi})^2}{\bar{\phi}^4} + Q_2(\alpha,\gamma)\frac{\nabla^\mu\bar{\phi}\nabla^\nu\bar{\phi}\nabla_\mu\nabla_\nu\bar{\phi}}{\bar{\phi}^3} + Q_3(\alpha,\gamma)\frac{\nabla^\mu\nabla^\nu\bar{\phi}\nabla_\mu\nabla_\nu\bar{\phi}}{\bar{\phi}^2} + \\ &+ 2\gamma\frac{\nabla_\mu\bar{\phi}\nabla^2\nabla^\mu\bar{\phi}}{\bar{\phi}^2} + Q_4(\alpha,\gamma)\frac{(\nabla\bar{\phi})^2\nabla^2\bar{\phi}}{\bar{\phi}^3} + Q_5(\alpha,\gamma)\frac{\nabla^2\bar{\phi}\nabla^2\bar{\phi}}{\bar{\phi}^2} + \\ &+ Q_6(\alpha,\gamma)\frac{R^{\mu\nu}\nabla_\mu\bar{\phi}\nabla_\nu\bar{\phi}}{\bar{\phi}^2} + Q_7(\alpha,\gamma)\frac{R^{\mu\nu}\nabla_\mu\nabla_\nu\bar{\phi}}{\bar{\phi}} + Q_8(\alpha,\gamma)R^{\mu\nu}R_{\mu\nu} + \\ &+ Q_9(\alpha,\gamma)\frac{R\nabla^\mu\bar{\phi}\nabla_\mu\bar{\phi}}{\bar{\phi}^2} + Q_{10}(\alpha,\gamma)R^2 + Q_{11}(\alpha,\gamma)\frac{R\nabla^2\bar{\phi}}{\bar{\phi}} + \frac{53}{45}R^{\mu\nu\alpha\beta}R_{\mu\nu\alpha\beta}\end{aligned} \tag{8.30}$$

[3]Note that in the $(1,1)$ element, the indices A, B can be mapped space-time indices μ, ν.

where

$$
\begin{aligned}
Q_1(\alpha,\gamma) &= \frac{16+108\alpha^2-8\gamma+96\alpha^2\gamma+4\gamma^2+18\alpha^2\gamma^2+\gamma^3+4\alpha^2\gamma^3}{\alpha^2}\\
Q_2(\alpha,\gamma) &= -\frac{2(96+405\alpha^2-48\gamma+390\alpha^2\gamma+13\gamma^2+57\alpha^2\gamma^2+3\gamma^3+12\alpha^2\gamma^3)}{9\alpha^2}\\
Q_3(\alpha,\gamma) &= \frac{48+81\alpha^2-24\gamma+102\alpha^2\gamma+\gamma^2+3\alpha^2\gamma^2}{9\alpha^2}\\
Q_4(\alpha,\gamma) &= -\frac{-102-378\alpha^2+96\gamma-420\alpha^2\gamma-44\gamma^2-60\alpha^2\gamma^2+3\gamma^3+12\alpha^2\gamma^3}{9\alpha^2}\\
Q_5(\alpha,\gamma) &= -\frac{-162+228\alpha^2-108\alpha^4-24\alpha^2\gamma+84\alpha^4\gamma+\alpha^2\gamma^2+3\alpha^4\gamma^2}{9\alpha^4}\\
Q_6(\alpha,\gamma) &= -\frac{-96-63\alpha^2+24\gamma-78\alpha^2\gamma-\gamma^2+15\alpha^2\gamma^2}{9\alpha^2}\\
Q_7(\alpha,\gamma) &= -\frac{4(4-3\alpha^2-\gamma+\alpha^2\gamma)}{3\alpha^2}\\
Q_8(\alpha,\gamma) &= -\frac{-120+361\alpha^2}{90\alpha^2}\\
Q_9(\alpha,\gamma) &= -\frac{11+24\alpha^2-6\gamma+32\alpha^2\gamma+3\gamma^2+6\alpha^2\gamma^2}{3\alpha^2}\\
Q_{10}(\alpha,\gamma) &= \frac{18-30\alpha^2+43\alpha^4}{36\alpha^4}\\
Q_{11}(\alpha,\gamma) &= \frac{-18+25\alpha^2-21\alpha^4-2\alpha^2\gamma+2\alpha^4\gamma}{3\alpha^4}
\end{aligned}
$$

It is worth mentioning that all the monomials including the scalar field diverge when $\bar{\phi}=0$. Naive power counting arguments cannot then be applied here. This fact also prevents the monomials that appear in the bare lagrangian to appear in the counterterm, representing the already known fact that we are dealing with a non-renormalizable theory. This physically means something that we already knew, namely that our calculation is restricted to the broken phase of the theory. When this is put on-shell by using the relations (7.68) and (7.74) derived in the previous chapter (particularized for the Weyl invariant value of the coupling ξ) all the gauge dependence on the parameters γ and α dissapears. This is a powerful check of the gauge independence of our result, complaining with Kallosh-DeWitt theorem. Moreover, we can also use the following relations

$$
\begin{aligned}
&E_4 = R_{\mu\nu\alpha\beta}R^{\mu\nu\alpha\beta}+R^2-4R_{\mu\nu}R^{\mu\nu}\\
&\int d^n x\sqrt{|g|}\,E_4 = \int d^n x\sqrt{|g|}\,C^2 - 2\int d^n x\sqrt{|g|}\,\left(R^{\mu\nu}R_{\mu\nu}-\frac{1}{3}R^2\right)
\end{aligned}
\tag{8.31}
$$

The first line is just the expression of the Gauss-Bonnet term in four dimensions, E_4 thus corresponding to the Euler density, whose integral gives the Euler characteristic of the manifold. By using it and the fact that the last term in the second line vanishes when using the equations of motion (7.3), we can express the on-shell value of our result as simply

$$\Gamma_{\mathbf{ren}}[g_{\mu\nu}, \bar{\phi}]\big|_{\text{on-shell}} = \frac{1}{16\pi^2} \log\left(\frac{\mu^2}{M^2}\right))\frac{53}{45} \int d^n x \sqrt{|g|}\, E_4 \tag{8.32}$$

This result is exactly the same as in General Relativity (including numerical factors). The reason is that, on one hand it is Weyl invariant, in order to comply with the gauge symmetries of the theory, but, since the transformation back to General Relativity is actually a conformal transformation

$$g_{\mu\nu} \to \frac{1}{M_p^2}\left(\frac{n-2}{8(n-1)}\right)^{\frac{2}{n-2}} \bar{\phi}^{\frac{4}{n-2}} g_{\mu\nu} \tag{8.33}$$

the value of the on-shell counterterm does not change and thus it is forced to have the same value for both theories related by this transformation. Indeed, this is no more than a new confirmation of the validity of Kallosh-DeWitt theorem that the on-shell values of the quantum effective action must be the transformation one of the other, identical in this case since the particular transformation is a symmetry of the theory.

9

Anomalous equivalence of frames

Up to here we have computed the corresponding one-loop counterterms for Dilaton Gravity both in the Weyl invariant and the non-Weyl invariant point. In chapter 6 we argued that the former was indeed just a field redefinition of Einstein-Hilbert action, which can be recovered from the CDG action (8.1) by just undoing the redefinition, which happens to be precisely a Weyl transformation

$$g_{\mu\nu} \to \frac{1}{M_p^2}\left(\frac{n-2}{8(n-1)}\right)^{\frac{2}{n-2}} \bar{\phi}^{\frac{4}{n-2}} g_{\mu\nu} \tag{9.1}$$

We also saw in expression (6.9) of chapter 6 that, at the light of Kallosh-DeWitt theorem, both quantum corrections, those computed in Einstein-Hilbert action and in CDG owed to preserve equivalence when taken on-shell. That is, S-matrix was ensured to be equivalent for both theories. This was because, in the background field formalism, we defined a new partition function (3.20)

$$\tilde{\mathcal{Z}}[J,\bar{\phi}] = \int [\mathcal{D}Q] e^{iS(\bar{\phi})} e^{i\langle Q,\hat{D}Q\rangle + i\langle J,Q\rangle} \tag{9.2}$$

and proved that its vacuum diagrams correspond to the effective action of the original theory of interest, a fact that at one-loop is reduced to the computation of a determinant

$$\Gamma[\bar{\phi}] = \frac{1}{2}\log\left[\det(\hat{D})\right] \tag{9.3}$$

which we obtained by using the Schwinger-DeWitt technique of chapter 3. Finally, by transforming this result using the transformation relating the fields in a generic way $Q' = Q'(Q)$ we find that

$$\Gamma^{'}[\bar{\phi}] = \frac{1}{2}\log\left[\det(\hat{D}') + \left(\frac{\partial Q'}{\partial Q}\right)^2 \frac{\partial^2 Q}{\partial Q'^2}\frac{\partial S}{\partial Q'}\right] \tag{9.4}$$

which shows that on-shell quantities must be the straightforward transformation one of the other.

However, there is a caveat in all this argument. If the transformation $Q' = Q'(Q)$ happens to open a symmetry in the new fields Q', as it happens in our case with Weyl invariance, then the integration measure $[\mathcal{D}Q]$ in (9.2) might not be equivalent and instead be anomalous. Equivalence is completely lost, since the divergence of the previously conserved current can now act as an operator producing a non-vanishing expectation value for some S-matrix amplitude which *is not present in the other frame.* This is of course what happens in principle in our case, because Weyl invariance, as explained in chapter 2 is generically anomalous.

The corresponding Ward identity (2.11) for the Weyl invariance of CDG reads

$$\left\langle 0 \;\middle|\; 2g_{\mu\nu}\frac{\delta\mathcal{L}}{\delta g_{\mu\nu}} + \frac{2-n}{2}\phi\frac{\delta\mathcal{L}}{\delta\phi} \;\middle|\; 0 \right\rangle = \mathcal{A} \tag{9.5}$$

where $\mathcal{A}$ is the possible anomalous term. It is identically satisfied at tree level, conveying with the classical result, but at one-loop it might get contributions from the non-invariance of the effective action.

If there were not anomalies, the Ward identity would also hold in the first frame (even if the symmetry is not there). However, anomalies spoil this fact and physical effects that depend on the observables related, *can be different* in both frames.

Let us particularize. The regularized effective action in the background field formalism and using zeta function regularization was given in (3.36) and reads

$$\Gamma_{reg}[\bar{\phi}]\big|_{s\to 0} = -\frac{1}{2}\left(\frac{1}{s} - \gamma_E + \log\left(\tilde{\mu}^2\right)\right)\zeta(0,1,\hat{D}) - \frac{1}{2}\frac{\partial}{\partial s}\,\zeta(s,1,\hat{D})\Big|_{s=0} \tag{9.6}$$

so that the pole term that we need to substract in the renormalization process (particularized for a effective action depending on $g_{\mu\nu}$ and the background dilation $\bar{\phi}$) is simply

$$\Gamma_{pole}[g_{\mu\nu},\bar{\phi}] = -\frac{1}{2s}\zeta(0,1,\hat{D}) \tag{9.7}$$

which, by using the standard relation (3.48) between the poles of the zeta function and the short-time expansion of the Heat Kernel, can be written

$$\Gamma_{pole}[g_{\mu\nu}, \bar{\phi}] = -\frac{1}{2s} a_n \left(1, \hat{D}\right) = \frac{1}{s}\frac{1}{16\pi^2}\frac{53}{45}\int d^n x \sqrt{|g|}\, E_4 \tag{9.8}$$

where in the second equality we have introduced the particular on-shell value for CDG, given in (8.32).

The easiest way to make the anomaly appear is to now substitute the divergent limit $s \to 0$ by the corresponding pole of dimensional regularization when $n \to 4$, having

$$\Gamma_{pole}[g_{\mu\nu}, \bar{\phi}] = \frac{1}{n-4}\frac{1}{16\pi^2}\frac{53}{45}\int d^n x \sqrt{|g|}\, E_4 \tag{9.9}$$

From here, it is straightforward to obtain the anomalous contribution $\mathcal{A}$ to the Ward identity. We just perform a Weyl transformation of the regularized effective action and, taking into account that the integrand transforms as

$$\sqrt{|g|}\, E_4 \to \Omega^{n-4}\sqrt{|g|}\, E_4 \tag{9.10}$$

we find, at the linear level, a finite contribution coming from a cancellation of the pole term

$$\delta\Gamma_{pole}[g_{\mu\nu}, \bar{\phi}] = \frac{1}{16\pi^2}\frac{53}{45}\int d^n x \sqrt{|g|}\, E_4\, \omega \tag{9.11}$$

so that

$$\mathcal{A} = \frac{1}{16\pi^2}\frac{53}{45} E_4 \tag{9.12}$$

Anomalies have always provoked some sort of discussion in the community. The reason is that sometimes, when using an inappropriate regularization scheme, something that looks like an anomaly arises and one has to be careful to not to break by hand the symmetry because of the scheme chosen. Thus, it is natural to ask ourselves if such scheme that preserves Weyl invariance exist. Indeed, in [25] it was suggested that when a dilaton, like our field $\bar{\phi}$, is present in the theory, one can modify dimensional regularization in such a way that the issue is solved. In our case, this amounts to cancelling the scaling dimension of the integrand by powers of ϕ. The corresponding one-loop pole term would be then

$$\Gamma_{pole}[g_{\mu\nu}, \bar{\phi}] = \frac{1}{n-4}\frac{1}{16\pi^2}\frac{53}{45}\int d^n x \sqrt{|g|}\, \bar{\phi}^{-\frac{2(n-4)}{n-2}}\, \hat{\mathcal{L}} \tag{9.13}$$

with $\hat{\mathcal{L}}$ as given in 8.30.

With this change, now the integrand is exactly Weyl invariant in any dimension and no anomaly arises. We are now preserving the Ward identity and saving the frame equivalence. A similar scheme can be implemented for higher loop corrections but as everything in life, this comes at a price. If we now perform the renormalization process, although the pole part is exactly the same as before, we find that *finite terms are modified.* The renormalized effective action receives now a new contribution

$$\sim \frac{1}{16\pi^2}\frac{53}{45}\int d^n x\sqrt{|g|}\ \log(\bar{\phi})\ \hat{\mathcal{L}} \tag{9.14}$$

which enters as a non-local function of the fields[1]. This implies that a corresponding non-local counterterm is required.

Of course, all this discussion is trivial for the one-loop counterterm since it is a total derivative, but it has a well-defined meaning if we go to higher loops. There, the counterterms will contain higher powers of the Riemann tensor which *are not total derivatives* and all this discussion applies. In particular, at two loops we find[2]

$$\Gamma_{pole}[g_{\mu\nu},\bar{\phi}] = \frac{1}{n-4}\frac{1}{(4\pi)^4}\frac{209}{2880}\left(\frac{n-2}{8(n-1)}\right)^{\frac{n-6}{n-2}}\int d^n x\sqrt{|g|}\ \bar{\phi}^{\,-\frac{2(n-6)}{n-2}}C^3 \tag{9.15}$$

where C^3 means the cube of the Weyl tensor.

From this analysis we can conclude two things. First, that the naïve expectation phrased by Duff that *"Spurious Weyl invariance is not anomalous"* is indeed only true if we get rid of locality notions when renormalizing. We have found a particular example in which this is realized.

But second, we have argued that the assumed equivalence of Einstein and Jordan frame must be taken carefully. Adding to the issue already noted by Vilkovisky quite time ago[67] that mean field equations *are not equivalent* we have found that even at the on-shell effective action level there could be

[1]It is important not to get confused by the appearance of logarithms of the fields in, for example, the Coleman-Weinberg potential. There, they appear because an identification of the μ scale with the effective mass of the field $\mu^2 = \frac{\delta\mathcal{L}}{\delta\bar{\phi}^2}$. Here, it appears *a priori* and no regularization scheme can avoid it.

[2]Assuming that the on-shell equivalence holds and using the result of [5] for the two-loops counterterm of General Relativity.

an issue when the transformation opens a new symmetry. Then, anomalies can spoil the equivalence by modifying physical observables in one of the frames. We have showed this in particular by using Weyl invariance and one can argue that this symmetry is not generically present in realistic cosmological applications. However, going to the Jordan frame always implies the substitution of Planck mass by a corresponding power of the dilaton field $M_p^2 \to \bar{\phi}^2$ and a scale invariance emerges. It is natural to question if quantum effects can break scale invariance and spoil the equivalence.

Part III

Unimodular Gravity

10

A new role for Λ

As we saw in chapter 1, the standard lore of effective field theory is compromised in the presence of gravity. The cosmological constant gets corrections from matter (and gravitational) loops in such a way that there is a huge discrepancy between naïve calculations and the observed value obtained by using different experimental methods. This reflects as a fine-tuning problem in equation (1.16), implying that the bare value of the cosmological constant had to be chosen

$$\Lambda + (\text{matter loop corrections}) \sim 10^{-47}\ \text{GeV}^4 \tag{10.1}$$

with extremely huge accuracy.

This introduces an extra scale in the theory which is not related at all with the UV scale M_p and wilsonian arguments about EFT's break completely.

The reason underneath this problem comes from the fact that that any potential energy couple to gravitation through a minimal coupling

$$S_V = \int d^4x \sqrt{|g|}\ V(\phi) \tag{10.2}$$

so that the zero mode of the metric, which is not suppressed at any scale, mediates the interaction and introduces corrections to Λ by means of vacuum diagrams.

Many possible solutions for this issue have been considered so far in the literature, being the standard lore that the IR behavior of GR must be modified in some manner, thus changing the way in which this zero mode behaves. This has lead in the recent years to a resurgence of massive gravity[71] and the consideration of bi-gravity[72] among many other modified gravitational theories that try to reconcile the wilsonian picture with the observed value for the cosmological constant.

Perhaps the most radical of these modifications is *Unimodular Gravity*, consisting on restricting the physically admissible metrics to those of unit determinant $|g| = 1$. By doing this, the zero mode does not couple to the potential term any more and any possible contribution to the cosmological constant drops out of the equations of motion. Thus, Unimodular Gravity (UG) is defined, in the absence of matter, as simply

$$S_{UG} = -M_p^2 \int d^4x \; R[|g| = 1] \tag{10.3}$$

Although this defines UG, it is not fully operationally comfortable. The degrees of freedom of the theory are not independent and one is forced to work with a restricted variational principle. Since we are constraining the physical metrics to be unimodular, the equations of motion of the theory must be obtained by using a traceless variation

$$\delta g^t_{\mu\nu} = \delta g_{\mu\nu} - \frac{1}{4} g_{\mu\nu} g^{\alpha\beta} \delta g_{\alpha\beta} \tag{10.4}$$

Therefore, when Unimodular Gravity is coupled minimally to matter with an arbitrary conserved energy momentum tensor

$$S_{UG+T} = \int d^4x \left\{ R[|g| = 1] + g_{\mu\nu} T^{\mu\nu} \right\} \tag{10.5}$$

the equations of motion will correspond to the traceless part of the Einstein equations[1]

$$R_{\mu\nu} - \frac{1}{4} R g_{\mu\nu} = T_{\mu\nu} - \frac{1}{4} T g_{\mu\nu} \tag{10.6}$$

It seems at a first glimpse that this theory fails to reproduce known IR physics and, in particular, Solar system physics, since the equations of motion do not reduce to the ones of General Relativity. Moreover, even if the wilsonian problem of the cosmological constant exists, we also know that our Universe has a non-vanishing value Λ and therefore this must be accommodated in some way in any realistic approach. Completely vanishing the presence of Λ is not an appropiate solution.

However, this is illusory and the trace degree of freedom can be taken back into game by using contracted Bianchi identities, which are always satisfied in any riemannian manifold. These are

$$\nabla_\mu R^{\mu\nu} - \frac{1}{2} \nabla^\nu R = 0 \tag{10.7}$$

[1] Actually, these equations could be named Einstein equations too, since they were already proposed by Einstein back in 1919[73] for completely different reasons than the ones presented here.

which, upon substitution of the equations of motion, imply

$$\nabla_\mu (R + T) = 0 \tag{10.8}$$

By integrating this we find a constraint that relates the scalar curvature with the trace of the energy momentum tensor and some integration constant that we cleverly label as Λ

$$(R + T) = -4\Lambda \tag{10.9}$$

Finally, combining this with the traceless equations of motion (10.6) we recover the trace degree of freedom and the full Einstein equations

$$R_{\mu\nu} - \frac{1}{2} g_{\mu\nu} R - \Lambda g_{\mu\nu} = T_{\mu\nu} \tag{10.10}$$

with the integration constant Λ taking the role of the cosmological constant. The main difference with the standard approach is that its value is here given in terms of initial conditions for the metric evolution, it is an integration constant after all, instead of being a dynamical coupling of the lagrangian. Vacuum energy generated by mater loops does not gravitate any more.

There is another main difference of UG with respect to GR. Since we are restricting ourselves to the set of metrics with a fixed determinant, the theory cannot be invariant under the full group of diffeomorphisms any more but only under those that preserve the value of the determinant. In other words, only those diffeomorphism which preserve the volume are symmetries of the theory. Those are given infinitesimally by

$$\delta g_{\mu\nu} = \pounds_\xi g_{\mu\nu} = \nabla_\mu \xi_\nu + \nabla_\nu \xi_\mu \tag{10.11}$$

where the generator of the diffeomorphism is constrained to be transverse

$$\nabla_\mu \xi^\mu = 0 \tag{10.12}$$

We dub this subgroup *transverse diffeomorphisms (TDiff)* and although one could think that reducing the symmetry group can compromise the perturbtive stabilty of the theory, this is not the case at hand. Surprisingly, one does not need full *Diff* invariance to protect the theory from ghost propagation and actually *TDiff* is the biggest subgroup of *Diff* that is able to kill all the unphysical polarizations of the graviton perturbation[**?**] $h_{\mu\nu} = g_{\mu\nu} - \eta_{\mu\nu}$ and thus we are safe.

The main question that arises here is whether this protection of the cosmological constant can be preserved in a Quantum Field Theory formulation of UG. Were not, then the wilsonian picture of effective field theory would be compromised anyway and UG would not solve anything.

It is important to note that the unimodular constraint is not a gauge fixing. Although it is true that in General Relativity one can always choose the condition $|g| = 1$ by means of the diffeomorphism invariance of the theory, in this case we are reducing the space of physical metrics *a priori* in the action, while a gauge fixing is done at the level of the equations of motion. In particular, in the second case, the coupling $g_{\mu\nu}\Lambda$ between the metric and the cosmological constant does not vanish in this gauge while for the unimodular theory it is always absent.

In order to construct a path integral, the action (10.3) is however not very useful. Using it we would have to integrate over those metric satisfying the unimodularity condition and an extra prescription is required. One possible option is to write the theory as GR and impose the unimodular condition by using a Lagrange multiplier ([74] and references therein). However, this is dangerous because the multiplier could acquire dynamics through quantum corrections, spoiling the constraint. Here instead we prefer to perform a field redefinition

$$\tilde{g}_{\mu\nu} = |g|^{-1/n} g_{\mu\nu} \tag{10.13}$$

which is explicitly unimodular, arriving to an action principle[75, 76, 77]

$$S = -\int d^n x |g|^{\frac{1}{n}} \left\{ R + \frac{(n-1)(n-2)}{4n^2} \frac{\nabla_\mu |g| \nabla^\mu |g|}{g^2} \right\} \tag{10.14}$$

which we write in arbitrary dimension n in order to use later dimensional regularization. Note that the appearance of the determinant of the metric here is not dangerous because it behaves as a scalar under *TDiff*, so the action is still covariant with respect to this symmetry.

In this form, the banning of the cosmological constant from the action happens because of the presence of an extra gauge symmetry in the form of Weyl invariance

$$g_{\mu\nu} \to \Omega^2 g_{\mu\nu} \tag{10.15}$$

with Ω a function of the spatial coordinates.

Then, any possible dimensionful constant and in particular a cosmological constant, is forbidden from the action principle. In particular, the coupling

$$\int d^4x\sqrt{|g|}\;\Lambda \tag{10.16}$$

would break Weyl invariance explicitly.

Additionally, and quite surprisingly, this action is the only other possible one, apart from General Relativity, which propagates a single spin two degree of freedom about flat space[77, 78].

In the new variables, the equations of motion are still traceless (as a requirement of the Noether constraint imposed by Weyl invariance) but they get new contributions proportional to the determinant of the metric

$$\begin{aligned} R_{\mu\nu} - \frac{1}{n}Rg_{\mu\nu} - \frac{(n-2)(2n-1)}{4n^2}\left(\frac{\nabla_\mu g\nabla_\nu g}{g^2} - \frac{1}{n}\frac{(\nabla g)^2}{g^2}g_{\mu\nu}\right) + \\ + \frac{n-2}{2n}\left(\frac{\nabla_\mu\nabla_\nu g}{g} - \frac{1}{n}\frac{\nabla^2 g}{g}g_{\mu\nu}\right) = \left(T_{\mu\nu} - \frac{1}{n}Tg_{\mu\nu}\right) \end{aligned} \tag{10.17}$$

The original theory is recovered by partialy fixing the gauge to $|g| = 1$ so that these reduce to the traceless Einstein equations (10.6).

In all the different formulations of Unimodular gravity, Bianchi identities ensure the full equivalence with General Relativity at the classical level by reintroducing the trace degree of freedom, recovering the full set of Einstein equations. However, this equivalence is not ensured at the quantum level [79].

If it is possible to preserve the unimodularity condition in the path integral formulation, then the naive expectation is that no possible renormalization of the cosmological constant may happen and that its property of being an integration constant is preserved when quantum corrections enter into game [**?**, 80]. There are some hints along this path in the context of the asymptotic safety scenario [81, 82] but their conclussions are not very clear.

In the following we will compute explicitly one-loop corrections to the action (10.14) by means of the generalized Schwinger-DeWitt technique of chapter 4. in order to answer this question. We will see how the on-shell counterterms do not contain any dynamical coupling to the cosmological constant which is hence fixed to an integration constant at any order in the loop expansion. Aditionally, we will check that no Weyl anomaly is generated so the algebraic gauge fixing $|g| = 1$ can be done at all levels without spoiling the equivalence between (10.3) and (10.14).

11
Quantizing Unimodular Gravity

In order to compute quantum corrections to Unimodular Gravity, we are going to exploit the techniques developed in chapters 3 and 4 to compute the divergences of the one-loop effective action in the background field approach. We start then by splitting our fields in background plus perturbation. However, here we will depart from the usual linear splitting. Since afterwards we pretend to fix the background gauge $|g| = 1$ in order to recover our original formulation for Unimodular Gravity, we will use the most useful splitting

$$\tilde{g}_{\mu\nu} = |g|^{\frac{1}{n}} \left(g_{\mu\nu} + h_{\mu\nu}\right) \tag{11.1}$$

which makes the background metric unimodular, rather than the usual $g_{\mu\nu} = \bar{g}_{\mu\nu} + h_{\mu\nu}$. Notice that we can convert the splitting in (11.1) into the usual splitting by performing a Weyl transformation of the quantum field together with one of the background metric.

Let us warn the reader that from now on the covariant derivative will be defined with respect to the metric $g_{\mu\nu}$ and that, unless explicitly said, we are dropping any symbol to denote background quantities in order to get cleaner formulas. Conceptually however it is important to keep in mind the difference between the full metric and the background one.

As it stands and after expanding it, S_{UG} will still contain the linearized realization of the gauge symmetries of the theory, *TDiff* and *Weyl*, and thus a proper definition requires a gauge fixing. In the BRST language of chapter 5, the action of these symmetries can be written as

$$\begin{aligned}
&s_D g_{\mu\nu} = s_W g_{\mu\nu} = 0 \\
&s_D h_{\mu\nu} = \nabla_\mu c^T_\nu + \nabla_\nu c^T_\mu + c^{T\rho}\nabla_\rho h_{\mu\nu} + \nabla_\mu c^{T\rho} h_{\rho\nu} + \nabla_\nu c^{T\rho} h_{\rho\mu} \\
&s_W h_{\mu\nu} = 2c\left(g_{\mu\nu} + h_{\mu\nu}\right)
\end{aligned} \tag{11.2}$$

where c and $c^{T\mu}$ are the anticommuting ghost fields for Weyl invariance and transverse diffeomorphisms, respectively. In this language, the transverse condition is satisfied by imposing $\nabla_\mu c^{T\mu} = 0$ on the ghost field. The superscript T thus means that the vector ghost satisfies this condition. The gauge fixing procedure of these gauge symmetries will be discussed next.

Recalling chapter 3, the quantum effective action in the background field approach was given by

$$\Gamma[\bar{\phi}] = \frac{1}{2}\log\left[\det(\hat{D})\right] \tag{11.3}$$

where $\hat{D}$ is the operator driving the one-loop quantum fluctuations, defined by the quadratic term in the expansion of the action around the background metric

$$S_2 = \int d^n x\, \mathcal{L}_2 = \int d^n x\, h^{\mu\nu}\hat{D}_{\mu\nu\rho\sigma}h^{\rho\sigma} \tag{11.4}$$

It is useful to write down the expression in arbitrary dimension as it would stand *before* the background metric is assumed to be unimodular

$$\begin{aligned}
\mathcal{L}_2 =& \frac{1}{4}h^{\mu\nu}\Box h_{\mu\nu} - \frac{n+2}{4n^2}h\Box h + \frac{1}{2}\left(\nabla_\mu h^{\mu\alpha}\right)\left(\nabla_\nu h^\nu_\alpha\right) - \frac{1}{n}\left(\nabla_\mu h\right)\left(\nabla_\nu h^{\mu\nu}\right) + \\
&+ \frac{1}{2n^2}h^2 R + \frac{2-n}{2n}\left(\nabla_\alpha \log g\right)\left(\frac{1}{2}h^{\beta\lambda}\nabla_\lambda h^\alpha_\beta + \frac{3}{2}h^{\alpha\beta}\nabla_\lambda h^\lambda_\beta - \frac{1}{n}h\nabla_\lambda h^{\alpha\lambda}\right) + \\
&+ \frac{(n-2)^2}{8n^3}\left(\nabla^\alpha \log g\right)\, h\nabla_\alpha h + \frac{n-2}{2n^2}\left(\nabla_\beta \log g\right)\, h^{\alpha\beta}\nabla_\alpha h + \frac{1}{2}h^{\alpha\beta}h^\mu_\beta R_{\mu\alpha} + \\
&+ (n^2-3n+2)\left(\frac{1}{8n^4}h^2(\nabla\log g)^2 - \frac{1}{8n^3}h_{\mu\nu}h^{\mu\nu}(\nabla\log g)^2 - \frac{1}{n}hh^{\mu\nu}R_{\mu\nu} - \right.\\
&\left. - \frac{1}{4n^3}hh_{\alpha\beta}\left(\nabla^\alpha \log g\right)\,\left(\nabla^\beta \log g\right)\right) + \frac{1}{2}h^{\mu\nu}h^{\alpha\beta}R_{\mu\alpha\nu\beta} - \frac{1}{2n}h^{\mu\nu}h_{\mu\nu}R - \\
&- \frac{8-6n+n^2}{8n^2}\left(\nabla^\alpha \log g\right)\, h^{\mu\nu}\nabla_\alpha h_{\mu\nu} + \frac{1}{4n^2}h^\lambda_\alpha h_{\beta\lambda}\left(\nabla^\alpha \log g\right)\left(\nabla^\beta \log g\right)
\end{aligned} \tag{11.5}$$

Of course, $\hat{D}$ will contain in principle zero modes coming from the gauge symmetries of the theory translated to the linear level which will make its determinant singular. This is solved by constructing an appropriate gauge fixing term using the BRST quantization method.

Finally, since we are using the splitting (11.1), the action for the one-loop quantum fluctuations greatly simplifies, since all terms depending on $\nabla_\mu g$ now vanish. Thus, we end up with

$$\begin{aligned}
\mathcal{L}_2 =& \frac{1}{4}h^{\mu\nu}\Box h_{\mu\nu} - \frac{n+2}{4n^2}h\Box h + \frac{1}{2}\left(\nabla_\mu h^{\mu\alpha}\right)\left(\nabla_\nu h^\nu_\alpha\right) - \frac{1}{n}\left(\nabla_\mu h\right)\left(\nabla_\nu h^{\mu\nu}\right) + \\
&+ \frac{1}{2}h^{\alpha\beta}h^\mu_\beta R_{\mu\alpha} - \frac{1}{n}hh^{\mu\nu}R_{\mu\nu} + \frac{1}{2}h^{\mu\nu}h^{\alpha\beta}R_{\mu\alpha\nu\beta} - \frac{1}{2n}h^{\mu\nu}h_{\mu\nu}R + \frac{1}{2n^2}h^2 R
\end{aligned} \tag{11.6}$$

11.1. Fixing the gauge freedom

To gauge-fix the symmetries in (11.2), we shall use the BRST technique in a similar way as explained in chapter 5 and introduce the following nilpotent BRST operator

$$\int = \mathbf{s}_D + \mathbf{s}_W \tag{11.7}$$

where $\mathbf{s}_D$ and $\mathbf{s}_W$ are defined in (11.2).

Here, have in mind that $\mathbf{s}_D$ is denoting diffeomorphisms that are transverse. The path integral over the ghost fields for *TDiff* c^T_μ must be then restricted to the subspace of transverse vectors. However, the definition of such a measure $[\mathcal{D}c^{T\mu}]$ is a notorious problem [83]. The way to come to grips with it chosen here is to parametrize this subspace in terms of unconstrained fields so that we can then integrate over the full space of c^μ, whose integration measure is well-defined. This we do by introducing an operator $\Theta_{\mu\nu}$[1]

$$c^T_\mu = \Theta_{\mu\nu}c^\nu = \left(g_{\mu\nu}\Box - \nabla_\mu\nabla_\nu - R_{\mu\nu}\right)c^\nu = \left(Q_{\mu\nu} - \nabla_\mu\nabla_\nu\right)c^\nu \tag{11.8}$$

which maps vectors into transverse vectors. In this way, the transversality condition over c^T_μ translates into a gauge symmetry for c_ν

$$c_\nu \to \nabla_\nu f \tag{11.9}$$

with f an arbitrary scalar function. Indeed, this transformation takes c_ν into a longitudinal vector, so that the $\Theta_{\mu\nu}$ operator annihilates it. Of course, in order to perform now the functional integration over c^μ we must gauge-fix this new gauge symmetry by introducing a non-trivial stairway of ghost levels with BRST transformations defined in such a way that the BRST algebra closes

$$\begin{aligned} \mathbf{s}_D^2 = \mathbf{s}_W^2 &= 0 \\ \{\mathbf{s}_D, \mathbf{s}_W\} &= 0 \end{aligned} \tag{11.10}$$

on all the different fields considered.

The systematic way to obtain this field content together with the appropriate BRST transformations is by using the Batalin-Vilkovisky[84]

[1] One can easily check that $\Theta_{\mu\nu}$ is an endomorphism in the space spanned by transverse vectors.

formalism. However, in our case, things are easy enough as to allow us to guess what the BRST transformations read, once the field content of the theory is chosen as done in [84] for first-stage reducible and irreducible gauge transformations. Notice that the gauge transformations in (11.2) generated by $\mathbf{s}_D$, with c_μ in (11.8), are first-stage reducible due to the gauge symmetry in (11.9). However, the gauge symmetries in (11.2) generated by $\mathbf{s}_W$ are irreducible. We introduce the following set of fields:

$$\begin{aligned}&h_{\mu\nu}^{(0,0)},\ c_\mu^{(1,1)},\ b_\mu^{(1,-1)},\ f_\mu^{(0,0)},\ \phi^{(0,2)},\\&\pi^{(1,-1)},\ \pi'^{(1,1)},\ \bar{c}^{(0,-2)},\ c'^{(0,0)},\\&c^{(1,1)},\ b^{(1,-1)},\ f^{(0,0)}\end{aligned} \tag{11.11}$$

where $c_\mu^{(1,1)}$ denotes c_μ, $h_{\mu\nu}^{(0,0)}$ stands for $h_{\mu\nu}$ and the superscript (n,m) carries the Grassmann number, n, (defined modulo two) and ghost number, m. In this language, the BRST operators $\mathbf{s}_D$ and $\mathbf{s}_W$ enjoy Grassmann number 1 and ghost number 1, each.

Here we have three families –displayed in three different lines– of fields. The first line includes the physical graviton field together with the usual ghost field content that would be naively necessary in order to gauge-fix an unrestricted *Diff* symmetry. In addition, there is a ϕ field which accounts for the transformation in (11.9). The second line represents the field content introduced to gauge fix the gauge symmetry in (11.9), together with the one that will be induced on $b_\mu^{(1,-1)}$ by contraction with $c_\mu^{(1,1)}$. Finally, the third line is the field content due to Weyl invariance.

Now, we define the action of $\mathbf{s}_D$ and $\mathbf{s}_W$ on the fields as shown in Table 11.1, where $\left(Q^{-1}\right)^\mu_\nu = g_{\nu\alpha}\left(Q^{-1}\right)^{\mu\alpha}$ denotes the inverse of the operator $Q_{\mu\nu} = g_{\mu\nu}\Box - R_{\mu\nu}$, which exists provided $Det(Q) \neq 0$. This is our case since $Q_{\mu\nu}$ is just a standard Laplacian-type operator acting on vector fields.

With these definitions, it can be readily shown that the equations in (11.10) hold. In doing so, it is advisable to show first that

$$\mathbf{s}_D c^{T\mu} = c^{T\rho}\nabla_\rho c^{T\nu} \tag{11.12}$$

if $c^{T\mu}$ is defined as in (11.8). This can be done by using the following results

$$\nabla_\mu(c^{\rho T}\nabla_\rho c^{T\mu}) = 0, \quad \nabla_\mu\left[\left(Q^{-1}\right)^\mu_\nu\left(c^{\rho T}\nabla_\rho c^{T\nu}\right)\right] = 0 \tag{11.13}$$

The path integral quantization of the theory is accomplished now by adding to the classical action the gauge-fixing action, $S_{gauge-fixing}$, which

field	$\mathbf{s}_D$	$\mathbf{s}_W$
$g_{\mu\nu}$	0	0
$h_{\mu\nu}$	$\nabla_\mu c^T_\nu + \nabla_\nu c^T_\mu + c^{\rho T}\nabla_\rho h_{\mu\nu} + \nabla_\mu c^{\rho T} h_{\rho\nu} + \nabla_\nu c^{\rho T} h_{\rho\mu}$	$2c^{(1,1)}\left(g_{\mu\nu}+h_{\mu\nu}\right)$
$c^{(1,1)\mu}$	$\left(Q^{-1}\right)^\mu_{\ \nu}\left(c^{\rho T}\nabla_\rho c^{T\nu}\right) + \nabla^\mu \phi^{(0,2)}$	0
$\phi^{(0,2)}$	0	0
$b^{(1,-1)}_\mu$	$f^{(0,0)}_\mu$	0
$f^{(0,0)}_\mu$	0	0
$\bar{c}^{(0,-2)}$	$\pi^{(1,-1)}$	0
$\pi^{(1,-1)}$	0	0
$c'^{\,(0,0)}$	$\pi'^{\,(1,1)}$	0
$\pi'^{\,(1,1)}$	0	0
$c^{(1,1)}$	$c^{T\rho}\nabla_\rho c^{(1,1)}$	0
$b^{(1,-1)}$	$c^{T\rho}\nabla_\rho b^{(1,-1)}$	$f^{(0,0)}$
$f^{(0,0)}$	$c^{T\rho}\nabla_\rho f^{(0,0)}$	0

Table 11.1: BRST transformations of the fields involved in the path integral.

is an appropriate BRST-exact term:

$$S_{gauge-fixing} = \int d^n x \, \mathfrak{s}\left(X_{TD} + X_W\right) \tag{11.14}$$

X_{TD} and X_W are polynomials of the quantum fields with ghost number -1 and Grassmann number equal to 1 and such that they give rise to free-kinetic terms that are invertible. Since we are only interested in one-loop computations, we shall further assume that X_{TD} and X_W are quadratic in the quantum fields. In the next sections we will construct the terms X_{TD} and X_W and derive the differential operators involved in the path integral whose contribution to the quantum effective action needs to be computed.

11.2. The *TDiff* ghost sector

Let us start with the function X_{TD} implementing the gauge fixing of the *TDiff* symmetry. With the field content introduced above and with the BRST transformations as given in table 11.1, one has the following general quadratic polynomial in the quantum fields

$$X_{TD} = b^{(1,-1)}_\mu\left[F^\mu + \rho_1 f^{\mu(0,0)}\right] + \bar{c}^{(0,-2)}\left[F^\mu_2 c_\mu + \rho_2 \pi^{\prime\,(1,1)}\right] + \\ + c'^{\,(0,0)}\left[F^\mu_1 b^{(1,-1)}_\mu + \rho_3 \pi^{(1,-1)}\right] \tag{11.15}$$

where F_μ is a function containing the graviton field that can be identified with the usual gauge fixing condition in the Faddeev-Popov technique and

F_1^μ, F_2^μ and the three ρ_i can be freely chosen as long as they do not vanish. This is enough to fix the *TDiff* symmetry with the minimal possible content of fields.

After applying the s operator, this gives a term in the action

$$\begin{aligned}\int d^n x \int X_{TD} = \int d^n x \Big\{ & f_\mu^{(0,0)} \left(F^\mu + \rho_1 f^{\mu(0,0)}\right) - b_\mu^{(1,-1)} sF^\mu + \\ & + \pi^{(1,-1)} \left(F_2^\mu c_\mu^{(1,1)} + \rho_2 \pi'^{\,(1,1)}\right) + \bar{c}^{(0,-2)} F_2^\mu \nabla_\mu \phi^{(0,2)} + \\ & + \pi'^{\,(1,1)} \left(F_{1\mu} b^{\mu(1,-1)} + \rho_3 \pi^{(1,-1)}\right) + c'^{\,(0,0)} F_1^\mu f_\mu^{(0,0)} \Big\} \end{aligned} \tag{11.16}$$

where we have already taken into account that in the expansion (11.1) the metric is unimodular.

Now, there are some simplifications that can be done. First, let us take the terms containing $f_\mu^{(0,0)}$

$$f_\mu^{(0,0)} \left(F^\mu + \rho_1 f^{\mu(0,0)}\right) + f_\mu^{(0,0)} \bar{F}_1^\mu c'^{\,(0,0)} \tag{11.17}$$

where we have introduced $\bar{F}_1^\mu$ using integration by parts as

$$\int d^n x \; a F_1^\mu b = \int d^n x \; b \bar{F}_1^\mu a \tag{11.18}$$

These can be rewritten completing the square as

$$\rho_1 \left(f_\mu^{(0,0)} + \frac{1}{2\rho_1} (F_\mu + \bar{F}_{1\mu} c'^{\,(0,0)}) \right)^2 - \frac{1}{4\rho_1} (F_\mu + \bar{F}_{1\mu} c'^{\,(0,0)})^2 \tag{11.19}$$

and by shifting the variable $f_\mu^{(0,0)}$ the first term just gives an overall normalization. We are left with the gauge fixing action

$$S_{hc'} = -\frac{1}{4\rho_1} \int d^n x \; (F_\mu + \bar{F}_{1\mu} c'^{\,(0,0)})^2 \tag{11.20}$$

where ρ_1 has been chosen to be a constant. This would be the outcome of a standard Faddeev-Popov procedure if this were all the story.

Now let us focus into the terms containing the fermionic π fields. Those read

$$\begin{aligned} & \pi^{(1,-1)} \left(F_2^\mu c_\mu^{(1,1)} + \rho_2 \pi'^{\,(1,1)}\right) + \pi'^{\,(1,1)} \left(F_1^\mu b_\mu^{(1,-1)} + \rho_3 \pi^{(1,-1)}\right) = \\ & = \left(\pi^{(1,-1)} - F_1^\mu b_\mu^{(1,-1)} (\rho_2 - \rho_3)^{-1}\right) (\rho_2 - \rho_3) \left(\pi'^{\,(1,1)} + (\rho_2 - \rho_3)^{-1} F_2^\mu c_\mu^{(1,1)}\right) + \\ & + F_1^\mu b_\mu^{(1,-1)} (\rho_2 - \rho_3)^{-1} F_2^\mu c_\mu^{(1,1)} \end{aligned} \tag{11.21}$$

and, again, by shifting the π fields we are left with a gauge fixing term plus an extra path integral depending on how we choose the operators ρ_2 and ρ_3

$$S_\pi + S_{gf}^{bc} = \int d^n x \, \left(\pi^{(1,-1)}(\rho_2 - \rho_3)\pi'^{\,(1,1)} + F_1^\mu b_\mu^{(1,-1)}(\rho_2 - \rho_3)^{-1} F_2^\mu c_\mu^{(1,1)}\right) \tag{11.22}$$

So that the BRST action for the *TDiff* sector is further simplified to

$$\int d^n x \; sX_{TD} = \int d^n x \left\{ - b_\mu^{(1,-1)} sF^\mu + \bar{c}^{(0,-2)} F_2^\mu \nabla_\mu \phi^{(0,2)} + \right.$$
$$\left. + \pi^{(1,-1)}(\rho_2 - \rho_3)\pi'^{\,(1,1)} + F_1^\mu b_\mu^{(1,-1)}(\rho_2 - \rho_3)^{-1} F_2^\mu c_\mu^{(1,1)} - \frac{1}{4\rho_1}(F_\mu + \bar{F}_{1\mu} c'^{\,(0,0)})^2 \right\} \tag{11.23}$$

As a next step, the function F_μ is chosen with two requirements in mind. First, that the term $F_\mu F^\mu$ is able to cancel the non-diagonal pieces of the operator for the graviton fluctuations in the original lagrangian and also that it is Weyl invariant so both gauge fixing sectors decouple and their ghost fields do not interact. With these two requirements, the choice is almost unique

$$F_\mu = \nabla^\nu h_{\mu\nu} - \frac{1}{n}\nabla_\mu h \tag{11.24}$$

and its variation under a transverse diffeomorphism is the equivalent to the application of the s operator

$$\mathbf{s}_D F_\mu = \Box c_\mu^T + \nabla^\nu \nabla_\mu c_\nu^T = \Box c_\mu^T + R_\mu^\nu c_\nu^T \tag{11.25}$$

where in the second step we have used Ricci identity $[\nabla_\nu, \nabla_\mu] c^\nu = R_{\mu\nu} c^\nu$ and the fact that, since we are performing a transverse diffeomorphism, c_μ^T satisfies $\nabla^\mu c_\mu^T = 0$.

Now, we have to rewrite c_μ^T in terms of an unconstrained field as explained before. We do this by introducing the operator $\Theta_{\mu\nu}$.

$$\begin{aligned}\mathbf{s}_D F_\mu = & \left(g_\mu^\alpha \Box + R_\mu^\alpha\right)\left(g_{\alpha\nu}\Box - \nabla_\alpha \nabla_\nu - R_{\alpha\nu}\right) c^{\nu(1,1)} = \\ & = \Box^2 c_\mu^{(1,1)} - \nabla_\mu \Box \nabla_\nu c^{\nu(1,1)} - 2R_{\mu\rho}\nabla^\rho \nabla_\nu c^{\nu(1,1)} - \Box R_{\mu\rho} c^{\rho(1,1)} - \\ & - 2\nabla_\sigma R_{\mu\rho}\nabla^\sigma c^{\rho(1,1)} - R_{\mu\rho}R^{\rho\nu} c_\nu^{(1,1)}\end{aligned} \tag{11.26}$$

The action for $b_\mu^{(1,-1)}$ and $c_\mu^{(1,1)}$ is then

$$\begin{aligned}S_{bc} = -\int d^n x \, b^{\mu\,(1,-1)} & \left(\Box^2 c_\mu^{(1,1)} - \nabla_\mu \Box \nabla_\nu c^{\nu(1,1)} - 2R_{\mu\rho}\nabla^\rho \nabla_\nu c^{\nu(1,1)} - \right. \\ & \left. - \Box R_{\mu\rho} c^{\rho(1,1)} - 2\nabla_\sigma R_{\mu\rho}\nabla^\sigma c^{\rho(1,1)} - R_{\mu\rho}R^{\rho\nu} c_\nu^{(1,1)}\right)\end{aligned} \tag{11.27}$$

The non-diagonal term with four derivatives can be canceled by an appropriate choice of the functions F_1^μ, F_2^μ, ρ_2 and ρ_3. We choose them to be

$$
\begin{aligned}
&F_1^\mu b_\mu^{(1,-1)} = -\nabla^\alpha b_\alpha^{(1,-1)} \\
&F_2^\mu c_\mu^{(1,1)} = \nabla^\mu c_\mu^{(1,1)} \\
&(\rho_2 - \rho_3)^{-1} = -\Box
\end{aligned}
\tag{11.28}
$$

Then

$$
F_1^\mu b_\mu^{(1,-1)}(\rho_2 - \rho_3)^{-1} F_2^\mu c_\mu^{(1,1)} = \left(\nabla^\nu b_\nu^{(1,-1)}\right)\Box\nabla^\mu c_\mu^{(1,1)} = -b_\nu^{(1,-1)}\nabla^\nu\Box\nabla^\mu c_\mu^{(1,1)}
\tag{11.29}
$$

where in the second step we have performed an integration by parts keeping in mind that we are always under an integral sign. The final action term for $b_\mu^{(1,-1)}$ and $c_\mu^{(1,1)}$ is then

$$
\begin{aligned}
S_{bc} + S_{gf}^{bc} = \int d^n x\, b^{\mu\,(1,-1)}\Big(&\Box^2 c_\mu^{(1,1)} - 2R_{\mu\rho}\nabla^\rho\nabla_\nu c^{\nu(1,1)} - \Box R_{\mu\rho} c^{\rho(1,1)} - \\
&-2\nabla_\sigma R_{\mu\rho}\nabla^\sigma c^{\rho(1,1)} - R_{\mu\rho}R^{\rho\nu} c_\nu^{(1,1)}\Big)
\end{aligned}
\tag{11.30}
$$

And with this choice of $(\rho_2 - \rho_3)$, the integration over the π fields is given by

$$
S_\pi = \int d^n x\, \pi^{(1,-1)}\Box^{-1}\pi'^{\,(1,1)}
\tag{11.31}
$$

The operator involving $c'^{\,(0,0)}$, and induced by this choice of fixing functions is

$$
\begin{aligned}
S_{hc'} &= -\int d^n x\, \frac{1}{4\rho_1}\left[\bar{F}_1^\mu c'^{(0,0)}\bar{F}_{1\mu} c'^{(0,0)} + 2F_\mu \bar{F}_1^\mu c'^{\,(0,0)} + F_\mu F^\mu\right] = \\
&= -\int d^n x\, \frac{1}{4\rho_1}\left[\nabla_\mu c'^{\,(0,0)}\nabla^\mu c'^{\,(0,0)} + 2F_\mu\nabla^\mu c'^{\,(0,0)} + F_\mu F^\mu\right]
\end{aligned}
\tag{11.32}
$$

which mixes with the operator of the graviton fluctuation due to the term containing F_μ and $c'^{\,(0,0)}$.

Finally, the operator for $\bar{c}^{(0,-2)}$ and $\phi^{(0,2)}$ is

$$
S_{\bar{c}\phi} = \int d^n x\, \bar{c}^{(0,-2)}\Box\phi^{(0,2)}
\tag{11.33}
$$

Summarizing, the BRST exact action for the TDIff symmetry is reduced to

$$\begin{aligned} S_{TDiff} = \int d^n x\, b^\mu & \Big(\Box^2 c_\mu^{(1,1)} - 2R_{\mu\rho}\nabla^\rho\nabla_\nu c^{\nu(1,1)} - \Box R_{\mu\rho} c^{\rho(1,1)} + \bar{c}^{(0,-2)}\Box\phi^{(0,2)} - \\ & -2\nabla_\sigma R_{\mu\rho}\nabla^\sigma c^{\rho(1,1)} - R_{\mu\rho}R^{\rho\nu}c_\nu^{(1,1)}\Big) + \pi^{(1,-1)}\Box^{-1}\pi'^{\,(1,1)} - \\ & - \frac{1}{4\rho_1}\Big(F_\mu F^\mu + \nabla_\mu c'^{\,(0,0)}\nabla^\mu c'^{\,(0,0)} + 2F_\mu\nabla^\mu c'^{\,(0,0)}\Big) = \\ & = S_{bc} + S_{gf}^{bc} + S_{\bar{c}\phi} + S_\pi + S_{hc'} \end{aligned} \tag{11.34}$$

The contribution of all these pieces to the quantum effective action will be computed in section 11.4.

11.3. The Weyl ghost sector

Now we turn our attention to the second part of the gauge fixing sector, corresponding to the Weyl invariance of the theory. We choose the function X_W to be

$$X_W = \nabla_\mu b^{(1,-1)}\nabla^\mu\Big(f^{(0,0)} - \alpha\, g(h)\Big) \tag{11.35}$$

with $g(h)$ being some function of the trace of the graviton fluctuation only, to ensure that it is invariant under a *TDiff* transformation and thus the ghosts do not interact with the *TDiff* sector. The parameter α we mean to keep arbitrary all along the computation. Aditionally, we know by Kallosh-DeWitt theorem that the on shell effective action should be independent of α (because it appears in a BRST exact piece), and this will be used as a nice partial check of our results.

After the application of $\mathbf{s}_W$, the BRST exact action is

$$S_{BRST}^{Weyl} = \int d^n x\, \Big[\nabla_\mu f^{(0,0)}\nabla^\mu\Big(f^{(0,0)} - \alpha\, g(h)\Big) - \alpha\nabla_\mu b^{(1,-1)}\nabla^\mu\left(sg(h)\right)\Big] \tag{11.36}$$

And we choose $g(h)$ to be the simplest choice

$$g(h) = h \tag{11.37}$$

The BRST term piece is then

$$\begin{aligned} S_{Weyl} = S_W + S_{hf} = & \int d^n x\, \nabla_\mu f^{(0,0)}\nabla^\mu\Big(f^{(0,0)} - \alpha\, h\Big) - 2n\alpha\nabla_\mu b^{(1,-1)}\nabla^\mu c^{(1,1)} = \\ = & \int d^n x\, \Big(-f^{(0,0)}\Box f^{(0,0)} + \frac{\alpha}{2}f^{(0,0)}\Box h + \frac{\alpha}{2}h\Box f^{(0,0)}\Big) + 2n\alpha\, b^{(1,-1)}\Box c^{(1,1)} \end{aligned} \tag{11.38}$$

This gives two contributions to the one-loop effective action. The first part needs to be added to the original action of Unimodular Gravity and mixes the gravitation perturbation with the auxiliar f field. The second piece is the corresponding ghost action.

11.4. The one-loop effective action of Unimodular Gravity

Once the gauge freedom is fixed completely, the computation of the one-loop counterterm of Unimodular Gravity is reduced to a computation of a set of determinants. By collecting all the terms defined in the previous sections, the pole part of the one-loop effective action will be given, as explained in chapter 3, by

$$\Gamma[g_{\mu\nu}] = \Gamma^{UG}[g_{\mu\nu}] + \Gamma^{bc}[g_{\mu\nu}] + \Gamma^{\pi}[g_{\mu\nu}] + \Gamma^{\bar{c}\phi}[g_{\mu\nu}] + \Gamma^{W}[g_{\mu\nu}] \qquad (11.39)$$

where each $\Gamma^i[g_{\mu\nu}]$ refers to the contribution to the pole given by the action labelled as S_i in the previous sections, with the only exception of $\Gamma^{UG}[g_{\mu\nu}]$ which is given by

$$S_{UG} = S_2 + S_{hc'} + S_{hf} \qquad (11.40)$$

All but one of the operators involved in our computation are *minimal* operators and thus their contribution to the quantum effective action can be computed by the techniques in chapter 3. The only exception is given by S_{bc} which contains four derivatives. It can be however treated by te techniques of chapter 4 and its contribution can be read from [53]. The operator contained in S_{UG} is however non-minimal and thus it requires to use the Generalized Schwinger-DeWitt technique of chapter 4. It reads

$$S^{(1)}_{UG} = S_2 + S_{hc'} + S_{hf} = \int d^n x\, \mathcal{L} \qquad (11.41)$$

with

$$\begin{aligned}\mathcal{L} = &\frac{1}{4}h^{\mu\nu}\Box h_{\mu\nu} - \frac{1}{4n}h\Box h + \frac{1}{2}h^{\alpha\beta}h^{\mu}_{\beta}R_{\mu\alpha} + \frac{1}{2}h^{\mu\nu}h^{\alpha\beta}R_{\mu\alpha\nu\beta} - \frac{1}{n}hh^{\mu\nu}R_{\mu\nu} - \\ &- \frac{1}{2n}h^{\mu\nu}h_{\mu\nu}R + \left(-f\Box f + \frac{\alpha}{2}f\Box h + \frac{\alpha}{2}h\Box f\right) + \frac{1}{2n^2}h^2R - \\ &- \frac{1}{2}\left(\nabla_\mu c'^{\,(0,0)}\nabla^\mu c'^{\,(0,0)} + 2\left(\nabla_\nu h^\nu_\mu - \frac{1}{n}\nabla_\mu h\right)\nabla^\mu c'^{\,(0,0)}\right)\end{aligned} \qquad (11.42)$$

where we have set $\rho_1 = \frac{1}{2}$ in order to cancel the non-diagonal parts in the kinetic term for $h_{\mu\nu}$.

To write it in the standard form, we identify

$$\Psi^A = \begin{pmatrix} h^{\mu\nu} \\ f \\ c' \end{pmatrix} \tag{11.43}$$

and the differential operator takes the form

$$F_{AB} = \gamma_{AB}\Box + J^{\mu\nu}_{AB}\nabla_\mu\nabla_\nu + M_{AB} \tag{11.44}$$

where the different matrices involved read

$$\gamma_{AB} = \begin{pmatrix} -\frac{1}{4}\left(\frac{1}{4}\mathcal{K}^{\alpha\beta}_{\mu\nu\rho\sigma} - \mathcal{P}^{\alpha\beta}_{\mu\nu\rho\sigma}\right) g_{\alpha\beta} & \frac{\alpha}{2} g_{\mu\nu} & -\frac{1}{8} g_{\mu\nu} \\ \frac{\alpha}{2} g_{\rho\sigma} & -1 & 0 \\ -\frac{1}{8} g_{\rho\sigma} & 0 & \frac{1}{2} \end{pmatrix} \tag{11.45}$$

$$J^{\alpha\beta}_{AB} = \begin{pmatrix} 0 & 0 & \frac{1}{4}\left(g^\alpha_\mu g^\beta_\nu + g^\alpha_\nu g^\beta_\mu\right) \\ 0 & 0 & 0 \\ \frac{1}{4}\left(g^\alpha_\rho g^\beta_\sigma + g^\alpha_\sigma g^\beta_\rho\right) & 0 & 0 \end{pmatrix} \tag{11.46}$$

$$M_{AB} = \begin{pmatrix} M_{hh} & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{pmatrix} \tag{11.47}$$

with

$$M_{hh} = \left(\frac{1}{2}\mathcal{P}^{\alpha\beta}_{\mu\nu\rho\sigma} - \frac{1}{4}\mathcal{K}^{\alpha\beta}_{\mu\nu\rho\sigma}\right) R_{\alpha\beta} - \frac{1}{8}\left(\mathcal{P}^{\alpha\beta}_{\mu\nu\rho\sigma} - \frac{1}{4}\mathcal{K}^{\alpha\beta}_{\mu\nu\rho\sigma}\right)\gamma_{\alpha\beta} R + \frac{1}{2} R_{(\mu\rho\nu\sigma)} \tag{11.48}$$

The round parenthesis for us mean complete symmetrization in all the enclosed indices unless otherwise stated. The tensors $\mathcal{P}^{\alpha\beta}_{\mu\nu\rho\sigma}$ and $\mathcal{K}^{\alpha\beta}_{\mu\nu\rho\sigma}$ are the same used in chapter 8

$$\mathcal{P}^{\alpha\beta}_{\mu\nu\rho\sigma} = \frac{1}{4}\left(g_{\mu\rho}\delta^{(\alpha}_\nu\delta^{\beta)}_\sigma + g_{\mu\sigma}\delta^{(\alpha}_\nu\delta^{\beta)}_\rho + g_{\nu\rho}\delta^{(\alpha}_\mu\delta^{\beta)}_\sigma + g_{\nu\sigma}\delta^{(\alpha}_\mu\delta^{\beta)}_\rho\right) \tag{11.49}$$

$$\mathcal{K}^{\alpha\beta}_{\mu\nu\rho\sigma} = \frac{1}{2}\left(g_{\mu\nu}\delta^{(\alpha}_\rho\delta^{\beta)}_\sigma + g_{\rho\sigma}\delta^{(\alpha}_\mu\delta^{\beta)}_\nu\right) \tag{11.50}$$

The contribution of this operator to the effective action can be computed by using the Generalized Schwinger-DeWitt technique of chapter 4 as explained there. All but one of the functional traces required for our result were contained in [53] but they are reproduced in chapter 4 for completeness. The computation has been performed with the help of the Mathematica software *xAct* [70]. A fair amount of computing time has been necessary in order to simplify the resulting expressions.

Finally, putting all together we find that the contribution to the pole part of the effective action of S_{UG} is

$$\Gamma[g_{\mu\nu}]_{UG} = \frac{1}{16\pi^2}\frac{1}{n-4}\int d^n x\; a_4$$

with

$$a_4 = \frac{16}{15}R_{\mu\nu\alpha\beta}R^{\mu\nu\alpha\beta} + \left(\frac{1}{6\alpha^2} - \frac{46}{15}\right)R_{\mu\nu}R^{\mu\nu} + \left(\frac{1}{3} - \frac{1}{24\alpha^2}\right)R^2 \qquad (11.51)$$

where we have neglected total derivatives in the integrand[2].

As has been already advertised, all the dependence on the gauge fixing, represented by the presence of the parameter α in the final result, disappears when we use the background equations of motion $R_{\mu\nu} = \frac{1}{4}Rg_{\mu\nu}$. This is as it should be because our gauge fixing is BRST exact.

11.5. The one-loop counterterm

After computing the contribution of the non-minimal operator, we are finally ready to write the pole part of the effective action of Unimodular Gravity, which reads

$$\Gamma[g_{\mu\nu}] = \Gamma[g_{\mu\nu}]_{UG} + \Gamma[g_{\mu\nu}]_{bc} + \Gamma[g_{\mu\nu}]_{\pi} + \Gamma[g_{\mu\nu}]_{\bar{c}\phi} + \Gamma[g_{\mu\nu}]_{W} \qquad (11.52)$$

Here $\Gamma[g_{\mu\nu}]_{UG}$ is the contribution we have computed in the last section while the rest of the contributions are given in appendix **??**. Adding everything, we find that the final result is

$$\Gamma[g_{\mu\nu}] = \frac{1}{16\pi^2}\frac{1}{n-4}\int d^n x\; A_4$$

where

$$A_4 = \frac{119}{90}R_{\mu\nu\alpha\beta}R^{\mu\nu\alpha\beta} + \left(\frac{1}{6\alpha^2} - \frac{359}{90}\right)R_{\mu\nu}R^{\mu\nu} + \frac{1}{72}\left(22 - \frac{3}{\alpha^2}\right)R^2 \qquad (11.53)$$

[2]This prevents us to compute the Weyl anomaly from our result, because total derivatives can give non-vanishing contributions to the anomalous Ward identity. However, this is irrelevant since, as it will be explained later, a regularization that preserves Weyl invariant can be implemented at all orders in the loop expansion.

Now we would like to focus on the issue of on-shell renormalizability. It is known that although General Relativity is one-loop finite in the absence of a cosmological constant, this property is lost in its presence. The on-shell counterterm in this case was obtained in [10]. It amounts to a renormalization of the cosmological constant and is proportional to

$$\Gamma_\infty^{GR} \equiv \frac{1}{16\pi^2(n-4)}\int \sqrt{|g|}d^4x\ \left(\frac{53}{45}\ C^2 - \frac{1142}{135}\Lambda^2\right) \tag{11.54}$$

Since the main attractive feature of Unimodular Gravity is precisely the different role that the cosmological constant plays with respect to GR, we would like to see what happens here with the renormalization group flow when we take the counterterm to be on-shell so that all external legs correspond to S-matrix states. In that case, the equations of motion for the $|g| = 1$ fixed background are the traceless Einstein equations

$$R_{\mu\nu} - \frac{1}{4}Rg_{\mu\nu} = 0 \tag{11.55}$$

which, altogether with Bianchi identities, imply the following for the operators appearing in the effective action

$$R_{\mu\nu\alpha\beta}R^{\mu\nu\alpha\beta} = E_4 \tag{11.56}$$

$$R_{\mu\nu}R^{\mu\nu} = \frac{1}{4}R^2 \tag{11.57}$$

$$R = \text{constant} \equiv \Lambda \tag{11.58}$$

The first line is nothing more than the statement of the Gauss-Bonet theorem when we take into account the equations of motion. E_4 is thus the Euler density, whose integral gives the Euler characteristic of the manifold.

By using these, we find that the on-shell effective action takes then the form

$$\Gamma[g_{\mu\nu}]^{\text{on-shell}} = \frac{1}{16\pi^2}\frac{1}{n-4}\int d^nx\left(\frac{119}{90}E_4 - \frac{83}{120}\Lambda^2\right) \tag{11.59}$$

Now we come to the issue of the renormalization of the cosmological constant. In principle here we have a divergence associated with vacuum energy that would require to add a counterterm of the form

$$\int d^nx\ \delta_\lambda\Lambda^2 \tag{11.60}$$

that renormalizes it. However, the point is that here this counterterm *does not couple* to gravity at all. The effective equations of motion that come from the variation of the effective action

$$\frac{\delta \Gamma}{\delta g_{\mu\nu}} = 0 \tag{11.61}$$

are blind to this parameter. Therefore, this renormalization is *non-dynamical* and completely irrelevant. What we call the cosmological constant in this setting is still given as an integration constant and its value is chosen classically by using Bianchi identities. This reduces the cosmological constant to the role of a standard coupling. Once its physical value is set, it stays as it is.

Actually, this effect is not specific to one-loop computations. We then conclude that the bare value of the cosmological constant is protected and quantum corrections do not modify it.

It could be thought that this effect is just a gauge artifact of our background choice $|g| = 1$. However, it can be easily argued that this is not the case. As we have commented before in this work, if we now want to obtain the effective action for an arbitrary background from the one with unimodular background metric, it is enough to make a change of variables so that

$$\tilde{g}_{\mu\nu} = g^{-\frac{1}{n}} g_{\mu\nu} \tag{11.62}$$

When doing this, we can see that the real reason of the cosmological constant not being renormalized is indeed the presence of Weyl invariance in our formalism, which protects the appearance of any mass scale in the effective action and, as a consequence, in the expectation value of the equations of motion. Therefore, our argument holds and the cosmological constant is protected and fixed to its bare value all along the renormalization group flow and at any loop order.

12

Protecting Λ

Summarizing our results, we have seen that the one-loop counterterm of Unimodular Gravity takes the form

$$\Gamma[g_{\mu\nu}]^{\text{on-shell}} = \frac{1}{16\pi^2}\frac{1}{n-4}\int d^n x \left(\frac{119}{90}E_4 - \frac{83}{120}\Lambda^2\right) \tag{12.1}$$

The first term is a topological invariant and it will give no contribution to the effective equations of motion. The same happens for the second term but for completely different reasons. It is a non-dynamical operator, which does not couple to any field and therefore its variation vanishes. We then conclude that Unimodular Gravity *in the presence of a cosmological constant* inherits one of the nicest properties of plain General Relativity: it is one-loop finite. The coupling constants of the theory are not renormalized and the effective equations of motion read

$$\left.\frac{\delta\Gamma}{\delta g_{\mu\nu}}\right|_{\text{one-loop}} = \frac{\delta S}{\delta g_{\mu\nu}} \tag{12.2}$$

This has deep implications from the point of view of the cosmological constant problem. First, it solves the problem of the vacuum energy. In this theory, vacuum energy *does not weight* neither clasically or through quantum corrections and the cosmological constant is stable under radiative corrections. It is demoted to the role of a simple coupling, whose value has to be set by an experiment, as with any other physical quantity like the electron mass in QED.

This also solves the tension with wilsonian arguments. Now all the higher order corrections of Unimodular Gravity will be weighted solely by the Planck mass, allowing us to construct a consistent EFT even when the cosmological constant does not vanish.

Of course, one could think that our result is a consequence of the particular background splitting we have chosen, where the background metric is unimodular by hand

$$|g| = 1 \tag{12.3}$$

However, as explained before, one can recover the result for arbitrary metrics by performing a Weyl transformations on this result

$$\tilde{g}_{\mu\nu} = |g|^{-\frac{1}{n}} g_{\mu\nu} \tag{12.4}$$

This is safe as long as there is not Weyl anomaly in the theory and this happens to be precisely the case. Let use remind that, as explained in chapter 2 and in the conclusions of the last section, one can obtain a regularization scheme which is safe and where Weyl anomalies do not appear by using a scalar dilaton field as a compensator. It amounts to do the following. If we were using plain dimensional regularization ,we would have a counterterm of the form

$$\Gamma_{CT} = \frac{1}{n-4} \int d^n x \; \mathcal{O} \tag{12.5}$$

so that, recalling that the anomaly appears because $\mathcal{O}$ is invariant only when $n = 4$, the new scheme consists on adding extra powers of the scalar field (let us call it ϕ)

$$\Gamma_{CT} = \frac{1}{n-4} \int d^n x \; \phi^{\frac{n-4}{\lambda_\phi}} \mathcal{O} \tag{12.6}$$

so that the integrand is invariant in any dimension. Here λ_ϕ is the scaling dimension of ϕ.

In UG we do not have any extra scalar field to use but, since the symmetry group of the theory is just *TDiff*, the determinant of the metric $|g|$ is a scalar quantity and can take the role of the compensator. Therefore, the anomalous safe one-loop counterterm for arbitrary metrics will be

$$\Gamma[g_{\mu\nu}]^{\text{on-shell}} = \frac{1}{16\pi^2} \frac{1}{n-4} \int d^n x \left(|g|^{\frac{2}{n}} \frac{119}{90} E_4 - \frac{83}{120} \Lambda^2 \right) \tag{12.7}$$

which is not only Weyl invariant in arbitrary dimension but also the cosmological constant remains decoupled at all orders in the perturbative expansion. This regularization scheme is not present in General Relativity or any other *Diff* invariant theory because then this would break precisely that symmetry, exchanging the Weyl anomaly by a *Diff* one. It is precisely the fact that the symmetry group is reduced to *TDiff* what makes the trick to work.

To conclude, although Unimodular Gravity it is non-renormalizable and must be still regarded as an effective field theory of the gravitational interactions, it solves the issue of the ambiguous description that appears in GR, where two scales, an UV cutoff M_p and an IR one Λ are present. Unimodular Gravity is then a *trustable* EFT.

Conclusions

The main conclusion of this work could be phrased simply by saying that *scale (Weyl) invariance* is still one of the most complex and poor understood symmetries in physics. The prospectives about the finiteness of quantum actions that it suggest are equally tarnished by the huge conceptual problems that it poses, particularly in the presence of dynamical gravitation.

While trying to understand some of the properties of this symmetry in the framework of Quantum Field Theory, we have studied two actions, corresponding to what we dub *Conformal Dilaton Gravity* in part II and to *Unimodular Gravity* in part III. By means of well-known techniques in modern theoretical physics, the Background Field method and the Schwinger-DeWitt technique (plus their generalizations) introduced in part I we have been able to shed light to two main questions where Weyl invariance might be relevant.

By using *Conformal Dilaton Gravity* we faced the problem of the equivalence between actions that are classically related by non-linear transformations. After computing quantum corrections for CDG both in and out the Weyl invariant point, we concluded that the divergences of the theory were related to the ones in the Einstein frame by the same transformations than the classical action. However, this is not true any more when we have an anomaly in one of the frames. Weyl anomalies in CDG imply the presence of a new operator whose vacuum expectation value is driven by the anomaly. In the Einstein frame, however, there is no anomaly, because there is no Weyl invariance or any equivalent non-exact symmetry, and the corresponding operator does have a vanishing vacuum expectation value. Even the naive expectation that S-matrix physics should be equivalent for both theories is broken in the presence of the anomalous term, that can mediate a scattering amplitude.

We related this situation to the fact that dimensional regularization, while it can be applied in both frames, is not as good in the CDG frame as in the Einstein frame. One can turn instead to an improved scheme, where a compensator field is introduced and anomalies disappear, but this comes at the cost of having to introduce non-meromorphic counterterms which apparently could also lead to an inequivalence of the S-matrix any way. These conclusions are however premature and more deep research is needed to assert to what extent one can trust in frame equivalence in order to extract physical universal results from the theory.

As a corollary of our work with CDG, we also see that since in a pure Weyl invariant theory there is no scale in the lagrangian, probably we never can decouple gravitational physics from the matter dynamics. Even when choosing a vacuum expectation value for the scalar field, thus working on the Higgs phase of the theory, Ward-Takahashi identities must still be non-linearly realized and quantization must be done along the same lines as with the non-broken theory. It is reasonable to think that the equivalence problems then could be originated by our poor understanding of some aspects of quantum gravitational fields mixed with the non-renormalizability of theory.

Later, we used Weyl invariance as a tool to address one of the most relevant problems in modern physics, *the active cosmological constant problem.* While Einstein gravity is a good behaved effective field theory in the absence of a cosmological constant, there is an infra-red problem when $\Lambda \neq 0$ that we addressed here in terms of *Unimodular Gravity.* By writing the action of such theory in a particular frame where the metric variable is Weyl invariant we were able to quantize the graviton fluctuations (although we had to use a very complicated ghost sector) and prove that regardless there is matter or not coupled to gravity, the unimodular theory solves the cosmological constant in a very elegant way. At every order in the perturbative expansion, the cosmological constant is banished to be a dynamical coupling of the theory and no operator in the renormalization process is able to produce it. It is reintroduced only when considering the effective equations of motion for the mean fields, where Bianchi identities allow for an extra integration constant to take the role of Λ at the time of solving the equations.

By using Unimodular Gravity, one can now construct a well-defined effective field theory for gravitational interactions, whose cut-off is given in terms of the Planck mass and no infra-red problems arise. Moreover, the cosmological constant is given, as if it were a marginal coupling, simply in terms of a physical measure. We believe that this is a huge improvement over the standard general relativistic lore.

Finally, to summarize all the content of this work in a single paragraph, we were able to prove the usefulness of Weyl invariance in allowing to relax fine-tuning problems related to gravitational operators. However, the quest for a finite theory of gravity still continues and although Weyl invariance is a promising idea, there are many questions that must be still answered, mostly related to the not so well understood Weyl anomalies. There is still many work to be done and while the XX century was the century of particle physics, it is reasonable to hope that the XXI century will be the century of quantum gravity.

Home is behind, the world ahead.

Conclusiones

La conclusión principal de este trabajo puede ser resumida diciendo simplemente que *la invariancia de Weyl* sigue siendo una de las simetrías más complejas y menos entendidas en física, estando las esperanzas sobre la finitud de acciones cuánticas que sugiere fuertemente frustradas por los grandes problemas conceptuales que induce, particularmente en la presencia de gravedad dinámica.

Mientras intentábamos comprender algunas de las propiedades de esta simetría en el marco de la Teoría Cuántica de Campos, hemos estudiado dos acciones, correspondientes a lo que hemos denominado Gravedad Dilatónica Conforme en la parte II y a Gravedad Unimodular en la parte III. Por medio de técnicas bien conocidas en la física teórica moderna, el método de campo de fondo y la técnica de Schwinger y DeWitt (y sus generalizaciones), introducidas en la parte I, hemos sido capaces de arrojar luz sobre dos cuestiones importantes en las que la invariancia de Weyl es relevante.

Utilizando la Gravedad Dilatónica Conforme, nos enfrentamos al problema de la equivalencia entre acciones que están relacionadas por transformaciones no lineales a nivel clásico. Tras calcular las correcciones cuánticas a GDC tanto en el punto invariante Weyl como en el no invariante, concluimos que las divergencias de la teorías están relacionadas con las calculadas en el referencial de Einstein bajo las mismas transformaciones que la acción clásica. Sin embargo, esto deja de ser cierto si tenemos una anomalía en uno de los referenciales. Las anomalías Weyl en GDC inducen la existencia de una nuevo operador cuyo valor esperado en el vacío viene dado precisamente en términos de la anomalía. En el referencial de Einstein, sin embargo, no hay anomalía, porque no hay ni invariancia Weyl ni ninguna otra simetría no exacta equivalente, y el valor esperado del operador correspondiente se anula. Incluso la esperanza ingenua de que la matriz S sea equivalente para ambos referenciales deja de ser cierta en presencia de la anomalía, que puede mediar en una amplitud de dispersión.

Hemos relacionado este efecto con el hecho de que la regularización dimensional, pese a que puede ser utilizada en ambos referenciales, no es tan óptima en el referencial de GDC como en el de Einstein. Uno siempre puede pasar a utilizar un esquema diferente, donde introducimos un campo compensador y la anomalía desaparece, pero a cambio debemos introducir contratérminos no meromorfos que aparentemente pueden llevarnos a la no equivalencia de la matriz S de todas formas. Pese a que estás conclusiones son prematuras, plantean la cuestión de hasta qué punto uno puede confiar en la equivalencia de referenciales para extraer resultados físicos universales a partir de la teoría.

Como corolario a nuestro trabajo en GDC también observamos que puesto que en una teoría puramente invariante Weyl no existe ninguna escala, probablemente no es posible desacoplar la dinámica gravitatoria de la de la materia nunca. Incluso si escogemos un valor esperado en el vacío no nulo para el campo escalar, pasando a trabajar por tanto en la fase de Higgs de la teoría, las identidades de Ward y Takahashi tienen que seguir cumpliéndose a nivel no lineal y la cuantización debe realizarse de forma similar a como se realiza con la teoría en la fase no rota. Es razonable, pues, pensar que los problemas de equivalencia podrían estar originados por nuestro pobre entendimiento de algunos aspectos de la física gravitatoria, mezclados con el hecho de que la teoría no sea renormalizable.

Más tarde, utilizamos la invariancia de Weyl como una herramienta para atacar uno de los problemas más relevantes de la física moderna, *el problema activo de la constante cosmológica.* Pese a que la gravedad de Einstein es una buena teoría efectiva en ausencia de esta constante, existe un problema infrarrojo cuando $\Lambda \neq 0$ que hemos afrontado en este trabajo a través de la Gravedad Unimodular. Gracias a que escribimos la acción en un referencial particular donde la variable de integración es estrictamente invariante Weyl, fuimos capaces de cuantizar las fluctuaciones de gravitón (pese a que tuvimos que usar un sector de fantasmas muy complicado) y demostrar que independientemente de que haya materia o no acoplada a la gravedad, la teoría unimodular resuelve el problema de la constante cosmológica de una forma muy elegante. A todo orden en la expansión en perturbaciones, la constante cosmológica no puede aparecer como un acoplo de la teoría y no existe ningún operador generado en el proceso de renormalización que pueda producirla. Reaparece sólo cuando consideramos las ecuaciones del movimiento efectivas para los campos medios, donde las identidades de Bianchi permiten que aparezca una constante de integración que toma el papel de Λ.

Utilizando Gravedad Unimodular, uno puede construir, por tanto, una teoría efectiva para las interacciones gravitatorias que está bien definida y cuyo corte viene dado exclusivamente en términos de la masa de Planck, sin que aparezca ningún problema infrarrojo. Creemos que este resultado representa una mejora sustancial sobre los procedimientos habituales en el caso relativista general.

Finalmente, podríamos resumir todo el contenido de este trabajo en un sencillo párrafo: hemos sido capaces de probar lo util de la invariancia de Weyl a la hora de relajar problemas de ajuste fino relacionados con operadores de origen gravitatorio. Sin embargo, la busqueda de una teoría finita de la gravedad sigue en pie y pese a que la invariancia de Weyl es una idea prometedora, aún quedan muchas preguntas abiertas que necesitan respuesta, la mayoría relacionadas con las anomalías Weyl, las cuales no comprendemos del todo. Todavia queda mucho trabajo por hacer y mientras que el siglo XX fue el siglo de la física de partículas, es razonable esperar que el siglo XXI sea el siglo de la gravedad cuántica.

Detrás queda el hogar, enfrente el mundo.

Appendices

A

Contributions to Γ by unimodular ghosts

Here we compute the different heat kernel coefficients corresponding to each of the minimal differential operator appearing in the path integral formulation of Unimodular Gravity.

A.1. The contribution of S_{bc}

The action term for the fields $b^{\mu(1,-1)}$ and $c^{\mu(1,1)}$ was defined in equation (11.30) and reads

$$\int d^n x\; b^\mu \Big\{ \Box^2 c_\mu^{(1,1)} - 2R_{\mu\rho}\nabla^\rho\nabla_\nu c^{\nu(1,1)} - \Box R_{\mu\rho} c^{\rho(1,1)} - 2\nabla_\sigma R_{\mu\rho}\nabla^\sigma c^{\rho(1,1)} - \\ - R_{\mu\rho}R^{\rho\nu} c_\nu^{(1,1)} \Big\} \tag{A.1}$$

This is a quartic operator of the standard form that we can find in [53] if we identify

$$J^{\mu\nu}_{\alpha\beta} = -2R^\mu_\alpha \delta^\nu_\beta \tag{A.2}$$

$$H^\mu_{\alpha\beta} = -2\nabla^\mu R_{\alpha\beta} \tag{A.3}$$

$$P_{\alpha\beta} = -\Box R_{\alpha\beta} - R_{\alpha\rho}R^\rho_\beta \tag{A.4}$$

here the bundle indices are just spacetime greek indices that we indicate with α and β.

And the field strength

$$[\nabla_\mu, \nabla_\nu] c^\alpha = \hat{F}_{\mu\nu}{}^{\alpha\beta} c_\beta = R_{\mu\nu}{}^{\alpha\beta} c_\beta \tag{A.5}$$

Its contribution to the effective action can be read from the results of [53] and reads

$$W^{bc}_\infty = \frac{1}{16\pi^2}\frac{1}{n-4}\int d^n x \left(\frac{11}{45} R_{\mu\nu\alpha\beta}R^{\mu\nu\alpha\beta} - \frac{41}{45} R_{\mu\nu}R^{\mu\nu} - \frac{1}{18}R^2 \right) \tag{A.6}$$

where we have set $|g| = 1$ and we have multiplied by minus two in order to take into account of the fact that there are two fermionic fields.

A.1.1 The contribution of $S_{\bar{c}\phi}$

The action term for $\bar{c}$ and ϕ was given in equation (11.33), reading

$$\int d^n x \; \bar{c}^{(0,-2)} \Box \phi^{(0,2)} \tag{A.7}$$

This is the simplest possible operator and its a_4 coefficient is given simply as

$$a_4^{\bar{c}\phi} = \frac{1}{180} \int d^n x \; \left(12\Box R + 5R^2 - 2R_{\mu\nu}R^{\mu\nu} + 2R_{\mu\nu\rho\sigma}R^{\mu\nu\rho\sigma}\right) \tag{A.8}$$

where a factor of two has been introduced to take into account that we have two fields. Again, remind that we have set $g = 1$.

Its contribution to the effective action is given by

$$W_\infty^{\bar{c}\phi} = \frac{1}{16\pi^2} \frac{1}{n-4} \frac{1}{180} \int d^n x \; \left(12\Box R + 5R^2 - 2R_{\mu\nu}R^{\mu\nu} + 2R_{\mu\nu\rho\sigma}R^{\mu\nu\rho\sigma}\right) \tag{A.9}$$

A.1.2 The contribution of S_π

The action term for the dynamics of the fermionic π fields was defined in (11.31) and reads

$$S_\pi = \int d^n x \; \pi^{(1,-1)} \Box^{-1} \pi'^{\,(1,1)} \tag{A.10}$$

Even if this is a pseudodifferential operator, its contribution to the pole part of the quantum effective action can be easily computed thanks to the fact that $\Box \times \Box^{-1} = 1$. This means that

$$Det(\Box) = Det(\Box^{-1})^{-1} \longrightarrow \log\left[Det(\Box)\right] = -\log\left[Det(\Box^{-1})\right] \tag{A.11}$$

if there is no multiplicative anomaly. This sums up into the fact that the corresponding Heat Kernel expansion of $\Box^{-1}$ will be minus the expansion of $\Box$. Therefore

$$a_4(\Box) = -\frac{1}{360} \int d^n x \; \left(12\Box R + 5R^2 - 2R_{\mu\nu}R^{\mu\nu} + 2R_{\mu\nu\rho\sigma}R^{\mu\nu\rho\sigma}\right) \tag{A.12}$$

where we have already set $g = 1$.

However, here we are integrating over two fermionic fields, which introduces another factor of minus two. Thus, we have that

$$a_4^\pi = \frac{1}{180}\int d^n x\ \left(12\Box R + 5R^2 - 2R_{\mu\nu}R^{\mu\nu} + 2R_{\mu\nu\rho\sigma}R^{\mu\nu\rho\sigma}\right) \tag{A.13}$$

and its contribution to the effective action is given by

$$W_\infty^\pi = \frac{1}{16\pi^2}\frac{1}{n-4}\frac{1}{180}\int d^n x\ \left(12\Box R + 5R^2 - 2R_{\mu\nu}R^{\mu\nu} + 2R_{\mu\nu\rho\sigma}R^{\mu\nu\rho\sigma}\right) \tag{A.14}$$

A.1.3 The contribution of S_W

The action term for the Weyl ghost field was given in (11.38) and reads

$$2n\alpha\int d^n x\ b\Box c \tag{A.15}$$

The global multiplicative constant will not contribute to the pole part of the quantum effective action, since it gives just an ultralocal contribution, so we can dismiss it, having just

$$\int d^n x\ b\Box c \tag{A.16}$$

Again, we are left the simplest possible operator and its a_4 reads

$$a_4^W = -\frac{1}{180}\int d^n x\ \left(12\Box R + 5R^2 - 2R_{\mu\nu}R^{\mu\nu} + 2R_{\mu\nu\rho\sigma}R^{\mu\nu\rho\sigma}\right) \tag{A.17}$$

where a factor of minus two has been introduced to take into account that we have two fermionic fields and we have set again $g = 1$.

Its contribution to the effective action is given by

$$W_\infty^W = -\frac{1}{16\pi^2}\frac{1}{n-4}\frac{1}{180}\int d^n x\ \left(12\Box R + 5R^2 - 2R_{\mu\nu}R^{\mu\nu} + 2R_{\mu\nu\rho\sigma}R^{\mu\nu\rho\sigma}\right) \tag{A.18}$$

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