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Lectures on Conformal Field Theories

in more than two dimensions

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Abstract

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1 Introduction

"It is difficult to overstate the importance of conformal field theories (CFTs)" [1]. That such a sentence might be written at the start of paper in 2014 is indicative of the genuine progress in our understanding of CFTs in more than two dimensions over the last few years and the resurgence of the conformal bootstrap as surprisingly accurate calculational tool. There are now insightful introductions to this rapidly developing field [2], [3], [4], [5], [6] and most comprehensively [7].

Two dimensional CFTs have been explored in great detail from the 1980's and partially classified, motivated by applications in string theory and statistical physics. This is described and reviewed in the compendious book [8]. However many of the techniques used in understanding two dimensional CFTs do not extend to higher dimensions. In two dimensions there is the infinite dimensional Virasoro algebra whereas in higher dimensions the conformal group is finite dimensional. The subject of CFTs in four, and later three, dimensions was revitalised in 2008 [9] when it was shown that the bootstrap equations, which follow just from crossing and unitarity, are tractable numerically and lead to significant constraints on the spectrum of operators and their dimensions and spins.

Conformal transformations may be defined as preserving angles and are more general than translations and rotations, or Lorentz transformations, which preserve lengths, or their relativistic equivalent space-time interval. They were considered in the 19th century, for a history see [10], but were first applied in a fundamental physics context by Bateman [11] who showed how the four dimensional scalar wave equation was invariant under conformal transformations and then Bateman [12] and Cunningham [13] extended this to Maxwell's equations for electromagnetic field.[1] However conformal transformations do not play a significant role in classical electrodynamics, they do not survive as a symmetry when coupled to matter and there are issues with causality.

In the modern context CFTs are relevant for quantum field theories at RG fixed points. For Lorentz invariant quantum field theories in d-dimensions the trace of the energy momentum tensor is a linear combinations of scalar operators of dimension d, with coefficients the RG β-functions, and also if present contributions involving lower dimension operators with derivatives and mass terms. At RG fixed points the β-functions vanish and if this leads to zero trace for the energy momentum tensor the fixed point defines a CFT. In these lectures we show how CFTs can be analysed using the basic requirements of conformal symmetry in association with the locality and unitarity conditions satisfied by quantum field theories to significantly constrain the spectrum of operators and their scale dimensions as well as other quantities which define a CFT. Although perhaps pedestrian these lectures are not intended just for pedestrians and cover in part different material from [2, 3, 4].

[1] Ebenezer Cunningham and Harry Bateman were both senior wranglers in the Cambridge mathematical tripos, Cunningham (St. Johns) in 1902 and Bateman (Trinity) in 1903, although he tied.

2 Conformal Transformations

The basic definition of a conformal transformations is a transformation of coordinates $x^\mu \to x'^\mu(x)$ such that infinitesimal line elements are invariant up to a local scale factor

$$\mathrm{d}x'^2 = \Omega(x)^2\, \mathrm{d}x^2\,, \qquad \mathrm{d}x^2 = \eta_{\mu\nu}\mathrm{d}x^\mu \mathrm{d}x^\nu\,, \tag{2.1}$$

with $\eta_{\mu\nu} = \mathrm{diag.}(-1, 1\ldots, 1)$ the Minkowski flat space metric or $\eta_{\mu\nu} = \delta_{\mu\nu}$ the usual Euclidean metric. For $\Omega = 1$ these reduce to translations and rotations with also Lorentz transformations in the Minkowski case. For the more general conformal transformations we consider infinitesimal transformations such that

$$x'^\mu = x^\mu + v^\mu(x)\,, \qquad \Omega(x) = 1 + \sigma(x)\,. \tag{2.2}$$

To first order in v, σ (2.1) requires

$$\partial_\mu v_\nu + \partial_\nu v_\mu = 2\,\sigma\,\eta_{\mu\nu}\,. \tag{2.3}$$

An immediate consequence is that v determines σ through

$$\partial_\mu v^\mu = d\,\sigma\,, \qquad d = \eta_\mu{}^\mu\,. \tag{2.4}$$

To find solutions of (2.3) we consider

$$\begin{aligned}&\tfrac{1}{2}\big(\partial_\rho(\partial_\mu v_\nu + \partial_\nu v_\mu) + \partial_\nu(\partial_\mu v_\rho + \partial_\rho v_\mu) - \partial_\mu(\partial_\rho v_\nu + \partial_\nu v_\rho)\big)\\ &= \partial_\rho\partial_\nu v_\mu = \partial_\rho\sigma\,\eta_{\mu\nu} + \partial_\nu\sigma\,\eta_{\mu\rho} - \partial_\mu\sigma\,\eta_{\rho\nu}\,.\end{aligned} \tag{2.5}$$

Acting with ∂^μ gives

$$(d-2)\,\partial_\rho\partial_\nu\sigma = -\eta_{\rho\nu}\,\partial^2\sigma \quad \Rightarrow \quad (d-1)\,\partial^2\sigma = 0 \quad \Rightarrow \quad (d-1)(d-2)\,\partial_\rho\partial_\nu\sigma = 0\,. \tag{2.6}$$

Clearly, except in one dimension which is trivial, $\partial^2\sigma = 0$ and then, so long as $d \neq 2$, $\partial_\rho\partial_\nu\sigma = 0$ and hence σ is linear in x,

$$\sigma = \kappa - 2\,b_\mu x^\mu\,. \tag{2.7}$$

From (2.5)

$$\partial_\rho\partial_\nu v_\mu = -2\,b_\rho\,\eta_{\mu\nu} + 2\,b_\mu\,\eta_{\rho\nu} - 2\,b_\nu\,\eta_{\mu\rho}\,, \tag{2.8}$$

which can be integrated, consistent with (2.4), to give

$$v_\mu(x) = \underbrace{a_\mu}_{\substack{\text{translations}\\ d}} - \underbrace{\omega_{\mu\nu}x^\nu}_{\substack{\text{rotations}\\ \frac{1}{2}d(d-1)}} + \underbrace{\kappa\,x_\mu}_{\substack{\text{scale}\\ 1}} + \underbrace{b_\mu\,x^2 - 2\,x_\mu\,b_\nu x^\nu}_{\substack{\text{special conformal}\\ d}}\,, \qquad \omega_{\mu\nu} = -\omega_{\nu\mu}\,. \tag{2.9}$$

The total number of parameters defining conformal transformations, so long as $d \neq 2$, is therefore $\frac{1}{2}(d+1)(d+2)$. The solutions of (2.3) are conformal Killing vectors.

Two dimensions are different. In the Euclidean case with complex coordinates $z, \bar{z}, \mathrm{d}x^2 = \mathrm{d}z\mathrm{d}\bar{z}$, so that $\eta_{z\bar{z}} = \frac{1}{2}$, then $z \to z' = f(z)$, $\bar{z} \to \bar{z}' = \bar{f}(\bar{z})$ for any f is a conformal transformation since $\mathrm{d}x'^2 = f'(z)\bar{f}'(\bar{z})\,\mathrm{d}x^2$. Infinitesimally $v^z = v(z)$, $v^{\bar{z}} = \bar{v}(\bar{z})$ and then

(2.3) gives $\sigma = \frac{1}{2}(v'(z) + \bar{v}'(\bar{z}))$ which is a general solution of $\partial^2\sigma = 0$ in two dimensions. A basis of two dimensional conformal Killing vectors is given by $v_n(z) = z^{n+2}$, $\bar{v}_n(\bar{z}) = \bar{z}^{n+2}$, $n \in \mathbb{Z}$. which is clearly infinite dimensional.

The generators of infinitesimal conformal transformations satisfy

$$\big[v^\mu\partial_\mu \, v'^\nu\partial_\nu\big] = [v,v']^\mu\partial_\mu\,, \qquad [v,v']^\mu = v^\nu\partial_\nu v'^\mu - v'^\nu\partial_\nu v^\mu\,. \tag{2.10}$$

It is convenient to define

$$\partial_\mu v_\nu = \sigma_v\,\eta_{\mu\nu} + \hat{\omega}_{v\,\mu\nu}\,, \qquad \hat{\omega}_{v\,\mu\nu} = -\hat{\omega}_{v\,\nu\mu}\,. \tag{2.11}$$

so that with (2.9) $\hat{\omega}_{v\,\mu\nu} = \omega_{\mu\nu} - 2(b_\mu x_\nu - b_\nu x_\mu)$. As a consequence of (2.5)

$$\partial_\rho\,\hat{\omega}_{v\,\mu\nu} = \partial_\mu\sigma_v\,\eta_{\rho\nu} - \partial_\nu\sigma_v\,\eta_{\rho\mu}\,. \tag{2.12}$$

From (2.10) also

$$\begin{aligned}
&v^\mu\partial_\mu\,\sigma_{v'} - v'^\mu\partial_\mu\,\sigma_v = \sigma_{[v,v']}\,,\\
&v^\mu\partial_\mu\,\hat{\omega}_{v'\,\mu\nu} - v'^\mu\partial_\mu\,\hat{\omega}_{v\,\mu\nu} + \hat{\omega}_{v\,\mu}{}^\alpha\,\hat{\omega}_{v'\,\alpha\nu} - \hat{\omega}_{v\,\nu}{}^\alpha\,\hat{\omega}_{v'\,\alpha\mu} = \hat{\omega}_{[v,v']\,\mu\nu}\,.
\end{aligned} \tag{2.13}$$

For an infinitesimal interval $\mathrm{d}x$ then

$$\delta_v\mathrm{d}x^\mu = \mathrm{d}v^\mu(x) = \sigma_v(x)\,\mathrm{d}x^\mu - \hat{\omega}_v{}^\mu{}_\nu(x)\,\mathrm{d}x^\nu\,, \tag{2.14}$$

and for two points x, y

$$\begin{aligned}
\delta_v(x-y)^2 &= \big(v^\mu(x)\partial_{x\mu} + v^\mu(y)\partial_{y\mu}\big)(x-y)^2\\
&= 2\big(v_\mu(x) - v_\mu(y)\big)(x-y)^\mu = \big(\partial_\nu v_\mu(x) + \partial_\nu v_\mu(y)\big)(x-y)^\nu(x-y)^\mu\\
&= \big(\sigma_v(x) + \sigma_v(y)\big)\,(x-y)^2\,.
\end{aligned} \tag{2.15}$$

Finite conformal transformations can be obtained by integrating

$$\frac{\mathrm{d}}{\mathrm{d}t}\,x_t{}^\mu = v^\mu(x_t)\,, \qquad x_0{}^\mu = x^\mu\,. \tag{2.16}$$

Writing

$$\mathrm{d}x_t{}^\mu = \Omega_t(x)\,R_t{}^\mu{}_\nu(x)\,\mathrm{d}x^\nu\,, \qquad \det[R_t{}^\mu{}_\nu] = 1\,, \tag{2.17}$$

then from (2.14) and (2.16)

$$\begin{aligned}
&\frac{\mathrm{d}}{\mathrm{d}t}\,\Omega_t(x) = \sigma_v(x_t)\,\Omega_t(x)\,, && \Omega_0(x) = 1\,,\\
&\frac{\mathrm{d}}{\mathrm{d}t}\,R_t{}^\mu{}_\nu(x) = -\,\hat{\omega}_v{}^\mu{}_\omega(x_t)\,R_t{}^\omega{}_\nu(x)\,, && R_0{}^\mu{}_\nu(x) = \delta^\mu{}_\nu\,.
\end{aligned} \tag{2.18}$$

The solutions satisfy the group property

$$\Omega_{t'+t}(x) = \Omega_{t'}(x_t)\,\Omega_t(x)\,, \qquad R_{t'+t}{}^\mu{}_\nu(x) = R_{t'}{}^\mu{}_\omega(x_t)\,R_t{}^\omega{}_\nu(x)\,. \tag{2.19}$$

From (2.15)

$$\frac{\mathrm{d}}{\mathrm{d}t}(x_t - y_t)^2 = \big(\sigma_v(x_t) + \sigma_v(y_t)\big)(x_t - y_t)^2 \quad \Rightarrow \quad (x_t - y_t)^2 = \Omega_t(x)\Omega_t(y)(x-y)^2 \, . \tag{2.20}$$

For v^μ in (2.9) restricted to just special conformal transformations (2.16) and (2.18) give[2]

$$x_t{}^\mu = \Omega_t(x)(x^\mu + t\, b^\mu\, x^2)\, , \qquad \Omega_t(x) = \frac{1}{1 + 2\,t\,b\cdot x + t^2\, b^2 x^2}\, . \tag{2.21}$$

For finite translations and scale transformations the solutions are trivially

$$x_t{}^\mu = x^\mu + t\, a^\mu\, , \qquad x_t{}^\mu = e^{t\kappa} x^\mu \, . \tag{2.22}$$

In general for conformal transformation $x \to x'$ then

$$\frac{\partial x'^\mu}{\partial x^\nu} = \Omega(x)\, R^\mu{}_\nu(x)\, , \qquad \eta_{\mu\nu} R^\mu{}_\rho(x) R^\nu{}_\tau(x) = \eta_{\rho\tau}\, . \tag{2.23}$$

so that R is an orthogonal, or pseudo-orthogonal, rotation matrix, and hence $R \in O(d)$ or $R \in O(d-1,1)$, and

$$(x' - y')^2 = \Omega(x)\Omega(y)\,(x-y)^2 \, . \tag{2.24}$$

To form a conformal invariant it is necessary to have at least four points. There are then invariant cross ratios

$$u_{ijkl} = \frac{x_{ij}{}^2\, x_{kl}{}^2}{x_{ik}{}^2\, x_{jl}{}^2}\, , \qquad i \ne j \ne k \ne l\, , \; x_{ij} = x_i - x_j\, , \tag{2.25}$$

since from (2.24) the factors $\Omega(x_i)$ all cancel. The $\{u_{ijkl}\}$ are not independent and obey various identities, such as $u_{ijkl} = u_{klij} = u_{jilk} = u_{ikjl}{}^{-1}$ and $u_{ijkm}/u_{ijkl} = u_{jlmk}$. For n points a basis is provided by u_{12kl}, $3 \le k < l \le n$, and u_{132l}, $4 \le l \le n$, which gives $\frac{1}{2}n(n-3)$ possible invariants. When $n = 4$ there are just two and in common terminology the invariants are $u = u_{1234}$, $v = u_{1423}$. However for any given dimension this overcounts invariants when $n > d+2$ since there are further relations between different $x_{ij}{}^2$. If the conformal group acts transitively on n points there are just $N_{d,n} = nd - \frac{1}{2}(d+1)(d+2)$ invariants. When $N_{d,n} < \frac{1}{2}n(n-3)$ the number of invariants is then reduced to $N_{d,n}$ but if $n \le d+2$ then $N_{d,n} \ge \frac{1}{2}n(n-3)$ and $\frac{1}{2}n(n-3)$ provides the correct counting. In this case there is a residual subgroup of the conformal group which leaves the n points invariant. Thus for $d = 4$ and $n = 4,5,6,7$ there are $2,5,9,13$ independent conformal invariants with the extra restrictions first relevant when $n = 7$.

[2]The solution for x_t may be obtained by writing $x_t{}^\mu = \alpha(t)\, x^\mu + \beta(t)\, b^\mu$ where $\alpha(0) = 1$, $\beta(0) = 0$. The differential equation reduces to $\dot\alpha = -2\alpha^2 b\cdot x - 2\alpha\beta\, b^2$, $\dot\beta = \alpha^2 x^2 - \beta^2 b^2$. These may be decoupled giving $\frac{\mathrm{d}}{\mathrm{d}t}(\beta + \lambda\alpha) = -b^2(\beta+\lambda\alpha)^2$ for $\lambda^2 b^2 = 2\lambda\, b\cdot x - x^2$ which integrates to $\beta + \lambda\alpha = \lambda/(1+\lambda^2 b^2 t)$. Eliminating β then $\dot\alpha = 2(\lambda\, b^2 - b\cdot x)\alpha^2 - 2\lambda\, b^2\, \alpha/(1+\lambda^2 b^2 t)$ and hence $\alpha = 1/(1 + 2\,t\,b\cdot x + t^2\, b^2 x^2)$ which with $\beta = \alpha\, x^2 t$ gives (2.21).

2.1 Inversion Tensor and Conformal Vectors

An important role is played by inversions for which

$$x^\mu \to x_i{}^\mu = \frac{x^\mu}{x^2}\,, \tag{2.26}$$

in which case

$$\mathrm{d}x_i{}^\mu = \frac{1}{x^2}\, I^\mu{}_\nu(x)\,\mathrm{d}x^\nu\,, \tag{2.27}$$

where the inversion tensor for vectors is given by

$$I^\mu{}_\nu(x) = \delta^\mu{}_\nu - 2\,\frac{x^\mu x_\nu}{x^2}\,. \tag{2.28}$$

satisfying $\eta_{\mu\nu} I^\mu{}_\alpha(x)\, I^\nu{}_\beta(x) = \eta_{\alpha\beta}$ so that $\mathrm{d}x_i{}^2 = \mathrm{d}x^2/(x^2)^2$ and, for two points,

$$(x-y)^2 \to (x_i - y_i)^2 = \frac{1}{x^2 y^2}\,(x-y)^2\,. \tag{2.29}$$

Inversions are therefore discrete conformal transformations not connected to the identity, since $\det[I^\mu{}_\nu] = -1$, but they can be used to generate finite special conformal transformations by considering an inversion followed by a translation and then another inversion,

$$x^\mu \to \frac{x^\mu}{x^2} \to \frac{x^\mu}{x^2} + b^\mu \to \frac{\frac{x^\mu}{x^2} + b^\mu}{(\frac{x}{x^2}+b)^2} = \frac{x^\mu + b^\mu\, x^2}{1 + 2\,b\cdot x + b^2\, x^2}\,, \tag{2.30}$$

identical to (2.21) for $t = 1$. To first order in b this reduces to the result for special conformal transformations in (2.9).

Since

$$-\tfrac{1}{2}\,\partial_{x\mu}\partial_{y\nu} \ln(x-y)^2 = \frac{I_{\mu\nu}(x-y)}{(x-y)^2}\,, \tag{2.31}$$

then as $\ln(x'-y')^2 = \ln(x-y)^2 + \ln\Omega(x) + \ln\Omega(y)$ we have from (2.23)

$$\begin{aligned} I_{\mu\nu}(x-y) &= I_{\rho\tau}(x'-y')\, R^\rho{}_\mu(x) R^\tau{}_\nu(y)\,, \\ I_{\mu\nu}(x-y) &= I_{\rho\tau}(x_i - y_i)\, I^\rho{}_\mu(x) I^\tau{}_\nu(y)\,. \end{aligned} \tag{2.32}$$

For infinitesimal transformations using (2.15)

$$\big(v^\rho(x)\partial_{x\rho} + v^\rho(y)\partial_{y\rho}\big) I_{\mu\nu}(x-y) = -\hat\omega_{v\,\mu}{}^\rho(x)\, I_{\rho\nu}(x-y) + I_{\mu\rho}(x-y)\,\hat\omega_v{}^\rho{}_\nu(y)\,. \tag{2.33}$$

For three points x, y, z we may define

$$\tfrac{1}{2}\,\partial_{x\mu} \ln\big((x-y)^2/(x-z)^2\big) = \frac{(x-y)_\mu}{(x-y)^2} - \frac{(x-z)_\mu}{(x-z)^2} = X_\mu\,. \tag{2.34}$$

From (2.24) under a conformal transformation $x, y, z \to x', y', z'$ then $X \to X'$ and also from (2.29) under an inversion $x, y, z \to x_i, y_i, z_i$ similarly $X \to X_i$ where

$$X_\mu = X'_\rho\, R^\rho{}_\mu(x)\Omega(x)\,, \qquad X_\mu = X_{i\rho}\, I^\rho{}_\mu(x)\frac{1}{x^2}\,. \tag{2.35}$$

so that X_μ transforms as vector at x. In a similar fashion

$$Y_\mu = \tfrac{1}{2}\,\partial_{y\mu} \ln\big((y-z)^2/(y-x)^2\big)\,, \qquad Z_\mu = \tfrac{1}{2}\,\partial_{z\mu} \ln\big((z-x)^2/(z-y)^2\big)\,, \tag{2.36}$$

are vectors at y, z. It is easy to see that

$$X^2 = \frac{(y-z)^2}{(x-y)^2(x-z)^2}\,, \quad Y^2 = \frac{(z-x)^2}{(y-x)^2(y-z)^2}\,, \quad Z^2 = \frac{(x-y)^2}{(z-x)^2(z-y)^2}\,. \tag{2.37}$$

Infinitesimally (2.35) corresponds to

$$\big(v(x)^\rho\partial_{x\rho} + v(y)^\rho\partial_{y\rho} + v(z)^\rho\partial_{z\rho}\big)X_\mu = -\sigma_v(x)\,X_\mu + X_\nu\,\hat\omega^\nu{}_\mu(x)\,. \tag{2.38}$$

Under $y \leftrightarrow z$, $X_\mu \to -X_\mu$, $Y_\mu \leftrightarrow -Z_\mu$.

Directly from (2.34)

$$X_\mu \overleftarrow{\partial}_{y\nu} = -\frac{1}{(x-y)^2}\,I_{\mu\nu}(x-y)\,, \tag{2.39}$$

so that by evaluating $X^2\overleftarrow{\partial}_{y\nu}$ we obtain

$$X^\mu I_{\mu\nu}(x-y) = -\frac{(y-z)^2}{(x-z)^2}\,Y_\nu\,. \tag{2.40}$$

By cyclically permuting x, y, z

$$Y^\mu I_{\mu\nu}(y-z) = -\frac{(z-x)^2}{(y-x)^2}\,Z_\nu\,, \qquad Z^\mu I_{\mu\nu}(z-x) = -\frac{(x-y)^2}{(z-y)^2}\,X_\nu\,. \tag{2.41}$$

Rewriting (2.40) as

$$\frac{1}{Y^2}\,Y_\nu = -(y-x)^2\,X^\rho\,I_{\rho\nu}(x-y) \tag{2.42}$$

and using

$$\partial_{z\mu}Y_\nu = -\frac{1}{(z-y)^2}\,I_{\mu\nu}(z-y)\,, \quad \partial_{z\mu}X_\rho = \frac{1}{(z-x)^2}\,I_{\mu\rho}(z-x)\,, \tag{2.43}$$

we obtain

$$I_\mu{}^\rho(z-y)\,I_{\rho\nu}(Y) = I_\mu{}^\rho(z-x)\,I_{\rho\nu}(x-y)\,. \tag{2.44}$$

Similarly

$$I_\mu{}^\rho(Z)\,I_{\rho\nu}(z-y) = I_\mu{}^\rho(z-x)\,I_{\rho\nu}(x-y)\,, \tag{2.45}$$

with other identities following from permuting x, y, z.

2.2 Conformal Transformations of Fields

Acting on fields ϕ_I, where I is a spin index for the rotation group $O(d)$ or $O(d-1,1)$, we can define the action of a conformal transformation such that $x \to x'$ on the field by

$$\phi_I \to \phi'_I\,, \qquad \phi'_I(x') = \Omega(x)^{-\Delta}\,\mathcal{R}_I{}^J(x)\,\phi_J(x)\,. \tag{2.46}$$

where Δ is the scale dimension of ϕ_I and $\mathcal{R}_I{}^J$ is the matrix corresponding to $R^\nu{}_\mu$ in the representation of $O(d)$ or $O(d-1,1)$ determined by ϕ_I. For an inversion (2.46) becomes

$$\phi'_I(x_i) = (x^2)^\Delta \, \mathcal{I}_I{}^J(x)\, \phi_J(x)\,, \tag{2.47}$$

where $\mathcal{I}_I{}^J(\lambda x) = \mathcal{I}_I{}^J(x)$ and $\mathcal{I}_I{}^K(x)\mathcal{I}_K{}^J(x) = \delta_I{}^J$. Fields satisfying (2.46) or (2.47) are called conformal primary fields. Crucially the derivative of a conformal primary field is not a conformal primary.

In general the transformation $\phi \to \phi'$ for an inversion may not be possible. If the inversion (2.31) is combined with a reflection $x^1 \to -x^1$ then the resulting transformation belongs to the identity component of the conformal group $SO(d+1,1)$ or $SO(d,2)$. To show this we may consider the combination of a special conformal and scale transformation with a translation given by

$$x_\lambda{}^\mu = \frac{(1+\lambda^2)\, x^\mu}{1+\lambda^2\, x^2 + 2\lambda\, x^1}\,, \qquad \mu \neq 1\,, \qquad x_\lambda{}^1 = -\lambda + \frac{(1+\lambda^2)(x^1+\lambda\, x^2)}{1+\lambda^2\, x^2+2\lambda\, x^1}\,. \tag{2.48}$$

Clearly $x_0{}^\mu = x^\mu$ and, as $\lambda \to \infty$, $x_\lambda{}^\mu \to r_1 x^\mu / x^2$ for $r_1 x^\mu = x^\mu$, $\mu \neq 1$, $r_1 x^1 = -x^1$. The corresponding action on fields is therefore defined in any CFT. For chiral CFTs when parity is not a symmetry neither is invariance under inversions.

Infinitesimally, when $\delta x^\mu = v^\mu$, the corresponding change in ϕ resulting from (2.46) is

$$\delta_v \phi_I = -v^\mu \partial_\mu \phi_I - \sigma_v\, \Delta\, \phi_I + \tfrac{1}{2}\, \hat{\omega}_v{}^{\mu\nu} (s_{\mu\nu})_I{}^J\, \phi_J\,, \tag{2.49}$$

with $s_{\mu\nu}$ the appropriate spin matrices satisfying

$$\big[s_{\mu\nu},\, s_{\rho\tau}\big] = \eta_{\mu\rho}\, s_{\nu\tau} - \eta_{\nu\rho}\, s_{\mu\tau} - \eta_{\mu\tau}\, s_{\nu\rho} + \eta_{\nu\tau} s_{\mu\rho}\,. \tag{2.50}$$

For a Euclidean metric, so that in (2.50) $\eta_{\mu\rho} \to \delta_{\mu\rho}$ and $s_{\mu\nu}$ are the generators for a representation of $SO(d)$, then we may take the spin matrices to be anti-hermitian, $s_{\mu\nu}{}^\dagger = -s_{\mu\nu}$. Together with (2.13), (2.50) ensures $[\delta_v, \delta_{v'}]\phi_I = \delta_{[v,v']}\phi_I$.

For an irreducible spin representation R the quadratic Casimir becomes

$$\tfrac{1}{2}\, (s_{\mu\nu}\, s^{\mu\nu})_I{}^J = -C_R\, \delta_I{}^J\,. \tag{2.51}$$

For a vector field A_σ the vector representation R_V is defined by spin matrices

$$(s_{\mu\nu})_\lambda{}^\rho = \delta_\mu{}^\rho\, \eta_{\nu\lambda} - \delta_\nu{}^\rho\, \eta_{\mu\lambda}\,, \qquad \tfrac{1}{2}\, (s_{\mu\nu}\, s^{\mu\nu})_\lambda{}^\rho = -C_V\, \delta_\lambda{}^\rho\,, \quad C_V = d-1\,. \tag{2.52}$$

If $s_{\mu\nu}$ is complex there is a conjugate spin matrix $\bar{s}_{\mu\nu}$, also satisfying (2.50), and the conjugate conformal primary transforms as

$$\delta_v \bar{\phi}_{\bar{I}} = -v^\mu \partial_\mu \bar{\phi}_{\bar{I}} - \sigma_v\, \bar{\phi}_{\bar{I}}\, \Delta - \tfrac{1}{2}\, \hat{\omega}_v{}^{\mu\nu} \bar{\phi}_{\bar{J}}\, (\bar{s}_{\mu\nu})^{\bar{J}}{}_{\bar{I}}\,. \tag{2.53}$$

In order to discuss spinor fields $\psi, \bar{\psi}$ we it is necessary to define gamma matrices. In even dimensions, $d = 2n$, we may define $2^{n-1} \times 2^{n-1}$ chiral gamma matrices such that

$$\gamma_\mu\, \bar{\gamma}_\nu + \gamma_\nu\, \bar{\gamma}_\mu = 2\, \eta_{\mu\nu}\, \mathbb{1}\,, \qquad \bar{\gamma}_\mu\, \gamma_\nu + \bar{\gamma}_\nu\, \gamma_\mu = 2\, \eta_{\mu\nu}\, \bar{\mathbb{1}}\,, \tag{2.54}$$

where $\gamma_\mu, \bar\gamma_\mu$ are inequivalent matrices and we distinguish the identity matrices for chiral and anti-chiral spinors. With a Minkowski metric we may impose the hermeticity conditions

$$\gamma_\mu{}^\dagger = -\gamma_\mu \, , \quad \bar\gamma_\mu{}^\dagger = -\bar\gamma_\mu \, , \qquad \bar\psi = \psi^\dagger \, , \tag{2.55}$$

and hence we must require $\mathbb{1}^\dagger = \bar{\mathbb{1}}$. For $n=1$, $d=2$ we can take $\gamma_\mu = i(1,1), \bar\gamma_\mu = i(1,-1)$ while for $n=2$, $d=4$ we can take $\gamma_\mu = i(1,\sigma_i), \bar\gamma_\mu = i(1,-\sigma_i)$ with σ_i the usual Pauli matrices. For odd dimensions, $d = 2n-1$, there are also $2^{n-1}\times 2^{n-1}$ gamma matrices γ_μ satisfying (2.54) with $\bar\gamma_\mu = \gamma_\mu$. For d odd it is important to recognise that γ_μ and $-\gamma_\mu$ define inequivalent representations of the Dirac algebra. In this case we may take

$$\beta \gamma_\mu{}^\dagger \beta = -\gamma_\mu \, , \qquad \bar\psi = \psi^\dagger \beta \, , \qquad \beta = \beta^\dagger = \beta^{-1} \, . \tag{2.56}$$

For a Euclidean metric, $\eta_{\mu\nu} = \delta_{\mu\nu}$, the hermeticity conditions become $\gamma_\mu{}^\dagger = \gamma_\mu$ for d odd and $\gamma_\mu{}^\dagger = \bar\gamma_\mu$ for d even but then in (2.54) we must take $\mathbb{1}^\dagger = \mathbb{1}$, $\bar{\mathbb{1}}^\dagger = \bar{\mathbb{1}}$.

The spin matrices for chiral/anti-chiral spinors are then for $d = 2n$

$$s_{\mu\nu} = -\tfrac{1}{2}\, \gamma_{[\mu}\, \bar\gamma_{\nu]} \, , \qquad \bar s_{\mu\nu} = -\tfrac{1}{2}\, \bar\gamma_{[\mu}\, \gamma_{\nu]} \, , \qquad s_{\mu\nu}{}^\dagger = -\bar s_{\mu\nu} \, , \tag{2.57}$$

which obey (2.50) by virtue of

$$s_{\mu\nu}\,\gamma_\rho - \gamma_\rho\, \bar s_{\mu\nu} = \eta_{\mu\rho}\,\gamma_\nu - \eta_{\nu\rho}\,\gamma_\mu = \gamma_\sigma\, (s_{\mu\nu})^\sigma{}_\rho \, , \qquad \bar s_{\mu\nu}\,\bar\gamma_\rho - \bar\gamma_\rho\, s_{\mu\nu} = \eta_{\mu\rho}\,\bar\gamma_\nu - \eta_{\nu\rho}\,\bar\gamma_\mu \, . \tag{2.58}$$

For $d = 2n-1$ identical relations are obtained so long as we identify $\bar s_{\mu\nu} = s_{\mu\nu}$ and now $s_{\mu\nu}{}^\dagger = -\beta s_{\mu\nu}\beta$. The quadratic Casimir from (2.51) becomes

$$\tfrac{1}{2}\, s_{\mu\nu} s^{\mu\nu} = -C_s\, \mathbb{1} \, , \quad \tfrac{1}{2}\, \bar s_{\mu\nu} \bar s^{\mu\nu} = -C_s\, \bar{\mathbb{1}} \, , \qquad C_s = \tfrac{1}{8} d(d-1) \, . \tag{2.59}$$

If $\psi, \bar\psi$ are conformal primary chiral spinors then under infinitesimal conformal transformations

$$\delta_v \psi = -v^\mu \partial_\mu \psi - \sigma_v\, \Delta\, \psi + \tfrac{1}{2}\, \hat\omega_v{}^{\mu\nu} s_{\mu\nu}\, \psi \, , \quad \delta_v \bar\psi = -v^\mu \partial_\mu \bar\psi - \sigma_v\, \bar\psi\, \Delta - \tfrac{1}{2}\, \hat\omega_v{}^{\mu\nu} \bar\psi\, \bar s_{\mu\nu} \, . \tag{2.60}$$

If we apply (2.60) for the conformal transformation in (2.48)

$$\begin{aligned} \frac{\mathrm{d}}{\mathrm{d}\lambda} \psi_\lambda(x) = \frac{1}{1+\lambda^2} \Big(\big((1-x^2)\,\partial_1 + 2\, x^1 x^\nu \partial_\nu \big) \psi_\lambda(x) + \Delta\, 2x^1\, \psi_\lambda(x) \\ - \tfrac{1}{2} \big(\gamma\cdot x\; \bar\gamma_1 - \gamma_1\; \bar\gamma \cdot x \big)\, \psi_\lambda(x) \Big) \, . \end{aligned} \tag{2.61}$$

Solving this equation gives

$$\psi_\lambda(x) = \bigg(\frac{1+\lambda^2 x^2 - 2\lambda\, x^1}{1+\lambda^2} \bigg)^{-\Delta - \frac{1}{2}} \frac{\mathbb{1} - \lambda\, \gamma \cdot x\; \bar\gamma_1}{(1+\lambda^2)^{\frac{1}{2}}}\, \psi(x_{-\lambda}) \, . \tag{2.62}$$

For $\lambda \to \infty$

$$\psi_\lambda(x) \to -(x^2)^{-\Delta - \frac{1}{2}}\, \gamma \cdot x\, \bar\gamma_1\, \psi(r_1 x / x^2) \, , \tag{2.63}$$

with r_1 reflection in the 1-direction.

When d is even for theories which are parity invariant then it is necessary for there to be spinors $\chi, \bar\chi$ of opposite chirality to $\psi, \bar\psi$ and which transform similarly to (2.60) but with $s_{\mu\nu} \leftrightarrow \bar s_{\mu\nu}$,

$$\delta_v \chi = -v^\mu \partial_\mu \chi - \sigma_v \, \Delta \, \chi + \tfrac{1}{2} \, \hat\omega_v{}^{\mu\nu} \bar s_{\mu\nu} \, \chi \, , \quad \delta_v \bar\chi = -v^\mu \partial_\mu \bar\chi - \sigma_v \, \bar\chi \, \Delta - \tfrac{1}{2} \, \hat\omega_v{}^{\mu\nu} \bar\chi \, s_{\mu\nu} \, . \tag{2.64}$$

The parity transformation is then $\psi(x) \to \gamma_1 \, \chi(r_1 x)$. In this case the action of inversion is given by

$$\psi(x) \to (x^2)^{-\Delta - \frac{1}{2}} \, \gamma \cdot x \, \chi(x/x^2) \, , \qquad \bar\psi(x) \to -(x^2)^{-\Delta-\frac{1}{2}} \, \bar\chi(x/x^2) \, \gamma \cdot x \, . \tag{2.65}$$

For d odd inversions can be defined on spinor fields in general by

$$\psi(x) \to (x^2)^{-\Delta-\frac{1}{2}} \, \gamma \cdot x \, \psi(x/x^2) \, , \quad \bar\psi(x) \to -(x^2)^{-\Delta-\frac{1}{2}} \, \bar\psi(x/x^2) \, \gamma \cdot x \, . \tag{2.66}$$

2.3 Derivative Constraints

In general derivatives of primary fields are not primary. From (2.49) and using (2.12)

$$\begin{aligned} \delta_v \, \partial_\lambda \phi_I = {} & - v^\mu \partial_\mu \partial_\lambda \phi_I - \sigma_v \, (\Delta + 1) \, \partial_\lambda \phi_I - \hat\omega_v \, {}_\lambda{}^\rho \, \partial_\rho \phi_I + \tfrac{1}{2} \, \hat\omega_v{}^{\mu\nu} (s_{\mu\nu})_I{}^J \, \partial_\lambda \phi_J \\ & - \big(\Delta \, \delta_I{}^J \, \eta_{\lambda\tau} + (s_{\lambda\tau})_I{}^J \big) \phi_J \, \partial^\tau \sigma_v \, . \end{aligned} \tag{2.67}$$

Only if the second line vanishes is $\partial_\lambda \phi_I$ also a conformal primary as well as ϕ_I. Imposing

$$f^{\lambda I} \big(\Delta \, \delta_I{}^J \, \delta_\lambda^\tau - M_{\lambda I}{}^{\tau J} \big) = 0 \ , \tag{2.68}$$

where, using the vector representation for the spin matrices in (2.52),

$$M_{\lambda I}{}^{\tau J} = -(s_\lambda{}^\tau)_I{}^J = \tfrac{1}{2} \, (s_{\mu\nu})_I{}^J (s^{\mu\nu})_\lambda{}^\tau \, , \tag{2.69}$$

gives an eigenvalue equation for Δ which ensures $f^{\lambda I} \partial_\lambda \phi_I$ are conformal primaries. The matrix M in (2.69) commutes with $SO(d)$, or alternatively $SO(d-1,1)$, so that to solve (2.68) it is sufficient to decompose $f^{\lambda I}$ into irreducible representations. If ϕ_I belongs to the representation space V_ϕ for a representation R_ϕ and

$$R_\phi \otimes R_V \simeq \oplus_i \, R_i \, , \tag{2.70}$$

then, acting on the tensor product $V_\phi \times V_V$,

$$- \tfrac{1}{2} \big((s_{\mu\nu})_I{}^K \delta_\lambda^\rho + \delta_I{}^K (s_{\mu\nu})_\lambda{}^\rho \big) \big((s^{\mu\nu})_K{}^J \delta_\rho^\tau + \delta_K{}^J (s^{\mu\nu})_\rho{}^\tau \big) = \textstyle\sum_i C_{R_i} \, (P_i)_{\lambda I}{}^{\tau J} \, , \tag{2.71}$$

with P_i the projector onto the irreducible representation space $V_i \subset V_\phi \times V_V$. The eigenvalue Δ_i corresponding to the irreducible representation R_i is, as a consequence of (2.68), determined by the Casimir eigenvalues according to

$$\Delta_i = \tfrac{1}{2} \big(C_\phi + C_V - C_i \big) \, . \tag{2.72}$$

Since $M_{\lambda I}{}^{\lambda I} = 0$ the eigenvalues Δ are in general both positive and negative as their sum, $\sum_i \dim V_i \, \Delta_i = 0$.

Simple cases can be analysed directly. For spinless fields trivially $\Delta = 0$. For a vector, $\phi \to A_\sigma$, (2.68) becomes

$$f^{\lambda\rho}\big(\Delta\, \delta_\rho{}^\sigma\, \eta_{\lambda\tau} + \delta_\lambda{}^\sigma\, \eta_{\rho\tau} - \delta_\tau{}^\sigma\, \eta_{\rho\lambda}\big) = 0\,, \tag{2.73}$$

which has solutions

$$f^{\lambda\rho} = \eta^{\lambda\rho}\,,\ \Delta = d-1\,, \quad f^{\lambda\rho} = f^{[\lambda\rho]}\,,\ \Delta = 1\,, \quad f^{\lambda\rho} = f^{(\lambda\rho)}\,,\ \eta_{\lambda\rho}f^{\lambda\rho} = 0\,,\ \Delta = -1\,. \tag{2.74}$$

Thus $\partial^\mu A_\mu$ is a conformal primary if $\Delta = d-1$ as is $\partial_\mu A_\nu - \partial_\nu A_\mu$ for $\Delta = 1$. The eigenvalue sum is then $d - 1 + \frac{1}{2}d(d-1) - \frac{1}{2}(d+2)(d-1) = 0$. For a rank two tensor field $T_{\sigma\rho}$ the equations become

$$f^{\lambda\mu\nu}\big(\Delta\, \delta_\mu{}^\sigma \delta_\nu{}^\rho\, \eta_{\lambda\tau} + \delta_\nu{}^\rho(\delta_\lambda{}^\sigma\, \eta_{\mu\tau} - \delta_\tau{}^\sigma\, \eta_{\mu\lambda}) + \delta_\mu{}^\sigma(\delta_\lambda{}^\rho\, \eta_{\nu\tau} - \delta_\tau{}^\rho\, \eta_{\nu\lambda})\big) = 0\,. \tag{2.75}$$

For $T_{\sigma\rho} = T_{\rho\sigma}$, $\eta^{\sigma\rho}T_{\sigma\rho} = 0$, this has solutions

$$\begin{aligned}
f^{\lambda\mu\nu} &= \eta^{\lambda\mu}\epsilon^\nu + \eta^{\lambda\nu}\epsilon^\mu - \frac{2}{d}\,\eta^{\mu\nu}\epsilon^\lambda\,, && \Delta = d\,,\\
f^{\lambda\mu\nu} &= f^{\lambda\nu\mu}\,,\ f^{(\lambda\mu\nu)} = 0\,,\ f^{\lambda\mu\nu}\eta_{\mu\nu} = 0\,, && \Delta = 1\,,\\
f^{\lambda\mu\nu} &= f^{(\lambda\mu\nu)}\,,\ f^{\lambda\mu\nu}\eta_{\mu\nu} = 0\,, && \Delta = -2\,.
\end{aligned} \tag{2.76}$$

Thus $\partial^\mu T_{\mu\nu}$ is a conformal primary for $\Delta = d$. For $T_{\sigma\rho} = -T_{\rho\sigma}$ the solutions are

$$\begin{aligned}
f^{\lambda\mu\nu} &= \eta^{\lambda\mu}\epsilon^\nu - \eta^{\lambda\nu}\epsilon^\mu\,, && \Delta = d-2\,,\\
f^{\lambda\mu\nu} &= f^{[\lambda\mu\nu]}\,, && \Delta = 2\,,\\
f^{\lambda\mu\nu} &= -\,f^{\lambda\nu\mu}\,,\ f^{[\lambda\mu\nu]} = 0\,,\ f^{\lambda\mu\nu}\eta_{\lambda\mu} = 0\,, && \Delta = -1\,,
\end{aligned} \tag{2.77}$$

with the first two conditions corresponding to $\partial^\mu T_{\mu\nu}$, $\partial_{[\lambda}T_{\mu\nu]}$, being conformal primaries.

If the derivative generates a conformal primary then it is possible to impose a constraint on the fields such that this vanishes. In the above examples $\partial^\mu T_{\mu\nu} = 0$ is the conservation equation for the symmetric traceless energy momentum tensor whereas $\partial_{[\lambda}F_{\mu\nu]} = 0$ is the Bianchi identity for the abelian field strength $F_{\mu\nu}$, the field equation $\partial^\mu F_{\mu\nu} = 0$ is then conformally consistent only for $d = 4$.

More generally if we consider symmetric traceless tensors of rank ℓ, corresponding to a representation $R_{(\ell)}$, corresponding to a Young tableau with just a single row of ℓ boxes $\boxed{1}\boxed{2}\boxed{}\boxed{}\boxed{}\boxed{\ell}$, then $R_V = R_{(1)}$ and

$$R_{(\ell)} \otimes R_{(1)} \simeq R_{(\ell-1)} \oplus R_{(\ell+1)} \oplus R_{(\ell,1)}\,, \tag{2.78}$$

where $R_{(\ell,1)}$ denotes the mixed symmetry representation corresponding to $\begin{array}{l}\boxed{1}\boxed{2}\boxed{}\boxed{}\boxed{}\boxed{\ell}\\ \boxed{}\end{array}$. Using

$$C_{(\ell)} = \ell(\ell+d-2)\,, \qquad C_{(\ell,1)} = (\ell+1)(\ell+d-3)\,, \tag{2.79}$$

(2.72) gives

$$\Delta_{(\ell-1)} = \ell + d - 2\,,\ \ell \geq 1\,, \qquad \Delta_{(\ell+1)} = -\ell\,, \qquad \Delta_{(\ell,1)} = 1\,,\ \ell \geq 1\,. \tag{2.80}$$

This of course corresponds to (2.76) when $\ell = 2$. For $d = 3$, $R_{(\ell,1)} \simeq R_{(\ell)}$.

For representations $R_{(n,m)}$ defined by mixed symmetry tensors corresponding to tableaux $\begin{array}{|c|c|c|c|c|}\hline 1 & 2 & \cdots & \cdots & n \\ \hline 1 & \cdots & m \\ \cline{1-3}\end{array}$, $m \le n$, then

$$R_{(n,m)} \otimes R_{(1)} \simeq R_{(n-1,m)} \oplus R_{(n+1,m)} \oplus R_{(n,m-1)} \oplus R_{(n,m+1)} \oplus R_{(n,m,1)} \,, \tag{2.81}$$

with $R_{(n,m,1)}$ corresponding to the tableaux $\begin{array}{|c|c|c|c|c|}\hline 1 & 2 & \cdots & \cdots & n \\ \hline 1 & \cdots & m \\ \cline{1-3} \\ \cline{1-1}\end{array}$. For these representations

$$C_{(n,m)} = n(n+d-2)+m(m+d-4) \,, \quad C_{(n,m,1)} = n(n+d-2)+(m+1)(m+d-5) \,. \tag{2.82}$$

Hence we get, assuming $n \ge m \ge 1$,

$$\begin{aligned} &\Delta_{(n-1,m)} = n+d-2 \,, \ n > m \,, \quad &&\Delta_{(n+1,m)} = -n \,, \qquad \Delta_{(n,m,1)} = 2 \,, \\ &\Delta_{(n,m-1)} = m+d-3 \,, &&\Delta_{(n,m+1)} = -m+1 \,, \ n > m \,. \end{aligned} \tag{2.83}$$

The results (2.83) reproduce (2.77) for $n = m = 1$. For $d = 4$ in (2.81) we may identify $R_{(n,1,1)} \simeq R_{(n)}$, $R_{(n,m,1)} = 0$, $m \ge 2$ whereas for $d = 5$, $R_{(n,m,1)} \simeq R_{(n,m)}$ and then (2.83) encompasses all tensorial representations.

2.4 Two Point Functions

For any CFT the natural observables are the correlation functions for arbitrarily many conformal primary fields. Conformal invariance determines the form of the two point function in terms of the scale dimension and spin. For a conformal primary and its conjugate satisfying (2.49) and (2.53) then

$$\big\langle \phi_I(x)\, \bar\phi_{\bar I}(y) \big\rangle = \frac{\mathcal{I}_{I\bar I}(x-y)}{((x-y)^2)^\Delta} \,, \tag{2.84}$$

where conformal invariance requires

$$\begin{aligned} &\big(v^\mu(x)\partial_{x\mu} + v^\mu(y)\partial_{y\mu}\big)\, \mathcal{I}_{I\bar I}(x-y) \\ &\qquad = \tfrac{1}{2}\, \hat\omega_v{}^{\mu\nu}(x)(s_{\mu\nu})_I{}^J\, \mathcal{I}_{J\bar I}(x-y) - \mathcal{I}_{I\bar J}(x-y)\, \tfrac{1}{2}\, \hat\omega_v{}^{\mu\nu}(y)(\bar s_{\mu\nu})^{\bar J}{}_{\bar I} \,. \end{aligned} \tag{2.85}$$

$\mathcal{I}_{I\bar I}(x-y)$ is an intertwiner between a spin representation and its conjugate at x, y. There is a corresponding dual $\bar{\mathcal{I}}^{\bar I I}(y-x)$ such that

$$\mathcal{I}_{I\bar J}(x-y)\, \bar{\mathcal{I}}^{\bar J J}(y-x) = \delta_I{}^J \,, \quad \bar{\mathcal{I}}^{\bar I J}(x-y)\, \mathcal{I}_{J\bar J}(y-x) = \delta^{\bar I}{}_{\bar J} \,. \tag{2.86}$$

For general representations formed from tensor products of vectors, which are self-conjugate, then from (2.33) $\mathcal{I}_{I\bar I}$ can be constructed in terms of corresponding products of the inversion tensor. For spinors we use, from (2.11) and (2.58),

$$\begin{aligned} \big(v^\mu(x)\partial_{x\mu} + v^\mu(y)\partial_{y\mu}\big)(x-y)^\rho\gamma_\rho &= \big(v^\rho(x) - v^\rho(y)\big)\gamma_\rho \\ &= \tfrac{1}{2}\big(\partial_\sigma v^\rho(x) + \partial_\sigma v^\rho(y)\big)(x-y)^\sigma\gamma_\rho \\ &= \tfrac{1}{2}\big(\sigma_v(x) + \sigma_v(y)\big)(x-y)^\rho\gamma_\rho \\ &\qquad + \tfrac{1}{2}\, \hat\omega_v{}^{\mu\nu}(x) s_{\mu\nu}\, (x-y)^\rho\gamma_\rho - (x-y)^\rho\gamma_\rho\, \tfrac{1}{2}\, \hat\omega_v{}^{\mu\nu}(y)\bar s_{\mu\nu} \,, \end{aligned} \tag{2.87}$$

since $\big(\hat{\omega}_v{}^{\mu\nu}(x) - \hat{\omega}_v{}^{\mu\nu}(y)\big)\big(s_{\mu\nu}\,(x-y)^\rho\gamma_\rho + (x-y)^\rho\gamma_\rho\,\bar{s}_{\mu\nu}\big) = 0$, together with its conjugate with $\bar{\gamma}_\rho \to \gamma_\rho$, $s_{\mu\nu} \leftrightarrow \bar{s}_{\mu\nu}$. For a self conjugate scalar

$$\big\langle \phi(x)\,\phi(y)\big\rangle = \frac{1}{((x-y)^2)^\Delta}\,, \tag{2.88}$$

while for a real vector V_μ and spinor fields $\psi, \bar{\psi}$

$$\big\langle V_\mu(x)\,V_\nu(y)\big\rangle = \frac{I_{\mu\nu}(x-y)}{((x-y)^2)^\Delta}\,, \qquad \big\langle \psi(x)\,\bar{\psi}(y)\big\rangle = \frac{(x-y)^\rho\gamma_\rho}{((x-y)^2)^{\Delta+\frac{1}{2}}}\,. \tag{2.89}$$

Two point functions for conformal primary fields belonging to more general spinorial or tensorial representations are formed from the reduction of the tensor products of the results in (2.89).

2.5 Dirac Algebra

We analyse here in more detail the structure of Dirac gamma matrices, defined by (2.54). Various properties have significant differences according to the dimension modulo 8.

To construct expressions for gamma matrices in even or odd dimensions it is convenient to start, for $n = 1, 2, \dots,$ from $2n-1$, $2^{n-1} \times 2^{n-1}$ dimensional, generalised Pauli matrices σ_i which are hermitian and traceless and obey[3]

$$\sigma_i\,\sigma_j + \sigma_j\,\sigma_i = 2\,\delta_{ij}\,\mathbb{1}\,. \tag{2.90}$$

Defining

$$\hat{\sigma} = \sigma_1\sigma_2\dots\sigma_{2n-1} = (-1)^{n-1}\sigma_{2n-1}\sigma_{2n-2}\dots\sigma_1\,, \tag{2.91}$$

then

$$[\hat{\sigma}, \sigma_i] = 0\,, \quad \hat{\sigma}^2 = (-1)^{n-1}\,\mathbb{1} \quad \Rightarrow \quad \hat{\sigma} = \rho\,\mathbb{1}\,, \quad \rho = \pm i^{n-1}\,. \tag{2.92}$$

The two choices for ρ are inequivalent and are related by taking $\sigma_i \to -\sigma_i$. From (2.90)

$$\mathrm{tr}(\sigma_i\,\sigma_j) = 2^{n-1}\,\delta_{ij}\,. \tag{2.93}$$

The $\binom{2n-1}{s}$ matrices

$$\sigma_{i_1\dots i_s} = \sigma_{[i_1}\dots\sigma_{i_s]}\,, \qquad s = 1, 2, \dots n-1\,, \tag{2.94}$$

are all linearly independent and traceless and, together with $\mathbb{1}$, span the space of $2^{n-1} \times 2^{n-1}$ matrices. Hence if $[\sigma_i, X] = 0$ then $X \propto \mathbb{1}$.

The gamma matrices for Minkowski signature are then given by, for $d = 2n-1$ odd,

$$\gamma_0 = i\,\sigma_d\,, \qquad \gamma_i = \sigma_i\,, \quad i = 1, \dots, 2n-1\,, \qquad \mathrm{tr}(\gamma_\mu\gamma_\nu) = 2^{n-1}\,\eta_{\mu\nu}\,. \tag{2.95}$$

and, for $d = 2n$ even,

$$\gamma_0 = \bar{\gamma}_0 = i\,\mathbb{1}\,, \qquad \gamma_i = -\bar{\gamma}_i = i\,\sigma_i\,, \quad i = 1, \dots, 2n-1\,, \qquad \mathrm{tr}(\gamma_\mu\bar{\gamma}_\nu) = 2^{n-1}\,\eta_{\mu\nu}\,, \tag{2.96}$$

[3] Such matrices can be constructed iteratively. If $\sigma_i{}^{(2n-1)}$ are matrices for $d = 2n-1$ we may define $\sigma_i{}^{(2n+1)} = \sigma_i{}^{(2n-1)} \otimes \left(\begin{smallmatrix} 0 & 1 \\ 1 & 0 \end{smallmatrix}\right)$ for $i = 1, \dots, 2n-1$ and $\sigma_{2n}{}^{(2n+1)} = \mathbb{1} \otimes \left(\begin{smallmatrix} 0 & -i \\ i & 0 \end{smallmatrix}\right)$, $\sigma_{2n+1}{}^{(2n+1)} = \mathbb{1} \otimes \left(\begin{smallmatrix} 1 & 0 \\ 0 & -1 \end{smallmatrix}\right)$. Beginning from $\sigma_1{}^{(1)} = 1$ this of course gives the usual Pauli matrices for $d = 3$. With this definition in (2.92) $\rho^{(2n+1)} = i\,\rho^{(2n-1)}$.

It is easy to see that with this basis $\gamma_\mu, \bar\gamma_\mu$ satisfy (2.55) and that (2.95) obeys (2.56) with $\beta = \sigma_{2n-1}$. From (2.92)

$$\gamma_0\,\bar\gamma_1\,\gamma_2\ldots\bar\gamma_{2n-1} = \rho\,\mathbb{1}\,, \qquad \bar\gamma_0\,\gamma_1\,\bar\gamma_2\ldots\gamma_{2n-1} = -\rho\,\bar{\mathbb{1}}\,. \tag{2.97}$$

To discuss charge conjugation and time reversal it is necessary to know the properties of the Dirac matrices under transposition and complex conjugation, which are linked by hermeticity. For the $2n-1$ matrices σ_i the charge conjugation C matrix satisfies[4]

$$C\sigma_i C^{-1} = (-1)^{n-1}\sigma_i{}^T\,, \tag{2.98}$$

where the sign $(-1)^{n-1}$ is determined by requiring $C\hat\sigma C^{-1} = \hat\sigma^T$ in accord with (2.92). Since σ_i are hermitian $[\sigma_i, C^\dagger C] = 0$ and also from (2.98) $[\sigma_i, C^{-1}C^T] = 0$ so that $C^\dagger C,\, C^{-1}C^T \propto \mathbb{1}$. For $C = \tau\, C^T$ it follows that $\tau^2 = 1$ or $\tau = \pm 1$. By rescaling C we may then require

$$CC^\dagger = \mathbb{1}\,, \qquad C = (-1)^{\frac12 n(n-1)}\, C^T\,. \tag{2.99}$$

The sign for τ assumed in (2.99) is determined by counting of symmetric/antisymmetric matrices. The matrices defined in (2.94) satisfy, from (2.98),

$$C\sigma_{i_1\ldots i_s}C^{-1} = (-1)^{\frac12 s(2n-1-s)}\,\sigma_{i_1\ldots i_s}{}^T\,, \tag{2.100}$$

and the set of all matrices $\{C, C\sigma_{i_1\ldots i_s} : s = 1,\ldots,n-1\}$ form a linearly independent basis, which are alternately either symmetric or antisymmetric, for $2^{n-1}\times 2^{n-1}$ matrices. As a consequence of

$$\sum_{s=0}^{n-1}\binom{2n-1}{s} = 2^{2(n-1)}\,, \qquad \sum_{s=0}^{n-1}(-1)^{\frac12 s(2n-1-s)}\binom{2n-1}{s} = (-1)^{\frac12 n(n-1)}\,2^{n-1}\,, \tag{2.101}$$

the requirement for there to be $\frac12\,2^{n-1}(2^{n-1}\pm 1)$ symmetric/antisymmetric matrices in this basis determines the sign in (2.99) (to verify the second binomial sum we may use $(-1)^{\frac12 s(2n-1-s)} = \sqrt2\cos\frac{\pi}{4}(2s+1)$, $n = 2,4,\ldots$, $\sqrt2\sin\frac{\pi}{4}(2s+1)$, $n = 1,3,\ldots$).

These results for C immediately demonstrate that for $d = 2n-1$ from (2.95)

$$C\gamma_\mu C^{-1} = (-1)^{n-1}\,\gamma_\mu{}^T\,, \quad Cs_{\mu\nu}C^{-1} = -s_{\mu\nu}{}^T\,, \quad C = (-1)^{\frac12 n(n-1)}\,C^T\,, \quad C\beta C^{-1} = (-1)^{n-1}\,\beta^T\,. \tag{2.102}$$

For $d = 2n$ from (2.96)

$$\begin{gathered} C\gamma_\mu\bar C^{-1} = \bar\gamma_\mu{}^T\,, \quad \bar C\bar\gamma_\mu C^{-1} = \gamma_\mu{}^T\,, \quad Cs_{\mu\nu}C^{-1} = -s_{\mu\nu}{}^T\,, \quad \bar C\bar s_{\mu\nu}\bar C^{-1} = -\bar s_{\mu\nu}{}^T\,, \\ C = (-1)^{\frac12 n}\,C^T\,, \quad \bar C = (-1)^{\frac12 n}\,\bar C^T\,, \qquad n = 2,4,\ldots\,, \end{gathered} \tag{2.103}$$

and

$$C\gamma_\mu C^{-1} = \gamma_\mu{}^T\,, \quad C\bar\gamma_\mu C^{-1} = \bar\gamma_\mu{}^T\,, \quad Cs_{\mu\nu}C^{-1} = -\bar s_{\mu\nu}{}^T\,, \quad C = (-1)^{\frac12(n-1)}C^T\,, \quad n = 1,3,\ldots\,. \tag{2.104}$$

Although (2.103), for $d = 4,8,\ldots$, is derived here for $C = \bar C$ it is convenient to generalise to distinct $C, \bar C$ with $C^\dagger = \bar C^{-1}$.

For $d = 2n$ and n even then defining

$$\bar\psi^C = \bar C^{-1}\bar\psi^T\,, \qquad \psi^C = \psi^T C\,, \tag{2.105}$$

[4]For the σ_i constructed in the previous footnote we may take $C^{(2n+1)} = C^{(2n-1)}\otimes\left(\begin{smallmatrix}0&1\\-1&0\end{smallmatrix}\right)$ for $n = 1,3,\ldots$ and $C^{(2n+1)} = C^{(2n-1)}\otimes\left(\begin{smallmatrix}1&0\\0&-1\end{smallmatrix}\right)$ for $n = 2,4,\ldots$.

gives charge conjugate spinors of opposite chirality satisfying (2.64) for $\chi = \bar\psi^C$, $\bar\chi = \psi^C$. Similarly $\chi^C = \chi^T \bar C$, $\bar\chi^C = C^{-1}\bar\chi^T$ transform as ψ, $\bar\psi$ and $(\bar\psi^C)^C = (-1)^{\frac{1}{2}n}\bar\psi$, $(\psi^C)^C = (-1)^{\frac{1}{2}n}\psi$. For $d = 2n$ and n odd (2.105), with $\bar C = C$, gives spinors of the same chirality as ψ, $\bar\psi$ and in this case acting twice $(\bar\psi^C)^C = (-1)^{\frac{1}{2}(n-1)}\bar\psi$, $(\psi^C)^C = (-1)^{\frac{1}{2}(n-1)}\psi$. For $d = 2 \mod 8$ then we may impose $\psi = \bar\psi^C$ giving Majorana-Weyl spinors in these dimensions.

For Lorentz transformations we can write

$$
\begin{aligned}
&g\,\gamma_\mu\,\bar g^{-1} = \gamma_\nu\,\Lambda^\nu{}_\mu\,, \quad \bar g\,\bar\gamma_\mu\,g^{-1} = \bar\gamma_\nu\,\Lambda^\nu{}_\mu\,, \quad \bar g^{-1} = g^\dagger\,, \qquad d \text{ even}\,,\\
&g\,\gamma_\mu\,g^{-1} = \gamma_\nu\,\Lambda^\nu{}_\mu\,, \quad g^{-1} = \beta g^\dagger \beta\,, \qquad d \text{ odd}\,,
\end{aligned} \tag{2.106}
$$

The traces $\mathrm{tr}(\gamma_\mu\,\bar\gamma_\nu) = 2^{n-1}\,\eta_{\mu\nu}$ for d even, or $\mathrm{tr}(\gamma_\mu\,\gamma_\nu) = 2^{n-1}\,\eta_{\mu\nu}$ for d odd, ensure that the metric is an invariant tensor, $\eta_{\mu\nu} = \eta_{\sigma\tau}\Lambda^\sigma{}_\mu\Lambda^\tau{}_\nu$. For $d = 2n$ and n even, or d odd, $g^T C g = C$ and $\det g = \pm 1$. When $d = 4$, $C^T = -C$, this implies, for $\det g = 1$, that $g \in Sl(2,\mathbb{C})$.

2.6 Three, Four, Six and Five Dimensions

For particular low dimensions the rotation groups $SO(d)$ or $SO(d-1,1)$ are isomorphic to other groups in which tensorial and spinorial representations are unified which allows significant simplifications.

With $\alpha, \beta = 1,2$ spinorial indices for $d = 3$ there are three independent γ-matrices $(\gamma_\mu)_\alpha{}^\beta$ and also three linearly independent spin matrices $(s_{\mu\nu})_\alpha{}^\beta = -\frac{1}{2}\,(\gamma_{[\mu}\gamma_{\nu]})_\alpha{}^\beta$ which form a basis for traceless 2×2 matrices and hence

$$
-\,(s_{\mu\nu})_\alpha{}^\beta\,(s^{\mu\nu})_\gamma{}^\delta = \tfrac{1}{2}\,(\gamma_\mu)_\alpha{}^\beta\,(\gamma^\mu)_\gamma{}^\delta = \delta_\alpha{}^\delta\,\delta_\gamma{}^\beta - \tfrac{1}{2}\,\delta_\alpha{}^\beta\,\delta_\gamma{}^\delta\,, \tag{2.107}
$$

which of course implies $-\frac{1}{2}\,(s_{\mu\nu}\,s^{\mu\nu})_\alpha{}^\beta = \frac{3}{4}\,\delta_\alpha{}^\beta$ as required by (2.59) when $d = 3$. Equivalently they are a basis for symmetric 2×2 matrices since

$$
(C\gamma_\mu)^{\alpha\beta} = (C\gamma_\mu)^{\beta\alpha}\,, \quad (Cs_{\mu\nu})^{\alpha\beta} = (Cs_{\mu\nu})^{\beta\alpha}\,, \qquad C^{\alpha\beta} = -C^{\beta\alpha}\,. \tag{2.108}
$$

For any spin representation matrices $\{(s_{\mu\nu})_I{}^J\}$ we can then define an equivalent basis

$$
\big(S_\alpha{}^\beta\big)_I{}^J = (s_{\mu\nu})_I{}^J(s^{\mu\nu})_\alpha{}^\beta\,, \qquad (s_{\mu\nu})_I{}^J = -\big(S_\alpha{}^\beta\big)_I{}^J(s_{\mu\nu})_\beta{}^\alpha\,. \tag{2.109}
$$

From (2.50) using (2.107) these have the commutation relations

$$
\big[S_\alpha{}^\beta,\,S_\gamma{}^\delta\big] = \delta_\gamma{}^\beta\,S_\alpha{}^\delta - \delta_\alpha{}^\delta\,S_\gamma{}^\beta\,, \tag{2.110}
$$

which corresponds to the Lie algebra $\mathfrak{sl}_2$. For

$$
\big[S_\alpha{}^\beta\big] = \begin{pmatrix} S_3 & S_+ \\ S_- & -S_3 \end{pmatrix}\,, \tag{2.111}
$$

then $S_3, S_\pm$ obey the usual angular momentum commutation relations and the Casimir operator becomes

$$
-\,\tfrac{1}{2}\,s_{\mu\nu}\,s^{\mu\nu} = \tfrac{1}{2}\,S_\alpha{}^\beta S_\beta{}^\alpha = S_3(S_3+1) + S_-S_+\,. \tag{2.112}
$$

Irreducible representations R_s are just labelled by $s = 0, \frac{1}{2}, 1 \ldots$, and can be described in terms of the representation space V_s formed by symmetric rank $2s$ spinors $\Psi_{\alpha_1 \ldots \alpha_{2s}} = \Psi_{(\alpha_1 \ldots \alpha_{2s})}$ where of course $\dim V_s = 2s + 1$ and the associated Casimir $C_s = s(s+1)$. The vector representation requires $s = 1$. Corresponding to (2.70) we now have

$$R_s \otimes R_1 \simeq R_{s-1} \oplus R_s \oplus R_{s+1} \,, \tag{2.113}$$

and the formula (2.72) then gives for the critical scaling dimensions for the three representations appearing in (2.113)

$$\Delta_{s-1} = 1 + s \,, \ s \geq 1 \,, \qquad \Delta_s = 1 \,, \ s \geq \tfrac{1}{2} \qquad \Delta_{s+1} = -s \,. \tag{2.114}$$

The first two cases correspond to $(C\gamma^\mu \partial_\mu)^{\alpha\beta} \Psi_{\alpha\beta\alpha_1 \ldots \alpha_{2s-2}}$ and $(\gamma^\mu \partial_\mu)_{(\alpha_1}{}^{\beta} \, \Psi_{\alpha_2 \ldots \alpha_{2s})\beta}$ being conformal primaries. Imposing them to be zero gives a conserved current and a solution of the free Dirac equation respectively. For $s = \ell = 0, 1, \ldots$ the conditions in (2.114) of course match (2.80) when $d = 3$.

For $d = 4$ there are two inequivalent chiral spinors with spinorial indices $\alpha, \dot\alpha = 1, 2$. The gamma matrices then become $(\gamma_\mu)_{\alpha\dot\alpha}$, $(\bar\gamma_\mu)^{\dot\alpha\alpha}$, $(\gamma_\mu)_{\alpha\dot\alpha} (\bar\gamma^\mu)^{\dot\beta\beta} = 2 \, \delta_\beta{}^\alpha \delta^{\dot\alpha}{}_{\dot\beta}$,[5] which construct, according to (2.57), two inequivalent 2×2 spin matrices $(s_{\mu\nu})_\alpha{}^\beta$, $(\bar s_{\mu\nu})^{\dot\alpha}{}_{\dot\beta}$. These satisfy analogous completeness relations to (2.107)

$$- (s_{\mu\nu})_\alpha{}^\beta \, (s^{\mu\nu})_\gamma{}^\delta = \delta_\alpha{}^\delta \, \delta_\gamma{}^\beta - \tfrac{1}{2} \, \delta_\alpha{}^\beta \, \delta_\gamma{}^\delta \,, \quad -(\bar s_{\mu\nu})^{\dot\alpha}{}_{\dot\beta} \, (\bar s^{\mu\nu})^{\dot\gamma}{}_{\dot\delta} = \delta^{\dot\alpha}{}_{\dot\delta} \, \delta^{\dot\gamma}{}_{\dot\beta} - \tfrac{1}{2} \, \delta^{\dot\alpha}{}_{\dot\beta} \, \delta^{\dot\gamma}{}_{\dot\delta} \,, \tag{2.115}$$

as well as $(s_{\mu\nu})_\alpha{}^\beta \, (\bar s^{\mu\nu})^{\dot\alpha}{}_{\dot\beta} = 0$. The six spin matrices $(s_{\mu\nu})_I{}^J$ can be rewritten just as in (2.109)

$$\big(S_\alpha{}^\beta\big)_I{}^J = \tfrac{1}{2} \, (s_{\mu\nu})_I{}^J (s^{\mu\nu})_\alpha{}^\beta \,, \quad \big(\bar S^{\dot\alpha}{}_{\dot\beta}\big)_I{}^J = \tfrac{1}{2} \, (s_{\mu\nu})_I{}^J (\bar s^{\mu\nu})^{\dot\alpha}{}_{\dot\beta} \,. \tag{2.116}$$

and satisfy from (2.50)

$$\big[S_\alpha{}^\beta , \, S_\gamma{}^\delta\big] = \delta_\gamma^\beta \, S_\alpha{}^\delta - \delta_\alpha{}^\delta \, S_\gamma{}^\beta \,, \quad \big[\bar S^{\dot\alpha}{}_{\dot\beta} , \, \bar S^{\dot\gamma}{}_{\dot\delta}\big] = \delta^{\dot\gamma}{}_{\dot\beta} \, \bar S^{\dot\alpha}{}_{\dot\delta} - \delta^{\dot\alpha}{}_{\dot\delta} \, \bar S^{\dot\gamma}{}_{\dot\beta} \,, \quad \big[S_\alpha{}^\beta , \, \bar S^{\dot\alpha}{}_{\dot\beta}\big] = 0 \,, \tag{2.117}$$

with the quadratic Casimir becoming

$$- \tfrac{1}{2} \, s_{\mu\nu} \, s^{\mu\nu} = S_\alpha{}^\beta S_\beta{}^\alpha + \bar S^{\dot\alpha}{}_{\dot\beta} \bar S^{\dot\beta}{}_{\dot\alpha} \,. \tag{2.118}$$

The Lie algebra is then isomorphic to $\mathfrak{sl}_2 \times \mathfrak{sl}_2$. As in (2.111)

$$\big[S_\alpha{}^\beta\big] = \begin{pmatrix} S_3 & S_+ \\ S_- & -S_3 \end{pmatrix} \,, \qquad \big[\bar S^{\dot\alpha}{}_{\dot\beta}\big] = \begin{pmatrix} \bar S_3 & \bar S_+ \\ \bar S_- & -\bar S_3 \end{pmatrix} \,, \tag{2.119}$$

define two commuting sets of angular momentum generators. Irreducible representations are given by $R_{[s,\bar s]}$, $s, \bar s = 0, \frac{1}{2}, 1, \ldots$, with a representation space $V_{[s,\bar s]}$ formed by spinors $\Psi_{\alpha_1 \ldots \alpha_{2s}, \dot\alpha_1 \ldots \dot\alpha_{2\bar s})} = \Psi_{(\alpha_1 \ldots \alpha_{2s}),(\dot\alpha_1 \ldots \dot\alpha_{2\bar s})}$, so that the dimension and the Casimir eigenvalue are then

$$\dim V_{(s,\bar s)} = (2s+1)(2\bar s + 1) \,, \qquad C_{[s,\bar s]} = 2s(s+1) + 2\bar s(\bar s + 1) \,. \tag{2.120}$$

The matrix corresponding to (2.69) here takes the form

$$M_{\alpha\dot\alpha I}{}^{\dot\beta\beta J} = - \big(S_\gamma{}^\delta\big)_I{}^J \, \big(s_\delta{}^\gamma\big)_\alpha{}^\beta \, \delta^{\dot\beta}{}_{\dot\alpha} - \big(\bar S^{\dot\gamma}{}_{\dot\delta}\big)_I{}^J \, \big(\bar s^{\dot\delta}{}_{\dot\gamma}\big)^{\dot\beta}{}_{\dot\alpha} \, \delta_\alpha{}^\beta \,, \tag{2.121}$$

[5] From (2.96), $(\gamma_\mu)_{\alpha\dot\alpha} = i \, (\sigma_\mu)_{\alpha\dot\alpha}$, $(\bar\gamma_\mu)^{\dot\alpha\alpha} = i \, (\bar\sigma_\mu)^{\dot\alpha\alpha}$ where σ_μ, $\bar\sigma_\mu$ are Wess and Bagger spin matrices.

and (2.70) becomes

$$R_{[s,\bar{s}]} \otimes R_{[\frac{1}{2},\frac{1}{2}]} \simeq R_{[s-\frac{1}{2},\bar{s}-\frac{1}{2}]} \oplus R_{[s-\frac{1}{2},\bar{s}+\frac{1}{2}]} \oplus R_{[s+\frac{1}{2},\bar{s}-\frac{1}{2}]} \oplus R_{[s+\frac{1}{2},\bar{s}+\frac{1}{2}]} \,. \tag{2.122}$$

In this case the eigenvalues are just

$$\Delta_{[s',\bar{s}']} = \tfrac{1}{2}\big(C_{[s,\bar{s}]} + 3 - C_{[s',\bar{s}']}\big)\,, \tag{2.123}$$

for $s' = s \pm \frac{1}{2}$, $\bar{s}' = \bar{s} \pm \frac{1}{2}$. This gives

$$\begin{aligned} &\Delta_{[s-\frac{1}{2},\bar{s}-\frac{1}{2}]} = 2 + s + \bar{s}\,,\ \ s,\bar{s} \geq \tfrac{1}{2}\,, && \Delta_{[s+\frac{1}{2},\bar{s}+\frac{1}{2}]} = -s - \bar{s}\,, \\ &\Delta_{[s-\frac{1}{2},\bar{s}+\frac{1}{2}]} = 1 + s - \bar{s}\,,\ \ s \geq \tfrac{1}{2}\,, && \Delta_{[s+\frac{1}{2},\bar{s}-\frac{1}{2}]} = 1 + \bar{s} - s\,,\ \ \bar{s} \geq \tfrac{1}{2}\,. \end{aligned} \tag{2.124}$$

The different conditions are related by elements of the Weyl group $\mathbb{Z}_2 \times \mathbb{Z}_2$ which is here generated by the reflections $[s,\bar{s}] \to [-s-1,\bar{s}]$ and $[s,\bar{s}] \to [s,-\bar{s}-1]$. Symmetric traceless tensorial representations correspond to $s = \bar{s}$ and then (2.124) coincides with (2.80) for $[s,s] \to (2s)$. The results for mixed symmetry tensors in (2.83), excluding $\Delta_{[n,m,1]}$, correspond to taking $[s,\bar{s}] \oplus [\bar{s},s] \to (s+\bar{s}, |s-\bar{s}|)$, $s \neq \bar{s}$, $s - \bar{s} \in \mathbb{Z}$.

In six dimensions Weyl spinors ψ_α have four components and the 15 spin matrices $(s_{\mu\nu})_\alpha{}^\beta$ form a basis for traceless $4{\times}4$ matrices. In consequence they satisfy the completeness relation

$$-\,(s_{\mu\nu})_\alpha{}^\beta\,(s^{\mu\nu})_\gamma{}^\delta = 2\,\delta_\alpha{}^\delta\,\delta_\gamma{}^\beta - \tfrac{1}{2}\,\delta_\alpha{}^\beta\,\delta_\gamma{}^\delta\,, \tag{2.125}$$

so that any matrix $X = [X_\alpha{}^\beta]$ can be expressed as $X = \frac{1}{4}\,\mathrm{tr}(X)\,\mathbb{1} - \frac{1}{2}\,\mathrm{tr}(s^{\mu\nu}X)\,s_{\mu\nu}$. Hence any six dimensional spin matrix can be written in a spinorial basis, $(s_{\mu\nu})_I{}^J \to (S_\alpha{}^\beta)_I{}^J$, by

$$\big(S_\alpha{}^\beta\big)_I{}^J = \tfrac{1}{2}\,(s_{\mu\nu})_I{}^J (s^{\mu\nu})_\alpha{}^\beta\,. \tag{2.126}$$

The Lie algebra (2.50) gives

$$\big[S_\alpha{}^\beta,\, S_\gamma{}^\delta\big]_I{}^J = (s^\mu{}_\tau)_\alpha{}^\beta\,(s^{\nu\tau})_\gamma{}^\delta\,(s_{\mu\nu})_I{}^J\,. \tag{2.127}$$

With the completeness condtiion (2.125) and using $s^\mu{}_\tau\, s^{\nu\tau} = -\frac{5}{4}\,\mathbb{1} + 2\,s^{\mu\nu}$, $s^\mu{}_\tau\, s_{\sigma\rho}\, s^{\nu\tau} = \delta_{[\sigma}{}^\mu \delta_{\rho]}{}^\nu\,\mathbb{1} - \frac{1}{4}\,\eta^{\mu\nu}\,s_{\sigma\rho} - 2\,\delta_{[\sigma}{}^{(\mu} s_{\rho]}{}^{\nu)}$, (2.127) becomes

$$\big[S_\alpha{}^\beta,\, S_\gamma{}^\delta\big] = \delta_\gamma^\beta\, S_\alpha{}^\delta - \delta_\alpha^\delta\, S_\gamma{}^\beta\,, \tag{2.128}$$

which corresponds to the Lie algebra $\mathfrak{sl}_4$.

It is convenient to decompose the $\mathfrak{sl}_4$ spin generators as

$$\begin{gathered} \big[S_\alpha{}^\beta\big] = \begin{pmatrix} \frac{1}{4}(3h_1{+}2h_2{+}h_3) & e_1 & e_{12} & e_{123} \\ f_1 & \frac{1}{4}(-h_1{+}2h_2{+}h_3) & e_2 & e_{23} \\ f_{12} & f_2 & -\frac{1}{4}(h_1{+}2h_2{-}h_3) & e_3 \\ f_{123} & f_{23} & f_3 & -\frac{1}{4}(h_1{+}2h_2{+}3h_3) \end{pmatrix}, \\ e_{12} = [e_1,e_2]\,,\quad e_{23} = [e_2,e_3]\,,\quad e_{123} = [e_1,[e_2,e_3]]\,,\quad [e_1,e_3] = 0\,, \\ f_{12} = -[f_1,f_2]\,,\quad f_{23} = -[f_2,f_3]\,,\quad f_{123} = [f_1,[f_2,f_3]]\,,\quad [f_1,f_3] = 0\,. \end{gathered} \tag{2.129}$$

In this basis h_i are the generators of the Cartan subalgebra and e_i correspond to the simple roots. In general $i = 1, \dots, r$ with r the rank, here $r = 3$ and $\{e_i, f_i, h_i\}$ satisfy the commutation relations, corresponding to a Chevalley basis, for the Lie algebra,

$$[h_i, h_j] = 0\,, \quad [e_i, f_j] = \delta_{ij}\, h_j\,, \quad [h_i, e_j] = e_j K_{ji}\,, \quad [h_i, f_j] = -f_j K_{ji}\,, \quad \text{no sum on } j\,, \tag{2.130}$$

with K_{ij} defining the $r \times r$ Cartan matrix, here

$$\big[K_{ij}\big] = \begin{pmatrix} 2 & -1 & 0 \\ -1 & 2 & -1 \\ 0 & -1 & 2 \end{pmatrix} . \tag{2.131}$$

Evidently $\{h_i, e_i, f_i\}$ generate a $\mathfrak{sl}_2$ subalgebra for each i.

The representation space for an irreducible representation are generated by the action of the lowering operators $\{f_i\}$ on a highest weight vector v satisfying

$$e_i\, v = 0\,, \quad h_i\, v = s_i\, v\,, \quad i = 1, \dots, r\,, \tag{2.132}$$

giving here, for $r = 3$, a representation space $V_{[s_1,s_2,s_3]}$ and spin representation $R_{[s_1,s_2,s_3]}$. For finite dimensional representations s_i are positive integers or zero and then

$$\dim V_{[s_1,s_2,s_3]} = \tfrac{1}{12}(s_1+s_2+s_3+3)(s_1+s_2+2)(s_2+s_3+2)(s_1+1)(s_2+1)(s_3+1)\,. \tag{2.133}$$

The quadratic Casimir becomes

$$\begin{aligned} -\tfrac{1}{2}\, s_{\mu\nu}\, s^{\mu\nu} = S_\alpha{}^\beta S_\beta{}^\alpha = \tfrac{3}{4}(h_1{}^2 + h_3{}^2) + \tfrac{1}{2}\, h_1 h_3 + 3(h_1 + h_3) + h_2(h_2 + h_1 + h_3 + 4) \\ + 2(f_1 e_1 + f_2 e_2 + f_3 e_3 + f_{12} e_{12} + f_{123} e_{123})\,. \end{aligned} \tag{2.134}$$

For representations defined by a highest weight vector as in (2.132)

$$C_{[s_1,s_2,s_3]} = s_1(s_1+3) + s_3(s_3+3) - \tfrac{1}{4}(s_1 - s_3)^2 + s_2(s_2 + s_1 + s_3 + 4)\,. \tag{2.135}$$

The vector representation has a highest weight $[0,1,0]$ and it decomposes under the action of the lowering operators f_i according to the weight diagram

$$\begin{array}{ccccc} & & [-1,0,1] & & \\ & {}^{f_1}\nearrow & & \searrow_{f_3} & \\ [0,1,0] \xrightarrow{f_2} [1,-1,1] & & & & [-1,1,-1] \xrightarrow{f_2} [0,-1,0]\,. \\ & {}_{f_3}\searrow & & \nearrow^{f_2} & \\ & & [1,0,-1] & & \end{array}$$

Consequently in general

$$\begin{aligned} R_{[s_1,s_2,s_3]} \otimes R_{[0,1,0]} \simeq R_{[s_1,s_2+1,s_3]} \oplus R_{[s_1+1,s_2-1,s_3+1]} \oplus R_{[s_1-1,s_2,s_3+1]} \\ \oplus R_{[s_1+1,s_2,s_3-1]} \oplus R_{[s_1-1,s_2+1,s_3-1]} \oplus R_{[s_1,s_2-1,s_3]}\,, \end{aligned} \tag{2.136}$$

with $R_{[s_1,s_2,s_3]} \simeq 0$ if any $s_i = -1$. Using (2.72) with (2.135) and (2.136) gives the critical scaling dimensions

$$\begin{aligned} \Delta_{[s_1,s_2+1,s_3]} &= -s_2 - \tfrac{1}{2}(s_1+s_3)\,, \quad \Delta_{[s_1,s_2-1,s_3]} = 4 + s_2 + \tfrac{1}{2}(s_1+s_3)\,,\ s_2 \geq 1\,, \\ \Delta_{[s_1-1,s_2,s_3+1]} &= 2 + \tfrac{1}{2}(s_1 - s_3)\,,\ s_1 \geq 1\,, \quad \Delta_{[s_1+1,s_2,s_3-1]} = 2 - \tfrac{1}{2}(s_1 - s_3)\,,\ s_3 \geq 1\,, \\ \Delta_{[s_1+1,s_2-1,s_3+1]} &= 1 - \tfrac{1}{2}(s_1+s_3)\,,\ s_2 \geq 1\,, \quad \Delta_{[s_1-1,s_2+1,s_3-1]} = 3 + \tfrac{1}{2}(s_1+s_3)\,,\ s_1, s_3 \geq 1\,. \end{aligned} \tag{2.137}$$

The Weyl group for $\mathfrak{sl}_4$ is the permutation group $\mathcal{S}_4$ and is generated by reflections r_i, where $r_i^2 = 1$, with respect to the simple roots, $[s_1, s_2, s_3]^{r_1} = [-s_1 - 2, s_1 + s_2 + 1, s_3]$, $[s_1, s_2, s_3]^{r_2} = [s_1 + s_2 + 1, -s_2 - 2, s_2 + s_3 + 1]$, $[s_1, s_2, s_3]^{r_3} = [s_1, s_2 + s_3 + 1, -s_3 - 2]$, $(r_1 r_3)^2 = (r_1 r_2)^3 = (r_2 r_3)^3 = 1$. Symmetric traceless rank ℓ tensors correspond to $[s_1, s_2, s_3] = [0, \ell, 0]$ and the results (2.83) for mixed symmetry (n, m) tensors correspond to $[s_1, s_2, s_3] = [m, n - m, m]$.

In five dimensions spinors ψ_α also have four components while the 10 spin matrices $(s_{\mu\nu} C^{-1})_{\alpha\beta}$, $(C s_{\mu\nu})^{\alpha\beta}$ are symmetric with the completeness relation

$$- (s_{\mu\nu} C^{-1})_{\alpha\beta} \, (C s^{\mu\nu})^{\gamma\delta} = \delta_\alpha{}^\gamma \, \delta_\beta{}^\delta + \delta_\alpha{}^\delta \, \delta_\beta{}^\gamma \, . \tag{2.138}$$

Hence we can express spin matrices in a spinorial basis by

$$\big(S_\alpha{}^\beta\big)_I{}^J = (s_{\mu\nu})_I{}^J (s^{\mu\nu})_\alpha{}^\beta \, , \tag{2.139}$$

The commutation relations (2.50) require

$$\big[S_\alpha{}^\beta , \, S_\gamma{}^\delta\big]_I{}^J = 4 \, (s^\mu{}_\tau)_\alpha{}^\beta \, (s^{\nu\tau})_\gamma{}^\delta \, (s_{\mu\nu})_I{}^J \, , \tag{2.140}$$

where

$$4 \, (s^{[\mu}{}_\tau)_\alpha{}^\beta \, (s^{\nu]\tau})_\gamma{}^\delta = \delta_\gamma{}^\beta \, (s^{\mu\nu})_\alpha{}^\delta - \delta_\alpha{}^\delta \, (s^{\mu\nu})_\gamma{}^\beta + C^{-1}{}_{\alpha\gamma} \, (C s^{\mu\nu})^{\beta\delta} + C^{\beta\delta} \, (s^{\mu\nu} C^{-1})_{\alpha\gamma} \, . \tag{2.141}$$

The form of the right hand side of (2.141) is dictated by the completeness of the symmetric matrices $s^{\mu\nu} C^{-1}$, and also that the relation must be invariant under $\gamma_\mu \to -\gamma_\mu$, and the coefficient may be determined from $s^\mu{}_\tau \, s^{\nu\tau} = -\eta^{\mu\nu} \, \mathbb{1} + \frac{3}{2} \, s^{\mu\nu}$. As a consequence of (2.141) (2.140) becomes

$$\big[S_\alpha{}^\beta , \, S_\gamma{}^\delta\big] = \delta_\gamma{}^\beta \, S_\alpha{}^\delta - \delta_\alpha{}^\delta \, S_\gamma{}^\beta + C^{-1}{}_{\alpha\gamma} \, (CS)^{\beta\delta} + C^{\beta\delta} \, (S C^{-1})_{\alpha\gamma} \, , \tag{2.142}$$

which defines the Lie algebra $\mathfrak{sp}(4)$.

Choosing a basis such that

$$\big[C^{-1}{}_{\alpha\beta}\big] = \begin{pmatrix} 0 & \mathbb{1} \\ -\mathbb{1} & 0 \end{pmatrix} , \qquad \big[C^{\alpha\beta}\big] = \begin{pmatrix} 0 & -\mathbb{1} \\ \mathbb{1} & 0 \end{pmatrix} , \tag{2.143}$$

then

$$\big[S_\alpha{}^\beta\big] = \begin{pmatrix} h_1 + h_2 & e_1 & e_{112} & e_{12} \\ f_1 & h_2 & e_{12} & 2e_2 \\ f_{112} & f_{12} & -h_1 - h_2 & -f_1 \\ f_{12} & 2f_2 & -e_1 & -h_2 \end{pmatrix} , \tag{2.144}$$

$$e_{12} = [e_1, e_2] \, , \quad e_{112} = [e_1, [e_1, e_2]] \, , \quad f_{12} = -[f_1, f_2] \, , \quad f_{112} = [f_1, [f_1, f_2]] \, . \tag{2.145}$$

In this basis e_i correspond to the simple roots with h_i the Cartan generators so that $\{e_i, f_i, h_i\}$ satisfy the commutation relations (2.130) for rank $r = 2$ and Cartan matrix

$$\big[K_{ij}\big] = \begin{pmatrix} 2 & -1 \\ -2 & 2 \end{pmatrix} . \tag{2.146}$$

As before the representation space $V_{[s_1,s_2]}$ for irreducible representations $R_{[s_1,s_2]}$ may be defined in terms of highest weight vectors satisfying (2.132). For finite dimensional representations, so that s_i are positive integers or zero,

$$\dim V_{[s_1,s_2]} = \tfrac{1}{6}(s_1+1)(s_2+1)(s_1+s_2+1)(s_1+2s_2+3)\,. \tag{2.147}$$

The quadratic Casimir is also

$$\begin{aligned} -\tfrac{1}{2}\, s_{\mu\nu}\, s^{\mu\nu} = \tfrac{1}{4}\, S_{\alpha\beta} S^{\alpha\beta} = {} & \tfrac{1}{2}\, h_1(h_1+4) + h_2(h_2+h_1+3) \\ & + f_1\, e_1 + 2\, f_2\, e_2 + f_{12}\, e_{12} + \tfrac{1}{2}\, f_{112}\, e_{112}\,, \end{aligned} \tag{2.148}$$

and therefore acting on $V_{[s_1,s_2]}$ the Casimir eigenvalue is

$$C_{[s_1,s_2]} = \tfrac{1}{2}\, s_1(s_1+4) + s_2(s_2+s_1+3)\,. \tag{2.149}$$

The vector representation is labelled by $[0,1]$ and the associated weight diagram becomes

$$[0,1] \xrightarrow{f_2} [2,-1] \xrightarrow{f_1} [0,0] \xrightarrow{f_1} [-2,1] \xrightarrow{f_2} [0,-1]\,. \tag{2.150}$$

Hence

$$R_{[s_1,s_2]} \otimes R_{[0,1]} \simeq R_{[s_1,s_2+1]} \oplus R_{[s_1+2,s_2-1]} \oplus R_{[s_1,s_2]} \oplus R_{[s_1-2,s_2+1]} \oplus R_{[s_1,s_2-1]}\,. \tag{2.151}$$

For $s_1 = 0$ the tensor product is truncated since $R_{[-2,s_2+1]} \simeq -R_{[0,s_2]}$ and also we take $R_{[-1,s_2]}, R_{[s_1,-1]} \simeq 0$. Implementing (2.72) gives

$$\begin{aligned} &\Delta_{[s_1,s_2+1]} = -\, s_2 - \tfrac{1}{2}\, s_1\,, \quad \Delta_{[s_1+2,s_2-1]} = 1 - \tfrac{1}{2}\, s_1\,,\ s_2 \ge 1\,, \quad \Delta_{[s_1,s_2]} = 2\,,\ s_1 \ge 1\,, \\ &\Delta_{[s_1-2,s_2+1]} = 2 + \tfrac{1}{2}\, s_1\,,\ s_1 \ge 2\,, \quad \Delta_{[s_1,s_2-1]} = 3 + s_2 + \tfrac{1}{2}\, s_1\,,\ s_2 \ge 1\,, \end{aligned} \tag{2.152}$$

for the critical dimensions when derivatives generate conformal primaries. The Weyl group for $\mathfrak{sp}(4)$, $D_4 \simeq \mathbb{Z}_2 \ltimes \mathbb{Z}_4$, is generated by reflections r_1, r_2 with respect to the short and long simple roots which give $[s_1,s_2]^{r_1} = [-s_1-2, s_1+s_2+1]$ and $[s_1,s_2]^{r_2} = [s_1+2s_2+2, -s_2-2]$, $(r_1\, r_2)^4 = 1$, links the different conditions in (2.152). Symmetric traceless rank ℓ tensors correspond to $[s_1,s_2] = [0,\ell]$ and for mixed symmetry $[n,m]$-tensors $[s_1,s_2] = [2m, n-m]$.

3 Embedding Space

The action of conformal transformations in d-dimensions on $x \in \mathbb{R}^d$ is nonlinear, as exemplified in (2.9) or (2.30). By extending to $X \in \mathbb{R}^{d+2}$ it is possible to define linear group transformations which reduce to conformal transformations under appropriate restrictions, [14, 15]. Defining coordinates

$$X^A\,, \quad A = 0,1,\dots,d-1,d+1,d+2\,, \tag{3.1}$$

then a d-dimensional space is obtained by imposing

$$\begin{aligned} 0 = \eta_{AB} X^A X^B &= \eta_{\mu\nu} X^\mu X^\nu + (X^{d+1})^2 - (X^{d+2})^2 \\ &= \eta_{\mu\nu} X^\mu X^\nu + X^+ X^-\,, \qquad X^\pm = X^{d+1} \pm X^{d+2}\,, \end{aligned} \tag{3.2}$$

and also requiring that the overall scale of X is arbitrary so that

$$X^A \sim \lambda\, X^A\,. \tag{3.3}$$

With these conditions $\{X^A\}$ are coordinates for a d-dimernsional projective null cone embedded in $\mathbb{R}^{d+2}$. The natural isometry group is clearly by

$$X^A \to G^A{}_B X^B\,, \quad \eta_{CD}\, G^C{}_A G^D{}_B = \eta_{AB} \quad \Rightarrow \quad \big[G^A{}_B\big] \in O(d+1,1) \text{ or } O(d,2)\,. \tag{3.4}$$

To make the connection with conformal transformations on x we define for $X^+ \neq 0$

$$x^\mu = \frac{X^\mu}{X^+}\,, \tag{3.5}$$

which is well defined since this is invariant under (3.3). Conversely

$$x^\mu \to X^A(x) \;\text{ for }\; (X^\mu, X^+, X^-) = X^+(x^\mu, 1, -x^2)\,, \quad X^+ \neq 0\,. \tag{3.6}$$

From (3.2)

$$2\,\eta_{\mu\nu} X^\mu \mathrm{d}X^\nu + \mathrm{d}X^+ X^- + X^+ \mathrm{d}X^- = 0\,, \tag{3.7}$$

and using this relation

$$\begin{aligned}\eta_{AB}\mathrm{d}X^A\mathrm{d}X^B &= (X^+)^2\,\eta_{\mu\nu}\mathrm{d}x^\mu\mathrm{d}x^\nu + 2\,\eta_{\mu\nu}X^\mu\mathrm{d}X^\nu\,\frac{\mathrm{d}X^+}{X^+} + \eta_{\mu\nu}X^\mu X^\nu\left(\frac{\mathrm{d}X^+}{X^+}\right)^2 + \mathrm{d}X^+\mathrm{d}X^- \\ &= (X^+)^2\,\eta_{\mu\nu}\mathrm{d}x^\mu\mathrm{d}x^\nu\,.\end{aligned} \tag{3.8}$$

Hence for any transformation $X^A \to X'^A$ as in (3.4) $\eta_{AB}\mathrm{d}X^A\mathrm{d}X^B$ is invariant and with $X'^A \to x'^\mu$, $X^A \to x^\mu$ defined by (3.5)

$$\eta_{\mu\nu}\mathrm{d}x'^\mu\mathrm{d}x'^\nu = \left(\frac{X^+}{X'^+}\right)^2 \eta_{\mu\nu}\mathrm{d}x^\mu\mathrm{d}x^\nu\,, \tag{3.9}$$

which demonstrates that $x^\mu \to x'^\mu$ determined by $[G^A{}_B]$ is a conformal transformation. Furthermore since $\dim O(d+1,1) = \dim O(d,2) = \frac{1}{2}(d+1)(d+2)$ the number of parameters match.

As particular cases we have the particular matrices

$$\big[G_t(a)^A{}_B\big] = \begin{pmatrix} \delta^\mu{}_\nu & a^\mu & a^\mu \\ -a_\nu & 1-\frac{1}{2}a^2 & -\frac{1}{2}a^2 \\ a_\nu & \frac{1}{2}a^2 & 1+\frac{1}{2}a^2 \end{pmatrix}\,, \quad \big[G_s(b)^A{}_B\big] = \begin{pmatrix} \delta^\mu{}_\nu & -b^\mu & b^\mu \\ b_\nu & 1-\frac{1}{2}b^2 & \frac{1}{2}b^2 \\ b_\nu & -\frac{1}{2}b^2 & 1+\frac{1}{2}b^2 \end{pmatrix}\,, \tag{3.10}$$

which correspond to translations, $x^\mu \to x^\mu + a^\mu$ and special conformal transformations as in (2.21) or (2.30). For scale tansformations and inversions

$$\big[G_d(\kappa)^A{}_B\big] = \begin{pmatrix} \delta^\mu{}_\nu & 0 & 0 \\ 0 & \cosh\kappa & -\sinh\kappa \\ 0 & -\sinh\kappa & \cosh\kappa \end{pmatrix}\,, \quad \big[G_I{}^A{}_B\big] = \begin{pmatrix} \delta^\mu{}_\nu & 0 & 0 \\ 0 & -1 & 0 \\ 0 & 0 & 1 \end{pmatrix}\,. \tag{3.11}$$

It is easy to see that $G_s(b) = G_I G_t(a) G_I$. The transformation given in (2.48) connected to the identity is just a rotation

$$X_\theta{}^{d+1} = \cos\theta\, X^{d+1} + \sin\theta\, X^1\,, \quad X_\theta{}^1 = \cos\theta\, X^1 - \sin\theta\, X^{d+1}\,, \tag{3.12}$$

with $\lambda = 2\tan\frac{1}{2}\theta$ which clearly demonstrated the simplicity of the embedding formalism.

From (3.6) we may define

$$e_\mu{}^A(x) = \frac{\partial}{\partial x^\mu} X^A(x)\,, \qquad \frac{\partial^2}{\partial x^\mu \partial x^\nu} X^A(x) = \eta_{\mu\nu} P^A\,, \tag{3.13}$$

and then $e_\mu{}^A(x) X_A(x) = 0$, $P^A X_A(x) == -(X^+)^2$. With these definitions

$$\begin{aligned} &e_\mu{}^A(x) e_\nu{}^B(x) \eta_{AB} = (X^+)^2 \eta_{\mu\nu}\,, \\ &\eta^{\mu\nu} e_\mu{}^A(x) e_\nu{}^B(x) = (X^+)^2 \eta^{AB} + X^A(x) P^B + P^A X^B(x)\,, \quad \partial_\mu e_\nu{}^A(x) = \eta_{\mu\nu} P^A\,. \end{aligned} \tag{3.14}$$

and

$$P^A = -2X^+\, \delta_-{}^A\,. \tag{3.15}$$

For two points X^A, Y^B on the null cone if

$$X \cdot Y \equiv -2\,\eta_{AB} X^A Y^B\,, \tag{3.16}$$

then

$$X \cdot Y = (x-y)^2\, X^+ Y^+\,. \tag{3.17}$$

Although $X \cdot Y$ is invariant under (3.4) it is of course not an invariant on the projective null cone. Such invariants need four points as in (2.25).

3.1 Scalar and Vector Fields on the Null Cone

Fields may also be extended to the null cone. For a scalar we may take

$$\phi(x) \to \Phi(X)\,, \tag{3.18}$$

with X satisfying (3.8), and to ensure it is defined on the projective null cone so that (3.3) holds then Φ is required to be homogeneous,

$$\Phi(\lambda X) = \lambda^{-\Delta}\, \Phi(X)\,, \tag{3.19}$$

where the scale dimension Δ determines the weight. For any such $\Phi(X)$ satisfying (3.19) then conversely $\Phi \to \phi$ by taking

$$\phi(x) = (X^+)^\Delta\, \Phi(X(x))\,. \tag{3.20}$$

Correspondingly for a vector field

$$V_\mu(x) \to V_A(X)\,, \qquad V_A(\lambda X) = \lambda^{-\Delta}\, V_A(X)\,, \tag{3.21}$$

but to reduce the spin degrees of freedom from $d+2$ to d it is also necessary to require

$$X^A V_A(X) = 0\,, \qquad V_A(X) \sim V_A(X) + X_A\, s(X)\,, \tag{3.22}$$

for arbitrary $s(X)$, $s(\lambda X) = \lambda^{-\Delta-1}\, s(X)$. The constraint and freedom in (3.22) reduce the degrees of freedom from $d+2$ to d. In this case V_μ is then given, with $e_\mu{}^A$ defined in (3.13), by

$$V_\mu(x) = (X^+)^{\Delta-1} e_\mu{}^A(x)\, V_A(X(x))\,. \tag{3.23}$$

Since $e_\mu{}^A(x) X_A(x) = 0$ it is easy to see that V_μ is invariant under the equivalence relation in (3.22) and also $X^A \to \lambda(x) X^A$ for arbitrary $\lambda(x)$.

Extending (3.14) to tensor fields there is a correspondence $T_{\mu\nu}(x) \to T_{AB}(X)$, where, with the obvious modification of(3.22), $X^A T_{AB} = 0$, $X^B T_{AB} = 0$ and $T_{AB} \sim T_{AB} + X_A v_B + v'_A X_B$ for arbitrary v_B, v'_A of the necessary homogeneity. As a consequence of (3.14) the traceless condition $\eta^{AB} T_{AB}(X) = 0$ is equivalent to $\eta^{\mu\nu} T_{\mu\nu}(x) = 0$.

For infinitesimal conformal transformations the action on the fields is just, for arbitrary $\omega^{AB} = -\omega^{BA}$,

$$\begin{aligned} \delta_\omega \Phi &= \tfrac{1}{2}\, \omega^{AB} L_{AB} \Phi\,, \qquad L_{AB} = -X_A \partial_B + X_B \partial_A\,, \\ \delta_\omega V_C &= \tfrac{1}{2}\, \omega^{AB} L_{AB} V_C - \omega_C{}^D\, V_D\,. \end{aligned} \tag{3.24}$$

The conformal algebra is then

$$\big[L_{AB},\, L_{CD}\big] = \eta_{AC}\, L_{BD} - \eta_{BC}\, L_{AD} - \eta_{AD}\, L_{BC} + \eta_{BD}\, L_{AC}\,. \tag{3.25}$$

Hence $\big[\tfrac{1}{2}\, \omega^{AB} L_{AB},\, \tfrac{1}{2}\, \omega'^{CD} L_{CD}\big] = -\tfrac{1}{2}\, [\omega, \omega']^{AB} L_{AB}$, where $[\omega, \omega']^{AB} = \omega^{AC} \omega'_C{}^D - \omega'^{AC} \omega_C{}^D$, and $[\delta_\omega, \delta_{\omega'}] = \delta_{[\omega,\omega']}$. Consistency with (3.22) follows from $\tfrac{1}{2}\, \omega^{AB} L_{AB} X^C = -X^A \omega_A{}^C$.

Since X is null it is necessary to be careful in the definition of ∂_A. For derivatives ∂_A acting on $f(X)$ restricted to the null cone $X^2 = 0$ the usual rules of differentiation lead to apparent inconsistencies as $\partial_A X^2 = 2\, X_A$ so that it is necessary to include additional contributions proportional to X_A beyond the result obtained by differentiation disregarding the constraint $X^2 = 0$. These extra terms disappear in L_{AB} and

$$\begin{aligned} L_{+-} X^A\big|_{X \to X(x)} &= -\tfrac{1}{2} (x^\mu \partial_\mu - X^+ \partial_+) X^A(x)\,, \quad L_{\mu\nu} X^A\big|_{X \to X(x)} = -(x_\mu \partial_\nu - x_\nu \partial_\mu) X^A(x)\,, \\ L_{-\mu} X^A\big|_{X \to X(x)} &= -\tfrac{1}{2}\, \partial_\mu X^A(x)\,, \\ L_{+\mu} X^A\big|_{X \to X(x)} &= \tfrac{1}{2} (-x^2 \partial_\mu + 2\, x_\mu\, x^\nu \partial_\nu - 2\, x_\mu X^+ \partial_+) X^A(x)\,. \end{aligned} \tag{3.26}$$

With these results $\delta_\omega \Phi$ coincides under the reduction (3.20) with $\delta_v \phi$ in (2.49) for v^μ given by (2.9) if $\omega^{-\mu} = 2\, a^\mu$, $\omega^{+-} = 2\, \kappa$, $\omega^{+\mu} = -2\, b^\mu$. For the reduction $\delta_\omega V_C \to \delta_v V_\mu$ it is necessary also to use $-v^\nu \partial_\nu\, e_\mu{}^C + e_\mu{}^D \omega_D{}^C = \hat\omega_\mu{}^\nu e_\nu{}^C + 2\, b_\mu X^C(x)$.

3.2 Spinors

To discuss spinor fields in the embedding formalism requires extending the usual d-dimensional gamma matrices to $d+2$ dimensions. For $d = 2n$ these are required to satisfy the Dirac algebra

$$\bar\Gamma_A\, \Gamma_B + \bar\Gamma_B\, \Gamma_A = 2\, \eta_{AB}\, \mathbb{1}\,, \qquad \Gamma_A\, \bar\Gamma_B + \Gamma_B\, \bar\Gamma_A = 2\, \eta_{AB}\, \mathbb{1}\,, \tag{3.27}$$

and may be obtained from $\gamma_\mu, \bar\gamma_\mu$ satisfying (2.54) by taking

$$\begin{gathered}\Gamma_\mu = \begin{pmatrix} \gamma_\mu & 0 \\ 0 & -\bar\gamma_\mu \end{pmatrix} , \qquad\qquad \bar\Gamma_\mu = \begin{pmatrix} \bar\gamma_\mu & 0 \\ 0 & -\gamma_\mu \end{pmatrix} , \\ \Gamma_{d+1} = \bar\Gamma_{d+1} = \begin{pmatrix} 0 & \mathbb{1} \\ \mathbb{1} & 0 \end{pmatrix} , \qquad \Gamma_{d+2} = \bar\Gamma_{d+2} = \begin{pmatrix} 0 & \mathbb{1} \\ -\mathbb{1} & 0 \end{pmatrix} ,\end{gathered} \tag{3.28}$$

and correspondingly

$$\Gamma_+ = \bar\Gamma_+ = \begin{pmatrix} 0 & \mathbb{1} \\ 0 & 0 \end{pmatrix} , \qquad \Gamma_- = \bar\Gamma_- = \begin{pmatrix} 0 & 0 \\ \mathbb{1} & 0 \end{pmatrix} . \tag{3.29}$$

Assuming (2.55) we may then require

$$\bar\Gamma_A = \mathcal{B}\,\Gamma_A{}^\dagger \mathcal{B} , \qquad \mathcal{B} = \begin{pmatrix} 0 & \mathbb{1} \\ \mathbb{1} & 0 \end{pmatrix} . \tag{3.30}$$

For $d = 2n - 1$ we have $\Gamma_A \Gamma_B + \Gamma_B \Gamma_A = 2\,\eta_{AB}\,\mathbb{1}$ and a representation is given by (3.28) with $\bar\gamma_\mu = \gamma_\mu$ and in this case from (2.56) with a Minkowski metric

$$\Gamma_A = \mathcal{B}\,\Gamma_A{}^\dagger\,\mathcal{B} , \qquad \mathcal{B} = \mathcal{B}^\dagger = \mathcal{B}^{-1} = \begin{pmatrix} 0 & \beta \\ \beta & 0 \end{pmatrix} . \tag{3.31}$$

For conformal transformations as in (3.4) when $d = 2n$

$$\mathcal{G}\,\Gamma_A\,\bar{\mathcal{G}}^{-1} = \Gamma_B\, G^B{}_A , \qquad \bar{\mathcal{G}}\,\bar\Gamma_A\,\mathcal{G}^{-1} = \bar\Gamma_B\, G^B{}_A , \tag{3.32}$$

where from (3.31)

$$\mathcal{G}^{-1} = \mathcal{B}\,\mathcal{G}^\dagger\,\mathcal{B} , \quad \bar{\mathcal{G}}^{-1} = \mathcal{B}\,\bar{\mathcal{G}}^\dagger\,\mathcal{B} , \quad \Rightarrow \quad \mathcal{G}\,\mathcal{B}\,\mathcal{G}^\dagger = \mathcal{B} , \qquad \bar{\mathcal{G}}\,\mathcal{B}\,\bar{\mathcal{G}}^\dagger = \mathcal{B} . \tag{3.33}$$

Alternatively for $d = 2n - 1$

$$\mathcal{G}\,\Gamma_A\,\mathcal{G}^{-1} = \Gamma_B\, G^B{}_A , \qquad \mathcal{G}^{-1} = \mathcal{B}\,\mathcal{G}^\dagger \mathcal{B} . \tag{3.34}$$

The associated conformal generators are defined similarly to (2.57)

$$S_{AB} = -\tfrac{1}{2}\,\Gamma_{[A}\,\bar\Gamma_{B]} , \qquad \bar S_{AB} = -\tfrac{1}{2}\,\bar\Gamma_{[A}\,\Gamma_{B]} , \tag{3.35}$$

and satisfy, analogously to (2.58),

$$S_{AB}\,\Gamma_C - \Gamma_C\,\bar S_{AB} = \eta_{AC}\,\Gamma_B - \eta_{BC}\,\Gamma_A , \qquad \bar S_{AB}\,\bar\Gamma_C - \bar\Gamma_C\,S_{AB} = \eta_{AC}\,\bar\Gamma_B - \eta_{BC}\,\bar\Gamma_A , \tag{3.36}$$

so that S_{AB}, $\bar S_{AB}$ have the same Lie algebra as L_{AB} in (3.25).

From (2.102) when $d = 2n - 1$

$$\mathcal{C}\,\Gamma_A\,\mathcal{C}^{-1} = (-1)^n\,\Gamma_A{}^T , \quad \mathcal{C}\,\mathcal{B}\,\mathcal{C}^{-1} = -\mathcal{B}^T , \qquad \mathcal{C} = \begin{pmatrix} 0 & C \\ (-1)^n C & 0 \end{pmatrix} , \tag{3.37}$$

where $\mathcal{C} = (-1)^{\frac{1}{2}n(n+1)}\,\mathcal{C}^T$ and $\mathcal{C}^\dagger\mathcal{C} = \mathbb{1}$, For $d = 2n$ from (2.103), (2.104)

$$\begin{aligned} \mathcal{C}\,\Gamma_A\,\bar{\mathcal{C}}^{-1} = {\Gamma_A}^T\,, \quad \bar{\mathcal{C}}\,\bar\Gamma_A\,\mathcal{C}^{-1} = {\bar\Gamma_A}{}^T\,, \quad \mathcal{C}\,\mathcal{B}\,\mathcal{C}^\dagger = \mathcal{B}\,, \quad n = 2,4,\dots\,, \\ \mathcal{C}\,\Gamma_A\,\mathcal{C}^{-1} = -{\bar\Gamma_A}{}^T\,, \quad \mathcal{C}\,\bar\Gamma_A\,\mathcal{C}^{-1} = -{\Gamma_A}^T\,, \quad \mathcal{C}\,\mathcal{B}\,\mathcal{C}^\dagger = -\mathcal{B}\,, \quad n = 1,3,\dots\,, \end{aligned} \tag{3.38}$$

where with the representation (3.28)

$$\mathcal{C} = \begin{pmatrix} 0 & \bar C \\ -C & 0 \end{pmatrix}, \quad \bar{\mathcal{C}} = \begin{pmatrix} 0 & -C \\ \bar C & 0 \end{pmatrix}, \quad n = 2,4,\dots\,, \qquad \mathcal{C} = \begin{pmatrix} 0 & C \\ -C & 0 \end{pmatrix}, \quad n = 1,3,\dots\,, \tag{3.39}$$

where $\mathcal{C} = (-1)^{\frac{1}{2}n}\bar{\mathcal{C}}^T, (-1)^{\frac{1}{2}(n+1)}\mathcal{C}^T$ respectively.

In (3.32) we may then take

$$\mathcal{C}^{-1}\,\mathcal{G}^T\,\mathcal{C} = \bar{\mathcal{G}}^{-1}\,, \ n = 2,4,\dots\,, \quad \mathcal{C}^{-1}\,\mathcal{G}^T\,\mathcal{C} = \mathcal{G}^{-1}\,, \ \mathcal{C}^{-1}\,\bar{\mathcal{G}}^T\,\mathcal{C} = \bar{\mathcal{G}}^{-1}\,, \ n = 1,3,\dots\,, \tag{3.40}$$

and in (3.34)

$$\mathcal{C}^{-1}\,\mathcal{G}^T\,\mathcal{C} = \mathcal{G}^{-1} \quad \Rightarrow \quad \mathcal{G}^T\,\mathcal{C}\,\mathcal{G} = \mathcal{C}\,. \tag{3.41}$$

For spinors an extension to corresponding spinor fields defined on the null cone and transforming covariantly

$$\psi(x) \to \Psi(X)\,, \ \bar\psi(x) \to \bar\Psi(X)\,, \quad \Psi(\lambda X) = \lambda^{-\Delta+\frac{1}{2}}\Psi(X)\,, \ \bar\Psi(\lambda X) = \lambda^{-\Delta+\frac{1}{2}}\bar\Psi(X)\,, \tag{3.42}$$

then requires doubling the number of spinor components. Assuming (3.30), or (3.31), requires $\bar\Psi = \Psi^\dagger\mathcal{B}$. The degrees of freedom of $\Psi, \bar\Psi$ are reduced to those for $\psi, \bar\psi$ by imposing

$$\bar\Gamma_A X^A\,\Psi(X) = 0\,, \qquad \bar\Psi(X)\,\Gamma_A X^A = 0\,. \tag{3.43}$$

With these conditions in the representation provided by (3.28),

$$\bar\Gamma_A X^A = \begin{pmatrix} \bar\gamma\cdot X & X^+ \\ X^- & -\gamma\cdot X \end{pmatrix}, \qquad \Gamma_A X^A = \begin{pmatrix} \gamma\cdot X & X^+ \\ X^- & -\bar\gamma\cdot X \end{pmatrix}. \tag{3.44}$$

the converse to (3.42), $\Psi(X) \to \psi(x)$, $\bar\Psi(X) \to \bar\psi(x)$, is given by

$$\begin{pmatrix} \psi(x) \\ -\bar\gamma\cdot x\,\psi(x) \end{pmatrix} = (X^+)^{\Delta-\frac{1}{2}}\,\Psi(X(x))\,, \quad \big(\bar\psi(x)\,\bar\gamma\cdot x \ \ \bar\psi(x)\big) = (X^+)^{\Delta-\frac{1}{2}}\,\bar\Psi(X(x))\,, \tag{3.45}$$

so that

$$\psi(x) = (X^+)^{\Delta-\frac{1}{2}}\,\bar\epsilon(x)\Psi(X(x))\,, \qquad \bar\psi(x) = (X^+)^{\Delta-\frac{1}{2}}\,\bar\Psi(X(x))\,\epsilon(x)\,. \tag{3.46}$$

with $\bar\epsilon$ defined by

$$e_\mu{}^A(x)\,\bar\epsilon(x)\Gamma_A = \gamma_\mu\,\bar\epsilon'(x)\,, \quad e_\mu{}^A(x)\,\bar\epsilon'(x)\bar\Gamma_A = \bar\gamma_\mu\,\bar\epsilon(x)\,, \quad \bar\epsilon(x)\Gamma_- = \bar\epsilon'(x)\bar\Gamma_- = 0\,, \tag{3.47}$$

and similarly for ϵ.

Alternatively to (3.42) spinor fields may be extended to the null cone as

$$\psi(x) \to \Psi'(X)\,, \quad \bar\psi(x) \to \bar\Psi'(X)\,, \quad \Psi'(\lambda X) = \lambda^{-\Delta-\frac{1}{2}}\Psi'(X)\,, \quad \bar\Psi'(\lambda X) = \lambda^{-\Delta-\frac{1}{2}}\bar\Psi'(X)\,, \tag{3.48}$$

where the degrees of freedom are now halved by imposing the equivalence relations

$$\Psi'(X) \sim \Psi'(X) + \bar{\Gamma}_A X^A \zeta(X)\,, \qquad \bar{\Psi}'(X) \sim \bar{\Psi}'(X) + \bar{\zeta}(X)\,\Gamma_A X^A\,, \tag{3.49}$$

for arbitrary spinors $\zeta(X)$, $\bar{\zeta}(X)$ of appropriate homogeneity. The equivalence to (3.42) is obtained by taking

$$\Psi(X) = \Gamma_A X^A \Psi'(X)\,, \qquad \bar{\Psi}(X) = \bar{\Psi}'(X)\bar{\Gamma}_A X^A\,. \tag{3.50}$$

The infinitesimal conformal transformation in (3.24) extends to spinor fields by taking

$$\delta_\omega \Psi = \tfrac{1}{2}\,\omega^{AB}\big(L_{AB} + S_{AB}\big)\Psi\,, \qquad \delta_\omega \bar{\Psi} = \tfrac{1}{2}\,\omega^{AB}\big(L_{AB}\bar{\Psi} - \bar{\Psi}\,S_{AB}\big)\,. \tag{3.51}$$

Under the reduction $\Psi \to \psi$, $\bar{\Psi} \to \bar{\psi}$ given by (3.45) $\frac{1}{2}\,\omega^{AB}\,S_{AB}\Psi \to \frac{1}{2}\,\omega_v{}^{\mu\nu} s_{\mu\nu}\psi - \frac{1}{2}\,\sigma_v\,\psi$, $\frac{1}{2}\,\omega^{AB}\,\bar{\Psi}\,S_{AB} \to \frac{1}{2}\,\omega_v{}^{\mu\nu}\bar{\psi}\,\bar{s}_{\mu\nu} + \frac{1}{2}\,\sigma_v\,\bar{\psi}$. Corresponding to (3.48)

$$\delta_\omega \Psi' = \tfrac{1}{2}\,\omega^{AB}\big(L_{AB} + \bar{S}_{AB}\big)\Psi'\,, \qquad \delta_\omega \bar{\Psi}' = \tfrac{1}{2}\,\omega^{AB}\big(L_{AB}\bar{\Psi}' - \bar{\Psi}'\,\bar{S}_{AB}\big)\,. \tag{3.52}$$

These transformations show that

$$V_A(X) = \bar{\Psi}(X)\Gamma_A\Psi'(X)\,, \tag{3.53}$$

transforms as a vector field as required in (3.24) with scale dimension 2Δ. Furthermore the constraint (3.43) and the arbitrariness (3.49) translate into (3.23) and $V_A \sim -\bar{\Psi}'\,\bar{\Gamma}_A\Psi$. The transformations (3.51) and (3.52) suggest $\bar{\Psi}\Psi$ and $\bar{\Psi}'\Psi'$ are scalars but $\bar{\Psi}\Psi = 0$ and $\bar{\Psi}'\Psi'$ is not invariant under (3.49) and so does not correspond to a scalar on the projective null cone. In odd dimensions, when Γ_A and $\bar{\Gamma}_A$ can be identified, $\bar{\Psi}\Psi' = \bar{\Psi}'\Psi$ is a scalar which is invariant under (3.49).

The transformations corresponding to (3.12) are obtained by a rotation in the 1-$(d+1)$ plane,

$$\Psi_\theta(X) = \big(\cos\tfrac{1}{2}\theta\,\mathbb{1} - \sin\tfrac{1}{2}\theta\;\Gamma_1\bar{\Gamma}_{d+1}\big)\,\Psi(X_{-\theta})\,. \tag{3.54}$$

It is easy to check that this is consistent with (3.43) and on reduction is identical with (2.62).

The two point function for Φ

$$\big\langle \Phi(X)\,\Phi(Y)\big\rangle = \frac{1}{(X\cdot Y)^{\Delta}}\,, \tag{3.55}$$

is determined by invariance under $O(d+1,1)$ or $O(d,2)$ and (3.19) and is easily seen to be equivalent to (2.88). Corresponding to (2.89)

$$\begin{aligned}\big\langle V_A(X)\,V_B(Y)\big\rangle &= \frac{\eta_{AB}\,X\cdot Y + 2\,Y_A X_B + \alpha\,X_A Y_B}{(X\cdot Y)^{\Delta+1}}\,,\\ \big\langle \Psi(X)\,\bar{\Psi}(Y)\big\rangle &= \frac{\Gamma_A X^A\,\bar{\Gamma}_B Y^B}{(X\cdot Y)^{\Delta+\frac{1}{2}}}\,,\end{aligned} \tag{3.56}$$

with α arbitrary. These results are equivalent to (3.21) and (3.43). To reduce the vector two point function we may note that $e_\mu{}^A(x)Y_A(y) = (y-x)_\mu$, $e_\nu{}^B(y)X_B(x) = (x-y)_\nu$.

3.3 Reduction to Low Dimensions

In the obviously interesting cases of three and four dimensions the embedding formalism and the special properties of Dirac matrices allow further simplifications.

In four dimensions from (3.40) $\bar{\mathcal{G}}^{-1} = \mathcal{C}^{-1}\,\mathcal{G}^T\,\mathcal{C}$ determines $\bar{\mathcal{G}}$ in terms of $\mathcal{G}$. Furthermore from (3.32) and (3.38)

$$\mathcal{G}\,\Gamma_A\mathcal{C}^{-1}\,\mathcal{G}^T = \Gamma_B\mathcal{C}^{-1}\,G^B{}_A\,, \qquad (\Gamma_A\mathcal{C}^{-1})^T = -\Gamma_A\mathcal{C}^{-1}\,, \tag{3.57}$$

requires no restriction on $\mathcal{G}$ since the six $\Gamma_A\mathcal{C}^{-1}$ form a basis for antisymmetric 4×4 matrices. Hence $\mathcal{G}$ is constrained just by (3.33), $\mathcal{G}\,\mathcal{B}\,\mathcal{G}^\dagger = \mathcal{B}$, and since, up to an equivalence, $\mathcal{B} \sim \left(\begin{smallmatrix} 1 & 0 \\ 0 & -1\end{smallmatrix}\right)$ this requires $\mathcal{G} \in SU(2,2)$ and the conformal group $SO(4,2) \simeq SU(2,2)/\mathbb{Z}_2$, with dimension 15.

With explicit indices $(\Gamma_A\mathcal{C}^{-1})_{ab} = -(\Gamma_A\mathcal{C}^{-1})_{ba}$, $a,b = 1,2,3,4$. Since ϵ^{abcd} is an invariant tensor for $SU(2,2)$ then we must have, by a choice of normalisation for Γ_A,

$$\tfrac{1}{8}\,\epsilon^{abcd}\,(\Gamma_A\mathcal{C}^{-1})_{ab}\,(\Gamma_B\mathcal{C}^{-1})_{cd} = \eta_{AB}\,. \tag{3.58}$$

This allows $\bar\Gamma_A$ to be defined by

$$(\mathcal{C}\bar\Gamma_A)^{ab} = -\tfrac{1}{2}\,\epsilon^{abcd}\,(\Gamma_A\mathcal{C}^{-1})_{cd}\,, \tag{3.59}$$

since

$$\begin{aligned} 0 &= \tfrac{5}{4}\,\delta_f{}^{[e}\,\epsilon^{abcd]}\,(\Gamma_A\mathcal{C}^{-1})_{ab}\,(\Gamma_B\mathcal{C}^{-1})_{cd} \\ &= 2\,\delta_f{}^e\,\eta_{AB} - (\Gamma_A\mathcal{C}^{-1})_{fb}\,(\mathcal{C}\bar\Gamma_B)^{be} - (\Gamma_B\mathcal{C}^{-1})_{fd}\,(\mathcal{C}\bar\Gamma_A)^{de}\,, \end{aligned} \tag{3.60}$$

verifying the Dirac algebra (3.27). For any six dimensional X^A then equivalently we can consider 4×4 antisymmetric matrices given by

$$\mathcal{X}_{ab} = X^A(\Gamma_A\mathcal{C}^{-1})_{ab}\,, \qquad \bar{\mathcal{X}}^{ab} = X^A(\mathcal{C}\bar\Gamma_A)^{ab} = -\tfrac{1}{2}\,\epsilon^{abcd}\,\mathcal{X}_{cd}\,, \tag{3.61}$$

where

$$\mathrm{Pf}(\mathcal{X}) \equiv \tfrac{1}{8}\,\epsilon^{abcd}\,\mathcal{X}_{ab}\mathcal{X}_{cd} = \eta_{AB}\,X^A X^B\,, \tag{3.62}$$

with Pf denoting the *Pfaffian*, satisfying $\mathrm{Pf}(\mathcal{G}\,\mathcal{X}\,\mathcal{G}^T) = \det\mathcal{G}\,\mathrm{Pf}(\mathcal{X})$. For X^A coordinates on the projective null cone then of course $\mathrm{Pf}(\mathcal{X}) = 0$. The Minkowski space reality condition requires

$$\bar{\mathcal{X}} = \mathcal{B}\,\mathcal{X}^*\,\mathcal{B}\,. \tag{3.63}$$

In three dimensions, so that there are five 4×4 matrices Γ_A,

$$\mathcal{G}\,\Gamma_A\mathcal{C}^{-1}\,\mathcal{G}^T = \Gamma_B\mathcal{C}^{-1}\,G^B{}_A\,, \quad \mathcal{G}\,\mathcal{C}^{-1}\,\mathcal{G}^T = \mathcal{C}^{-1}\,, \qquad \mathcal{C}^T = -\mathcal{C}\,, \quad (\Gamma_A\mathcal{C}^{-1})^T = -\Gamma_A\mathcal{C}^{-1}\,, \tag{3.64}$$

where now $\Gamma_A\mathcal{C}^{-1}, \mathcal{C}^{-1}$ form a basis for 4×4 antisymmetric matrices. The condition $\mathcal{G}\,\mathcal{C}^{-1}\,\mathcal{G}^T = \mathcal{C}^{-1}$ then implies $\mathcal{G} \in Sp(4)$. In addition in this case

$$\mathcal{G} = \mathcal{B}\mathcal{C}^{-1}\,\mathcal{G}^*\,\mathcal{C}\mathcal{B}\,, \qquad (\mathcal{C}\mathcal{B})^T = \mathcal{C}\mathcal{B}\,, \tag{3.65}$$

which ensures that the representation is equivalent to a real representation, essentially taking $\mathcal{C}\mathcal{B} = \mathbb{1}$, belonging to $Sp(4,\mathbb{R})$, and hence $SO(3,2) \simeq Sp(4,\mathbb{R})/\mathbb{Z}_2$, with dimension 10. Since $\Gamma_A = \mathcal{B}\mathcal{C}^{-1}\,\Gamma_A{}^*\,\mathcal{C}\mathcal{B}$ the Gamma matrices are real if $\mathcal{C}\mathcal{B} = \mathbb{1}$.

4 Energy Momentum Tensor

In any local quantum field theory the energy momentum tensor plays an important role. In an arbitrary CFT such a local field may not be present but if the CFT is derived from a conformally invariant action then Noether's theorem provides a construction of the energy momentum tensor. To show this we assume an action $S[\varphi]$ which is a local functional of fundamental fields φ and various derivatives is conformally invariant so that for a conformal Killing vector v_μ there is an action $\delta_v\varphi$ so that $\delta_v S[\varphi] = 0$. Conformal primary fields constructed in terms of φ then transform as in (2.49) or (2.53).

The energy momentum tensor can be constructed, using a version of Noether's theorem, by extending conformal transformations $\delta_v\varphi$ so as to allow v_μ to be unconstrained and also ${\omega_v}^{\mu\nu} \to \omega^{\mu\nu}$, $\sigma_v \to \sigma$, for arbitrary $\omega^{\mu\nu}(x) = -\omega^{\nu\mu}(x)$, $\sigma(x)$, so that (2.49) becomes

$$\delta_{v,\omega,\sigma}\phi_I = -v^\mu\partial_\mu\phi_I - \sigma\,\Delta\,\phi_I + \tfrac{1}{2}\,\omega^{\mu\nu}(s_{\mu\nu})_I{}^J\phi_J\,, \tag{4.1}$$

where Δ may be a matrix. In this case we must have

$$\begin{aligned}\delta_{v,\omega,\sigma}S[\varphi] = \int \mathrm{d}^d x\, \big((\partial_\mu v_\nu - \omega_{\mu\nu} - \sigma\,\eta_{\mu\nu})&T_c^{\mu\nu} \\ + (\partial_\rho\,\omega_{\mu\nu} - \partial_\mu\sigma\,\eta_{\rho\nu} + \partial_\nu\sigma\,\eta_{\rho\mu})&X^{\rho\mu\nu} \\ + (\partial_\rho\partial_\nu v_\mu - \eta_{\mu\rho}\,\partial_\nu\sigma - \eta_{\mu\nu}\,\partial_\rho\sigma + \eta_{\rho\nu}\,\partial_\mu\sigma)&Y^{\mu\rho\nu} + \partial_\mu\partial_\nu\sigma\,Z^{\mu\nu}\big)\,, \\ X^{\rho\mu\nu} = -X^{\rho\nu\mu}\,, \qquad Y^{\mu\rho\nu} = Y^{\mu\nu\rho}\,, \qquad & Z^{\mu\nu} = Z^{\nu\mu}\,,\end{aligned} \tag{4.2}$$

since the right side vanishes when restricted to an infinitesimal conformal transformation as a consequence of (2.11), (2.12), (2.5) and (2.6). The equations of motion are obtained by requiring $\delta S[\phi_I] = 0$ for arbitrary $\delta\phi_I$. Hence varying $v^\mu, \omega^{\mu\nu}, \sigma$ independently gives

$$\begin{aligned}\partial_\mu T_c^{\mu\nu} &= \partial_\rho\partial_\mu Y^{\nu\rho\mu}\,, \\ T_c^{[\mu\nu]} &= -\,\partial_\rho X^{\rho\mu\nu}\,, \\ \eta_{\mu\nu}T_c^{\mu\nu} &= 2\,\partial_\mu X^{\rho\mu\nu}\eta_{\rho\nu} + 2\,\partial_\mu Y^{\rho\nu\mu}\eta_{\rho\nu} - \partial_\mu Y^{\mu\rho\nu}\eta_{\rho\nu} + \partial_\mu\partial_\nu Z^{\mu\nu}\,,\end{aligned} \tag{4.3}$$

subject to ϕ satisfying its equations of motion. In general $T_c^{\mu\nu}$, which may be regarded as the canonical energy momentum tensor, is neither symmetric or traceless or indeed conserved in general. However, defining

$$T^{\mu\nu} = T_c^{\mu\nu} + \partial_\rho\big(X^{\rho\mu\nu} - X^{\mu\rho\nu} - X^{\nu\rho\mu} + Y^{\rho\mu\nu} - Y^{\mu\nu\rho} - Y^{\nu\mu\rho}\big) + \mathcal{D}^{\mu\nu\sigma\rho}Z_{\sigma\rho}\,, \tag{4.4}$$

with the differential operator

$$\begin{aligned}\mathcal{D}^{\mu\nu\sigma\rho} = {}& \frac{1}{d-2}\,\big(\eta^{\mu(\sigma}\partial^{\rho)}\partial^\nu + \eta^{\nu(\sigma}\partial^{\rho)}\partial^\mu - \eta^{\mu(\sigma}\eta^{\rho)\nu}\partial^2 - \eta^{\mu\nu}\partial^\sigma\partial^\rho\big) \\ & - \frac{1}{(d-2)(d-1)}\,\big(\partial^\mu\partial^\nu - \eta^{\mu\nu}\partial^2\big)\eta^{\sigma\rho}\,,\end{aligned} \tag{4.5}$$

which is constructed to satisfy $\partial_\mu\mathcal{D}^{\mu\nu\sigma\rho} = 0$ and $\eta_{\mu\nu}\mathcal{D}^{\mu\nu\sigma\rho} = -\partial^\sigma\partial^\rho$, ensures, subject to the equations of motion,

$$\partial_\mu T^{\mu\nu} = 0\,, \qquad T^{[\mu\nu]} = 0\,, \qquad \eta_{\mu\nu}T^{\mu\nu} = 0\,. \tag{4.6}$$

In a CFT taking $T^{\mu\nu}$ as the energy momentum tensor ensures that it is symmetric and traceless as well as conserved, the additional contributions involving X, Y, Z are 'improvement' terms. With the improvement (4.4), (4.2) becomes

$$\delta_{v,\omega,\sigma} S[\phi_I] = \int \mathrm{d}^d x \, (\partial_\mu v_\nu - \omega_{\mu\nu} - \sigma\, \eta_{\mu\nu}) T^{\mu\nu} \,, \tag{4.7}$$

which directly implies (4.6) subject to the equations of motion.

For any conformal Killing vector satisfying (2.3), (4.6) implies that there is an associated conserved current

$$J_v^\mu = T^{\mu\nu} v_\nu \quad \Rightarrow \quad \partial_\mu J_v^\mu = T^{\mu\nu} \partial_\mu v_\nu = 0 \,. \tag{4.8}$$

An alternative prescription for determining the energy momentum tensor is possible if the theory is extended to an arbitrary curved background with metric $g_{\mu\nu}$ so that $S[\varphi] \to S[\varphi, g_{\mu\nu}]$. In this case we may define a symmetric energy momentum tensor by

$$T^{\mu\nu} = 2 \, \frac{1}{\sqrt{-g}} \frac{\delta}{\delta g_{\mu\nu}} S[\varphi, g_{\rho\tau}] \,. \tag{4.9}$$

Invariance under diffeomorphisms, $\delta_v \varphi = -\mathcal{L}_v \varphi$, $\delta_v g_{\mu\nu} = \nabla_\mu v_\nu + \nabla_\nu v_\mu$, for $\mathcal{L}_v$ an appropriate Lie derivative, requires

$$\int \mathrm{d}^d x \, \Big(- \mathcal{L}_v \varphi \, \frac{\delta}{\delta \varphi} + (\nabla_\mu v_\nu + \nabla_\nu v_\mu) \, \frac{\delta}{\delta g_{\mu\nu}} \Big) S[\varphi, g_{\rho\tau}] = 0 \,. \tag{4.10}$$

Varying v ensures conservation

$$\nabla_\mu T^{\mu\nu} = 0 \,, \tag{4.11}$$

up to terms which vanish on the equations of motion. For Weyl invariant theories S is invariant under local Weyl rescalings of the metric so that $S[e^{-\sigma\Delta}\varphi, e^{2\sigma} g_{\mu\nu}] = S[\varphi, g_{\mu\nu}]$. Infinitesimally

$$\int \mathrm{d}^d x \, \sigma \Big(- \Delta_\varphi \varphi \, \frac{\delta}{\delta \varphi} + 2 \, g_{\mu\nu} \, \frac{\delta}{\delta g_{\mu\nu}} \Big) S[\varphi, g_{\rho\tau}] = 0 \,, \tag{4.12}$$

which implies by varying σ

$$g_{\mu\nu} \, T^{\mu\nu} = 0 \,, \tag{4.13}$$

so long as ϕ satisfies the equations of motion. If (4.12) holds up to contributions involving two derivatives of σ then these can generally be removed, so as to restore (4.12), by adding appropriate curvature dependent contributions to S.

Weyl invariant theories reduce to CFTs on flat space since for conformal Killing vectors the metric variation can be eliminated between (4.10) and (4.12)

$$\nabla_\mu v_\nu + \nabla_\nu v_\mu = 2 \, \sigma \, g_{\mu\nu} \quad \Rightarrow \quad \int \mathrm{d}^d x \, \big(- \mathcal{L}_v \varphi + \sigma \, \Delta_\varphi \, \varphi \big) \frac{\delta}{\delta \varphi} \, S[\varphi, g_{\mu\nu}] = 0 \,. \tag{4.14}$$

Reduced to flat space, $g_{\mu\nu} = \eta_{\mu\nu}$, this ensures that S is conformally invariant.

For a free scalar field φ with $\Delta_\varphi = \frac{1}{2}(d-2)$ then the extension of (4.29) to a Weyl invariant action on curved space has the form

$$S[\varphi, g_{\mu\nu}] = - \int \mathrm{d}^d x \sqrt{-g} \, \tfrac{1}{2} \big(\partial^\mu \varphi \partial_\mu \varphi + \tfrac{d-2}{4(d-1)} \, R \, \varphi^2 \big) \,, \tag{4.15}$$

with R the scalar curvature. Under a Weyl rescaling $g_{\mu\nu} \to e^{2\sigma} g_{\mu\nu}$, $\sqrt{-g} \to e^{d\sigma}\sqrt{-g}$, then $R \to e^{-2\sigma}(R - 2(d-1)\,\nabla^2\sigma - (d-1)(d-2)\,\partial^\mu\sigma\partial_\mu\sigma)$ and, for $\varphi \to e^{-\frac{1}{2}(d-2)\sigma}\varphi$,

$$\partial^\mu\varphi\partial_\mu\varphi \to e^{-d\sigma}\big(\partial^\mu\varphi\partial_\mu\varphi + \tfrac{1}{2}(d-2)\big(\nabla^2\sigma\,\varphi^2 - \nabla_\mu(\partial^\mu\sigma\,\varphi^2)\big) + \tfrac{1}{4}(d-2)^2\,\partial^\mu\sigma\partial_\mu\sigma\,\varphi^2\big)\,, \tag{4.16}$$

which is sufficient to verify the invariance of S in (4.15). For variations of the metric $\delta_g\sqrt{-g} = \frac{1}{2}\,g^{\mu\nu}\delta g_{\mu\nu}\sqrt{-g}$, $\delta_g R = (\nabla^\mu\nabla^\nu - g^{\mu\nu}\nabla^2 - R^{\mu\nu})\delta g_{\mu\nu}$ and it is easy to see that (4.9) for S given by (4.15) gives an identical result to (4.31) for the energy momentum tensor on reduction to flat space.

If the theory defined by $S[\varphi]$ on flat space is just scale invariant, in addition to invariance under translations and rotations or Lorentz transformations, then it is necessary to restrict σ in (2.3) to be just a constant, so that in (2.9) $b_\mu = 0$. In such a case (4.2) is relaxed to

$$\delta_{v,\omega,\sigma}S[\varphi]_{\text{Scale}} = \delta_{v,\omega,\sigma}S[\varphi] - \int \mathrm{d}^d x\,\partial_\mu\sigma\,V^\mu\,, \tag{4.17}$$

for some vector field V^μ termed the virial current and where in (4.2) we set $Y^{\mu\rho\nu} = Z^{\mu\nu} = 0$. The trace condition now becomes

$$\eta_{\mu\nu}T^{\mu\nu} = \partial_\mu V^\mu\,. \tag{4.18}$$

In this case there is a conserved current associated with scale transformations

$$J^\mu_{\text{Scale}} = T^{\mu\nu}x_\nu - V^\mu\,, \tag{4.19}$$

but no associated current associated with special conformal transformations. If

$$V^\mu = \partial_\nu L^{\mu\nu}\,, \qquad L^{\mu\nu} = L^{\nu\mu}\,. \tag{4.20}$$

then (4.17) becomes equivalent to (4.2) with now $Z^{\mu\nu} = L^{\mu\nu}$ so that

$$T^{\mu\nu}_{\text{improved}} = T^{\mu\nu} + \mathcal{D}^{\mu\nu\sigma\rho}L_{\sigma\rho}\,, \tag{4.21}$$

with $\mathcal{D}^{\mu\nu\sigma\rho}$ defined in (4.5), is an improved conserved traceless energy momentum tensor and scale invariance extends to the full conformal group.

4.1 Ward Identities

In a CFT correlation functions involving the energy momentum tensor $\langle T^{\mu\nu}(x)\ldots\rangle$ satisfy Ward identities. In two dimensional CFTs the energy momentum tensor, when $T_{\mu\nu}(x) \to T(z) = T_{zz}(x)$, $\bar{T}(\bar{z}) = T_{\bar{z}\bar{z}}(x)$, $\langle T(z)\,X\rangle$ and $\langle\bar{T}(\bar{z})\,X\rangle$ are fully determined in terms of $\langle X\rangle$ by the extended Virasoro identities for any X which is a product of conformal primary fields. However for $d > 2$ Ward identities still provide constraints without determining $\langle T^{\mu\nu}(x)\ldots\rangle$ completely although the identities do give a precise prescription for the normalisation of $T^{\mu\nu}$.

In a Lagrangian theory the correlation functions are determined by functional integrals, continuing here to a Euclidean metric (the Minkowski identities are obtained by letting $\delta^d(x-y) \to i\,\delta^d(x-y)$),

$$\big\langle \phi_I(x) \dots \big\rangle = \int \mathrm{d}[\varphi]\; e^S\, \phi_I(x) \dots \,, \tag{4.22}$$

We assume that the functional measure is invariant under $\delta_{v,\omega,\sigma}\varphi$ (of course classical conformal invariance of S is generally broken by quantum anomalies but here we assume the theory is at a fixed point) and then considering variations as in (4.1) and (4.7) we have the identity

$$\int \mathrm{d}^d x\, \big(\partial_\mu v_\nu(x) - \omega_{\mu\nu}(x) - \sigma(x)\,\eta_{\mu\nu}\big) \big\langle T^{\mu\nu}(x)\, \phi_I(y) \dots \big\rangle + \big\langle \delta_{v,\omega,\sigma}\phi_I(y) \dots \big\rangle = 0\,. \tag{4.23}$$

Varying v, ω, σ gives three independent Ward identities

$$\begin{aligned}
\partial_{x\mu} \big\langle T^{\mu\nu}(x)\, \phi_I(y) \dots \big\rangle &= -\delta^d(x-y)\; \partial_y{}^\nu \big\langle \phi_I(y) \dots \big\rangle\,, \\
\big\langle T^{[\mu\nu]}(x)\, \phi_I(y) \dots \big\rangle &= \delta^d(x-y)\; \tfrac{1}{2} (s^{\mu\nu})_I{}^J \big\langle \phi_J(y) \dots \big\rangle\,, \\
\eta_{\mu\nu} \big\langle T^{\mu\nu}(x)\, \phi_I(y) \dots \big\rangle &= -\Delta\, \delta^d(x-y)\; \big\langle \phi_I(y) \dots \big\rangle\,.
\end{aligned} \tag{4.24}$$

However Ward identities involving the energy momentum tensor are not unique since $T^{\mu\nu}(x)\,\phi_I(y)$ is arbitrary, as a result of $T^{\mu\nu}$ having dimension d, up to contact terms proportional to $\delta^d(x-y)$. Letting

$$T^{\mu\nu}(x)\, \phi_I(y) \to T^{\mu\nu}(x)\, \phi_I(y) + \delta^d(x-y)\, \big(\tfrac{1}{2}(s^{\mu\nu})_I{}^J - (X^{\mu\nu})_I{}^J\big) \phi_J(y)\,, \tag{4.25}$$

for some $(X^{\mu\nu})_I{}^J = (X^{\nu\mu})_I{}^J$ satisfying $[X^{\mu\nu}, \tfrac{1}{2}\,\omega^{\sigma\rho} s_{\sigma\rho}] = \omega^\mu{}_\lambda X^{\lambda\nu} + \omega^\nu{}_\lambda X^{\mu\lambda}$, requiring then $\eta_{\mu\nu}(X^{\mu\nu})_I{}^J = X\,\delta_I{}^J$. With the change (4.25), (4.24) becomes

$$\begin{aligned}
\partial_{x\mu} \big\langle T^{\mu\nu}(x)\, \phi_I(y) \dots \big\rangle =&\; -\delta^d(x-y)\; \partial_y{}^\nu \big\langle \phi_I(y) \dots \big\rangle \\
&\; - \partial_{x\mu}\delta^d(x-y) \big(\tfrac{1}{2}(s^{\mu\nu})_I{}^J - (X^{\mu\nu})_I{}^J\big) \big\langle \phi_J(y) \dots \big\rangle\,, \\
\eta_{\mu\nu} \big\langle T^{\mu\nu}(x)\, \phi_I(y) \dots \big\rangle =&\; (X-\Delta)\, \delta^d(x-y)\; \big\langle \phi_I(y) \dots \big\rangle\,,
\end{aligned} \tag{4.26}$$

with $T^{[\mu\nu]} = 0$ including contact terms. The freedom arising from the choice of $(X^{\mu\nu})_I{}^J$ can be used to recast the Ward identities in various different forms. For scalars taking $X^{\mu\nu} = \eta^{\mu\nu}$ we get

$$\begin{aligned}
\partial_{x\mu} \big\langle T^{\mu\nu}(x)\, \phi(y) \dots \big\rangle &= \partial_x{}^\nu \delta^d(x-y)\; \big\langle \phi(x) \dots \big\rangle\,, \\
\eta_{\mu\nu} \big\langle T^{\mu\nu}(x)\, \phi(y) \dots \big\rangle &= (d-\Delta)\, \delta^d(x-y)\; \big\langle \phi(y) \dots \big\rangle\,,
\end{aligned} \tag{4.27}$$

and for vectors if $(X^{\mu\nu})_\rho{}^\lambda = \eta^{\mu\nu}\delta_\rho^\lambda - \tfrac{1}{2}(\eta^{\mu\lambda}\delta_\rho{}^\nu + \eta^{\nu\lambda}\delta_\rho{}^\mu)$ (4.26) simplifies to

$$\begin{aligned}
\partial_{x\mu} \big\langle T^{\mu\nu}(x)\, V_\rho(y) \dots \big\rangle &= \partial_x{}^\nu \delta^d(x-y)\; \big\langle V_\rho(x) \dots \big\rangle - \delta_\rho{}^\nu\, \partial_x{}^\lambda \delta^d(x-y)\; \big\langle V_\lambda(x) \dots \big\rangle\,, \\
\eta_{\mu\nu} \big\langle T^{\mu\nu}(x)\, V_\rho(y) \dots \big\rangle &= (d-1-\Delta)\, \delta^d(x-y)\; \big\langle V_\rho(y) \dots \big\rangle\,.
\end{aligned} \tag{4.28}$$

(4.27) and (4.28) are the Ward identities obtained starting from curved space with the definition (4.9) and considering diffeomorphisms and Weyl scale transformations $\delta g_{\mu\nu} = \nabla_\mu v_\nu + \nabla_\nu v_\mu + 2\sigma\, g_{\mu\nu}$ and extending (4.10), (4.12) to include sources or couplings dual to ϕ_I. In (4.28) the trace identity vanishes if V_ρ is conserved so that $\Delta = d-1$.

4.2 Free Fields

As an illustration of the construction of the energy momentum tensor we consider a free spinless scalar field with an action

$$S_\varphi[\varphi] = -\int \mathrm{d}^d x \, \tfrac{1}{2}\, \partial^\mu \varphi \, \partial_\mu \varphi \, , \qquad \Delta_\varphi = \tfrac{1}{2}(d-2) \, . \tag{4.29}$$

Under transformations of φ as in (4.1) the variation of S can be expressed in the form (4.2) with

$$T_{\varphi,c}^{\mu\nu} = \partial^\mu \varphi \, \partial^\nu \varphi - \tfrac{1}{2}\, \eta^{\mu\nu}\, \partial^\rho \varphi \partial_\rho \varphi \, , \qquad Z_\varphi^{\mu\nu} = -\tfrac{1}{2}(d-2)\, \eta^{\mu\nu} \varphi^2 \, . \tag{4.30}$$

The construction (4.4) then gives for the free scalar field

$$T_\varphi^{\mu\nu} = \partial^\mu \varphi \, \partial^\nu \varphi - \tfrac{1}{2}\, \eta^{\mu\nu}\, \partial^\rho \varphi \partial_\rho \varphi - \frac{d-2}{4(d-1)} \big(\partial^\mu \partial^\nu - \eta^{\mu\nu}\, \partial^2 \big) \varphi^2 \, , \tag{4.31}$$

from which it is easy to verify

$$\partial_\mu T_\varphi^{\mu\nu} = \partial^2 \varphi \, \partial^\nu \varphi \, , \qquad \eta_{\mu\nu} T_\varphi^{\mu\nu} = \tfrac{1}{2}(d-2)\, \partial^2 \varphi \, \varphi \, , \tag{4.32}$$

which of course vanish when $\partial^2 \varphi = 0$.

A less trivial example is the higher derivative non unitary scalar theory with an action

$$S_{\varphi,4}[\varphi] = -\int \mathrm{d}^d x \, \tfrac{1}{2}\, \partial^2 \varphi \, \partial^2 \varphi \, , \qquad \Delta_\varphi = \tfrac{1}{2}(d-4) \, . \tag{4.33}$$

In this case, using $\partial_\mu \varphi \, \partial^2 \varphi = \partial^\rho (\partial_\mu \varphi \, \partial_\rho \varphi) - \frac{1}{2}\, \partial_\mu (\partial^\rho \varphi \partial_\rho \varphi)$,

$$\begin{aligned} T_{\varphi,4,c}^{\mu\nu} &= 2\, \partial^\mu \partial^\nu \varphi \, \partial^2 \varphi - \tfrac{1}{2}\, \eta^{\mu\nu}\, \partial^2 \varphi \partial^2 \varphi \, , \qquad Y_{\varphi,4}^{\mu\rho\nu} = \eta^{\rho\nu}\, \partial^\mu \varphi \, \partial^2 \varphi \, , \\ Z_{\varphi,4}^{\mu\nu} &= 2\, \partial^\mu \varphi \, \partial^\nu \varphi - \eta^{\mu\nu}\, \partial^\rho \varphi \partial_\rho \varphi + \tfrac{1}{2}(d-4)\, \eta^{\mu\nu}\, \partial^2 \varphi \, \varphi \, . \end{aligned} \tag{4.34}$$

From (4.4)

$$\begin{aligned} T_{\varphi,4}^{\mu\nu} = {} & 2\, \partial^\mu \partial^\nu \varphi \, \partial^2 \varphi - \tfrac{1}{2}\, \eta^{\mu\nu}\, \partial^2 \varphi \partial^2 \varphi - \partial^\mu (\partial^\nu \varphi \, \partial^2 \varphi) - \partial^\nu (\partial^\mu \varphi \, \partial^2 \varphi) + \eta^{\mu\nu}\, \partial_\rho (\partial^\rho \varphi \, \partial^2 \varphi) \\ & + 2\, \mathcal{D}^{\mu\nu\sigma\rho} \big(\partial_\sigma \varphi \, \partial_\rho \varphi \big) - \frac{1}{d-1} \big(\partial^\mu \partial^\nu - \eta^{\mu\nu} \partial^2 \big) \big(\partial^\rho \varphi \, \partial_\rho \varphi - \tfrac{1}{2}(d-4)\, \partial^2 \varphi \, \varphi \big) \, , \end{aligned} \tag{4.35}$$

and then

$$\partial_\mu T_{\varphi,4}^{\mu\nu} = -\partial^2 \partial^2 \varphi \, \partial^\nu \varphi \, , \qquad \eta_{\mu\nu} T_{\varphi,4}^{\mu\nu} = -\tfrac{1}{2}(d-4)\, \partial^2 \partial^2 \varphi \, \varphi \, , \tag{4.36}$$

with $\partial^2 \partial^2 \varphi = 0$ the equation of motion.

When $d = 2$ this result breaks down since the operator $\mathcal{D}^{\mu\nu\sigma\rho}$ no longer exists. If in (4.35) the second line is dropped then

$$T_\varphi^{\mu\nu} = -\partial^\nu \varphi \, \partial^\mu \partial^2 \varphi - \partial^\mu \varphi \, \partial^\nu \partial^2 \varphi + \tfrac{1}{2}\, \eta^{\mu\nu}\, \partial^2 \varphi \partial^2 \varphi + \eta^{\mu\nu}\, \partial^\rho \varphi \, \partial_\rho \partial^2 \varphi \, , \tag{4.37}$$

and

$$\partial_\mu T_\varphi^{\mu\nu} = -\partial^2 \partial^2 \varphi \, \partial^\nu \varphi \, , \quad \eta_{\mu\nu} T_\varphi^{\mu\nu} = \partial_\mu D^\mu + \partial^2 \partial^2 \varphi \, \varphi \, , \quad D^\mu = \partial^2 \varphi \, \partial^\mu \varphi - \varphi \, \partial^\mu \partial^2 \varphi \, . \tag{4.38}$$

Free fermions also provide an example of a CFT for any dimension d. In this case we take

$$S_\psi[\psi,\bar\psi] = -\int \mathrm{d}^d x\, \bar\psi\, \bar\gamma^\mu \partial_\mu \psi\,, \qquad \Delta_\psi = \Delta_{\bar\psi} = \tfrac{1}{2}(d-1)\,, \tag{4.39}$$

where ψ, $\bar\psi$ transform as in (2.60). Extending (2.60) as in (4.1) then the variation of S, using (2.58), is in the form (4.2) with

$$T_{\psi,c}^{\mu\nu} = \bar\psi\, \bar\gamma^\mu \partial^\nu \psi - \eta^{\mu\nu}\, \bar\psi\, \bar\gamma\cdot\partial\, \psi\,, \qquad X_\psi^{\rho\mu\nu} = -\tfrac{1}{2}\, \bar\psi\, \bar\gamma^\rho s^{\mu\nu}\, \psi\,. \tag{4.40}$$

Since now $X_\psi^{\rho\mu\nu} - X_\psi^{\mu\rho\nu} - X_\psi^{\nu\rho\mu} = -\frac{1}{4}\, \bar\psi\big(\bar\gamma^\rho s^{\mu\nu} + \bar s^{\mu\nu}\, \bar\gamma^\rho\big)\psi - \frac{1}{2}\, \eta^{\rho\nu}\, \bar\psi\, \bar\gamma^\mu\, \psi + \frac{1}{2}\, \eta^{\mu\nu}\, \bar\psi\, \bar\gamma^\rho\, \psi$ it follows that (4.4) gives

$$T_\psi^{\mu\nu} = \tfrac{1}{2}\, \bar\psi\, \big(\bar\gamma^\mu\, \overleftrightarrow{\partial}^\nu + \bar\gamma^\nu\, \overleftrightarrow{\partial}^\mu\big)\, \psi - \eta^{\mu\nu}\, \bar\psi\, \bar\gamma\cdot \overleftrightarrow{\partial}\, \psi - \tfrac{1}{2}\, \bar\psi\big(\bar\gamma\cdot\overleftarrow{\partial}\, s^{\mu\nu} + \bar s^{\mu\nu}\, \bar\gamma\cdot\partial\big)\psi\,, \tag{4.41}$$

for $\overleftrightarrow{\partial} = \frac{1}{2}\big(\partial - \overleftarrow{\partial}\big)$. The last term, antisymmetric in μ,ν, vanishes on the equations of motion. From (4.41)

$$\partial_\mu T_\psi^{\mu\nu} = \bar\psi\big(\bar\gamma\cdot\overleftarrow{\partial}\, \partial^\nu - \overleftarrow{\partial}^\nu\, \bar\gamma\cdot\partial\big)\psi\,, \qquad \eta_{\mu\nu} T_\psi^{\mu\nu} = -(d-1)\, \bar\psi\, \bar\gamma\cdot\overleftrightarrow{\partial}\, \psi\,. \tag{4.42}$$

In $d=2n$ dimensions free conformal field theories are obtained in terms of $(n-1)$-forms $A_{\mu_1\dots\mu_{n-1}}$ with an action

$$S_A[A] = -\int \mathrm{d}^{2n}x\, \frac{1}{2n!}\, F^{\mu_1\dots\mu_n} F_{\mu_1\dots\mu_n}\,, \quad F_{\mu_1\dots\mu_n} = n\, \partial_{[\mu_1} A_{\mu_2\dots\mu_n]}\,, \quad \Delta_A = n-1\,. \tag{4.43}$$

(4.1) becomes $\delta_{v,\omega,\sigma} A_{\mu_1\dots\mu_{n-1}} = -\big(v^\mu \partial_\mu + (n-1)\,\sigma\big) A_{\mu_1\dots\mu_{n-1}} - (n-1)\, \omega_{[\mu_1}{}^\rho A_{\rho\mu_2\dots\mu_{n-1}]}$ and in (4.2)

$$\begin{aligned} T_{A,c}^{\mu\nu} &= \tfrac{1}{(n-1)!}\, F^{\mu\mu_1\dots\mu_{n-1}} \partial^\nu A_{\mu_1\dots\mu_{n-1}} - \tfrac{1}{2n!}\, \eta^{\mu\nu}\, F^{\mu_1\dots\mu_n} F_{\mu_1\dots\mu_n}\,, \\ X_A^{\rho\mu\nu} &= \tfrac{1}{2(n-2)!}\, \big(F^{\rho\mu\mu_1\dots\mu_{n-2}} A^\nu{}_{\mu_1\dots\mu_{n-2}} - F^{\rho\nu\mu_1\dots\mu_{n-2}} A^\mu{}_{\mu_1\dots\mu_{n-2}}\big)\,. \end{aligned} \tag{4.44}$$

Since $X_A^{\rho\mu\nu} - X_A^{\mu\rho\nu} - X_A^{\nu\rho\mu} = \frac{1}{(n-2)!}\, F^{\rho\mu\mu_1\dots\mu_{n-2}} A^\nu{}_{\mu_1\dots\mu_{n-2}}$ (4.4) gives

$$\begin{aligned} T_A^{\mu\nu} = {} & \tfrac{1}{(n-1)!}\big(F^{\mu\mu_1\dots\mu_{n-1}} F^\nu{}_{\mu_1\dots\mu_{n-1}} - \tfrac{1}{2n}\, \eta^{\mu\nu}\, F^{\mu_1\dots\mu_n} F_{\mu_1\dots\mu_n}\big) \\ & + \tfrac{1}{(n-2)!}\, \partial_\rho F^{\rho\mu\mu_1\dots\mu_{n-2}} A^\nu{}_{\mu_1\dots\mu_{n-2}}\,. \end{aligned} \tag{4.45}$$

The last term vanishes subject to the equations of motion, $\partial_\rho F^{\rho\mu_1\dots\mu_{n-1}} = 0$, leaving then a symmetric traceless energy momentum tensor. From (4.45)

$$\partial_\mu T_A^{\mu\nu} = \frac{1}{(n-1)!}\, \partial_\mu F^{\mu\mu_1\dots\mu_{n-1}}\, \partial^\nu A_{\mu_1\dots\mu_{n-1}}\,, \quad \eta_{\mu\nu} T_A^{\mu\nu} = \frac{1}{(n-2)!}\, \partial_\mu F^{\mu\mu_1\dots\mu_{n-1}} A_{\mu_1\dots\mu_{n-1}}\,. \tag{4.46}$$

4.3 Two Point Function

The energy momentum tensor is present as a conformal primary in all local CFTs. The correlation functions involving $T_{\mu\nu}$ are then of critical interest. The two point function is determined by conformal invariance as in (2.84) which here takes the form

$$S_d^2 \big\langle T_{\mu\nu}(x)\, T_{\sigma\rho}(0)\big\rangle = C_T \, \frac{1}{(x^2)^d} \, \mathcal{I}_{\mu\nu,\sigma\rho}(x) \, , \tag{4.47}$$

for the inversion tensor in this case given by

$$\mathcal{I}_{\mu\nu,\sigma\rho}(x) = \tfrac{1}{2}\big(I_{\mu\sigma}(x) I_{\nu\rho}(x) + I_{\mu\rho}(x) I_{\nu\sigma}(x)\big) - \frac{1}{d}\, \delta_{\mu\nu}\, \delta_{\sigma\rho} \, , \tag{4.48}$$

where $I_{\mu\sigma}$ is defined in (2.28). The coefficient C_T in (4.47) is an intrinsic property of the CFT since the normalisation is prescribed by requiring the Ward identity (4.26). In (4.47)

$$S_d = \frac{2\pi^{\frac{1}{2}d}}{\Gamma(\frac{1}{2}d)} \, , \tag{4.49}$$

with the factor on the left hand side introduced for later convenience.

For free scalar fields the basic two point function determined by (4.29) is

$$\big\langle \varphi(x)\, \varphi(0)\big\rangle = \frac{1}{(d-2)S_d} \, \frac{1}{(x^2)^{\frac{1}{2}(d-2)}} \, . \tag{4.50}$$

Since φ^2 is a conformal primary it follows that $\langle T_\varphi^{\mu\nu}(x)\, \partial^\sigma \partial^\rho \varphi^2(0)\rangle = 0$. To calculate (4.47) it is therefore sufficient to evaluate $\langle T_\varphi^{\mu\nu}(x)\, \partial^\sigma \varphi \partial^\rho \varphi(0)\rangle$ with $T_\varphi^{\mu\nu}$ given by (4.31) and also taking $\partial^2 \varphi = 0$. This gives in this case

$$C_{T,\varphi} = \frac{d}{d-1} \, . \tag{4.51}$$

For free fermions then from (4.39)

$$\big\langle \psi(x)\, \bar\psi(0)\big\rangle = \frac{1}{S_d} \, \frac{\bar\gamma \cdot x}{(x^2)^{\frac{1}{2}d}} \, , \tag{4.52}$$

and (4.41) gives

$$\big\langle T_\psi^{\mu\nu}(x)\, T_\psi^{\sigma\rho}(y)\big\rangle = -\mathrm{tr}\bigg(\gamma^{(\mu} \overleftrightarrow{\partial}_{\!x}{}^{\nu)} \, \frac{\bar\gamma\cdot(x-y)}{\big((x-y)^2\big)^{\frac{1}{2}d}} \, \gamma^{(\sigma} \overleftrightarrow{\partial}_{\!y}{}^{\rho)} \, \frac{\bar\gamma\cdot(y-x)}{\big((y-x)^2\big)^{\frac{1}{2}d}}\bigg) \, . \tag{4.53}$$

Hence

$$C_{T,\psi} = \tfrac{1}{2}\, d\, \mathrm{tr}(\mathbb{1}) \, , \qquad \mathrm{tr}(\mathbb{1}) = 2^{n-1} \quad \text{for } d = 2n \text{ or } d = 2n-1 \, , \tag{4.54}$$

with the even dimensional case corresponding to chiral, or equivalently Weyl, fermions.

The theory of free $(n-1)$-forms in $(2n)$-dimensions with an action (4.43) is a gauge theory and it is necessary to fix the gauge. In a Feynman type gauge

$$S_A[A] \to -\frac{1}{2(n-1)!} \int \mathrm{d}^{2n}x \; \partial^\lambda A^{\mu_1 \dots \mu_{n-1}} \, \partial_\lambda A_{\mu_1 \dots \mu_{n-1}} \, , \tag{4.55}$$

so that

$$\big\langle A^{\mu_1\dots\mu_{n-1}}(x)\, A_{\nu_1\dots\nu_{n-1}}(0)\big\rangle = \frac{(n-2)!}{2\,S_{2n}}\,\frac{1}{(x^2)^{n-1}}\,\delta^{[\mu_1}{}_{\nu_1}\dots\delta^{\mu_{n-1}]}{}_{\nu_{n-1}}\,. \tag{4.56}$$

Then, with F defined in (4.43), the two point function for F, which is gauge independent, is given by

$$\big\langle F^{\mu_1\dots\mu_n}(x)\, F_{\nu_1\dots\nu_n}(0)\big\rangle = \frac{n\,n!}{S_{2n}}\,\frac{1}{(x^2)^n}\,I^{[\mu_1}{}_{\nu_1}(x)\dots I^{\mu_n]}{}_{\nu_n}(x)\,. \tag{4.57}$$

This has the expected form according to (2.84) for $F^{\mu_1\dots\mu_n}$ a conformal primary. Using (4.57) with the expression (4.45) for the energy momentum tensor and using the identity $a^{[\mu}\,b_{[\nu}\,\delta^{\mu_1}{}_{\mu_1}\dots\delta^{\mu_{n-1}]}{}_{\mu_{n-1}]} = \big(a^\mu b_\nu\,(d-n)+\delta^\mu{}_\nu\,a\cdot b\,(n-1)\big)(d-2)(d-3)\dots(d-n+1)/(n\,n!)$ for $d=2n$ then

$$C_{T,A} = \frac{2\,n^2(2n-2)!}{(n-1)!^2}\,. \tag{4.58}$$

For $n=1$ this coincides with the result for a free scalar as expected. The results (4.51), (4.54) were obtained in [16] and (4.58) in [17].

In four dimensions then for n_S free scalars, n_W Weyl fermions and n_A gauge vector fields

$$C_T = \tfrac{4}{3}\,n_S + 4\,n_W + 16\,n_A\,. \tag{4.59}$$

For the higher derivative theory defined by (4.33) the two point function for the scalar field is now

$$\big\langle\varphi(x)\,\varphi(0)\big\rangle = \frac{1}{2(d-4)(d-2)S_d}\,\frac{1}{(x^2)^{\frac{1}{2}(d-4)}}\,. \tag{4.60}$$

Using the expression for the energy momentum tensor in (4.35) (it is useful to note that $\partial^\mu(\partial^\nu\varphi\,\partial^2\varphi) = \partial^\mu\partial_\sigma(\partial^\nu\varphi\,\partial^\sigma\varphi) - \frac{1}{2}\,\partial^\mu\partial^\nu(\partial_\sigma\varphi\,\partial^\sigma\varphi)$) determining the energy momentum tensor two point function can be reduced to calculating

$$\big\langle T^{\mu\nu}_{\varphi,4}(x)\,T^{\sigma\rho}_{\varphi,4}(0)\big\rangle = \big\langle T^{\mu\nu}_{\varphi,4}(x)\,2\,\partial^\sigma\partial^\rho\varphi\partial^2\varphi(0)\big\rangle\,, \tag{4.61}$$

since, as $T^{\mu\nu}_{\varphi,4}$ is a conformal primary, derivatives of lower dimension conformal primaries in $T^{\sigma\rho}_{\varphi,4}$ can be dropped and also contributions involving $\partial^2\partial^2\varphi$ vanish at non coincident points. The result is in accord with (4.47) with

$$C_{T,\varphi,4} = -\frac{2d(d+4)}{(d-2)(d-1)}\,. \tag{4.62}$$

The negative sign reflects the fact that the theory defined by (4.33) is non unitary, it also fails to give a well defined energy momentum tensor when $d=2$.

5 Operators and States

5.1 Conformal Lie Algebra

In a quantum field theory the fields are operators and the symmetry generators also become operators obeying the appropriate Lie algebra. For the conformal group as $O(d+1,1)$ or

$O(d,2)$ the Lie algebra generators belong to $\mathfrak{so}(d+1,1)$ or $\mathfrak{so}(d,2)$, which has dimension $\frac{1}{2}(d+1)(d+2)$ and a basis $\mathcal{M}_{AB} = -\mathcal{M}_{BA}$. For a scalar $\Phi(X)$ the generators act as

$$\big[\mathcal{M}_{AB}, \Phi(X)\big] = L_{AB}\Phi(X)\,. \tag{5.1}$$

with L_{AB} defined in (3.24). (5.1) implies the Lie algebra

$$\big[\mathcal{M}_{AB}, \mathcal{M}_{CD}\big] = \eta_{AC}\,\mathcal{M}_{BD} - \eta_{BC}\,\mathcal{M}_{AD} - \eta_{AD}\,\mathcal{M}_{BC} + \eta_{BD}\,\mathcal{M}_{AC}\,. \tag{5.2}$$

The generators $\mathcal{M}_{AB}$ can be decomposed as

$$[\mathcal{M}_{AB}] = \begin{array}{c|ccc} {}_A{}^{B\to}\ \downarrow & \nu & d{+}1 & d{+}2 \\ \hline \mu & M_{\mu\nu} & -\frac{1}{2}(P_\mu + K_\mu) & \frac{1}{2}(P_\mu - K_\mu) \\ d{+}1 & \frac{1}{2}(P_\nu + K_\nu) & 0 & -D \\ d{+}2 & -\frac{1}{2}(P_\nu - K_\nu) & D & 0 \end{array}\,, \tag{5.3}$$

or, with components $+,-$ defined as in (3.2),

$$\mathcal{M}_{-\mu} = -\mathcal{M}_{\mu-} = \tfrac{1}{2}\,P_\mu\,, \qquad \mathcal{M}_{+\mu} = -\mathcal{M}_{\mu+} = \tfrac{1}{2}\,K_\mu\,, \qquad \mathcal{M}_{+-} = -\mathcal{M}_{-+} = \tfrac{1}{2}\,D\,, \tag{5.4}$$

Using $\eta_{+-} = \eta_{-+} = \frac{1}{2}$,

$$[\mathcal{M}^A{}_B] = \begin{array}{c|ccc} {}_A{}^{B\to}\ \downarrow & \nu & + & - \\ \hline \mu & M^\mu{}_\nu & -\frac{1}{2}\,K^\mu & -\frac{1}{2}\,P^\mu \\ + & P_\nu & -D & 0 \\ - & K_\nu & 0 & D \end{array}\,. \tag{5.5}$$

The algebra (5.2) gives

$$\big[M_{\mu\nu}, M_{\rho\tau}\big] = \eta_{\mu\rho}\,M_{\nu\tau} - \eta_{\nu\rho}\,M_{\mu\tau} - \eta_{\mu\tau}\,M_{\nu\rho} + \eta_{\nu\tau}\,M_{\mu\rho}\,, \tag{5.6}$$

which is just the Lie algebra for $\mathfrak{so}(d)$ or $\mathfrak{so}(d-1,1)$, and also the commutators

$$\begin{gathered} \big[M_{\mu\nu}, M_{\pm\rho}\big] = \eta_{\mu\rho}\,M_{\pm\nu} - \eta_{\nu\rho}\,M_{\pm\mu}\,, \qquad \big[M_{\mu\nu}, M_{+-}\big] = 0\,, \\ \big[M_{+-}, M_{\mp\mu}\big] = \pm\,\tfrac{1}{2}\,M_{\mp\mu}\,, \qquad \big[M_{+\mu}, M_{-\nu}\big] = \eta_{\mu\nu}\,M_{+-} + \tfrac{1}{2}\,M_{\mu\nu}\,, \end{gathered} \tag{5.7}$$

or

$$\begin{gathered} \big[M_{\mu\nu}, P_\rho\big] = \eta_{\mu\rho}\,P_\nu - \eta_{\nu\rho}\,P_\mu\,, \quad \big[M_{\mu\nu}, K_\rho\big] = \eta_{\mu\rho}\,K_\nu - \eta_{\nu\rho}\,K_\mu\,, \quad \big[M_{\mu\nu}, D\big] = 0\,, \\ \big[D, P_\mu\big] = P_\mu\,, \quad \big[D, K_\mu\big] = -K_\mu\,, \qquad \big[K_\mu, P_\nu\big] = 2\,\eta_{\mu\nu}\,D + 2\,M_{\mu\nu}\,. \end{gathered} \tag{5.8}$$

In applications an important role is played by Casimir operators, most significantly the quadratic Casimir

$$C_2 = \tfrac{1}{2}\,\mathrm{tr}(\mathcal{M}^2) = -\tfrac{1}{2}\,\mathcal{M}_{AB}\mathcal{M}^{AB} = D(D-d) + \tfrac{1}{2}\,\mathrm{tr}(M^2) - P^\mu K_\mu\,. \tag{5.9}$$

This of course commutes with all conformal generators and it is easy to check $[C_2,\, K_\alpha] = 0$. There is also a 4th order Casimir with a more involved decomposition

$$
\begin{aligned}
C_4 &= \tfrac{1}{2}\,\mathrm{tr}(\mathcal{M}^4) \\
&= D^2(D-d)^2 + \tfrac{1}{2}d(d-1)\,D(D-d) + \tfrac{1}{2}\,\mathrm{tr}(M^4) - \tfrac{1}{2}\,\mathrm{tr}(M^2) \\
&\quad + \tfrac{1}{2}\,P^\mu P^\nu K_\mu K_\nu + \tfrac{1}{2}\,P^2 K^2 - 2\big(P^\mu D^2 K_\mu - P^\mu D M_\mu{}^\nu K_\nu + P^\mu (M^2)_\mu{}^\nu K_\nu\big) \\
&\quad + 3(d-1)\big(P^\mu D K_\nu - P^\mu M_\mu{}^\nu K_\nu)\big) - \tfrac{1}{2}(d-1)(3d-4)\,P^\mu K_\mu\,. \qquad (5.10)
\end{aligned}
$$

For this result we use $\mathrm{tr}(M^3) = -\frac{1}{2}(d-2)\,\mathrm{tr}(M^2)$. We may directly verify, albeit tediously[6], that $[C_4,\, K_\alpha] = 0$.

For any eigenstate of D with eigenvalue Δ it is easy to see that K_μ on the state decreases the eigenvalue by 1 whereas P_μ increased it by 1. If $|\Delta, R\rangle$ is a vector in a representation with a minimum eigenvalue Δ and belonging to representation R of $\mathfrak{so}(d-1,1)$ or $\mathfrak{so}(d)$ for the Lie algebra generated by $M_{\mu\nu}$ then we must have

$$
K_\mu|\Delta, R\rangle = 0\,, \qquad C_2|\Delta, R\rangle = c_{\Delta,R}|\Delta, R\rangle\,, \quad C_4|\Delta, R\rangle = d_{\Delta,R}|\Delta, R\rangle\,, \qquad (5.11)
$$

with

$$
c_{\Delta,R} = \Delta(\Delta - d) + c_R\,, \quad d_{\Delta,R} = \Delta^2(\Delta-d)^2 + \tfrac{1}{2}d(d-1)\Delta(\Delta-d) + d_R - c_R \qquad (5.12)
$$

where c_R, d_R are the corresponding eigenvalues for $\frac{1}{2}\,\mathrm{tr}(M)^2,\ \frac{1}{2}\,\mathrm{tr}(M^4)$ depending on the representation R. The infinite dimensional representation is spanned by the products of arbitrarily many momentum operators acting on $|\Delta, R\rangle$.

With the representation (3.28) and the decomposition (5.3) we may write

$$
-\tfrac{1}{2}\,\Gamma^A \bar{\Gamma}^B \mathcal{M}_{AB} = \begin{pmatrix} s^{\mu\nu} M_{\mu\nu} + \mathbb{1}\,D & \gamma^\mu P_\mu \\ -\bar{\gamma}^\mu K_\mu & \bar{s}^{\mu\nu} M_{\mu\nu} + \bar{\mathbb{1}}\,D \end{pmatrix}. \qquad (5.13)
$$

5.2 Action on Fields

The action of the conformal group generators on conformal primary fields ϕ_I may be decomposed as

$$
\begin{aligned}
\big[M_{\mu\nu}, \phi_I(x)\big] &= L_{\mu\nu}\phi_I(x) - (s_{\mu\nu})_I{}^J \phi_J(x)\,, \quad \big[D, \phi_I(x)\big] = (x\cdot\partial + \Delta)\phi_I(x)\,, \\
\big[P_\mu, \phi_I(x)\big] &= \partial_\mu \phi_I(x)\,, \\
\big[K_\mu, \phi_I(x)\big] &= \big(-x^2\partial_\mu + 2\,x_\mu\, x\cdot\partial + 2\,\Delta\, x_\mu\big)\phi_I(x) - 2x^\nu (s_{\mu\nu})_I{}^J \phi_J(x)\,. \qquad (5.14)
\end{aligned}
$$

Under finite scale transformations

$$
e^{\lambda\,D}\phi_I(x)\,e^{-\lambda D} = e^{\lambda\Delta}\phi_I(e^\lambda x)\,. \qquad (5.15)
$$

[6] A long winded calculation gives

$$
[\mathrm{tr}(M)^4, K_\alpha] = -\big(4(M^3)_\alpha{}^\mu + 6(d-1)(M^2)_\alpha{}^\mu + (3d^2-7d+6)M_\alpha{}^\mu + \tfrac{1}{2}(d-1)(d^2-3d+4)\delta_\alpha{}^\mu\big)K_\mu
$$

For the conjugate field $\bar{\phi}_{\bar{I}}$

$$\begin{aligned}
\big[M_{\mu\nu}, \bar{\phi}_{I}(x)\big] &= L_{\mu\nu}\bar{\phi}_{I}(x) + \bar{\phi}_{\bar{J}}(x)\,(\bar{s}_{\mu\nu})^{\bar{J}}{}_{\bar{I}}\,, \quad \big[D, \bar{\phi}_{\bar{I}}(x)\big] = (x\cdot\partial + \Delta)\bar{\phi}_{\bar{I}}(x)\,,\\
\big[P_{\mu}, \bar{\phi}_{\bar{I}}(x)\big] &= \partial_\mu \bar{\phi}_{\bar{I}}(x)\,,\\
\big[K_{\mu}, \bar{\phi}_{\bar{I}}(x)\big] &= \big(-x^2\partial_\mu + 2\,x_\mu\,x\cdot\partial + 2\,\Delta\,x_\mu\big)\bar{\phi}_{\bar{I}}(x) + \bar{\phi}_{\bar{J}}(x)\,(\bar{s}_{\mu\nu})^{\bar{J}}{}_{\bar{I}}\,2x^\nu\ . \qquad (5.16)
\end{aligned}$$

Although in general inversion is not a symmetry in CFTs we may use the inversion tensor to define a dual or conjugate field by

$$\tilde{\phi}^{\bar{I}}(x) = (x^2)^{-\Delta}\,\mathcal{I}^{\bar{I}I}(x)\,\phi_I(x/x^2)\,, \quad \tilde{\bar{\phi}}^{I}(x) = (x^2)^{-\Delta}\,\bar{\phi}_{\bar{I}}(x/x^2)\,\mathcal{I}^{\bar{I}I}(x)\,, \qquad (5.17)$$

where $\mathcal{I}^{\bar{I}I}(x)$ is the inverse of $\mathcal{I}_{I\bar{I}}(x/x^2) = \mathcal{I}_{I\bar{I}}(x)$. From (2.85)

$$\begin{aligned}
&x^\mu\partial_\mu\,\mathcal{I}^{\bar{I}I}(x) = 0\,, \qquad L_{\mu\nu}\,I^{\bar{I}I}(x) = (\bar{s}_{\mu\nu})^{\bar{I}}{}_{\bar{J}}\,\mathcal{I}^{\bar{J}I}(x) - \mathcal{I}^{\bar{I}J}(x)\,(s_{\mu\nu})_J{}^I\,,\\
&x^2\partial_\mu\,\mathcal{I}^{\bar{I}I}(x) = \,-2x^\nu(\bar{s}_{\mu\nu})^{\bar{I}}{}_{\bar{J}}\,\mathcal{I}^{\bar{J}I}(x) = \mathcal{I}^{\bar{I}J}(x)\,(s_{\mu\nu})_J{}^I\,2x^\nu\,, \qquad (5.18)
\end{aligned}$$

so that

$$\begin{aligned}
\big[M_{\mu\nu}, \tilde{\phi}^{\bar{I}}(x)\big] &= L_{\mu\nu}\tilde{\phi}^{\bar{I}}(x) - (\bar{s}_{\mu\nu})^{\bar{I}}{}_{\bar{J}}\,\tilde{\phi}^{\bar{J}}(x)\,, \quad \big[D, \tilde{\phi}^{\bar{I}}(x)\big] = -(x\cdot\partial + \Delta)\tilde{\phi}^{\bar{I}}(x)\,,\\
\big[P_{\mu}, \tilde{\phi}^{I}(x)\big] &= \big(x^2\partial_\mu - 2\,x_\mu\,x\cdot\partial - 2\,\Delta\,x_\mu\big)\tilde{\phi}^{I}(x) + 2x^\nu\,(\bar{s}_{\mu\nu})^{\bar{I}}{}_{\bar{J}}\,\tilde{\phi}^{\bar{J}}(x)\,,\\
\big[K_{\mu}, \tilde{\phi}^{\bar{I}}(x)\big] &= \,-\partial_\mu\tilde{\phi}^{\bar{I}}(x)\,, \qquad (5.19)
\end{aligned}$$

with similar results for $\tilde{\bar{\phi}}^{I}$.

For any CFT there is a correspondence between conformal primary fields and states. For the field $\bar{\phi}_{\bar{I}}$ the associated state is defined by the ket vector

$$|\bar{\phi}_{\bar{I}}\rangle = \bar{\phi}_{\bar{I}}(0)|0\rangle\,, \qquad (5.20)$$

and then from (5.16)

$$D|\bar{\phi}_{\bar{I}}\rangle = \Delta|\bar{\phi}_{\bar{I}}\rangle\,, \qquad M_{\mu\nu}|\bar{\phi}_{\bar{I}}\rangle = |\bar{\phi}_{\bar{J}}\rangle\,(\bar{s}_{\mu\nu})^{\bar{J}}{}_{\bar{I}}\,, \qquad K_\mu|\bar{\phi}_{\bar{I}}\rangle = 0\,. \qquad (5.21)$$

For any such conformal primary the associated infinite dimensional representation space is the Verma module, in integer d dimensions, defined in this case by

$$\mathcal{V}_{\Delta,R} = \mathrm{span}\,\big\{\textstyle\prod_{\mu=1}^{d} P_\mu{}^{n_\mu}|\bar{\phi}_{\bar{I}}\rangle : n_\mu = 0,1,\ldots\big\}\,. \qquad (5.22)$$

Here R labels the spin representation in d dimensions to which $\bar{\phi}_{\bar{I}}$ belongs. The states in the basis in (5.22), for $n_\mu > 0$ for some μ, are the descendants of the conformal primary with $\sum_\mu n_\mu$ the level. The module $\mathcal{V}_{\Delta,R}$ is obviously closed under the action of P_μ and the action of the conformal generators D, $M_{\mu\nu}$, K_μ on any state in $\mathcal{V}_{\Delta,R}$ is determined using the commutators (5.8) and (5.21). Thus at level N all states have scale dimension $\Delta + N$. Assuming the spin representation is irreducible so that

$$-\tfrac{1}{2}\,M_{\mu\nu}M^{\mu\nu}\,|\bar{\phi}_{\bar{I}}\rangle = c_R\,|\bar{\phi}_{\bar{I}}\rangle\,, \qquad (5.23)$$

then for the Casimir operator in (5.9) all states in the Verma module have the same eigenvalue so that

$$C_2\,\mathcal{V}_{\Delta,R} = c_{\Delta,R}\,\mathcal{V}_{\Delta,R}\,, \tag{5.24}$$

with $c_{\Delta,R}$ as in (5.12).

Corresponding to (5.20) there is then an associated dual bra vector given by

$$\langle\tilde\phi^{\bar I}| = \langle 0|\tilde\phi^{\bar I}(0)\,, \tag{5.25}$$

where taking the limit $x \to 0$ is well defined in the Euclidean regime. With the definition (5.25)

$$\langle\tilde\phi^{I}|D = \Delta\,\langle\tilde\phi^{I}|\,, \qquad \langle\tilde\phi^{I}|M_{\mu\nu} = (\bar s_{\mu\nu})^{\bar I}{}_{\bar J}\,\langle\tilde\phi^{\bar J}|\,, \qquad \langle\tilde\phi^{I}|P_\mu = 0\,, \tag{5.26}$$

which define a dual conformal primary. The corresponding dual Verma module is then given by

$$\bar{\mathcal{V}}_{\Delta,R} = \operatorname{span}\big\{\textstyle\prod_{\mu=1}^d \langle\tilde\phi^{\bar I}|K_\mu{}^{n_\mu} : n_\mu = 0,1,\dots\big\}\,. \tag{5.27}$$

The two point function (2.84) determines a scalar product so that

$$\langle\tilde\phi^{\bar I}|\bar\phi_{\bar J}\rangle \equiv \big\langle\tilde\phi^{\bar I}(0)\,\bar\phi_{\bar J}(0)\big\rangle = \delta^{\bar I}{}_{\bar J}\,. \tag{5.28}$$

This can be extended to all descendants defining a scalar product on $\bar{\mathcal{V}}_{\Delta,R}\times\mathcal{V}_{\Delta,R}$ assuming

$$K_\mu{}^\dagger = P_\mu\,, \qquad D^\dagger = D\,, \qquad M_{\mu\nu}{}^\dagger = -M_{\mu\nu}\,. \tag{5.29}$$

which requires the spin matrices $s_{\mu\nu}$, $\bar s_{\mu\nu}$ to be anti-hermitian. These hermeticity conditions in (2.55) are of course consistent with the commutation relations (5.6) and (5.8).

For scalar fields the two point function for ϕ and its dual becomes

$$\big\langle\tilde\phi(x)\,\phi(y)\big\rangle = \frac{1}{(1+x^2\,y^2 - 2\,x\cdot y)^\Delta}\,, \tag{5.30}$$

which reduces to (5.28) for $x, y \to 0$.

5.3 Conformal Generators in a Spinorial Basis

In subsection 2.6 it was shown how for $d = 3,4,5,6$ the spin generators belonging to $\mathfrak{so}(d-1,1)$, or $\mathfrak{so}(d)$, may be expressed in equivalent spinorial bases in each case using the special properties of the Dirac matrices for the particular dimension. Here we extend these results to the conformal generators satisfying (5.6) and (5.8).

In three dimensions we may re-express $M_{\mu\nu}, P_\mu, K_\mu$ by writing

$$\begin{aligned} &M_\alpha{}^\beta = M_{\mu\nu}\,(s^{\mu\nu})_\alpha{}^\beta\,, \\ &P_{\alpha\beta} = P_{\beta\alpha} = \tfrac12\,P_\mu\,(\gamma^\mu C^{-1})_{\alpha\beta}\,, \qquad K^{\alpha\beta} = K^{\beta\alpha} = \tfrac12\,K_\mu\,(C\gamma^\mu)^{\alpha\beta}\,. \end{aligned} \tag{5.31}$$

Using completeness conditions in the form

$$(C\gamma^\mu)^{\gamma\delta}\,(\gamma_\mu C^{-1})_{\alpha\beta} = \delta_\alpha{}^\gamma\,\delta_\beta{}^\delta + \delta_\beta{}^\gamma\,\delta_\alpha{}^\delta\,, \qquad -C^{-1}{}_{\alpha\beta}\,C^{\gamma\delta} = \delta_\alpha{}^\gamma\,\delta_\beta{}^\delta - \delta_\beta{}^\gamma\,\delta_\alpha{}^\delta\,, \tag{5.32}$$

and $M_{\mu\nu} = -M_\alpha{}^\beta (s_{\mu\nu})_\beta{}^\alpha$ then (5.6), (5.8) are equivalent to

$$\begin{aligned}
&\big[M_\alpha{}^\beta, M_\gamma{}^\delta\big] = \delta_\gamma{}^\beta M_\alpha{}^\delta - \delta_\alpha{}^\delta M_\gamma{}^\beta\,, \qquad \big[M_\alpha{}^\beta, P_{\gamma\delta}\big] = \delta_\gamma{}^\beta P_{\alpha\delta} + \delta_\delta{}^\beta P_{\gamma\alpha} - \delta_\alpha{}^\beta P_{\gamma\delta}\,,\\
&\big[M_\alpha{}^\beta, K^{\gamma\delta}\big] = -\,\delta_\alpha{}^\gamma K^{\beta\delta} - \delta_\alpha{}^\delta K^{\gamma\beta} + \delta_\alpha{}^\beta K^{\gamma\delta}\,, \qquad \big[D, P_{\alpha\beta}\big] = P_{\alpha\beta}\,, \quad \big[D, K^{\alpha\beta}\big] = -K^{\alpha\beta}\,,\\
&\qquad\qquad \big[K^{\gamma\delta}, P_{\alpha\beta}\big] = \delta_{(\alpha}{}^\gamma \delta_{\beta)}{}^\delta D + \delta_{(\alpha}{}^{(\gamma} M_{\beta)}{}^{\delta)}\,,
\end{aligned} \tag{5.33}$$

which extends (2.110). For a Euclidean metric $s_{\mu\nu}{}^\dagger = -s_{\mu\nu}$, $(C\gamma_\mu)^\dagger = \gamma_\mu C^{-1}$ then, corresponding to (5.29), the hermeticity conditions in this spinorial basis become

$$K^{\alpha\beta\dagger} = P_{\alpha\beta}\,, \qquad M_\alpha{}^{\beta\dagger} = M_\beta{}^\alpha\,, \qquad\qquad D^\dagger = D\,. \tag{5.34}$$

For $a = (\alpha, \alpha')$, $b = (\beta, \beta')$ the conformal generators are expressible as a 4×4 matrix

$$\big[\mathcal{M}_a{}^b\big] = \begin{pmatrix} M_\alpha{}^\beta + \delta_\alpha{}^\beta D & 2P_{\alpha\beta'} \\ -2\,K^{\alpha'\beta} & M^{\alpha'}{}_{\beta'} - \delta^{\alpha'}{}_{\beta'} D \end{pmatrix}, \qquad M^\alpha{}_\beta = -M_\beta{}^\alpha\,, \tag{5.35}$$

which satisfies

$$\mathcal{M}_{ab} = \mathcal{M}_a{}^c\, \mathcal{C}^{-1}{}_{cb} = \mathcal{M}_{ba}\,, \qquad \mathcal{M}^{ab} = \mathcal{C}^{ac}\mathcal{M}_c{}^b = \mathcal{M}^{ba}\,, \tag{5.36}$$

for

$$\big[\mathcal{C}^{-1}{}_{ab}\big] = \begin{pmatrix} 0 & \delta_\alpha{}^{\beta'} \\ -\delta^{\alpha'}{}_\beta & 0 \end{pmatrix}, \qquad \big[\mathcal{C}^{ab}\big] = \begin{pmatrix} 0 & -\delta^\alpha{}_{\beta'} \\ \delta_{\alpha'}{}^\beta & 0 \end{pmatrix}. \tag{5.37}$$

The commutation relations (5.33) are then equivalent to

$$\big[\mathcal{M}_a{}^b, \mathcal{M}_c{}^d\big] = \delta_c{}^b\, \mathcal{M}_a{}^d - \delta_a{}^d\, \mathcal{M}_c{}^b + \mathcal{C}^{bd}\, \mathcal{M}_{ac} + \mathcal{C}^{-1}{}_{ac}\, \mathcal{M}^{bd}\,. \tag{5.38}$$

This defines the Lie algebra for $\mathfrak{sp}(4)$.

In four dimensions we may define

$$\begin{aligned}
M_\alpha{}^\beta &= \tfrac12\, \mathcal{M}_{\mu\nu}\, (s^{\mu\nu})_\alpha{}^\beta\,, \qquad & \bar M^{\dot\alpha}{}_{\dot\beta} &= \tfrac12\, M_{\mu\nu}\, (\bar s^{\mu\nu})^{\dot\alpha}{}_{\dot\beta}\,,\\
P_{\alpha\dot\beta} &= \tfrac12\, P_\mu (\gamma^\mu)_{\alpha\dot\beta}\,, & K^{\dot\alpha\beta} &= \tfrac12\, K_\mu (\bar\gamma^\mu)^{\dot\alpha\beta}\,.
\end{aligned} \tag{5.39}$$

Using (2.115) as well as

$$(\gamma_\mu)_{\alpha\dot\alpha}\, (\bar\gamma^\mu)^{\dot\beta\beta} = 2\, \delta_\alpha{}^\beta\, \delta^{\dot\beta}{}_{\dot\alpha}\,, \tag{5.40}$$

the non zero commutators in this spinorial basis which follow from (5.6), (5.8) are then

$$\begin{aligned}
\big[M_\alpha{}^\beta, M_\gamma{}^\delta\big] &= \delta_\gamma{}^\beta M_\alpha{}^\delta - \delta_\alpha{}^\delta M_\gamma{}^\beta\,, & \big[\bar M^{\dot\alpha}{}_{\dot\beta}, \bar M^{\dot\gamma}{}_{\dot\delta}\big] &= -\delta^{\dot\alpha}{}_{\dot\delta}\, \bar M^{\dot\gamma}{}_{\dot\beta} + \delta^{\dot\gamma}{}_{\dot\beta}\, \bar M^{\dot\alpha}{}_{\dot\delta}\,,\\
\big[M_\alpha{}^\beta, P_{\gamma\dot\gamma}\big] &= \delta_\gamma{}^\beta P_{\alpha\dot\gamma} - \tfrac12\, \delta_\alpha{}^\beta P_{\gamma\dot\gamma}\,, & \big[M_\alpha{}^\beta, K^{\dot\gamma\gamma}\big] &= -\delta_\alpha{}^\gamma K^{\dot\gamma\beta} + \tfrac12\, \delta_\alpha{}^\beta K^{\dot\gamma\gamma}\,,\\
\big[\bar M^{\dot\alpha}{}_{\dot\beta}, P_{\gamma\dot\gamma}\big] &= -\,\delta^{\dot\alpha}{}_{\dot\gamma}\, P_{\gamma\dot\beta} + \tfrac12\, \delta^{\dot\alpha}{}_{\dot\beta}\, P_{\gamma\dot\gamma}\,, & \big[\bar M^{\dot\alpha}{}_{\dot\beta}, K^{\dot\gamma\gamma}\big] &= \delta^{\dot\gamma}{}_{\dot\beta}\, K^{\dot\alpha\gamma} - \tfrac12\, \delta^{\dot\alpha}{}_{\dot\beta}\, K^{\dot\gamma\gamma}\,,\\
\big[D, P_{\alpha\dot\beta}\big] &= P_{\alpha\dot\beta}\,, & \big[D, K^{\dot\alpha\beta}\big] &= -K^{\dot\alpha\beta}\,,
\end{aligned} \tag{5.41}$$

as well as

$$\big[K^{\dot\alpha\beta}, P_{\alpha\dot\beta}\big] = \delta_\alpha{}^\beta \delta^{\dot\alpha}{}_{\dot\beta}\, D + \delta^{\dot\alpha}{}_{\dot\beta}\, M_\alpha{}^\beta - \delta_\alpha{}^\beta\, \bar M^{\dot\alpha}{}_{\dot\beta}\,. \tag{5.42}$$

Corresponding to (5.29) the hermeticity conditions in this spinorial basis, with for a Euclidean metric $(s_{\mu\nu})^\dagger = -s_{\mu\nu}$, ${\gamma_\mu}^\dagger = \bar\gamma_\mu$, become

$$K^{\dot\alpha\beta\dagger} = P_{\beta\dot\alpha} \, , \qquad {M_\alpha}^{\beta\dagger} = {M_\beta}^\alpha \, , \qquad {\bar M^{\dot\alpha}}{}_{\dot\beta}{}^\dagger = {\bar M^{\dot\beta}}{}_{\dot\alpha} \, , \qquad D^\dagger = D \, . \tag{5.43}$$

Defining

$$\big[{\mathcal M_a}^b\big] = \begin{pmatrix} {M_\alpha}^\beta + \frac12 \, {\delta_\alpha}^\beta \, D & P_{\alpha\dot\beta} \\ -K^{\dot\alpha\beta} & \bar M^{\dot\alpha}{}_{\dot\beta} - \frac12 \, \delta^{\dot\alpha}{}_{\dot\beta} \, D \end{pmatrix} , \qquad \big[{\delta_a}^b\big] = \begin{pmatrix} {\delta_\alpha}^\beta & 0 \\ 0 & \delta^{\dot\alpha}{}_{\dot\beta} \end{pmatrix} , \tag{5.44}$$

where $a=(\alpha,\dot\alpha)$, $b=(\beta,\dot\beta)$, then the commutators (5.41), (5.42) are equivalent to

$$\big[{\mathcal M_a}^b , \, {\mathcal M_c}^d\big] = {\delta_c}^b \, {\mathcal M_a}^d - {\delta_a}^d \, {\mathcal M_c}^b \, . \tag{5.45}$$

This defines the Lie algebra $\mathfrak{sl}_4$, which is therefore equivalent to (5.2) in this case.

In six dimensions from (2.104) the six matrices $C\bar\gamma_\mu$, $\gamma_\mu C^{-1}$ form a dual basis for antisymmetric 4×4 matrices with the completeness relation

$$-\tfrac12 \, (C\bar\gamma^\mu)^{\gamma\delta} \, (\gamma_\mu C^{-1})_{\alpha\beta} = {\delta_\alpha}^\gamma \, {\delta_\beta}^\delta - {\delta_\beta}^\gamma \, {\delta_\alpha}^\delta \, . \tag{5.46}$$

The spinorial basis for the conformal generators is given by

$$\begin{aligned} &{M_\alpha}^\beta = \tfrac12 \, M_{\mu\nu} \, {(s^{\mu\nu})_\alpha}^\beta \, , \\ &P_{\alpha\beta} = -P_{\beta\alpha} = \tfrac12 \, P_\mu \, (\gamma^\mu C^{-1})_{\alpha\beta} \, , \qquad K^{\alpha\beta} = -K^{\beta\alpha} = -\tfrac12 \, K_\mu \, (C\bar\gamma^\mu)^{\alpha\beta} \, . \end{aligned} \tag{5.47}$$

Using $M_{\mu\nu} = -{M_\alpha}^\beta {(s_{\mu\nu})_\beta}^\alpha$ then (5.6), (5.8) are equivalent in this case to

$$\begin{aligned} &\big[{M_\alpha}^\beta, {M_\gamma}^\delta\big] = {\delta_\gamma}^\beta {M_\alpha}^\delta - {\delta_\alpha}^\delta {M_\gamma}^\beta \, , \qquad \big[{M_\alpha}^\beta, P_{\gamma\delta}\big] = {\delta_\gamma}^\beta P_{\alpha\delta} + {\delta_\delta}^\beta P_{\gamma\alpha} - \tfrac12 \, {\delta_\alpha}^\beta P_{\gamma\delta} \, , \\ &\big[{M_\alpha}^\beta, K^{\gamma\delta}\big] = -{\delta_\alpha}^\gamma K^{\beta\delta} - {\delta_\alpha}^\delta K^{\gamma\beta} + \tfrac12 \, {\delta_\alpha}^\beta K^{\gamma\delta} \, , \quad \big[D, P_{\alpha\beta}\big] = P_{\alpha\beta} \, , \quad \big[D, K^{\alpha\beta}\big] = -K^{\alpha\beta} \, , \\ &\qquad\qquad \big[K^{\gamma\delta}, P_{\alpha\beta}\big] = 2 \, \delta_{[\alpha}{}^\gamma \delta_{\beta]}{}^\delta D + 4 \, \delta_{[\alpha}{}^{[\gamma} M_{\beta]}{}^{\delta]} \, . \end{aligned} \tag{5.48}$$

The hermeticity conditions are identical to (5.34)

In a similar fashion to three dimensions, but interchanging symmetric and antisymmetric, the conformal generators can be encoded in terms of an 8×8 matrix

$$\big[{\mathcal M_a}^b\big] = \begin{pmatrix} {M_\alpha}^\beta + \frac12 \, {\delta_\alpha}^\beta \, D & P_{\alpha\beta'} \\ K^{\alpha'\beta} & M^{\alpha'}{}_{\beta'} - \frac12 \, \delta^{\alpha'}{}_{\beta'} \, D \end{pmatrix} , \qquad M^\alpha{}_\beta = -{M_\beta}^\alpha \, , \tag{5.49}$$

which satisfies

$$\mathcal M_{ab} = (\mathcal M \, \mathcal C^{-1})_{ab} = -\mathcal M_{ba} \, , \qquad \mathcal M^{ab} = (\mathcal C \mathcal M)^{ab} = -\mathcal M^{ba} \, , \tag{5.50}$$

for

$$\big[\mathcal C^{-1}{}_{ab}\big] = \begin{pmatrix} 0 & {\delta_\alpha}^{\beta'} \\ \delta^{\alpha'}{}_\beta & 0 \end{pmatrix} , \qquad \big[\mathcal C^{ab}\big] = \begin{pmatrix} 0 & \delta^\alpha{}_{\beta'} \\ {\delta_{\alpha'}}^\beta & 0 \end{pmatrix} . \tag{5.51}$$

The commutation relations (5.48) are then equivalent to

$$\big[\mathcal{M}_a{}^b, \mathcal{M}_c{}^d\big] = \delta_c{}^b\, \mathcal{M}_a{}^d - \delta_a{}^d\, \mathcal{M}_c{}^b - \mathcal{C}^{bd}\, \mathcal{M}_{ac} - \mathcal{C}^{-1}{}_{ac}\, \mathcal{M}^{bd}\,, \tag{5.52}$$

or

$$\big[\mathcal{M}_{ab}, \mathcal{M}_{cd}\big] = -\mathcal{C}^{-1}{}_{ac}\, \mathcal{M}_{bd} + \mathcal{C}^{-1}{}_{bc}\, \mathcal{M}_{ad} + \mathcal{C}^{-1}{}_{ad}\, \mathcal{M}_{bc} - \mathcal{C}^{-1}{}_{bd}\, \mathcal{M}_{ac}\,. \tag{5.53}$$

This is just the standard form for the Lie algebra of $\mathfrak{so}(8)$ albeit in a spinorial basis, which is a reflection of $SO(8)$ triality, the vector and the two chiral spinor representations each have dimension 8.

In five dimensions $(C\gamma_\mu, C)$, $(\gamma_\mu C^{-1}, C^{-1})$ form a dual basis of antisymmetric matrices with the completeness relations

$$-\tfrac{1}{2}\big((C\gamma^\mu)^{\gamma\delta}\,(\gamma_\mu C^{-1})_{\alpha\beta} + C^{\gamma\delta}\, C^{-1}{}_{\alpha\beta}\big) = \delta_\alpha{}^\gamma\, \delta_\beta{}^\delta - \delta_\beta{}^\gamma\, \delta_\alpha{}^\delta\,, \tag{5.54}$$

implying $-\frac{1}{2}\big((\gamma^\mu)_\gamma{}^\delta\,(\gamma_\mu)_\alpha{}^\beta + \delta_\gamma{}^\delta\,\delta_\alpha{}^\beta\big) = C^{-1}{}_{\gamma\alpha}\, C^{\delta\beta} - \delta_\gamma{}^\beta\,\delta_\alpha{}^\delta$, and also

$$\begin{aligned} &(\gamma^\mu C^{-1})_{[\alpha\beta}\,(\gamma^\nu C^{-1})_{\gamma\delta]} = \lambda\, \eta^{\mu\nu} \varepsilon_{\alpha\beta\gamma\delta}\,, \quad C^{-1}{}_{[\alpha\beta}\, C^{-1}{}_{\gamma\delta]} = -\lambda\, \varepsilon_{\alpha\beta\gamma\delta}\,, \\ &(\gamma^\mu C^{-1})_{[\alpha\beta}\,(C^{-1})_{\gamma\delta]} = 0\,, \end{aligned} \tag{5.55}$$

where $\lambda = -\frac{1}{3}\,\mathrm{Pf}(C^{-1})$, with the Pfaffian here defined just as in (3.62). The generators in a spinorial basis are given by

$$\begin{aligned} &M_\alpha{}^\beta = M_{\mu\nu}\,(s^{\mu\nu})_\alpha{}^\beta\,, \qquad (CMC^{-1})^\alpha{}_\beta = -M_\beta{}^\alpha\,, \\ &P_{\alpha\beta} = -\,P_{\beta\alpha} = \tfrac{1}{2}\, P_\mu\,(\gamma^\mu C^{-1})_{\alpha\beta}\,, \qquad K^{\alpha\beta} = -K^{\beta\alpha} = -\tfrac{1}{2}\, K_\mu\,(C\gamma^\mu)^{\alpha\beta}\,, \end{aligned} \tag{5.56}$$

with commutators

$$\begin{aligned} \big[M_\alpha{}^\beta, M_\gamma{}^\delta\big] &= \delta_\gamma{}^\beta M_\alpha{}^\delta - \delta_\alpha{}^\delta M_\gamma{}^\beta + C^{\beta\delta}\,(MC^{-1})_{\alpha\gamma} + C^{-1}{}_{\alpha\gamma}\,(CM)^{\beta\delta}\,, \\ \big[M_\alpha{}^\beta, P_{\gamma\delta}\big] &= \delta_\gamma{}^\beta P_{\alpha\delta} + \delta_\delta{}^\beta P_{\gamma\alpha} - C^{-1}{}_{\alpha\delta}\,(CP)^\beta{}_\gamma + C^{-1}{}_{\alpha\gamma}\,(CP)^\beta{}_\delta\,, \\ \big[M_\alpha{}^\beta, K^{\gamma\delta}\big] &= -\,\delta_\alpha{}^\gamma K^{\beta\delta} - \delta_\alpha{}^\delta K^{\gamma\beta} + C^{\beta\delta}\,(KC^{-1})^\gamma{}_\alpha - C^{\beta\gamma}\,(KC^{-1})^\delta{}_\alpha\,, \\ \big[D, P_{\alpha\beta}\big] &= P_{\alpha\beta}\,, \qquad \big[D, K^{\alpha\beta}\big] = -K^{\alpha\beta}\,, \\ \big[K^{\gamma\delta}, P_{\alpha\beta}\big] &= 2\big(\delta_{[\alpha}{}^\gamma \delta_{\beta]}{}^\delta + \tfrac{1}{4}\, C^{-1}{}_{\alpha\beta}\, C^{\gamma\delta}\big) D + 2\,\delta_{[\alpha}{}^{[\gamma} M_{\beta]}{}^{\delta]}\,. \end{aligned} \tag{5.57}$$

5.4 Positivity

For unitary representations of conformal symmetry it is necessary that the norm of all descendants in the Verma module $\mathcal{V}_{\Delta,R}$ defined in (5.22) should have positive definite norm. This is only possible if there are restrictions on Δ. The bounds on Δ are associated with singular vectors which are descendants $|\mathcal{O}_{\Delta',I'}\rangle$ such that, for some $\Delta' > \Delta$

$$|\mathcal{O}_{\Delta',I'}\rangle \in \mathcal{V}_{\Delta,\ell}\,, \qquad K_\mu |\mathcal{O}_{\Delta',I'}\rangle = 0\,. \tag{5.58}$$

The space spanned by $\{|\mathcal{O}_{\Delta',I'}\rangle\}$ is invariant under the action of $M_{\mu\nu}$ and so must form the representation space for a spin representation R'. The states $\{|\mathcal{O}_{\Delta',I'}\rangle\}$ therefore satisfy the conditions (5.21) defining a conformal primary. Hence, just as in (5.22), there is an associated Verma module $\mathcal{V}_{\Delta',R'}$. Clearly

$$\mathcal{V}_{\Delta',R'} \subset \mathcal{V}_{\Delta,R} \,, \tag{5.59}$$

forms an invariant subspace under the action of the conformal generators. In consequence $\mathcal{V}_{\Delta,R}$ is a reducible representation space but this reducibility is eliminated by taking the vector space quotient[7]

$$\mathcal{V}_{\Delta,R}/\mathcal{V}_{\Delta',R'} \,. \tag{5.60}$$

With the scalar product (5.28) extended to $\bar{\mathcal{V}}_{\Delta,R} \times \mathcal{V}_{\Delta,R}$ it is evident that

$$\langle \mathcal{O}_{\Delta',I'} | \mathcal{O}_{\Delta',J'} \rangle = 0 \,, \tag{5.61}$$

and consequently $|\mathcal{O}_{\Delta',J'}\rangle$, and all descendants in in $\mathcal{V}_{\Delta',R'}$, have zero norm or are null. For a unitary representation, with all states having positive norm, it is necessary to set $|\mathcal{O}_{\Delta',I'}\rangle = 0$.

For a conformal primary with arbitrary Δ and spin representation the norms of all descendants can be computed using the commutation relations (5.8), with $\eta_{\mu\nu} \to \delta_{\mu\nu}$. To illustrate we consider first a conformal primary scalar $|\phi\rangle$. Then at level one

$$\langle \phi | K_\mu \, P_\nu | \phi \rangle = 2 \, \langle \phi | (M_{\mu\nu} + \delta_{\mu\nu} \, D) | \phi \rangle = 2 \, \delta_{\mu\nu} \, \Delta \,. \tag{5.62}$$

Positivity requires

$$\Delta > 0 \,, \tag{5.63}$$

and if $\Delta = 0$ this implies $P_\nu|\phi\rangle = 0$, so that $|\phi\rangle \to |0\rangle$, the translationally invariant vacuum $|0\rangle$. At level two

$$\langle \phi | K_\sigma K_\rho \, P_\mu P_\nu | \phi \rangle = 4\Delta\big((\Delta+1)(\delta_{\sigma\mu}\,\delta_{\rho\nu} + \delta_{\sigma\nu}\,\delta_{\rho\mu}) - \delta_{\sigma\rho}\,\delta_{\mu\nu} \big) \,. \tag{5.64}$$

The eigenvectors are $\delta^{\mu\nu}$ and $\epsilon^{\mu\nu} = \epsilon^{\nu\mu}$, $\delta_{\mu\nu}\epsilon^{\mu\nu} = 0$, with eigenvalues $4\Delta(2\Delta + 2 - d)$ and $8\Delta(\Delta+1)$. Hence for (5.64) to give a positive definite norm we must have

$$\Delta > \tfrac{1}{2}(d-2) \,. \tag{5.65}$$

Furthermore we must have

$$\Delta = \tfrac{1}{2}(d-2) \quad \Rightarrow \quad P^2|\phi\rangle = 0 \,, \tag{5.66}$$

which corresponds to ϕ being a free scalar $\partial^2\phi = 0$. For $\Delta = -1$ there is also a null vector given by $\epsilon^{\mu\nu} P_\mu P_\nu |\phi\rangle$.

For states with spin we have at level one

$$\langle \tilde{\bar{\phi}}^{\bar{I}} | K^\sigma \, P_\rho \, | \bar{\phi}_{\bar{J}} \rangle = 2 \big(\delta^{\bar{I}}{}_{\bar{J}} \, \delta^\sigma{}_\rho \, \Delta - \bar{M}^{\sigma \bar{I}}{}_{\rho \bar{J}} \big) \,, \qquad \bar{M}^{\sigma \bar{I}}{}_{\rho \bar{J}} = -(\bar{s}^\sigma{}_\rho)^{\bar{I}}{}_{\bar{J}} = \tfrac{1}{2} (\bar{s}_{\mu\nu})^{\bar{I}}{}_{\bar{J}} \, (\bar{s}^{\mu\nu})^\sigma{}_\rho \,, \tag{5.67}$$

with $(\bar{s}^{\mu\nu})^\sigma{}_\rho = -\delta^{\mu\sigma}\,\delta^\nu{}_\rho + \delta^{\nu\sigma}\,\delta^\mu{}_\rho$ spin matrices for the vector representation. The matrix $\bar{M}^{\sigma\bar{I}}{}_{\rho\bar{J}}$ is just the conjugate of that defined in (2.69) and positivity requires in (5.67) that

[7] For a vector space V with a subspace U the quotient $V/U = \big\{ |v\rangle/\!\sim \, : |v\rangle \sim |v'\rangle \text{ if } |v\rangle - |v'\rangle \in U \big\}$, where it is easy to verify that V/U is a vector space.

$\Delta > \Delta_i$ the maximum eigenvalue determined by (2.72). Hence from the results in (2.80) and (2.83) then for symmetric traceless tensors of rank ℓ in general dimensions

$$\Delta > d - 2 + \ell \, , \quad \ell \geq 1 \, , \tag{5.68}$$

and for mixed symmetry tensors corresponding to a (n, m) Young tableaux

$$\Delta > d - 2 + n \, , \quad n > m \geq 1 \, , \qquad \Delta > d - 3 + m \, , \quad n = m \geq 1 \, . \tag{5.69}$$

In general the positivity restrictions require separate considerations for each dimension for spinorial representations. Here we list results for $d = 3, 4, 5, 6$ excluding the spin zero case where the bound just reduces to (5.63). From (2.114) in three dimensions for spin-s spinors

$$\Delta > \begin{cases} 1 + s \, , & s \geq 1 \, , \\ 1 \, , & s = \frac{1}{2} \, . \end{cases} \tag{5.70}$$

In four dimensions from (2.124) for $(s, \bar{s})$ spinors

$$\Delta > \begin{cases} 2 + s + \bar{s} \, , & s, \bar{s} \geq \frac{1}{2} \, , \\ 1 + s \, , & s \geq \frac{1}{2} \, , \, \bar{s} = 0 \, , \\ 1 + \bar{s} \, , & \bar{s} \geq \frac{1}{2} \, , \, s = 0 \, , \end{cases} \tag{5.71}$$

as was first shown by Mack [18]. When these bounds are saturated, and the representation becomes reducible, then the corresponding fields are required to obey differential constraints to ensure irreducibility. This for $\Delta = \ell + d - 2$ the tensor is conserved, and in four dimensions $\Delta = 1 + s$ for $\bar{s} = 0$ necessitates that the chiral spinor obeys the Dirac equation.

In six dimensions the bounds for the scale dimension of a primary operator with spin representation $[s_1, s_2, s_3]$ follow from (2.137)

$$\Delta > \begin{cases} 4 + s_2 + \frac{1}{2}(s_1 + s_3) \, , & s_2 \geq 1 \, , \\ 3 + \frac{1}{2}(s_1 + s_3) \, , & s_2 = 0 \, , \, s_1, s_3 \geq 1 \, , \\ 2 + \frac{1}{2} \, s_1 \, , & s_2 = s_3 = 0 \, , \, s_1 \geq 1 \, , \\ 2 + \frac{1}{2} \, s_3 \, , & s_2 = s_1 = 0 \, , \, s_3 \geq 1 \, . \end{cases} \tag{5.72}$$

In five dimensions from (2.152)

$$\Delta > \begin{cases} 3 + s_2 + \frac{1}{2} \, s_1 \, , & s_2 \geq 1 \, , \\ 2 + \frac{1}{2} \, s_1 \, , & s_2 = 0 \, , \, s_1 \geq 2 \, , \\ 2 \, , & s_2 = 0 \, , \, s_1 = 1 \, . \end{cases} \tag{5.73}$$

Positivity restrictions equivalent to (5.72), (5.73) were obtained in [19] and for general d in [20] and [21].

6 Conformal and $SO(d)$ Representations

We discuss here some aspects of the representations of the conformal and rotation groups in various dimensions which are relevant in the understanding of CFTs. The Verma module defined in (5.22) provides a representation space for the conformal group which has positive energy in the sense that the scale dimension eigenvalues are bounded below by a minimum Δ. Since the conformal group is non compact unitary representations are necessarily infinite dimensional but the Verma module may give rise for particular Δ to a reducible representation.

6.1 Singular Vectors

The Verma module gives a reducible representation when there are singular vectors solving (5.58).[8] The determination of singular vectors in general requires finding all vectors in the Verma module, at an arbitrary level, annihilated by K_μ subject to conditions on Δ and the spin of the conformal primary state. Here we determine singular vectors in the Verma module $\mathcal{V}_{\Delta,\ell}$, formed from a conformal primary which is a symmetric traceless tensor of rank ℓ and therefore satisfies

$$D|\mathcal{O}_{\sigma_1\dots\sigma_\ell}\rangle = \Delta\,|\mathcal{O}_{\sigma_1\dots\sigma_\ell}\rangle\,, \qquad K_\mu|\mathcal{O}_{\sigma_1\dots\sigma_\ell}\rangle = 0\,,$$
$$M_{\mu\nu}|\mathcal{O}_{\sigma_1\dots\sigma_\ell}\rangle = \textstyle\sum_{s=0}^{\ell}\big(-\delta_{\nu\sigma_s}\,|\mathcal{O}_{\sigma_1\dots\mu\dots\sigma_\ell}\rangle + \delta_{\mu\sigma_s}\,|\mathcal{O}_{\sigma_1\dots\nu\dots\sigma_\ell}\rangle\big)\,, \tag{6.1}$$

so that $-\frac{1}{2}M_{\mu\nu}M_{\mu\nu}|\mathcal{O}_{\sigma_1\dots\sigma_\ell}\rangle = C_{[\ell]}\,|\mathcal{O}_{\sigma_1\dots\sigma_\ell}\rangle$ with $C_{[\ell]} = \ell(\ell+d-2)$. For these states the Casimir eigenvalue is then $C(\Delta,\ell) = \Delta(\Delta-d)+\ell(\ell+d-2)$. For arbitrary d the Verma module in this case can be expressed as a sum over the level N so that $\mathcal{V}_{\Delta,\ell} \simeq \oplus_N \mathcal{V}_{\Delta,\ell,N}$ where

$$\mathcal{V}_{\Delta,\ell,N} = \mathrm{span}\,\big\{P_{\mu_1}\dots P_{\mu_N}|\mathcal{O}_{\sigma_1\dots\sigma_\ell}\rangle\big\}\,. \tag{6.2}$$

For simplicity we determine here singular vectors which are symmetric traceless tensors of rank k and are then expressible as

$$|\mathcal{O}^{(\ell)}{}_{k,n,r}(t)\rangle = (P\cdot t)^{k-r}\,(P^2)^n\,t_{\sigma_1}\dots t_{\sigma_r}\,P_{\sigma_{r+1}}\dots P_{\sigma_\ell}\,|\mathcal{O}_{\sigma_1\dots\sigma_\ell}\rangle\,,$$
$$n,k,r = 0,1,\dots\,, \qquad r\le k,\ell\,, \tag{6.3}$$

where t_μ is a d-dimensional null vector, $t^2=0$. It is easy to check $D|\mathcal{O}^{(\ell)}{}_{k,n+r,r}(t)\rangle = \Delta_{k,n}|\mathcal{O}^{(\ell)}{}_{k,n+r,r}(t)\rangle$ for $\Delta_{k,n} = \Delta+\ell+k+2n$.

From the commutation relations (5.8)

$$\big[K_\mu,(P\cdot t)^k\big] = 2k\,(P\cdot t)^{k-1}\big(t_\nu M_{\mu\nu} + t_\mu\,(D+k-1)\big)\,,$$
$$\big[K_\mu,(P^2)^n\big] = 4n\,(P^2)^{n-1}\big(P_\nu M_{\mu\nu} + P_\mu\,(D+n-\tfrac{1}{2}d)\big)\,. \tag{6.4}$$

[8] Singular vectors are crucial in the representation theory for Lie algebras. For the simple case of $\mathfrak{sl}(2)$, with $[J_+,J_-]=2J_3,\ [J_3,J_\pm]=\pm J_\pm$, the highest weight vectors satisfy $J_+|j,j\rangle=0$ and then ${J_-}^{2j+1}|j.j\rangle$ is a singular vector if $2j=0,1,2,\dots$ since then $J_+{J_-}^{2j+1}|j.j\rangle=0$. Finite dimensional unitary representations, with $J_-={J_+}^\dagger$, $J_3={J_3}^\dagger$, as usual are obtained by setting ${J_-}^{2j+1}|j,j\rangle=0$.

and

$$K_\mu\, P_{\sigma_{r+1}} \dots P_{\sigma_\ell}\, |\mathcal{O}_{\sigma_1\dots\sigma_\ell}\rangle = 2(\ell-r)(\Delta-d-r+1)\, P_{\sigma_{r+1}} \dots P_{\sigma_{\ell-1}}\, |\mathcal{O}_{\mu\sigma_1\dots\sigma_{\ell-1}}\rangle\,, \tag{6.5}$$

we may obtain

$$\begin{aligned} K_\mu\, |\mathcal{O}^{(\ell)}{}_{k,n,r}(t)\rangle = {}& 4n(\Delta+\ell+n-r-\tfrac{1}{2}d)\, P_\mu|\mathcal{O}^{(\ell)}{}_{k,n-1,r}(t)\rangle \\ & + 2(k-r)(\Delta+k+\ell+2n-r-1)\, t_\mu|\mathcal{O}^{(\ell)}{}_{k-1,n,r}(t)\rangle \\ & + 4nr\, t_\mu|\mathcal{O}^{(\ell)}{}_{k-1,n-1,r-1}(t)\rangle \\ & + 2(\ell-r)(\Delta-d-r+1)\, |\mathcal{O}^{(\ell)}{}_{\mu,k,n,r}(t)\rangle \\ & - 4nr\, |\mathcal{O}^{(\ell)}{}_{\mu,k,n-1,r-1}(t)\rangle\,, \end{aligned} \tag{6.6}$$

for

$$\begin{aligned} |\mathcal{O}^{(\ell)}{}_{\sigma,k,n,r}(t)\rangle = (P\cdot t)^{k-r}\, (P^2)^n\, t_{\sigma_1}\dots t_{\sigma_r}\, P_{\sigma_{r+1}}\dots P_{\sigma_{\ell-1}}\, |\mathcal{O}_{\sigma\,\sigma_1\dots\sigma_{\ell-1}}\rangle\,, \\ n\ge 0\,,\ \ell\ge 1,\ 0\le r\le \ell-1, k\,. \end{aligned} \tag{6.7}$$

Using (6.6) there are singular vectors in $\mathcal{V}_{\Delta,\ell}$ when

$$\Delta = \tfrac{1}{2}d - m\,, \qquad k=\ell\,,\ m=\ell+n-r = 1,2,\dots\,, \tag{6.8a}$$

$$\Delta = 1-\ell-m\,, \qquad n=0\,,\ r=\ell\,,\ m=k-\ell=1,2,\dots\,, \tag{6.8b}$$

$$\Delta = d+k-1\,, \qquad n=0\,,\ r=k\,,\ \ k=0,1,\dots,\ell-1\,. \tag{6.8c}$$

The case $n=0$, $r=k=\ell$ of course corresponds to the original conformal primary. The two solutions given by (6.8b), (6.8c) are evident from (6.6). To obtain (6.8a) it is sufficient to assume the singular vectors are of the form

$$|\mathcal{O}_s^{(\ell)}(t)\rangle = \textstyle\sum_{-n,0\le r\le \ell} \epsilon_r\, |\mathcal{O}^{(\ell)}{}_{\ell,n+r,r}(t)\rangle\,, \quad \Delta_s = m-\tfrac{1}{2}d\,, \tag{6.9}$$

where ϵ_r are required to satisfy

$$\begin{aligned} (\ell-r)(\Delta+2\ell+2n+r-1)\,\epsilon_r + 2(n+r+1)(r+1)\,\epsilon_{r+1} = 0\,, \\ (\ell-r)(\Delta-d-r+1)\,\epsilon_r - 2(n+r+1)(r+1)\,\epsilon_{r+1} = 0\,. \end{aligned} \tag{6.10}$$

Consistency requires $\Delta = \frac{1}{2}d-\ell-n$, agreeing with (6.8a) for $n\to n-r$ and ensuring also that the coefficient of $P_\mu|\mathcal{O}^{(\ell)}{}_{k,n-1,r}(t)\rangle$ vanishes in (6.6). The solution for the singular vector in (6.9) is then given, up to an arbitrary factor, by

$$\epsilon_r = \frac{(-1)^r\,(\frac{1}{2}d+\ell+n-1)_r}{2^r\, r!\,(\ell-r)!\,(n+r)!}\,. \tag{6.11}$$

As a special case for $\ell=0$ $(\partial^2)^n|\mathcal{O}\rangle$ is a singular vector for $\Delta=\frac{1}{2}d-n$. For (6.8b), (6.8c) the singular vectors are just

$$\begin{aligned} |O_s^{(k)}(t)\rangle &= (P\cdot t)^m\, t_{\sigma_1}\dots t_{\sigma_\ell}\, |\mathcal{O}_{\sigma_1\dots\sigma_\ell}\rangle\,, \quad k=\ell+m\,,\ \ \Delta_s = 1-\ell\,, \\ |O_s^{(k)}(t)\rangle &= t_{\sigma_1}\dots t_{\sigma_k}\, P_{\sigma_{k+1}}\dots P_{\sigma_\ell}\, |\mathcal{O}_{\sigma_1\dots\sigma_\ell}\rangle\,, \quad \Delta_s = d+\ell-1\,, \end{aligned} \tag{6.12}$$

and we may directly verify $C(\Delta_s,k) = C(\Delta,\ell)$ in each case. For (6.8a), when from (6.9) $\Delta_s = d-\Delta$, $\Delta(d-\Delta)$ is invariant. The results for singular vectors in (6.8) were obtained in [22].

At level one the conditions in (6.8b), (6.8c) match the results obtained directly in (2.80), while (6.8a) for $\ell=0$, $n=1$ corresponds to (5.66).

6.2 $SO(d)$ Tensorial Representations and Null Vectors

Tensorial representations of $SO(d)$ can be defined for arbitrary d. Irreducible representations are defined in terms of representations of the permutation group and are summarised in terms of Young tableaux but it is also necessary to impose tracelessness under contraction of tensorial indices. For symmetrisation and removal of the traces for a set of tensorial indices it is very convenient to contract them with d-dimensional null vectors.

For the simplest case of totally symmetric rank k tensors it is sufficient to use a single null vector t_μ, as was done in (6.3) to ensure that $|\mathcal{O}^{(\ell)}{}_{k,n,r}(t)\rangle$ transformed under the action of $M_{\mu\nu}$ as an irreducible representation of $\mathfrak{so}(d)$. The action of the Lie algebra $\mathfrak{so}(d)$ on arbitrary $f(t)$ is given by the differential operator

$$L_{\mu\nu} = -t_\mu\,\partial_\nu + t_\nu\,\partial_\mu\,. \tag{6.13}$$

The action of $L_{\mu\nu}$ preserves the constraint $t^2 = 0$ but more generally to define a derivative acting on $f(t)$ restricted to $t^2 = 0$ it is necessary to extend the usual derivative $\partial_\mu \to \nabla_\mu$ [23]. For applications here ∇_μ is defined by

$$\nabla_\mu\, f(t) = \partial_\mu\, f(t) - t_\mu\,\frac{1}{2n+d-4}\,\partial\cdot\partial f(t)\,, \tag{6.14}$$

assuming $f(\lambda t) = \lambda^n f(t)$ and where ∂_μ is the conventional derivative, $\partial_\mu t_\nu = \delta_{\mu\nu}$, without regard to the constraint $t^2 = 0$. Taking $n = 2$, (6.14) gives $\nabla_\mu\, t^2 = 0$. With the prescription (6.14) the standard Leibnitz rule for differentiation of a product is then modified to

$$\nabla_\mu\big(f(t)\,g(t)\big) = \nabla_\mu f(t)\,g(t) + f(t)\,\nabla_\mu g(t) - t_\mu\,\frac{2}{2n+d-4}\,\nabla_\nu f(t)\,\nabla_\nu g(t)\,,\quad n = n_f + n_g\,. \tag{6.15}$$

Furthermore

$$\big[\nabla_\mu, \nabla_\nu\big] = 0\,, \qquad \nabla\cdot\nabla = 0\,, \tag{6.16}$$

and

$$[L_{\mu\nu},\, t_\sigma] = \delta_{\mu\sigma}\,t_\nu - \delta_{\nu\sigma}\,t_\mu\,, \qquad [L_{\mu\nu},\, \nabla_\sigma] = \delta_{\mu\sigma}\,\nabla_\nu - \delta_{\nu\sigma}\,\nabla_\mu\,. \tag{6.17}$$

Directly from (6.14) and (6.15)

$$t_\mu\nabla_\mu f(t) = n\,f(t)\,,\quad \nabla_\mu\big(t_\mu f(t)\big) = a_n\,f(t)\,,\quad a_n = \frac{(2n+d)(n+d-2)}{2n+d-2}\,, \tag{6.18}$$

and also

$$t_\nu\nabla_\mu\,t_\nu\,f(t) = b_n\,t_\mu\,f(t)\,,\quad \nabla_\nu\,t_\mu\nabla_\nu\,f(t) = b_{n-1}\,\nabla_\mu\,f(t)\,,\quad b_n = \frac{d-2}{2n+d-2}\,. \tag{6.19}$$

For $SO(d)$ there are $\lfloor\frac{1}{2}d\rfloor$ Casimir operators. There is always a quadratic Casimir operator

$$C_{R,2} = \tfrac{1}{2}L_{\mu\nu}L_{\nu\mu}\,, \tag{6.20}$$

and acting on symmetric traceless tensors from (6.13)

$$C_{R,2}\,f(t) = \big(t_\nu\nabla_\mu\ - t_\mu\nabla_\nu\big)t_\mu\nabla_\nu f(t) = c_{(n)}\,f(t)\,, \tag{6.21}$$

the associated eigenvalue is then given using (6.18) and (6.19) by

$$c_{(n)} = n\big(a_{n-1} - b_{n-1}\big) = n(n+d-2)\,, \tag{6.22}$$

just as in (2.79). The quartic Casimir is defined by

$$C_{R,4} = \tfrac{1}{2}L_{\nu\mu}L_{\mu\sigma}L_{\sigma\rho}L_{\rho\nu}\,, \tag{6.23}$$

and acting on symmetric traceless tensors

$$C_{R,4}\, f(t) = d_{(n)}\, f(t)\,, \qquad d_{(n)} = c_{(n)}\big(c_{(n)} + \tfrac{1}{2}(d-2)(d-3)\big)\,. \tag{6.24}$$

To evaluate $d_{(n)}$ using it is sufficient to use (6.17) and (6.18) to obtain

$$\begin{aligned} L_{\mu\sigma} t_\rho \nabla_\mu\, f(t) ={}& -\big((n-1)\, t_\rho \nabla_\sigma - t_\sigma \nabla_\rho + n\, \delta_{\sigma\rho}\big) f(t)\,, \\ L_{[\mu|\sigma} L_{\sigma\rho} L_{\rho|\nu]}\, f(t) ={}& \big(c_{(n)} + \tfrac{1}{2}(d-2)(d-3)\big) L_{\mu\nu}\, f(t)\,. \end{aligned} \tag{6.25}$$

For the conformal Casimir C_4 acting on symmetric traceless fields as in (5.12) then the contribution of the $SO(d)$ Casimirs is $d_{(n)} - c_{(n)} = {c_{(n)}}^2 + \frac{1}{2}(d-1)(d-4)\, c_{(n)}$. For symmetric traceless tensors where the representation depends only on the rank n the quartic Casimir eigenvalue is not independent of the quadratic eigenvalue.

The vector space $V_{(n)}$ formed from symmetric rank n tensors after subtracting all traces corresponds exactly to homogeneous functions $f(t)$ of null vectors t with degree n so that $f(t) \in V_{(n)}$. The formal dimension (this takes integer values for integer d) is then

$$\dim V_{(n)} = N_{d,(n)} = \tfrac{1}{n!}\, (d)_n - \tfrac{1}{(n-2)!}\, (d)_{n-2} = \tfrac{1}{n!}\, (2n+d-2)\, (d-1)_{n-1}\,, \tag{6.26}$$

with the Pochhammer symbol $(a)_n$ defined by

$$(a)_n = \frac{\Gamma(a+n)}{\Gamma(a)}\,, \qquad (a)_n = a(a+1)\dots(a+n-1)\,, \quad n \ge 1\,. \tag{6.27}$$

From (6.26) $N_{3,(n)} = 2n+1$, $N_{4,(n)} = (n+1)^2$ and, for integer d, $N_{d,(n)}$ is polynomial in n of degree $d-2$. The results satisfy the symmetry for integer d

$$c_{\sigma(n)} = c_{(n)}\,, \quad d_{\sigma(n)} = d_{(n)}\,, \quad N_{d,\sigma(n)} = (-1)^d N_{d,(n)}\,, \qquad \sigma(n) = (-d-n+2)\,, \tag{6.28}$$

where $\sigma^2 = e$.

With the definition (6.3) $|\mathcal{O}^{(\ell)}{}_{k,n,r}(t)\rangle$ is a scalar under rotations generated by $M_{\mu\nu}$ and $L_{\mu\nu}$, $(M_{\mu\nu} + L_{\mu\nu})|\mathcal{O}^{(\ell)}{}_{k,n,r}(t)\rangle = 0$, so that

$$-\tfrac{1}{2}\, M_{\mu\nu} M_{\mu\nu} |\mathcal{O}^{(\ell)}{}_{k,n,r}(t)\rangle = -\tfrac{1}{2}\, L_{\mu\nu} L_{\mu\nu} |\mathcal{O}^{(\ell)}{}_{k,n,r}(t)\rangle = c_{(k)}\, |\mathcal{O}^{(\ell)}{}_{k,n,r}(t)\rangle\,. \tag{6.29}$$

Additional irreducible representations of $\mathfrak{so}(d)$ for general d can be constructed with extra null vectors. For two such vectors t, s then

$$L_{\mu\nu} = L_{t,\mu\nu} + L_{s,\mu\nu}\,. \tag{6.30}$$

For homogeneous functions $f(t,s) \in V_{(n)} \otimes V_{(m)}$, $f(\lambda t, \mu s) = \lambda^n \mu^m f(t,s)$, then imposing the conditions

$$\nabla_t \cdot \nabla_s f(t,s) = t \cdot \nabla_s f(t,s) = 0\,, \tag{6.31}$$

ensures $f(t,s) \in V_{(n,m)}$ forming the representation space for a mixed symmetry irreducible representation corresponding to the Young tableaux $(n, m, 0, \dots)$ for $m \le n$, $\begin{array}{|c|c|c|c|c|c|c|} \hline 1 & 2 & & & & & n \\ \hline 1 & & & m \\ \cline{1-4} \end{array}$. To verify that, subject to these conditions $f(t,s)$, defines an irreducible representation we may consider the action of the quadratic Casimir $C_{R,2} = \frac{1}{2}\, L_{\nu\mu} L_{\mu\nu}$. Since

$$\begin{aligned} L_{t,\nu\mu} L_{s,\mu\nu}\, f(t,s) ={}& -2\, s \cdot \nabla_s f(t,s) + 2\, s \cdot \nabla_t \big(t \cdot \nabla_s f(t,s)\big) - 2\, \frac{2n+d-4}{2n+d-2}\, t \cdot s\, \nabla_t \cdot \nabla_s f(t,s) \\ ={}& -2m\, f(t,s) \end{aligned} \tag{6.32}$$

after applying (6.31), and using (6.21) the Casimir has a unique eigenvalue as befits an irreducible representation.

$$C_{R,2}\, f(t,s) = c_{(n,m)}\, f(t,s)\,, \qquad c_{(n,m)} = c_{(n)} + \tilde{c}_{(m)}\,, \qquad \tilde{c}_{(m)} = m(m+d-4)\,, \tag{6.33}$$

with $c_{(n,1)}$ identical to (2.79). For the corresponding quartic Casimir a rather lengthy calculation gives

$$C_{R,4}\, f(t,s) = d_{(n,m)}\, f(t,s)\,, \qquad d_{(n,m)} = d_{(n)} + \tilde{c}_{(m)}\big(\tilde{c}_{(m)} + \tfrac{1}{2}(d-3)(d-6)\big)\,. \tag{6.34}$$

For $d=3$, $c_{(n,1)} = c_{(n,0)} = n(n+1)$, $d_{(n,1)} = d_{(n,0)} = n^2(n+1)^2$.

For $f(t,s) \in V_{(n)} \otimes V_{(m)}$ then $t\cdot\nabla_s f(t,s) \in V_{(n+1)} \otimes V_{(m-1)}$ and $\nabla_t\cdot\nabla_s f(t,s) \in V_{(n-1)} \otimes V_{(m-1)}$. To determine the dimension of this mixed symmetry representation $V_{(n,m)}$ it is necessary to note that the conditions in (6.31) are not completely independent as a consequence of

$$\nabla_t\cdot\nabla_s\, t\cdot\nabla_s\, f(t,s) = \frac{1}{2n+d-2}\, t\cdot\nabla_s\, \nabla_t\cdot\nabla_s f(t,s) \in V_{(n)} \otimes V_{(m-2)}\,. \tag{6.35}$$

This ensures the dimension counting is given by

$$\begin{aligned} \dim V_{(n,m)} = N_{(n,m)} &= N_{d,(n)}\, N_{d,(m)} - N_{d,(n+1)}\, N_{d,(m-1)} - N_{d,(n-1)}\, N_{d,(m-1)} + N_{d,if(n)}\, N_{d,(m-2)} \\ &= \tfrac{1}{(n+1)!\, m!}\, (n+1-m)(n+m+d-3)(2n+d-2)(2m+d-4) \\ &\qquad\qquad \times (d-1)_{n-2}(d-3)_{m-1}\,. \end{aligned} \tag{6.36}$$

For integer d this can be rewritten as

$$N_{(n,m)} = \tfrac{1}{(d-4!\,(d-2)!}\, (n+1-m)(n+m+d-3)(2n+d-2)(2m+d-4)(n+2)_{d-5}(m+1)_{d-5}\,. \tag{6.37}$$

Of course $V_{(n,0)} \simeq V_{(n)}$ and $N_{(n,0)} = N_{d,(n)}$ though for $d=3,4$ it is necessary to set $m=0$ in $N_{(n,m)}$ and then $d \to 3,4$. For d an integer $N_{(n,m)}$ is just a product of linear factors in n,m. For $d=4$ (6.36) gives $N_{(n,m)} = 2(n+m+1)(n-m+1)$ for $m>0$ and the representation corresponds to $R_{[s,\bar{s}]} \oplus R_{[\bar{s},s]}$ with $n = s+\bar{s}$, $m = s-\bar{s}$ so that the results are equivalent to those given in (2.120). For $d=5,6$ (6.36) agrees with (2.147), (2.133) for $s_1 = 2m$, $s_2 = n-m$ and $s_1 = s_3 = m$, $s_2 = n-m$ respectively. In three dimensions mixed symmetry representations are reduced by the identifications

$$d = 3\,, \quad V_{(n,m)} \simeq 0,\ m \ge 2\,, \quad V_{(n,1)} \simeq V_{(n)}\,, \quad N_{(n,0)} = N_{(n,1)} = 2n+1\,, \tag{6.38}$$

where in (6.36) $(0)_{-1} = -1$.

The results in (6.28) may be extended by defining, for integer d,

$$a(n,m) = (-m-d+3, n+1)\,, \quad b(n,m) = (m-1, n+1)\,, \qquad a^4 = b^2 = e\,, \ \ ab = ba^3\,, \tag{6.39}$$

so that

$$\begin{aligned} c_{a(n,m)} &= c_{b(n,m)} = c_{(n,m)}\,, & d_{a(n,m)} &= d_{b(n,m)} = d_{(n,m)}\,, \\ N_{a(n,m)} &= (-1)^{d-1} N_{(n,m)}\,, & N_{b(n,m)} &= -N_{(n,m)}\,. \end{aligned} \tag{6.40}$$

(a,b) generate the 8 element dihedral group D_4[9].

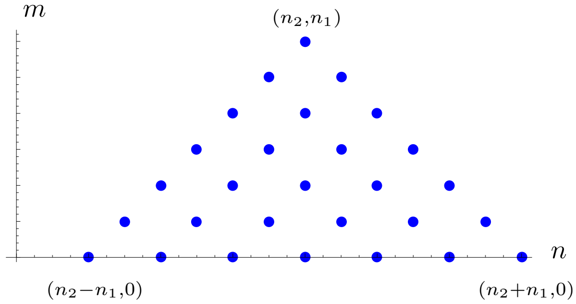
Tensor products of symmetric traceless tensors can be reduced to a sum over mixed symmetry irreducible representations $V_{(n,m)}$,

$$V_{(n_1)} \otimes V_{(n_2)} \simeq \bigoplus_{m=0}^{\min(n_1,n_2)} \ \bigoplus_{r=0}^{\min(n_1,n_2)-m} V_{(n_1+n_2-m-2r,m)}\,. \tag{6.41}$$

[9] $a^2(n,m)=(2-d-n,4-d-m)$, $a^3(n,m)=(m-1,3-d-n)$, $ab(n,m)=(2-d-n,m)$, $a^2b(n,m)=(3-d-m,3-d-n)$, $a^3b(n,m)=(n,4-d-m)$.

Possible (n,m) are restricted by $|n_1-n_2|+m \leq n \leq n_1+n_2-m$ and n_1+n_2-n-m even. Replacing $\min(n_1,n_2)$ just by n_1 in (6.41) the possible (n,m) appearing in the tensor product can be illustrated by

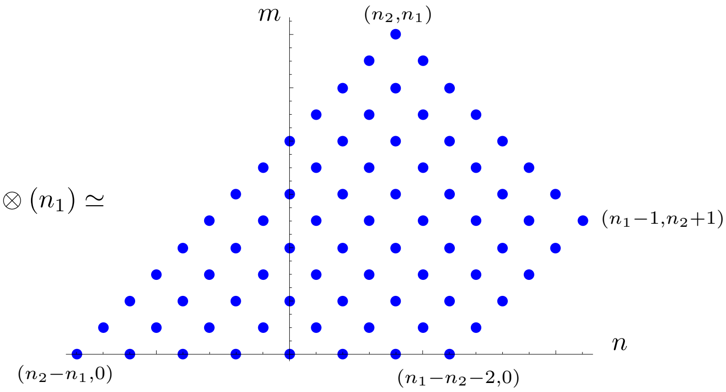
$$(n_1)\otimes(n_2) \simeq \tag{6.42}$$



For $n_1 \leq n_2$ all (n,m) satisfy $n \geq m \geq 0$. With the dimension formuls (6.36) we may verify $N_{d,(n_1)}N_{d,(n_2)} = \sum_{m=0}^{n_1}\sum_{r=0}^{n_1-m} \dim V_{(n_1+n_2-m-2r,m)}$ irrespective of the sign of n_2-n_1. For $d=3$, (6.41) using (6.38) reduces to $V_{(n_1)}\otimes V_{(n_2)} \simeq \oplus_{r=|n_1-n_2|}^{n_1+n_2} V_{(r)}$. The contributions to the sum $V_{(n_1+n_2-r)}$ for r odd derive in (6.41) from $V_{(n_1+n_2-1-2r,1)}$ using (6.38).

For $n_1 > n_2$ (6.42) contains points (n,m) with $n<0$. The difference then with the restrictions of the sums to $n_2 < n_1$ has the form

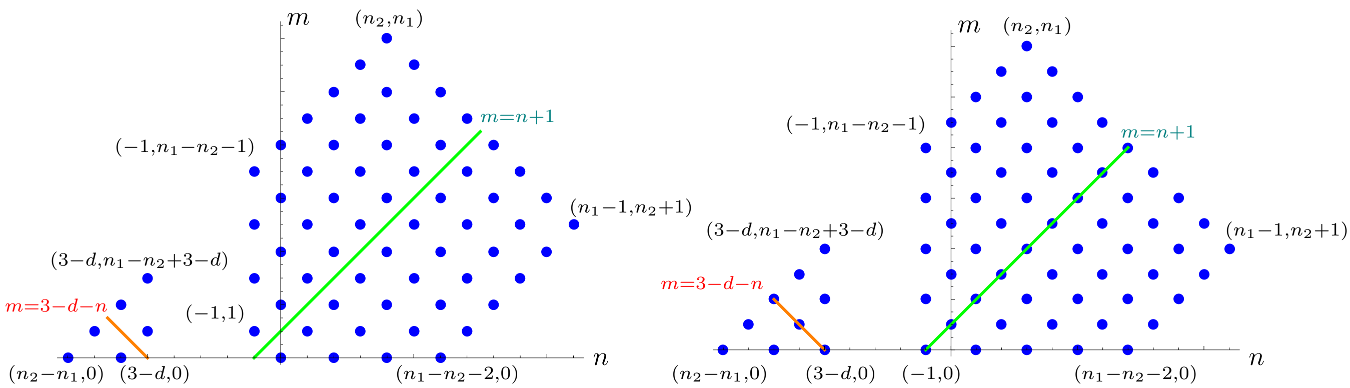
$$n_1 > n_2 \qquad (n_1)\otimes(n_2) - (n_2)\otimes(n_1) \simeq \tag{6.43}$$



The associated contribution can be shown to vanish. First we may impose

$$V_{(n,m)} \simeq 0\,, \qquad n = 4-d,\ldots,-2\,, \quad d \geq 6\,. \tag{6.44}$$

If $n_1-n_2+3-d \geq 0$ removing such points (6.43) reduces to



(6.45)

where the two diagrams correspond respectively to $n_1 - n_2, d$ even and $n_1 - n_2, d$ odd. Alternatively if $n_1 - n_2 + 3 - d < 0$ the points in (6.45) with $n < -1$ are absent. The associated contributions vanish by imposing

$$V_{(n,m)} \simeq -V_{(m-1,n+1)}\,, \quad V_{(n,m)} \simeq -V_{(3-d-m,3-d-n)}\,, \quad V_{(n,n+1)} \simeq V_{(n,3-d-n)} \simeq 0\,, \tag{6.46}$$

where the points are related by reflection in the green, orange lines. For $d = 4$ there is no gap in (6.45) but we can then require $V_{(-1,m)} \simeq -2\,V_{(m-1,0)} \simeq -2\,V_{(-1-m,0)}$.

By introducing more null vectors further mixed symmetry representations can be discussed. For three such vectors and extending the constraints (6.31) irreducible representations of $SO(d)$ corresponding to Young tableaux with three rows $\begin{smallmatrix} 1 & 2 & \cdots & & n \\ 1 & \cdots & m & & \\ 1 & \cdots \, l & & & \end{smallmatrix}$ can be obtained. Labelling these by (n, m, l), with $l \le m \le n$, this gives for the Casimir eigenvalues

$$c_{(n,m,l)} = n(n + d - 2) + m(m + d - 4) + l(l + d - 6)\,, \tag{6.47}$$

which agrees with (2.82) for $l = 1$. For the dimension of the representation

$$\begin{aligned} \dim V_{(n,m,l)} = {} & \tfrac{1}{(n+2)!\,(m+1)!\,l!}\,(n + 1 - m)(n - l + 2)(m - l + 1) \\ & \times (n + m + d - 3)(n + l + d - 4)(m + l + d - 5) \\ & \times (2n + d - 2)(2m + d - 4)(2l + d - 6)\,(d - 1)_{n-3}(d - 3)_{m-2}(d - 5)_{l-1}\,. \end{aligned} \tag{6.48}$$

For $l = 0$ this reduces to (6.36). For $d = 4, 5$ the decomposition of tensor products as a direct sum of irreducible spaces can be reduced with the identities

$$d = 4\,, \quad V_{(n,m,l)} \simeq 0\,,\, l \ge 3\,, \quad V_{(n,m,1)} \simeq \begin{cases} 0\,, & m \ge 2 \\ V_{(n)}\,, & m = 1 \end{cases}\,, \quad V_{(n,m,2)} \simeq -V_{(n,m)}\,,\ m \ge 2\,, \tag{6.49a}$$

$$d = 5\,, \quad V_{(n,m,l)} \simeq 0\,,\, l \ge 2\,, \quad V_{(n,m,1)} \simeq V_{(n,m)}\,. \tag{6.49b}$$

For $d = 6$ the tensorial representation (n, m, l) is related to $SO(6)$ representations with Dynkin or highest weight labels $[s_1, s_2, s_3]$ by $(n, m, 0) \to [n - m, m, m]$ and for $l > 0$ by $(n, m, l) \to [n - m, m + l, m - l] + [n - m, m - l, m + l]$.

The representation spaces $V_{(n,m,l)}$ arise in tensor products involving $V_{(n,m)}$ and $V_{(\ell)}$ but now there is a non trivial multiplicity,

$$V_{(n,m)} \otimes V_{(\ell)} \simeq \bigoplus_{t=0}^{\lfloor \frac{1}{2}\ell \rfloor} \ \bigoplus_{r,s,\,|r|+|s| \le \ell - 2t} (1 + t)\, V_{(n+r,m+s,\ell-2t-|r|-|s|)}\,. \tag{6.50}$$

The representations $V_{(n',m',l)}$ appearing in the tensor product satisfy $n' \ge m' \ge l$ if $n - m, m \ge \ell$.

6.3 Representation Space

The Verma module defines a representation space for the conformal algebra. Instead of (6.7) we define

$$\begin{aligned} & |\hat{\mathcal{O}}^{(\ell)}{}_{\sigma,k,n,r}(t)\rangle \\ & \quad = |\mathcal{O}^{(\ell)}{}_{\sigma,k,n,r}(t)\rangle - P_\sigma |\mathcal{O}^{(\ell)}{}_{k,n,r+1}(t)\rangle \\ & \qquad + \frac{1}{d + k - 3}\big((k - r - 1)\, t_\sigma |\mathcal{O}^{(\ell)}{}_{k-1,n+1,r+1}(t)\rangle - (k - 2r - 1)\, t_\sigma |\mathcal{O}^{(\ell)}{}_{k-1,n,r}(t)\rangle\big)\,. \end{aligned} \tag{6.51}$$

This satisfies $\nabla_\sigma|\hat{\mathcal{O}}^{(\ell)}{}_{\sigma,k,n,r}(t)\rangle = t_\sigma|\hat{\mathcal{O}}^{(\ell)}{}_{\sigma,k,n,r}(t)\rangle = 0$ as required by (6.31) for $M = 1$ and so belongs to the $[k, 1, 0, \dots]$ irreducible representation.

With the definition (6.51)

$$
\begin{aligned}
P_\mu|\mathcal{O}^{(\ell)}{}_{k,n+r,r}(t)\rangle = {}& \frac{1}{k+1}\Big(\nabla_\mu|\mathcal{O}^{(\ell)}{}_{k+1,n+r,r}(t)\rangle - r\,|\hat{O}^{(\ell)}{}_{\mu,k,n+r,r-1}(t)\rangle\Big) \\
& + \alpha_r^{(k)}\, t_\mu|\mathcal{O}^{(\ell)}{}_{k-1,n+r+1,r}(t)\rangle + \alpha_{r,-}^{(k)}\, t_\mu|\mathcal{O}^{(\ell)}{}_{k-1,n+r,r-1}(t)\rangle\,,
\end{aligned} \tag{6.52}
$$

for $r \le k, \ell$ where

$$
\alpha_r^{(k)} = \frac{(k-r)(d+k+r-3)}{(d+2k-2)(d+k-3)}\,, \qquad \alpha_{r,-}^{(k)} = r\,\frac{d+2r-4}{(d+2k-2)(d+k-3)}\,. \tag{6.53}
$$

Similarly

$$
\begin{aligned}
& K_\mu|\mathcal{O}^{(\ell)}{}_{k,n+r,r}(t)\rangle \\
& = \frac{1}{k+1}\Big(\gamma_r^{(n)}\,\nabla_\mu|\mathcal{O}^{(\ell)}{}_{k+1,n+r-1,r}(t)\rangle + \delta_{r,-}^{(k,n)}\,|\hat{O}^{(\ell)}{}_{\mu,k,n+r-1,r-1}(t)\rangle \\
& \qquad + \gamma_{r,+}\big(\nabla_\mu|\mathcal{O}^{(\ell)}{}_{k+1,n+r,r+1}(t)\rangle + (k-r)\,|\hat{O}^{(\ell)}{}_{\mu,k,n+r,r}(t)\rangle\big)\Big) \\
& \quad + \beta_{r,+}^{(k,n)}\, t_\mu|\mathcal{O}^{(\ell)}{}_{k-1,n+r+1,r+1}(t)\rangle + \beta_r^{(k,n)}\, t_\mu|\mathcal{O}^{(\ell)}{}_{k-1,n+r,r}(t)\rangle \\
& \quad + \beta_{r,-}^{(k,n)}\, t_\mu|\mathcal{O}^{(\ell)}{}_{k-1,n+r-1,r-1}(t)\rangle\,,
\end{aligned} \tag{6.54}
$$

for $r \le k, \ell$ and

$$
\begin{aligned}
\gamma_r^{(n)} &= 4(n+r)(\Delta+\ell+n-r-\tfrac{1}{2}d)\,, \qquad \gamma_{r,+} = 2(\ell-r)(\Delta-d-r+1)\,, \\
\delta_{r,-}^{(k,n)} &= -4(n+r)r\,(\Delta+\ell+k+n-r+1-\tfrac{1}{2}d)\,, \\
\beta_{r,+}^{(k,n)} &= -2\,\frac{(\ell-r)(k-r)(k-r-1)(\Delta-d-r+1)}{(d+2k-2)(d+k-3)}\,, \\
\beta_{r,-}^{(k,n)} &= 4(n+r)r\,\frac{(d+2r-4)(\Delta+\ell+2k+n-r-2+\tfrac{1}{2}d)}{(d+2k-2)(d+k-3)}\,, \\
\beta_r^{(k,n)} &= \frac{4(k-r)}{(d+2k-2)(d+k-3)}\,\big((\ell-r)(\tfrac{1}{2}d+k-r-2)(\Delta-d-r+1) \\
&\qquad\qquad + (n+r)r\,(\Delta+\ell+n+k-r+1-\tfrac{1}{2}d)\big) \\
&\quad + \frac{4(k-r)}{d+2k-2}\,(\tfrac{1}{2}d+k+n+r-1)(\Delta+\ell+n+k-1)\,.
\end{aligned} \tag{6.55}
$$

The scalar product of symmetric traceless tensor descendants

$$
\langle\mathcal{O}^{(\ell)}{}_{k',n'+r',r'}(t)\,|\,\mathcal{O}^{(\ell)}{}_{k,n+r,r}(t)\rangle = (\bar{t}\cdot t)^k\,\delta_{k'k}\delta_{n'n}\,\mathcal{N}_{r'r}^{(k,n)}\,, \tag{6.56}
$$

defines a symmetric $(\ell+1)\times(\ell+1)$, or $(k+1)\times(k+1)$, if $k \le \ell$ and $n \ge 0$, normalisation matrix $\underline{\mathcal{N}}^{(k,n)} = \big[\mathcal{N}_{r'r}^{(k,n)}\big]$, if $n < 0$ then $-n \le r, r' \le k, \ell$. The basic normalisation is chosen so that

$$
\langle\mathcal{O}^{(\ell)}{}_{\ell,0,\ell}(t)\,|\,\mathcal{O}^{(\ell)}{}_{\ell,0,\ell}(t)\rangle = (\bar{t}\cdot t)^\ell\,, \qquad \mathcal{N}_{\ell\ell}^{(\ell,-\ell)} = 1\,, \tag{6.57}
$$

or equivalently, using $\bar{\nabla}\cdot\nabla\,(\bar{t}\cdot t)^r = r(d+r-3)\,\frac{d+2r-2}{d+2r-4}\,(\bar{t}\cdot t)^{r-1}$,

$$
\langle\mathcal{O}_{\mu_1\dots\mu_\ell}|\mathcal{O}_{\mu_1\dots\mu_\ell}\rangle = \frac{1}{(\ell!)^2}\,(\bar{\nabla}\cdot\nabla)^\ell\,(\bar{t}\cdot t)^\ell = n_{d\ell}\,, \tag{6.58}
$$

with $n_{d\ell}$ given by (6.26). Directly from (6.5)

$$\langle \mathcal{O}^{(\ell)}{}_{0,0,0}\,|\,\mathcal{O}^{(\ell)}{}_{0,0,0}\rangle = 2^\ell \ell!\,(\Delta-d-\ell+2)_\ell\,\langle \mathcal{O}_{\mu_1\dots\mu_\ell}|\mathcal{O}_{\mu_1\dots\mu_\ell}\rangle\,, \tag{6.59}$$

giving

$$\mathcal{N}^{(0,0)}_{00} = 2^\ell \ell!\,(\Delta-d-\ell+2)_\ell\,n_{d\ell}\,. \tag{6.60}$$

By considering $\langle \mathcal{O}^{(\ell)}{}_{k,n+r,r}(t)|K_\mu|\mathcal{O}^{(\ell)}{}_{k+1,n+s,s}(t)\rangle$ and $\langle \mathcal{O}^{(\ell)}{}_{k+1,n+s-1,s}(t)|K_\mu|\mathcal{O}^{(\ell)}{}_{k,n+r,r}(t)\rangle$ and using (6.54) and the conjugate of (6.52) we may obtain the matrix recurrence relations

$$\underline{\mathcal{P}}\,\underline{\mathcal{N}}^{(k+1,n)} = \underline{\mathcal{N}}^{(k,n)}\,\underline{\mathcal{B}}^{(k,n)}\,, \qquad \underline{\mathcal{A}}^{(k)}\,\underline{\mathcal{N}}^{(k,n)} = \underline{\mathcal{N}}^{(k+1,n-1)}\,\underline{\mathcal{C}}^{(n)}\,, \tag{6.61}$$

for

$$\begin{gathered}
\mathcal{A}^{(k)}_{sr} = \alpha^{(k+1)}_s\,\delta_{sr} + \alpha^{(k+1)}_{s,-}\,\delta_{s-1\,r}\,, \qquad \mathcal{C}^{(n)}_{sr} = \gamma^{(n)}_r\,\delta_{sr} + \gamma_{r+}\,\delta_{s\,r+1}\,,\\
\mathcal{B}^{(k,n)}_{rs} = \beta^{(k+1,n)}_{s,-}\,\delta_{r\,s-1} + \beta^{(k+1,n)}_{s}\,\delta_{rs} + \beta^{(k+1,n)}_{s,+}\,\delta_{r\,s+1}\,, \quad \mathcal{P}_{rs} = \delta_{rs}\,,\\
0,-n\le r\le k,\ell\,, \qquad 0,-n\le s\le k+1,\ell\,.
\end{gathered} \tag{6.62}$$

The relations (6.61) determine $\underline{\mathcal{N}}^{(k,n)}$ for all k,n although there is no simple general expression. Simpler results are obtained by considering determinants. For $k\ge \ell, n\ge 0$ all matrices in (6.61) are $(\ell+1)\times(\ell+1)$ giving for this case

$$\begin{aligned}
\frac{\det \underline{\mathcal{N}}^{(k+1,n)}}{\det \underline{\mathcal{N}}^{(k,n)}} &= 2^{\ell+1}\,\frac{(k-\ell+1)_{\ell+1}\,(\frac12 d+k+n)_{\ell+1}\,(d+k-2)_{\ell+1}\,(\Delta+k+n)_{\ell+1}}{(\frac12 d+k)^{\ell+1}\,(d+k-2)^{\ell+1}}\,,\\
\frac{\det \underline{\mathcal{N}}^{(k+1,n)}}{\det \underline{\mathcal{N}}^{(k,n+1)}} &= \frac{1}{2^{3(\ell+1)}}\,\frac{(k-\ell+1)_{\ell+1}\,(d+k-2)_{\ell+1}}{(n+1)_{\ell+1}\,(\Delta+n+1-\frac12 d)_{\ell+1}\,(\frac12 d+k)^{\ell+1}\,(d+k-2)^{\ell+1}}\,.
\end{aligned} \tag{6.63}$$

For $k\le\ell, n>0$ taking the determinant of $\underline{\mathcal{P}}\,\underline{\mathcal{A}}^{(k)}\,\underline{\mathcal{N}}^{(k,n)} = \underline{\mathcal{N}}^{(k,n-1)}\,\underline{\mathcal{B}}^{(k,n-1)}\,\underline{\mathcal{C}}^{(n)}$ gives

$$\frac{\det \underline{\mathcal{N}}^{(k,n)}}{\det \underline{\mathcal{N}}^{(k,n-1)}} = 2^{4(k+1)}\,(n)_{k+1}\,(\tfrac12 d+\ell+n-1)_{k+1}\,(\Delta+n-\tfrac12 d)_{k+1}\,(\Delta+\ell+n-1)_{k+1}\,. \tag{6.64}$$

When $0\le -n\le k<\ell$,

$$\frac{\det \underline{\mathcal{N}}^{(k,n)}}{\det \underline{\mathcal{N}}^{(k+1,n-1)}} = 2^{k+n+1}(\tfrac12 d+k)^{k+n+1}(d+k-2)^{k+n+1}\,\frac{(\ell-k)_{k+n+1}\,(\Delta-d-k+1)_{k+n+1}}{(1-n)_{k+n+1}\,(\frac12 d-n-1)_{k+n+1}}\,, \tag{6.65}$$

and if $0\le -n\le \ell\le k$,

$$\frac{\det \underline{\mathcal{N}}^{(k+1,n)}}{\det \underline{\mathcal{N}}^{(k,n)}} = 2^{\ell+n+1}\,\frac{(k-\ell+1)_{\ell+n+1}\,(\frac12 d+k)_{\ell+n+1}\,(d+k-2)_{\ell+n+1}\,(\Delta+k)_{\ell+n+1}}{(\frac12 d+k)^{\ell+n+1}\,(d+k-2)^{\ell+n+1}}\,. \tag{6.66}$$

The results for $\det \underline{\mathcal{N}}^{(k,n)}$ split into four similar cases according to whether k is larger or less than ℓ and n is positive or negative. For $k\le \ell$, $n\ge 0$ we may combine (6.61) to obtain for $k=0,1,2,\dots$, assuming (6.60),

$$\begin{aligned}
\det \underline{\mathcal{N}}^{(k,n)} = {}& \bigg(\frac{2^{4n+2k+\ell}}{(\frac12 d-1)_{k+1}}\bigg)^{k+1}\\
&\times \prod_{r=0}^{k}\frac{1}{\ell!}(\ell-r)!\,(n+r)!\,(k-r)!^2\,(d+k-2)_{\ell-k+r}\,\big((\tfrac12 d-1)_r\big)^2\,(\tfrac12 d+\ell-1)_{n+r+1}\\
&\qquad \times(\Delta-1)_r\,(\Delta-d-\ell+2)_{\ell-r}\,(\Delta+\ell)_{n+r}\,(\Delta+1-\tfrac12 d)_{n+r}\,.
\end{aligned} \tag{6.67}$$

Starting from $k=\ell$ then for any $k \geq \ell$, $n \geq 0$ (6.63) implies

$$\det \underline{\mathcal{N}}^{(k,n)} = \bigg(\frac{2^{4n+k+2\ell}}{(\frac{1}{2}d-1)_\ell} \bigg)^{\ell+1} \\ \times \prod_{r=0}^{\ell} \frac{1}{\ell!\, r!} (\ell-r)!^3\, (n+r)!\, (k-\ell+r)!\, (d+k-2)_r \, \big((\tfrac{1}{2}d-1)_r \big)^2 \, (\tfrac{1}{2}d+k)_{n+r} \\ \times (\Delta-1)_r \, (\Delta-d-\ell+2)_{\ell-r} \, (\Delta+\ell)_{n+k-\ell+r} \, (\Delta+1-\tfrac{1}{2}d)_{n+r} \, . \qquad (6.68)$$

For $n \leq 0$ and $k \leq \ell$ using (6.65)

$$\det \underline{\mathcal{N}}^{(k,n)} = \bigg(\frac{2^{3n+2k+\ell}}{(\frac{1}{2}d-1)_{k+1}} \bigg)^{k+n+1} \\ \times \prod_{r=0}^{k+n} \frac{1}{\ell!} (\ell+n-r)!\, r!\, (k-r)!\, (k+n-r)!\, (d+k-2)_{\ell-k+r} \, (\tfrac{1}{2}d-1)_{r-n} \, (\tfrac{1}{2}d-1)_r \\ \times (\tfrac{1}{2}d+\ell-1)_{r+1} \, (\Delta-1)_r \, (\Delta-d-\ell+2)_{\ell+n-r} \, (\Delta+\ell)_r \, (\Delta+1-\tfrac{1}{2}d)_r \, , \qquad (6.69)$$

and for $k \geq \ell$,

$$\det \underline{\mathcal{N}}^{(k,n)} = \bigg(\frac{2^{3n+k+2\ell}}{(\frac{1}{2}d-1)_\ell} \bigg)^{\ell+n+1} \\ \times \prod_{r=0}^{\ell+n} \frac{1}{\ell!} (\ell+n-r)!\, r!\, (\ell-r)!\, (k+n-r)!\, (d+k-2)_r \, (\tfrac{1}{2}d-1)_{r-n} \, (\tfrac{1}{2}d-1)_r \\ \times (\tfrac{1}{2}d+\ell-1)_r \, (\Delta-1)_r \, (\Delta-d-\ell+2)_{\ell+n-r} \, (\Delta+\ell)_{k-\ell+r} \, (\Delta+1-\tfrac{1}{2}d)_r \, . \qquad (6.70)$$

6.4 Singular Vectors for Three Dimensional CFTs

There is a general theory [24] for singular vectors in the representations of non compact groups which extends that for compact groups initiated by Verma. For low dimensional CFTs such results can also be obtained by brute force.

In three dimensions with the spinorial basis in (5.31) and writing ${M_\alpha}^\beta$ in terms of angular momentum operators, just as in (2.111),

$$\big[{M_\alpha}^\beta \big] = \begin{pmatrix} J_3 & J_+ \\ J_- & -J_3 \end{pmatrix} , \qquad (6.71)$$

we assume irreducible representations of the conformal algebra (5.33) are defined in terms of lowest weight vectors satisfying

$$D|\Delta,s\rangle = \Delta|\Delta,s\rangle \, , \quad J_3|\Delta,s\rangle = -s|\Delta,s\rangle \, , \quad J_-|\Delta,s\rangle = 0 \, , \quad K^{22}|\Delta,s\rangle = 0 \, , \qquad (6.72)$$

which as $[J_, K^{22}] = 2\,K^{12}$, $[J_, K^{12}] = K^{11}$ is equivalent to (5.21) in this case. The associated Verma module is defined by

$$\mathcal{V}_{\Delta,s} = \mathrm{span}\big\{ {P_{11}}^v \, {P_{12}}^u \, {P_{22}}^t \, {J_+}^r |\Delta,s\rangle : v,u,t,r = 0,1,2,\dots \big\} \, . \qquad (6.73)$$

Finite dimensional spin representations dictate $s = 0, \frac{1}{2}, 1, \dots$.

The action of J_-, K^{22}, which correspond to short, long simple roots for $\mathfrak{sp}(4)$, on this basis in (6.73) is determined by

$$\begin{aligned}
&\big[J_-, {P_{11}}^v\big] = 2v\, {P_{11}}^{v-1} P_{12}\,, \quad \big[J_-, {P_{12}}^u\big] = u\, P_{11}{P_{12}}^{u-1}\,, \quad \big[J_-, {P_{22}}^t\big] = 0\,,\\
&\big[J_-, {J_+}^r\big] = \, -r\, {J_+}^{r-1}(2J_3 + r - 1)\,,\\
&\big[K^{22}, {P_{11}}^v\big] = 0\,, \qquad \big[K^{22}, {P_{12}}^u\big] = \tfrac{1}{2}u\, {P_{12}}^{u-1} J_+ + \tfrac{1}{4}u(u-1)\, P_{11}{P_{12}}^{u-2}\,,\\
&\big[J_+, {P_{22}}^t\big] = 2t\, P_{12}{P_{22}}^{t-1}\,, \quad \big[K^{22}, {P_{22}}^t\big] = t\, {P_{22}}^{t-1}(D - J_3 + t - 1)\,,\\
&\big[K^{22}, {J_+}^r\big] = 0\,.
\end{aligned} \tag{6.74}$$

To obtain singular vectors we consider an eigenvector of D, J_3 with eigenvalues $\Delta', -s'$ of the form

$$\begin{aligned}
&|\Delta', s'\rangle = \textstyle\sum_{r,t\geq 0} \epsilon_{rt}\, {P_{11}}^v\, {P_{12}}^u\, {P_{22}}^t\, {J_+}^r |\Delta, s\rangle\,, \quad u = N - 2t + r \geq 0\,,\ v = S + t - r \geq 0\,,\\
&N = \Delta' + s' - \Delta - s \in \mathbb{Z}\,,\ S = s - s' \in \mathbb{Z}\,, \quad N + S \geq 0\,,\ N + 2S \geq 0\,.
\end{aligned} \tag{6.75}$$

Imposing the conditions that this is annihilated by J_-, K^{22}, so that (6.72) defines a descendant satisfying the conditions in (6.72) for $\Delta \to \Delta'$, $s \to s'$, gives

$$\begin{aligned}
&2(S + t - r)\, \epsilon_{r\,t} + (N - 2t + r + 2)\, \epsilon_{r\,t-1} + (r+1)(2s - r)\, \epsilon_{r+1\,t} = 0\,,\\
&\tfrac{1}{4}(N - 2t + r)(N - 2t + r - 1)\, \epsilon_{r\,t} + \tfrac{1}{2}(N - 2t + r - 1)\, \epsilon_{r-1\,t}\\
&\qquad\qquad + (t+1)(\Delta' + s' - t - 2)\, \epsilon_{r\,t+1} = 0\,.
\end{aligned} \tag{6.76}$$

There are two immediate solutions

$$\epsilon_{r\,t} = \delta_{r,0}\, \delta_{t,-S}\,, \quad N = -2S\,, \quad \Delta' + s' + S = 1\,, \tag{6.77}$$

so that (6.75) gives the singular vectors

$$|1{-}s, s{+}n\rangle = {P_{22}}^n\, |1{-}s{-}n, s\rangle\,, \quad s \geq 0\,, \quad n = 1, 2, \dots\,, \tag{6.78}$$

and

$$\epsilon_{r\,t} = \delta_{r,2s+1}\, \delta_{t,0}\,, \quad S = -N = 2s + 1\,, \quad |\Delta, -s{-}1\rangle = {J_+}^{2s+1} |\Delta, s\rangle\,. \tag{6.79}$$

This singular descendant is set to zero for finite dimensional unitary representations of the spin group.

Otherwise solving (6.76) requires integrability conditions to be imposed since ϵ_{rt} for $r, t > 0$ can be obtained by more than one route starting from ϵ_{r0} and ϵ_{0t}. For $\epsilon_{00} \neq 0$ by considering ϵ_{11} we obtain the constraints

$$\Delta' = \tfrac{3}{2} + \tfrac{1}{2} N \qquad \text{or} \qquad N = 0\,, \tag{6.80}$$

and then from the results for ϵ_{21} we obtain

$$\Delta' = \tfrac{3}{2} + \tfrac{1}{2} N \ \Rightarrow\ S = 0 \ \text{ or } N = -1 - 2s\,, \qquad N = 0 \ \Rightarrow\ \Delta' = 2 + s\,. \tag{6.81}$$

There are thus potential solutions, for $S = 0$,

$$\epsilon_{rt} = \frac{2^{r-2t}}{r!\,(t-r)!}\,\frac{N!}{(N-2t+r)!}\,\frac{1}{\big(\tfrac{1}{2}(1-N)-s\big)_t}\,, \quad s \geq \tfrac{1}{2}\,, \quad N = 1, 2, \dots\,, \tag{6.82}$$

and for $N = 0$,

$$\epsilon_{rt} = \frac{2^{r-2t}}{t!\,(r-2t)!}\,\frac{S!}{(S+t-r)!}\,\frac{1}{(-2s)_r}\,, \quad 2t \leq r \leq 2s\,, \quad s \geq \tfrac{1}{2}\,, \quad S = 1, 2, \dots, 2s\,, \tag{6.83}$$

and finally

$$\epsilon_{rt} = \frac{2^{r-2t}}{r!\;t!}\,\frac{2^N\,(N+S)!\,(N+2S)!}{(N-2t+r)!\,(S+t-r)!}\,, \quad s \geq 0\,, \quad N = -1-2s,\ S+N = 1, 2, \dots\,. \tag{6.84}$$

For $S > s$, corresponding to $s' = -\frac{1}{2}, -\frac{3}{2}, \dots, -s$ or $s' = -1, -2, \dots, -s$, in (6.82) the singular vectors can be discarded by restricting to finite dimensional spin representations so that (6.82) and (6.83) then give respectively singular vectors

$$\begin{aligned} &|\tfrac{3}{2} + \tfrac{1}{2}\,N, s\rangle \in \mathcal{V}_{\frac{3}{2}-\frac{1}{2}N,s}\,, \quad s \geq \tfrac{1}{2}\,, \quad N = 1, 2, \dots\,, \\ &|2+s, s-S\rangle \in \mathcal{V}_{2+s-S,s}\,, \quad s \geq 1\,, \quad S = 1, 2, \dots, \lfloor s \rfloor\,. \end{aligned} \tag{6.85}$$

The results given by (6.78), (6.85) correspond to (2.114) for $n, N, S = 1$ and to (6.8b), (6.8a), (6.8c) for $s = \ell$ and $n = m+1$, $N = 2m$, $S = m+1$ respectively. The solution (6.84) corresponds to a singular vector

$$|1{-}s, -1{-}s{-}n\rangle = J_+{}^{2(s+n)+1} P_{22}{}^n\,|1{-}s{-}n, s\rangle\,, \quad n = 1, 2, \dots\,, \tag{6.86}$$

the descendant of (6.9) analogous to (6.79), which is set to zero for finite dimensional spin representations.

7 Three Point Functions

A crucial consequence of conformal symmetry is that the three point functions of conformal primary fields are determined uniquely, up to an overall coefficient, if two of the fields are spinless and in general for all fields with non zero spin there are a finite number of possibilities. This is a result of the fact that conformal transformations map any three points on $\mathbb{R}^d$ to any other three points and there are therefore no conformal invariants. If the three points are on a line then $SO(d+1,1) \to SO(2,1) \times SO(d-1)$, where $SO(2,1)$ corresponds to conformal transformations along the line and $SO(d-1)$ rotations in the transverse space. For three points in d dimensions then $3d - \dim SO(d+1,1) + \dim(SO(2,1) \times SO(d-1)) = 0$.

For three conformal primary scalars ϕ_i with scale dimensions Δ_i then it is easy to see that the three point function is given by

$$\begin{aligned} \big\langle \phi_1(x_1)\,\phi_2(x_2)\,\phi_3(x_3)\big\rangle = {}& \frac{C_{123}}{(x_{12}{}^2)^{\frac{1}{2}(\Delta_1+\Delta_2-\Delta_3)}\,(x_{23}{}^2)^{\frac{1}{2}(\Delta_2+\Delta_3-\Delta_1)}\,(x_{31}{}^2)^{\frac{1}{2}(\Delta_3+\Delta_1-\Delta_2)}}\,, \\ & x_{ij} = x_i - x_j\,. \end{aligned} \tag{7.1}$$

Assuming the normalisation of ϕ_i is fixed by the two-point function then the coefficients C_{123} are, along with the scale dimensions Δ_i, fundamental properties of the CFT. Of course symmetries may entail $C_{123} = 0$. These results may be quite easily extended to the case when one of the fields have spin. In d-dimensions for a symmetric traceless tensor conformal primary $\phi_{\mu_1\dots\mu_\ell}$ we may define

$$\phi^{(\ell)}(x,t) = \phi_{\mu_1\dots\mu_\ell}(x)\, t^{\mu_1}\dots t^{\mu_\ell}\,, \qquad t^2 = 0\,, \qquad \phi^{(\ell)}(x,\lambda t) = \lambda^\ell \phi^{(\ell)}(x,t)\,. \tag{7.2}$$

Applying the results (2.35) for the conformal vectors defined in (2.34) it is also straightforward to see that, if $\phi^{(\ell)}$ is a conformal primary with scale dimension Δ,

$$\begin{aligned}&\big\langle \phi_1(x_1)\,\phi_2(x_2)\,\phi^{(\ell)}(x_3,t)\big\rangle \\ &\quad = \frac{C_{12,(\Delta,\ell)}}{(x_{12}{}^2)^{\frac12(\Delta_1+\Delta_2-\Delta+\ell)}\,(x_{23}{}^2)^{\frac12(\Delta+\Delta_2-\Delta_1-\ell)}\,(x_{31}{}^2)^{\frac12(\Delta+\Delta_1-\Delta_2-\ell)}}\,(\mathcal{X}_3\cdot t)^\ell\,,\end{aligned} \tag{7.3}$$

where

$$\mathcal{X}_{3\mu} = \frac{1}{x_{13}{}^2}\,x_{31\mu} - \frac{1}{x_{23}{}^2}\,x_{32\mu}\,. \tag{7.4}$$

For fields extended to the embedding space then there are equivalent expressions for three point functions [25]. For scalars, $\phi_i(x) \to \Phi_i(X)$ satisfying (3.19) for $\Delta = \Delta_i$, then (7.1) is equivalent to

$$\begin{aligned}&\big\langle \Phi_1(X_1)\,\Phi_2(X_2)\,\Phi_3(X_3)\big\rangle \\ &\quad = \frac{C_{123}}{(X_1\cdot X_2)^{\frac12(\Delta_1+\Delta_2-\Delta_3)}\,(X_2\cdot X_3)^{\frac12(\Delta_2+\Delta_3-\Delta_1)}\,(X_3\cdot X_1)^{\frac12(\Delta_3+\Delta_1-\Delta_2)}}\,,\end{aligned} \tag{7.5}$$

with $X_i \cdot X_j$ defined according to (3.16). It is easy to verify that (7.5) satisfies the crucial homegeneity properties (7.5). For symmetric traceless tensors then (3.21), (3.22) may be extended to tensors on the null cone

$$\phi_{\mu_1\dots\mu_\ell}(x) \to \Phi_{A_1\dots A_\ell}(X)\,, \tag{7.6}$$

subject to

$$X^{A_i}\Phi_{A_1\dots A_\ell}(X) = 0\,,\ \Phi_{A_1\dots A_\ell}(X) \sim \Phi_{A_1\dots A_\ell}(X) + X_{A_i}\,\chi_{A_1\dots\hat{A}_i\dots A_\ell}(X)\,,\ i = 1,\dots,\ell\,, \tag{7.7}$$

with $\hat{A}_i$ denoting that this index is omitted. Corresponding to (7.2)

$$\phi^{(\ell)}(x,t) \to \Phi^{(\ell)}(X,T) = \Phi_{A_1\dots A_\ell}(X)\,T^{A_1}\dots T^{A_\ell}\,, \quad T^2 = X\cdot T = 0\,, \tag{7.8}$$

where $\Phi^{(\ell)}$ satisfies

$$\Phi^{(\ell)}(X,T) = \Phi^{(\ell)}(X, T+\lambda\,X)\,, \qquad \Phi^{(\ell)}(\lambda X, \rho\, T) = \lambda^{-\Delta}\rho^\ell\,\Phi^{(\ell)}(X,T)\,. \tag{7.9}$$

which is necessary for $\Phi^{(\ell)}(X,T)$ to be unambiguous since T^A is arbitrary up to $T^A \to T^A + \lambda\,X^A$. For $X^A \to X^A(x)$ as in (3.6) we may take

$$T^A \to T^A(x,t) = (t^\mu, 0, -x\cdot t)\,, \tag{7.10}$$

which obeys the conditions in (7.8). The result (7.3) is then equivalent to

$$
\begin{aligned}
&\big\langle \Phi_1(X_1)\,\Phi_2(X_2)\,\Phi^{(\ell)}(X_3,T_3)\big\rangle \\
&\quad = \frac{C_{12,(\Delta,\ell)}}{(X_1\cdot X_2)^{\frac12(\Delta_1+\Delta_2-\Delta+\ell)}\,(X_2\cdot X_3)^{\frac12(\Delta+\Delta_2-\Delta_1-\ell)}\,(X_3\cdot X_1)^{\frac12(\Delta+\Delta_1-\Delta_2-\ell)}}\,\mathcal{Z}_{12}(X_3,T_3)^{\ell}\,,
\end{aligned}
\tag{7.11}
$$

where, with the definition (3.16),

$$
\mathcal{Z}_{12}(X_3,T_3) = \frac{X_1\cdot T_3}{X_1\cdot X_3} - \frac{X_2\cdot T_3}{X_2\cdot X_3}\,, \tag{7.12}
$$

is determined by requiring $\mathcal{Z}_{12}(X_3,T_3)=\mathcal{Z}_{12}(X_3,T_3+\lambda\,X_3)$.

For three point correlation functions for conformal primaries with spin results are more involved. Obtaining an explicit form is just a linear algebraic problem although the number of solutions may depend on the dimension. For primary fields transforming as in (2.49) the basic conformal Ward identity has the form

$$
\begin{aligned}
\textstyle\sum_{i=1,2,3}&\big(v^\mu(x_i)\partial_{i\mu}+\Delta_i\,\sigma_v(x_i)\big)\big\langle\phi_{1,I_1}(x_1)\,\phi_{2,I_2}(x_2)\,\phi_{3,I_3}(x_3)\big\rangle \\
&= \tfrac12\,\hat\omega_v{}^{\mu\nu}(x_1)(s_{1\mu\nu})_{I_1}{}^J\big\langle\phi_{1,J}(x_1)\,\phi_{2,I_2}(x_2)\,\phi_{3,I_3}(x_3)\big\rangle \\
&\quad + \tfrac12\,\hat\omega_v{}^{\mu\nu}(x_2)(s_{2\mu\nu})_{I_2}{}^J\big\langle\phi_{1,I_1}(x_1)\,\phi_{2,J}(x_2)\,\phi_{3,I_3}(x_3)\big\rangle \\
&\quad + \tfrac12\,\hat\omega_v{}^{\mu\nu}(x_3)(s_{3\mu\nu})_{I_3}{}^J\big\langle\phi_{1,I_1}(x_1)\,\phi_{2,I_2}(x_2)\,\phi_{3,J}(x_3)\big\rangle\,.
\end{aligned}
\tag{7.13}
$$

Clearly in the spinless case this is satisfied by (7.1). Solutions for particular cases in general dimensions can be constructed using the inversion tensor and the conformal vectors defined in (2.34), (2.36) and more generally using the embedding formalism [25, 26]. The identity (7.13) may be simplified [16, 27] by using the intertwiners $\mathcal{I}$ defined by the two point function in (2.84) to write the three point function in the form

$$
\big\langle\phi_{1,I_1}(x_1)\,\phi_{2,I_2}(x_2)\,\phi_{3,I_3}(x_3)\big\rangle = \frac{\mathcal{I}_{1,I_1\bar I_1}(x_{13})\,\mathcal{I}_{2,I_2\bar I_2}(x_{23})}{(x_{13}{}^2)^{\Delta_1}\,(x_{23}{}^2)^{\Delta_2}}\,C_{12,3}{}^{\bar I_1\bar I_2}{}_{I_3}(\mathcal{X}_3)\,, \tag{7.14}
$$

with $\mathcal{X}_3$ defined in (7.4). To verify the result (7.14) we use, as a consequence of (2.85),

$$
\begin{aligned}
\Big(\textstyle\sum_{i=1,3}&v^\mu(x_i)\partial_{i\mu}+\Delta_1\,\sigma_v(x_1)\Big)\frac{\mathcal{I}_{1,I_1\bar I_1}(x_{13})}{(x_{13}{}^2)^{\Delta_1}} - \tfrac12\,\hat\omega_v{}^{\mu\nu}(x_1)(s_{1\mu\nu})_{I_1}{}^J\,\frac{\mathcal{I}_{1,J\bar I_1}(x_{13})}{(x_{13}{}^2)^{\Delta_1}} \\
&= -\,\Delta_1\,\sigma_v(x_3)\,\frac{\mathcal{I}_{1,I_1\bar I_1}(x_{13})}{(x_{13}{}^2)^{\Delta_1}} - \frac{\mathcal{I}_{1,I_1\bar J}(x_{13})}{(x_{13}{}^2)^{\Delta_1}}\,\tfrac12\,\hat\omega_v{}^{\mu\nu}(x_3)(\bar s_{1\mu\nu})^{\bar J}{}_{\bar I_1}\,,
\end{aligned}
\tag{7.15}
$$

and also the corresponding result for $\mathcal{I}_{2,I_2\bar I_2}(x_{23})/(x_{23}{}^2)^{\Delta_2}$, to rewrite the conformal identity (7.13) just in terms of $C_{12,3}{}^{\bar I_1\bar I_2}{}_{I_3}$

$$
\begin{aligned}
&\textstyle\sum_{i=1,2,3} v^\mu(x_i)\partial_{i\mu}\,C_{12,3}{}^{\bar I_1\bar I_2}{}_{I_3}(\mathcal{X}_3) \\
&\quad = (\Delta_1+\Delta_2-\Delta_3)\,\sigma_v(x_3)\,C_{12,3}{}^{\bar I_1\bar I_2}{}_{I_3}(\mathcal{X}_3) + \tfrac12\,\hat\omega_v{}^{\mu\nu}(x_3)(s_{3\mu\nu})_{I_1}{}^J C_{12,3}{}^{\bar I_1\bar I_2}{}_J(\mathcal{X}_3) \\
&\qquad + \tfrac12\,\hat\omega_v{}^{\mu\nu}(x_3)(\bar s_{1\mu\nu})^{\bar I_1}{}_{\bar J}\,C_{12,3}{}^{\bar J\bar I_2}{}_{I_3}(\mathcal{X}_3) + \tfrac12\,\hat\omega_v{}^{\mu\nu}(x_3)(\bar s_{2\mu\nu})^{\bar I_2}{}_{\bar J}\,C_{12,3}{}^{\bar I_1\bar J}{}_{I_3}(\mathcal{X}_3)\,.
\end{aligned}
\tag{7.16}
$$

This is satisfied, by extending the transformation rules (2.38) to $\mathcal{X}_3$, if

$$C_{12,3}{}^{\bar I_1 \bar I_2}{}_{I_3}(\lambda\, y) = \lambda^{-\Delta_1-\Delta_2+\Delta_3}\, C_{12,3}{}^{\bar I_1 \bar I_2}{}_{I_3}(y)\,, \tag{7.17}$$

and

$$L_{y\,\mu\nu} C_{12,3}{}^{\bar I_1 \bar I_2}{}_{I_3}(y) = (s_{3\mu\nu})_{I_1}{}^{J} C_{12,3}{}^{\bar I_1 \bar I_2}{}_{J}(y) + (\bar s_{1\mu\nu})^{\bar I_1}{}_{\bar J}\, C_{12,3}{}^{\bar J \bar I_2}{}_{I_3}(y) + (\bar s_{2\mu\nu})^{\bar I_2}{}_{\bar J}\, C_{12,3}{}^{\bar I_1 \bar J}{}_{I_3}(y)\,. \tag{7.18}$$

This condition just implies that $C_{12,3}{}^{\bar I_1 \bar I_2}{}_{I_3}(y)$ is rotationally covariant under $SO(d)$ transformations. Up to the freedom allowed by (7.17), (7.18) the solution is unique since there are no conformal invariants. For the spinless case (7.1) is reproduced if we take $C(\mathcal{X}_3) = C_{123}\,({\mathcal{X}_3}^2)^{-\frac{1}{2}(\Delta_1+\Delta_2-\Delta_3)}$.

Under finite transformations (7.18) integrates to

$$C_{12,3}{}^{\bar I_1 \bar I_2}{}_{I_3}(y) = \bar{\mathcal{R}}_1{}^{\bar I_1}{}_{\bar J_1}\, \bar{\mathcal{R}}_2{}^{\bar I_2}{}_{\bar J_2}\, \mathcal{R}_3{}_{I_3}{}^{J_3}\, C_{12,3}{}^{\bar J_1 \bar J_2}{}_{J_3}(y\,R)\,, \tag{7.19}$$

with $\bar{\mathcal{R}}_1$, $\bar{\mathcal{R}}_2$, $\mathcal{R}_3$ corresponding to the rotation $R\in SO(d)$ in the appropriate representations. Choosing

$$\begin{aligned} \bar{\mathcal{R}}_i{}^{\bar I_i}{}_{\bar J_i} &= \bar{\mathcal{I}}_{i,}{}^{\bar I_i K}(x_{31})\,\mathcal{I}_{i,K\bar J_i}(x_{12}) = \bar{\mathcal{I}}_{i,}{}^{\bar I_i K}(x_{32})\,\mathcal{I}_{i,K\bar J_i}(\mathcal{X}_2)\,, \quad i=1,2\,,\\ \mathcal{R}_3{}_{I_3}{}^{J_3} &= \mathcal{I}_{3,\,I_3\bar K}(x_{32})\,\bar{\mathcal{I}}_{3,}{}^{\bar K J_3}(\mathcal{X}_2)\,, \qquad \mathcal{X}_{2\mu} = \frac{1}{{x_{23}}^2}\,x_{23\mu} - \frac{1}{{x_{12}}^2}\,x_{21\mu}\,, \end{aligned} \tag{7.20}$$

which depend on (2.44), and using, from (2.41),

$$\mathcal{X}_{3\nu}\, I^{\nu\rho}(x_{32})\, I_{\rho\mu}(\mathcal{X}_2) = \frac{{x_{12}}^2}{{x_{13}}^2}\,\mathcal{X}_{2\mu}\,, \tag{7.21}$$

the result (7.14) is equivalent to

$$\big\langle \phi_{1,I_1}(x_1)\,\phi_{2,I_2}(x_2)\,\phi_{3,I_3}(x_3)\big\rangle = \frac{\mathcal{I}_{1,I_1\bar I_1}(x_{12})\,\mathcal{I}_{3,\,I_3\bar I_3}(x_{32})}{({x_{12}}^2)^{\Delta_1}\,({x_{23}}^2)^{\Delta_3}}\, C_{13,2}{}^{\bar I_1 \bar I_3}{}_{I_2}(\mathcal{X}_2)\,, \tag{7.22}$$

for

$$C_{13,2}{}^{\bar I_1 \bar I_3}{}_{I_2}(y) = (y^2)^{\Delta_2-\Delta_3}\,\mathcal{I}_{2,I_2\bar I_2}(y)\,\bar{\mathcal{I}}_{3,}{}^{\bar I_3 I_3}(y)\, C_{12,3}{}^{\bar I_1 \bar I_2}{}_{I_3}(y)\,. \tag{7.23}$$

An illustration of these results is given by the three point function involving two scalars and the energy momentum tensor. The general construction easily gives

$$\begin{aligned} \big\langle \phi(x_1)\,\phi(x_2)\,T_{\mu\nu}(x_3)\big\rangle &= \frac{1}{({x_{12}}^2)^{\Delta_\phi-\frac{1}{2}d}\,({x_{13}}^2\,{x_{23}}^2)^{\frac{1}{2}d}}\, C_{\mu\nu}(\mathcal{X}_3)\,,\\ C_{\mu\nu}(\mathcal{X}_3) = \mathcal{I}_{\mu\nu,\sigma\rho}(x_{32})\, C_{\mu\nu}(\mathcal{X}_2) &= C_{\phi\phi T}\bigg(\frac{\mathcal{X}_{3\mu}\,\mathcal{X}_{3\nu}}{{\mathcal{X}_3}^2} - \frac{1}{d}\,\eta_{\mu\nu}\bigg)\,, \end{aligned} \tag{7.24}$$

with the inversion tensor as in (4.48). The coefficient $C_{\phi\phi T}$ in (7.24) is later determined by Ward identities.

Following the definitions in (5.20) and (5.25), with a Euclidean metric, the three point function (7.14) is equivalent to

$$\begin{aligned} \big\langle \tilde\phi_1{}^{\bar I}\big|\,\phi_{2,J}(y)\,\big|\phi_{3,K}\big\rangle &= \lim_{x\to\infty}(x^2)^{\Delta_1}\bar{\mathcal{I}}_{1,}{}^{\bar I I}(x)\,\big\langle \phi_{1,I}(x)\,\phi_{2,J}(y)\,\phi_{3,K}(0)\big\rangle\\ &= \frac{1}{(y^2)^{\Delta_2}}\,\mathcal{I}_{2,J\bar J}(y)\, C_{12,3}{}^{\bar I \bar J}{}_{K}(y/y^2)\,, \end{aligned} \tag{7.25}$$

7.1 Three Point Function for Symmetric Traceless Tensors

As an application we may consider the three point function for three symmetric traceless conformal primaries encoded in terms of three null vectors t_i, $t_i^2 = 0$, as in (7.2). In this case the general expression (7.14) becomes

$$\big\langle \phi_1^{(\ell_1)}(x_1,t_1)\,\phi_2^{(\ell_2)}(x_2,t_2)\,\phi^{(\ell_3)}(x_3,t_3)\big\rangle = \frac{1}{({x_{13}}^2)^{\Delta_1}\,({x_{23}}^2)^{\Delta_2}}\,C_{12,3}\big(t'_1,t'_2,t_3,\mathcal{X}_3\big)\,, \tag{7.26}$$

for

$$t'_{i\mu} = t_{i\nu}\,I_{\nu\mu}(x_{i3})\,, \quad i=1,2\,, \quad {t'_i}^2 = 0\,. \tag{7.27}$$

and $C_{12,3}\big(t_1,t_2,t_3,y\big)$ constrained by

$$\begin{aligned} &C_{12,3}\big(\lambda_1\,t_1,\lambda_2\,t_2,\lambda_3\,t_3,\rho\,y\big) = {\lambda_1}^{\ell_1}{\lambda_2}^{\ell_2}{\lambda_3}^{\ell_3}\rho^{-(\Delta_1+\Delta_2-\Delta_3)}\,C_{12,3}\big(t_1,t_2,t_3,y\big)\,,\\ &\big(L_{t_1\,\mu\nu}+L_{t_2\,\mu\nu}+L_{t_3\,\mu\nu}+L_{y\,\mu\nu}\big)C_{12,3}\big(t_1,t_2,t_3,y\big)=0\,, \quad L_{y\,\mu\nu} = -y_\mu\,\partial_{y\nu}+y_\nu\,\partial_{y\mu}\,, \end{aligned} \tag{7.28}$$

and $L_{t\,\mu\nu}$ as in (6.13). In general with these conditions

$$\begin{aligned} C_{12,3}\big(t_1,t_2,t_3,y\big) &= \frac{(t_1\cdot y)^{\ell_1}\,(t_2\cdot y)^{\ell_2}\,(t_3\cdot y)^{\ell_3}}{(y^2)^{\frac12(\Delta_1+\Delta_2-\Delta_3+\ell_1+\ell_2+\ell_3)}}\,H_{12,3}(\xi_1,\xi_2,\xi_3)\,,\\ H_{12,3}(\xi_1,\xi_2,\xi_3) \ &= \sum_{\{n_1,n_2,n_3\}\in\mathcal{V}_\xi} C_{n_1n_2n_3}\,{\xi_1}^{n_1}\,{\xi_2}^{n_3}\,{\xi_3}^{n_3}\,, \end{aligned} \tag{7.29}$$

where

$$\xi_1 = \frac{t_2\cdot t_3\;y^2}{2\,t_2\cdot y\;t_3\cdot y}\,, \qquad \xi_2 = \frac{t_1\cdot t_3\;y^2}{2\,t_1\cdot y\;t_3\cdot y}\,, \qquad \xi_3 = \frac{t_1\cdot t_2\;y^2}{2\,t_1\cdot y\;t_2\cdot y}\,, \tag{7.30}$$

and

$$\mathcal{V}_\xi = \big\{n_i\ge 0, n_1+n_2\le\ell_3, n_1+n_3\le\ell_2, n_2+n_3\le\ell_1\big\}\,, \quad |\ell_1-\ell_2|\le\ell_3\le\ell_1+\ell_2\,. \tag{7.31}$$

Under interchange

$$\begin{aligned} &H_{21,3}(\xi_2,\xi_1,\xi_3) = H_{12,3}(\xi_1,\xi_2,\xi_3)\,,\\ &H_{13,2}(\xi_1,\xi_3,\xi_2) = (-1)^{\ell_2+\ell_3}H_{12,3}(\xi_1,1-\xi_2,1-\xi_3)\,. \end{aligned} \tag{7.32}$$

For $\ell_1=\ell_2=\ell_3=\ell$ then a symmetric three point function is obtained by requiring $H_{123}(\xi_1,\xi_2,\xi_3) = H_{12,3}(\xi_1,\xi_2,1-\xi_3)$ to be symmetric. The number of independent terms contributing to $H_{123}(\xi_1,\xi_2,\xi_3)$ is

$$N_{\rm sym}(\ell) = \begin{cases} \frac{1}{24}(\ell+2)(\ell+3)(\ell+4)\,, & \ell \text{ even}\,,\\ \frac{1}{24}(\ell+1)(\ell+3)(\ell+5)\,, & \ell \text{ odd}\,. \end{cases} \tag{7.33}$$

To verify this inductively, starting from $N_{\rm sym}(0)=1$, we consider the additional symmetric polynomials which are present when $\ell\to\ell+1$. These contain terms ${\xi_1}^{n_1}{\xi_2}^{n_2}{\xi_3}^{n_3}$, $n_i\ge 0$, where $n_i+n_j=\ell+1$ for some $i\ne j$. For ℓ even these symmetric polynomials are of the form ${\xi_1}^{\frac12\ell-n}\,{\xi_2}^{\frac12\ell-r}\,{\xi_3}^{\frac12\ell+1+r}$ + permutations where $r=0,\dots,n$ and $n=0,\dots,\frac12\ell$ so that the total number of such polynomials is $\sum_{n=0}^{\frac{1}{2}\ell}(n+1) = \frac{1}{8}(\ell+2)(\ell+4)$. For ℓ odd the extra polynomials are ${\xi_1}^{\frac{1}{2}(\ell+1)-n}\,{\xi_2}^{\frac{1}{2}(\ell+1)-r}\,{\xi_3}^{\frac{1}{2}(\ell+1)+r}$ + permutations and $r=0,\dots,n$, $n=0,\dots,\frac{1}{2}(\ell+1)$ with a corresponding sum $\sum_{n=0}^{\frac{1}{2}(\ell+1)}(n+1) = \frac{1}{8}(\ell+3)(\ell+5)$. It is easy to check that these results agree with $N_{\rm sym}(\ell+1) - N_{\rm sym}(\ell)$ from (7.33).

An equivalent form to (7.29) with (7.30) was given in [25]. The necessary connection may be obtained by defining

$$V_i = t_i\cdot\mathcal{X}_i\,, \qquad H_{ij} = t_i\cdot I(x_{ij})\cdot t_j/{x_{ij}}^2\,, \tag{7.34}$$

so that

$$\begin{aligned} &t'_1\cdot\mathcal{X}_3 = -\frac{{x_{12}}^2}{{x_{23}}^2}\,V_1\,, \qquad t'_2\cdot\mathcal{X}_3 = -\frac{{x_{12}}^2}{{x_{13}}^2}\,V_2\,, \qquad t_3\cdot\mathcal{X}_3 = V_3\,, \\ &t'_1\cdot t_3 = {x_{13}}^2 H_{13}\,, \quad t'_2\cdot t_3 = {x_{23}}^2 H_{23}\,, \quad t'_1\cdot t'_2 = {x_{12}}^2(H_{12} + 2\,V_1V_2)\,. \end{aligned} \tag{7.35}$$

The result given by (7.26) and (7.29) can then be expressed as

$$\begin{aligned} &\big\langle \phi_1^{(\ell_1)}(x_1,t_1)\,\phi_2^{(\ell_2)}(x_2,t_2)\,\phi^{(\ell_3)}(x_3,t_3)\big\rangle \\ &\quad = \frac{(-V_1)^{\ell_1}\,(-V_2)^{\ell_2}\,{V_3}^{\ell_3}}{({x_{12}}^2)^{\frac{1}{2}(\tau_1+\tau_2-\tau_3)}\,({x_{13}}^2)^{\frac{1}{2}(\tau_1+\tau_3-\tau_2)}\,({x_{23}}^2)^{\frac{1}{2}(\tau_2+\tau_3-\tau_1)}} \\ &\quad\quad \times \sum_{\{n_1,n_2,n_3\}\in\mathcal{V}_\xi} C_{n_1n_2n_3}\bigg(1+\frac{H_{12}}{2\,V_1\,V_2}\bigg)^{n_3}\bigg(-\frac{H_{13}}{2\,V_1\,V_3}\bigg)^{n_2}\bigg(-\frac{H_{23}}{2\,V_2\,V_3}\bigg)^{n_1}\,, \end{aligned} \tag{7.36}$$

with

$$\tau_i = \Delta_i - \ell_i\,. \tag{7.37}$$

When $d=3$, t_1,t_2,t_3,y are linearly dependent which leads to the relation

$$(\xi_1+\xi_2-\xi_3)^2 = 4\,\xi_1\xi_2(1-\xi_3)\,. \tag{7.38}$$

Otherwise each choice of n_1,n_2,n_3 in (7.29) compatible with ℓ_1,ℓ_2,ℓ_3 gives linearly independent contributions to $H_{12,3}(\xi_1,\xi_2,\xi_3)$. The number of independent terms in (7.26) is then given by

$$N_{\ell_1\ell_2\ell_3} = \sum_{\{n_1,n_2,n_3\}\in\mathcal{V}_\xi} 1\,. \tag{7.39}$$

Manifestly

$$N_{\ell_1\ell_2 0} = \min(\ell_1,\ell_2)+1\,. \tag{7.40}$$

The three dimensional polyhedron $\mathcal{V}_\xi$ defined by the 6 planes in (7.31) depends on ℓ_1,ℓ_2,ℓ_3. For $\ell_1=\ell_2=\ell_3=\ell$, $\mathcal{V}_\xi$ is a triangular bipyramid with five vertices $(0,0,0)$, $(0,0,\ell)$, $(0,\ell,0)$, $(\ell,0,0)$ and $(\frac{1}{2}\ell,\frac{1}{2}\ell,\frac{1}{2}\ell)$. For $\ell_1<\ell_2<\ell_3$, $\mathcal{V}_\xi$ is a non regular polyhedron with 12 edges and eight vertices $v_1=(0,0,0)$, $v_2=(0,0,\ell_1)$, $v_3=(0,\ell_1,0)$, $v_4=(\ell_2,0,0)$, $v_5=(\ell_2-\ell_1,0,\ell_1)$, $v_6=(\ell_3-\ell_1,\ell_1,0)$, $v_7=(\ell_2,\ell_3-\ell_2,0)$ and $v_8=(\frac{1}{2}(\ell_2+\ell_3-\ell_1),\frac{1}{2}(\ell_1+\ell_3-\ell_2),\frac{1}{2}(\ell_1+\ell_2-\ell_3))$. The faces are two pentagons $v_1v_3v_6v_7v_4$, $v_2v_5v_8v_6v_3$, two quadrilaterals $v_1v_4v_5v_2$, $v_4v_7v_8v_5$ and two triangles $v_1v_2v_3$, $v_6v_7v_8$.

The sum in (7.39) may be calculated directly but equivalently the number of independent contributions to $C_{12,3}(t_1,t_2,t_3,y)$ is determined by the numbers of ways in which $V_{(n,m)}$ representations can appear in both the tensor products $V_{(\ell_1)}\otimes V_{(\ell_2)}$ and $V_{(\ell_3)}\otimes V_{(\ell)}$ for $\ell=0,1,2,\dots$ using the tensor product (6.41). Here $V_{(\ell)}$ corresponds to the symmetric traceless tensor representations which can be formed from a vector y for any ℓ. The (n,m) allowed by (6.41) are constrained by

$$\begin{aligned} &|\ell_1-\ell_2|+m\le n\le \ell_1+\ell_2-m\,, \quad (-1)^{\ell_1+\ell_2-m-n}=1\,, \quad 0\le m\le \min(\ell_1,\ell_2)\,,\\ &|\ell_3-n|+m\le \ell\le \ell_3+n-m\,, \quad (-1)^{\ell_3+\ell-m-n}=1\,. \end{aligned} \tag{7.41}$$

The sum over possible ℓ for fixed (n,m) is straightforward

$$d_{n,\ell_3,m}=\textstyle\sum_{\ell=|\ell_3-n|+m}^{\ell_3+n-m}\frac12\big(1+(-1)^{\ell_3+\ell-m-n}\big)=\min\big(n-m+1,\ell_3-m+1\big)\,. \tag{7.42}$$

The sum over n then gives a total

$$\textstyle\sum_{n=|\ell_1-\ell_2|+m}^{\ell_1+\ell_2-m}\frac12\big(1+(-1)^{\ell_1+\ell_2-m-n}\big)\,d_{n,\ell_3,m}\,. \tag{7.43}$$

Choosing

$$\ell_1\le\ell_2\le\ell_3\,,\qquad p=\ell_1+\ell_2-\ell_3\,, \tag{7.44}$$

the sum (7.43) becomes, ($\theta(p)=1$, $p\ge 0$, 0, $p<0$),

$$\begin{aligned} &\textstyle\sum_{n=|\ell_1-\ell_2|+m}^{\ell_1+\ell_2-m}\frac12\big(1+(-1)^{\ell_1+\ell_2-m-n}\big)(n-m+1)\\ &\qquad\qquad -\theta(p-m)\textstyle\sum_{n=\ell_3}^{\ell_1+\ell_2-m}\frac12\big(1+(-1)^{\ell_1+\ell_2-m-n}\big)(n-\ell_3)\\ &=(\ell_1-m+1)(\ell_2-m+1)-\theta(p-m)\big(\tfrac14(p-m)(p-m+2)+\tfrac18(1-(-1)^{p-m})\big)\,. \end{aligned} \tag{7.45}$$

Finally summing over $m=0,\dots,\ell_1$ gives, subject to (7.44) so that $p-\ell_1\le 0$ and in the sum over the second term $m=0,\dots,p$,

$$N_{\ell_1\ell_2\ell_3}=\tfrac16(\ell_1+1)(\ell_1+2)(3\ell_2-\ell_1+3)-\theta(p)\big(\tfrac1{24}\,p(p+2)(2p+5)+\tfrac1{16}(1-(-1)^p)\big)\,, \tag{7.46}$$

as was obtained in [25].

For $d=3$ the sum over m is restricted to just $m=0,1$ which gives

$$N^+{}_{\ell_1\ell_2\ell_3}=2\ell_1\ell_2+\ell_1+\ell_2+1-\theta(p)\,\tfrac12 p(p+1)\,. \tag{7.47}$$

As a consequence of the identification $V_{(n,0)}\simeq V_{(n,1)}$ in (6.38) there are additional independent contributions to the three point function in three dimensions by combining the result from (7.42) with $m=1$ together with the sum over n in (7.43) where $m=0$ and also conversely with $m=0$ in (7.42) and $m=1$ in (7.43). This results in the counting

$$\begin{aligned} N^-{}_{\ell_1\ell_2\ell_3}&=\textstyle\sum_{n=|\ell_1-\ell_2|}^{\ell_1+\ell_2}\frac12\big(1+(-1)^{\ell_1+\ell_2-n}\big)\,d_{n,\ell_3,1}+\sum_{n=|\ell_1-\ell_2|+1}^{\ell_1+\ell_2-1}\frac12\big(1-(-1)^{\ell_1+\ell_1-n}\big)\,d_{n,\ell_3,0}\\ &=2\ell_1\ell_2+\ell_1+\ell_2-\theta(p)\,\tfrac12 p(p+1)\,, \end{aligned} \tag{7.48}$$

assuming (7.44) again. These terms involve the ε-tensor and hence are parity odd.

7.2 Conservation Conditions

For $\Delta = \ell + d - 2$, $\partial_x \cdot \nabla_t \phi^{(\ell)}(x,t)$ is also a conformal primary with $\Delta \to \Delta + 1,\ \ell \to \ell - 1$. As a reflection of this

$$\begin{aligned}
&\partial_{x_1}\!\cdot\nabla_{t_1}\big\langle \phi_1^{(\ell_1)}(x_1,t_1)\,\phi_2^{(\ell_2)}(x_2,t_2)\,\phi^{(\ell_3)}(x_3,t_3)\big\rangle\big|_{\Delta_1=\ell_1+d-2}\\
&\qquad\qquad = \frac{1}{(x_{13}{}^2)^{\ell_1+d-1}\,(x_{23}{}^2)^{\Delta_2}}\ \partial_{\mathcal{X}_3}\cdot\nabla_{t'_1} C_{12,3}\big(t'_1,t'_2,t_3,\mathcal{X}_3\big)\,, \qquad (7.49a)\\
&\partial_{x_2}\!\cdot\nabla_{t_2}\big\langle \phi_1^{(\ell_1)}(x_1,t_1)\,\phi_2^{(\ell_2)}(x_2,t_2)\,\phi^{(\ell_3)}(x_3,t_3)\big\rangle\big|_{\Delta_2=\ell_2+d-2}\\
&\qquad\qquad = \frac{1}{(x_{13}{}^2)^{\Delta_1}\,(x_{23}{}^2)^{\ell_2+d-1}}\ \partial_{\mathcal{X}_3}\cdot\nabla_{t'_2} C_{12,3}\big(t'_1,t'_2,t_3,\mathcal{X}_3\big)\,, \qquad (7.49b)\\
&\partial_{x_3}\!\cdot\nabla_{t_3}\big\langle \phi_1^{(\ell_1)}(x_1,t_1)\,\phi_2^{(\ell_2)}(x_2,t_2)\,\phi^{(\ell_3)}(x_3,t_3)\big\rangle\big|_{\Delta_3=\ell_3+d-2}\\
&= \frac{1}{(x_{13}{}^2)^{\Delta_1}\,(x_{23}{}^2)^{\Delta_2}}\,\big(-(\Delta_1-\Delta_2)\mathcal{X}_3\cdot\nabla_{t_3} + \mathcal{X}_{3\mu}(L_{t'_1\mu\nu} - L_{t'_2\mu\nu})\nabla_{t_3\nu}\big) C_{12,3}\big(t'_1,t'_2,t_3,\mathcal{X}_3\big)\,.\\
&\qquad\qquad\qquad\qquad\qquad\qquad\qquad\qquad\qquad\qquad\qquad\qquad\qquad\qquad (7.49c)
\end{aligned}$$

We may then impose conservation equations by requiring the vanishing of the right hand side. For $\ell_i = 1$ this is just the usual conservation of a vector current with $\Delta_i = d-1$ while for $\ell_i = 2$ and $\Delta_i = d$ this is related to the conserved energy momentum tensor. Conserved higher spin currents arise in free theories.

7.2.1 Conserved Currents with One Scalar

For the conservation equation $\partial_{x_1}\cdot\nabla_{t_1}\big\langle \phi_1^{(\ell_1)}(x_1,t_1)\,\phi_2^{(\ell_2)}(x_2,t_2)\,\phi^{(0)}(x_3)\big\rangle\big|_{\Delta_1=\ell_1+d-2} = 0$, where $\ell_3 = 0$, it is sufficient to take

$$C_{12,3}(t_1,t_2,y) = \frac{(t_1\cdot y)^{\ell_1}\,(t_2\cdot y)^{\ell_2}}{(y^2)^{\frac12(2\ell_1+\ell_2+d-2+\Delta_2-\Delta_3)}}\ H_{12,3}(\xi_3)\,, \quad H_{12,3}(\xi) = \textstyle\sum_{n=0}^{\min(\ell_1,\ell_2)} C_n\,\xi^n\,. \qquad (7.50)$$

From (7.49a) it is necessary to require $\partial_y\cdot\nabla_{t_1}C_{12,3}(t_1,t_2,y) = 0$ which leads to the three term relations

$$a_n\,C_n + b_{n+1}\,C_{n+1} + c_{n-1}\,C_{n-1} = 0\,, \quad C_{-1} = 0\,, \quad n = 0,\dots,\min(\ell_1,\ell_2)\,, \qquad (7.51)$$

where

$$\begin{aligned}
a_n &= (\ell_1-n)\big(\ell_1\ell_2 - \ell_2 - (2\ell_1+3\ell_2)n + 4n^2 - (\ell_1+d-3+n)(\Delta_2-\Delta_3)\big)\,,\\
b_n &= -\tfrac12 n(d-4+2n)(d-2+2\ell_1+\ell_2+\Delta_2-\Delta_3-2n)\,,\\
c_n &= -2(\ell_1-n)(\ell_1-n-1)(\ell_2-n)\,. \qquad (7.52)
\end{aligned}$$

For $\ell_2 \geq \ell_1$ and with $c_{\ell_1} = 0$, there are apparently ℓ_1+1 equations in (7.51), taking $C_{\ell_1+1} = C_{\ell_1+2} = 0$, but as $a_{\ell_1} = c_{\ell_1-1} = 0$ the equation where $n = \ell_1$ is redundant and the remaining equations can be solved without constraints for all C_n in terms of C_{ℓ_1} or C_0. For $\ell_1 > \ell_2$ it is necessary that $C_n = 0$ for $n > \ell_2$ and with $c_{\ell_2} = 0$ there remain ℓ_2+1 equations for $C_0, \dots, C_{\ell_2}$. For a non trivial solution the associated determinant must vanish which gives $\ell_2 + 1$ possibilities

$$\Delta_2 - \Delta_3 = 0, \pm 2, \pm 4, \dots, \pm \ell_2\,, \quad \ell_2 \text{ even}\,, \qquad \Delta_2 - \Delta_3 = \pm 1, \pm 3, \dots, \pm \ell_2\,, \quad \ell_2 \text{ odd}\,. \tag{7.53}$$

For $\ell_2 = 0$ so there is a conserved current and two scalars the condition is just $\Delta_2 = \Delta_3$. As a consequence if $\partial_{x_1} \cdot \nabla_{t_1} \langle \phi_1^{(\ell)}(x_1, t_1)\, \partial_{x_2} \cdot \nabla_{t_2} \phi_2^{(\ell)}(x_2, t_2)\, \phi^{(0)}(x_3) \rangle \big|_{\Delta_1 = \Delta_2 = \ell + d - 2} = 0$, then generically $\langle \phi_1^{(\ell)}(x_1, t_1)\, \partial_{x_2} \cdot \nabla_{t_2} \phi_2^{(\ell)}(x_2, t_2)\, \phi^{(0)}(x_3) \rangle \big|_{\Delta_1 = \Delta_2 = \ell + d - 2} = 0$ unless (7.53) applies.

If $\partial_{x_3} \cdot \nabla_{t_3} \langle \phi_1^{(\ell_1)}(x_1, t_1)\, \phi_2^{(0)}(x_2)\, \phi^{(\ell_3)}(x_3, t_3) \rangle \big|_{\Delta_3 = \ell_3 + d - 2} = 0$ and instead of (7.50) then $C_{12,3}(t_1, t_3, y) = (t_1 \cdot y)^{\ell_1} (t_3 \cdot y)^{\ell_3} (y^2)^{-\lambda} \sum_{n=0}^{\min(\ell_1, \ell_3)} C_n\, \xi_2{}^n$ where in this case $2\lambda = \Delta_1 + \Delta_2 + \ell_1 + \ell_2 - d + 2$ then requiring (7.49c) to be zero gives the same relations as in (7.51) and (7.52) so long as $\ell_1 \to \ell_3,\ \ell_2 \to \ell_1,\ \Delta_2 \to \Delta_1,\ \Delta_3 \to \Delta_1$.

For two conserved currents and a scalar we require from (7.49a), (7.49b)

$$\partial_y \cdot \nabla_{t_1} C_{12,3}(t_1, t_2, y) \big|_{\Delta_1 = \ell_1 + d - 2, \Delta_2 = \ell_2 + d - 2} = \partial_y \cdot \nabla_{t_2} C_{12,3}(t_1, t_2, y) \big|_{\Delta_1 = \ell_1 + d - 2, \Delta_2 = \ell_2 + d - 2} = 0\,. \tag{7.54}$$

For $\ell_1 = \ell_2$ the two conditions are identical and there is a solution for any Δ_3. For $\ell_1 > \ell_2$ then (7.53) is relevant and the two conservation conditions hold only if $\Delta_2 - \Delta_3 = \ell_2$ or $\Delta_3 = d - 2$.

In general the recurrence relations (7.51) translate into the third order differential equation

$$\mathcal{D}_\xi H_{12,3}(\xi) = 0\,, \qquad 2\,\mathcal{D}_\xi = a_\theta + \frac{1}{\xi}\, b_\theta + \xi c_\theta\,, \quad \theta = \xi \frac{\mathrm{d}}{\mathrm{d}\xi}\,. \tag{7.55}$$

In a special case the third order operator factorises

$$\begin{aligned} \mathcal{D}_\xi \big|_{\Delta_1 = \ell_1 + d - 2, \Delta_2 = \ell_2 + d - 2, \Delta_3 = d - 2} &= \big((1 - \xi)\, \theta + \tfrac{1}{2}(d - 2) + (\ell_1 - 1)\, \xi \big) \mathcal{D}_{2,\xi}\,, \\ \mathcal{D}_{2,\xi} &= \frac{1}{\xi}\, \theta \big(\theta - \tfrac{1}{2}(d - 2) - \ell_1 - \ell_2 \big) - (\theta - \ell_1)(\theta - \ell_2)\,. \end{aligned} \tag{7.56}$$

Solving $\mathcal{D}_{2,\xi} H_{12,3}(\xi) = 0$ is straightforward giving

$$\begin{aligned} H_{12,3}(\xi) &= F(-\ell_1, -\ell_2; -\tfrac{1}{2}d + 2 - \ell_1 - \ell_2; \xi) \\ &= \tfrac{\Gamma(-\frac{1}{2}d + 2 - \ell_1 - \ell_2)\, \Gamma(-\frac{1}{2}d + 4)}{\Gamma(-\frac{1}{2}d + 2 - \ell_1)\, \Gamma(-\frac{1}{2}d + 2 - \ell_2)}\ F(-\ell_1, -\ell_2; \tfrac{1}{2}d - 1; 1 - \xi)\,. \end{aligned} \tag{7.57}$$

This solution is a polynomial of degree $\min(\ell_1, \ell_2) + 1$, is obviously symmetric in ℓ_1, ℓ_2 and so realises both conservation equations when $\Delta_3 = d - 2$ including also $\ell_1 = \ell_2$. There is a corresponding generating function

$$\sum_{\ell_1, \ell_2 \geq 0} \frac{x^{\ell_1} y^{\ell_2}}{l_1!\, l_2!}\, F(-\ell_1, -\ell_2; \tfrac{1}{2}d - 1; 1 - \xi) = e^{x + y}\, {}_0F_1\big(\tfrac{1}{2}d - 1; x\, y (1 - \xi) \big)\,. \tag{7.58}$$

7.2.2 General Conservation Equations

For three conserved tensor fields with spins ℓ_1, ℓ_2, ℓ_3 and $\Delta_i = \ell_i + d - 2$ then starting from (7.29) and (7.49a), (7.49c) we have

$$\mathcal{D}_1 H_{12,3}(\xi_1, \xi_2, \xi_3) = 0\,, \qquad \mathcal{D}_3 H_{12,3}(\xi_1, \xi_2, \xi_3) = 0\,, \tag{7.59}$$

with $\mathcal{D}_1, \mathcal{D}_3$ third order differential operators defined by

$$\mathcal{D}_1 = a + \frac{1}{\xi_2}\, b_2 + \frac{1}{\xi_3}\, b_3 + \xi_2\, c_2 + \xi_3\, c_3 + \frac{\xi_1}{\xi_2}\, d_2 + \frac{\xi_1}{\xi_3}\, d_3 + \frac{\xi_3}{\xi_2}\, g_2 + \frac{\xi_2}{\xi_3}\, g_3 + \frac{\xi_1}{\xi_2\, \xi_3}\, e\,, \tag{7.60}$$

and

$$\begin{aligned} \mathcal{D}_3 = \hat{a} + \frac{1}{\xi_1}\, \hat{b}_1 + \frac{1}{\xi_2}\, \hat{b}_2 + \frac{1}{\xi_3}\, \hat{b}_3 + \xi_1\, \hat{c}_1 + \xi_2\, \hat{c}_2 + \frac{\xi_3}{\xi_1}\, \hat{d}_1 + \frac{\xi_3}{\xi_2}\, \hat{d}_3 + \frac{\xi_2}{\xi_1}\, \hat{g}_1 + \frac{\xi_1}{\xi_2}\, \hat{g}_2 + \frac{\xi_1}{\xi_2\, \xi_3}\, \hat{e} \\ + \frac{\xi_1}{\xi_3}\, \hat{j}_1 + \frac{\xi_2}{\xi_3}\, \hat{j}_2 + \frac{\xi_2}{\xi_1\, \xi_3}\, \hat{k}_1 + \frac{\xi_1}{\xi_2\, \xi_3}\, \hat{k}_2\,, \end{aligned} \tag{7.61}$$

with the coefficients cubic polynomials in the derivatives $\theta_i = \xi_i \frac{\partial}{\partial \xi_i}$. Explicitly in (7.60)

$$\begin{aligned} b_2 &= \theta_2\big(\theta_2 + \theta_3 + \tfrac{1}{2}d - 2\big)\big(\theta_1 + \theta_2 + \theta_3 - \ell_1 - \ell_2 + 1 - \tfrac{1}{2}d\big)\,, \\ b_3 &= \theta_3\big(\theta_2 + \theta_3 + \tfrac{1}{2}d - 2\big)\big(\theta_1 + \theta_2 + \theta_3 - \ell_1 - \ell_2 + 1 - \tfrac{1}{2}d\big)\,, \\ c_2 &= -\,\kappa_1(\kappa_1 - 1)\kappa_3\,, \qquad\qquad c_3 = -\kappa_1(\kappa_1 - 1)\kappa_2\,, \\ d_2 &= \theta_2\, \kappa_2\big(\ell_1 - \theta_3 + \tfrac{1}{2}d - 2\big)\,, \qquad d_3 = \theta_3\, \kappa_3\big(\ell_1 - \theta_2 + \tfrac{1}{2}d - 2\big)\,, \\ g_2 &= -\,\theta_2\, \kappa_1\, \kappa_2\,, \qquad\qquad g_3 = -\theta_3\, \kappa_1\, \kappa_3\,, \\ e &= -\,\theta_2\, \theta_3\big(\theta_1 + \theta_2 + \theta_3 - \ell_1 - \ell_2 + 1 - \tfrac{1}{2}d\big)\,, \\ a &= -\,\kappa_1\,\big(\ell_1(\theta_1 + \theta_2 + \theta_3) + \ell_2(\theta_2 + 2\,\theta_3 + \tfrac{1}{2}d - 1) \\ &\qquad + \ell_3(\theta_2 - \ell_1 - \tfrac{1}{2}d + 2) - \theta_1 - 2(\theta_1 + \theta_2)(\theta_2 + \theta_3) - 2\,\theta_3{}^2\big)\,, \end{aligned} \tag{7.62}$$

with $\kappa_1 = \ell_1 - \theta_2 - \theta_3\,, \kappa_2 = \ell_2 - \theta_1 - \theta_3\,, \kappa_3 = \ell_3 - \theta_1 - \theta_2$. Further in (7.61).

$$\begin{aligned} \hat{b}_1 &= \theta_1\big((\ell_3 + \tfrac{1}{2}d - 2)(\kappa_1 - \ell_2 - \tfrac{1}{2}d + 1) - \kappa_3(\kappa_1 - \kappa_2 - 1) + \theta_2(\theta_3 + 1)\big)\,, \\ \hat{b}_2 &= -\,\theta_2\big((\ell_3 + \tfrac{1}{2}d - 2)(\kappa_2 - \ell_1 - \tfrac{1}{2}d + 1) + \kappa_3(\kappa_1 - \kappa_2 + 1) + \theta_1(\theta_3 + 1)\big)\,, \\ \hat{b}_3 &= -\,\theta_3(\theta_1 - \theta_2)\kappa_3\,, \\ \hat{c}_1 &= -\,\kappa_2\, \kappa_3(\kappa_3 - 1)\,, \qquad \hat{c}_2 = \kappa_1\kappa_3(\kappa_3 - 1)\,, \\ \hat{d}_1 &= -\,\theta_1\, \kappa_1\big(\ell_3 - \theta_2 + \tfrac{1}{2}d - 2\big)\,, \qquad \hat{d}_2 = \theta_2\, \kappa_2\big(\ell_3 - \theta_1 + \tfrac{1}{2}d - 2\big)\,, \\ \hat{g}_1 &= \theta_1\, \kappa_1\, \kappa_3\,, \qquad \hat{g}_2 = -\theta_2\, \kappa_2\, \kappa_3\,, \\ \hat{j}_1 &= -\,\hat{j}_2 = -\theta_3\, \kappa_3(\kappa_3 - 1)\,, \qquad \hat{k}_1 = \theta_1\, \theta_3\, \kappa_3\,, \qquad \hat{k}_2 = -\theta_2\, \kappa_3\, \kappa_3\,, \\ \hat{e} &= -\,\theta_2\, \theta_3(\kappa_1 - \kappa_2)\,, \\ \hat{a} &= \kappa_3\,\big((\ell_1 - \ell_2)(\theta_1 + \theta_2 + \tfrac{1}{2}d - 1) - (\theta_1 - \theta_2)(\kappa_3 - \theta_1 - \theta_2 - \theta_3) + \ell_1\, \theta_2 - \ell_2\theta_1)\big)\,, \end{aligned} \tag{7.63}$$

which reflect the antisymmetry under $1 \leftrightarrow 2$. The operators satisfy

$$\begin{aligned} &\mathcal{D}_1 H_{12,3}(\xi_1, \xi_2, \xi_3) = -\,\mathcal{D}_1 H_{13,2}(\xi_1, 1 - \xi_3, 1 - \xi_2)\,, \quad \mathcal{D}_2 H_{12,3}(\xi_1, \xi_2, \xi_3) = \mathcal{D}_2 H_{21,3}(\xi_2, \xi_1, \xi_3)\,, \\ &\mathcal{D}_3 H_{12,3}(\xi_1, \xi_2, \xi_3) = -\,\mathcal{D}_3 H_{21,3}(\xi_2, \xi_1, \xi_3) = \mathcal{D}_3 H_{32,1}(1 - \xi_3, 1 - \xi_2, \xi_1)\,. \end{aligned} \tag{7.64}$$

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