xConf: conformal transformations with xAct

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Abstract

The small package xConf, which works on top of on the abstract tensor algebra suite xTensor which belongs to xAct [1], allows to perform Weyl rescalings of the metric and its associated curvature tensors, along with other primary tensors. Weyl rescalings, often called losely conformal transformations, form an Abelian group, and the package provides both a method to actively rescale expressions, or to express passively a given tensor in terms of rescaled images of the tensors appearing in it. We summarize the properties of Weyl rescalings and provide a few examples which are also distributed with the package.

Keywords: Conformal transformation, Weyl rescaling, Computer tensor algebra

PROGRAM SUMMARY
Program Title: xConf
Developer's repository link: https://www2.iap.fr/users/pitrou/xconf.htm
Licensing provisions: GPLv3
Programming language: Mathematica

1. Weyl rescalings

TODO. say what we do for volume forms, determinants etc ?

1.1. Conformally related metrics

A Weyl rescaling is a mapping from one metric to a related metric. It is not a coordinate transformation as we relate two different metrics. It is casually called a conformal transformation, although this is an improper naming, as in e.g. appendix D of Wald's book [2]. Here, we closely follow this reference, except that we choose Greek letters $\alpha, \beta, \mu, \nu, \ldots$ for the abstract indices. The conformal relation between two metrics is

$$\mathcal{T}_S(g_{\mu\nu}) \equiv S^2 g_{\mu\nu}\,, \qquad \mathcal{T}_S(g^{\mu\nu}) \equiv S^{-2} g^{\mu\nu} \tag{1}$$

where S is a scalar field on the manifold. In short hand notation we write $\mathcal{T}_S g$ instead of $\mathcal{T}_S(g_{\mu\nu})$. For a given metric, there is a class of equivalence $\{\boldsymbol{g}\}$ of conformally related metrics. The lengths defined by conformally related metrics differ, but angles are invariant, and the causal structure are identical as the nature of vectors (spacelike, null, or timelike) is preserved.

It is common for any vector V^μ to build a related form by $V_\mu = g_{\mu\nu}V^\nu$, and conversely from a form ω_μ we associate the tensor $\omega^\mu = g^{\mu\nu}\omega_\nu$, where $g^{\mu\nu}$ is the inverse metric, such that $g_{\mu\alpha}g^{\alpha\nu} = g_{\mu\alpha}g^{-1,\alpha\nu} = \delta^\nu_\mu$. If we choose a metric for raising and lowering indices, then noting $\tilde{g}_{\mu\nu} \equiv \mathcal{T}_S(g_{\mu\nu})$ and $\tilde{g}^{\mu\nu} \equiv \mathcal{T}_S(g^{\mu\nu})$, we find

$$\tilde{g}^{-1,\mu\nu} = S^{-2}_{g\to\tilde{g}} g^{\mu\nu} = S^{-4}_{g\to\tilde{g}} \tilde{g}^{\mu\nu} \quad \text{where} \quad \tilde{g}^{\mu\nu} \equiv g^{\mu\alpha}g^{\nu\beta}\tilde{g}_{\alpha\beta} \equiv g^{-1,\mu\alpha}g^{-1,\nu\beta}\tilde{g}_{\alpha\beta}\,, \tag{2}$$

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which shows that possible confusions can arise. It is thus advised to always write explicitely the metrics.

1.2. Transformation of Levi-Civita connection

We only consider Levi-Civita connections throughout, which are torsionless. As we work with several metrics, we must specify with an upper index the metric to which they are associated. The Levi-Civita connection $\overset{\mathcal{T}_S g}{\nabla}$ associated with $\mathcal{T}_S(g_{\mu\nu})$ ($\overset{\mathcal{T}_S g}{\nabla}_\alpha \mathcal{T}_S(g_{\mu\nu}) = 0$) is different from the Levi-Civita connection $\overset{g}{\nabla}$ associated with $g_{\mu\nu}$ ($\overset{g}{\nabla}_\alpha g_{\mu\nu} = 0$). They are related via

$$\overset{\mathcal{T}_S g}{\nabla}_\mu V^\nu = \overset{g}{\nabla}_\mu V^\nu + \Gamma[\overset{\mathcal{T}_S g}{\nabla}, \overset{g}{\nabla}]^\nu_{\mu\alpha} V^\alpha \,, \tag{3}$$

$$\overset{\mathcal{T}_S g}{\nabla}_\mu \omega_\nu = \overset{g}{\nabla}_\mu \omega_\nu - \Gamma[\overset{\mathcal{T}_S g}{\nabla}, \overset{g}{\nabla}]^\alpha_{\mu\nu} \omega_\alpha \,. \tag{4}$$

From the respective metric compatibility conditions, we get that the Christoffel tensors[1] relating the connections is

$$\Gamma[\overset{\mathcal{T}_S g}{\nabla}, \overset{g}{\nabla}]^\alpha_{\mu\nu} = -\Gamma[\overset{g}{\nabla}, \overset{\mathcal{T}_S g}{\nabla}]^\alpha_{\mu\nu} = 2\delta^\alpha_{(\mu}\partial_{\nu)} \ln S - g_{\mu\nu} g^{\alpha\beta} \partial_\beta \ln S \,. \tag{5}$$

Note that the Christoffel tensors only depend on the conformal class of equivalence $\{g\}$ and on the rescaling factor since $g_{\mu\nu} g^{\alpha\beta}$ is conformally invariant. This means that the correct objects are the

$$\Gamma[\{g\}, S]^\alpha_{\mu\nu} \equiv 2\delta^\alpha_{(\mu}\partial_{\nu)} \ln S - g_{\mu\nu} g^{\alpha\beta} \partial_\beta \ln S \tag{6}$$

which we shall use herafter to emphasize that any metric in the class of equivalence can be used in (5). A fundamental property related to the presence of $\ln S$ is

$$\Gamma[\{g\}, S_1 S_2]^\alpha_{\mu\nu} = \Gamma[\{g\}, S_1]^\alpha_{\mu\nu} + \Gamma[\{g\}, S_2]^\alpha_{\mu\nu} \,. \tag{7}$$

1.3. Group of Weyl rescalings

The Weyl rescalings of the metric form an Abelian group, which is the natural action of the group of scalar functions on the manifold equipped with the multiplication. For a given metric, its aforemoentioned class of equivalence associated to a metric is its orbit under the Weyl rescaling group action. It is obvious that

$$\mathcal{T}_{S_1} \circ \mathcal{T}_{S_2}(g_{\mu\nu}) = \mathcal{T}_{S_2} \circ \mathcal{T}_{S_1}(g_{\mu\nu}) = \mathcal{T}_{S_1 S_2}(g_{\mu\nu}) \,, \qquad \mathcal{T}_S^{-1}(g_{\mu\nu}) = \mathcal{T}_{1/S}(g_{\mu\nu}) \,. \tag{8}$$

It is a priori not obvious that the group structure exists when considering conformal transformation on other tensors, or composite expressions involving the Levi-Civita derivatives of tensors. In order to check it is the case, we first need to explicit how a Weyl rescaling acts on products and sums of tensors, and on covariant derivatives applied to tensors.

1.4. Rescaling of primary tensors

We ask the following rules for tensors

$$\mathcal{T}_S(V^{\alpha_1\dots\alpha_p}_{\beta_1\dots\beta_q} + W^{\mu_1\dots\mu_r}_{\nu_1\dots\nu_s}) = \mathcal{T}_S(V^{\alpha_1\dots\alpha_p}_{\beta_1\dots\beta_q}) + \mathcal{T}_S(W^{\mu_1\dots\mu_r}_{\nu_1\dots\nu_s}) \tag{9}$$

$$\mathcal{T}_S(V^{\alpha_1\dots\alpha_p}_{\beta_1\dots\beta_q} W^{\mu_1\dots\mu_r}_{\nu_1\dots\nu_s}) = \mathcal{T}_S(V^{\alpha_1\dots\alpha_p}_{\beta_1\dots\beta_q}) \mathcal{T}_S(W^{\mu_1\dots\mu_r}_{\nu_1\dots\nu_s}) \tag{10}$$

[1] As we rely on abstract indices exclucively and we specify the connection related by a Christoffel symbol, these are truely tensors. Hence we adopt the point of view of [2], which is also the one of the abstract tensor manipulation in `xTensor`.

The primary tensors are those which can also be assigned a homogeneous transformation rule

$$\mathcal{T}_S(V^{\alpha_1\dots\alpha_p}_{\beta_1\dots\beta_q}) = S^{w_T^{(p,q)}} T^{\alpha_1\dots\alpha_p}_{\beta_1\dots\beta_q}\,. \tag{11}$$

The action of the Weyl rescaling on such tensors has properties similar to (8), hence it is the action of an Abelian group.

The tensor weight depends on the positioning of indices. Since indices are lowered or raised by a metric, then from the rule (10) we must have for a given primary tensor

$$w_T^{(p\mp 1,q\pm 1)} = w_T^{(p,q)} \pm 2\,. \tag{12}$$

Hence there exists w_T such that

$$w_T^{(p,q)} = w_T + (q-p)\,. \tag{13}$$

We decide to call the $w_T^{(p,q)}$ the dressed weights of a tensor with $p+q$ indices, whereas the unique w_T is its naked weight. There are several ways to be dressed, but only one way to be naked. In practice, each tensor has a natural positioning of indices and a natural weight for this positioning. Then the naked weight w_T is deduced from the previous relation. For instance for all metrics $w_g^{(0,2)} = 2$, hence the naked weights $w_g = 0$. Finally the conformal factors must all be weightless, that is for any rescaling factor S

$$w_S = 0\,. \tag{14}$$

The total naked weight of a product of tensors is

$$w_{\boldsymbol{A}\otimes\boldsymbol{B}} = w_{\boldsymbol{A}} + w_{\boldsymbol{B}}\,. \tag{15}$$

Furthemore the δ^μ_ν tensor must be weightless since

$$w_\delta = w_\delta^{(1,1)} = w_g^{(1,1)} = 0\,. \tag{16}$$

In practice, the two previous relations imply that we can contract indices of a tensor without changing its dressed weight.

The linearity of the conformal transformation with respect to a multiplicative weightless scalar function is deduced from (10). However, even though we have the rule (9) for a sum of tensors, there is in general no meaning for the weight of a sum of tensors where each term can have in general a different weight, hence the group of Weyl rescaling is not a morphism on the algebra of tensors. Furthermore not all tensors are primary tensors.

1.5. Weyl rescaling of covariant derivatives

The most notable non-primary tensors are the ones obtained by applying a covariant derivative on a primary tensor, hence we must also specify a rule for the Weyl rescaling of such expressions. Covariant derivatives transform non homogeneously (except on scalars) as

$$\mathcal{T}_S(\overset{g}{\nabla}_\mu V^{\alpha_1\dots\alpha_p}_{\beta_1\dots\beta_q}) = \overset{\mathcal{T}_S g}{\nabla}_\mu \mathcal{T}_S(V^{\alpha_1\dots\alpha_p}_{\beta_1\dots\beta_q})\,, \tag{17}$$

and in particular for weightless scalars ($w_S = 0$) we must have

$$\mathcal{T}_S(\overset{g}{\nabla}_\mu S) = \overset{\mathcal{T}_S g}{\nabla}_\mu S\,, \tag{18}$$

which is compatible with the fact that all derivatives are equivalent to an ordinary derivative. In particular, from (6), and using that $g_{\mu\nu}g^{\alpha\beta}$ is a rescaling invariant, the Christoffels associated to conformal transformations must be primary tensors, and their naked weights are

$$w_\Gamma = -1\,. \tag{19}$$

From (3) and (5), we see that the covariant derivative of a tensor is not a primary tensor. Nonetheless, thanks to (7) and (19) we deduce that the action of the Weyl rescaling on such objects is also the action of an Abelian group, hence as in (8).

An equivalent formulation which emphasizes that a conformal transformation is entirely determined by the scale factor S is

$$\begin{aligned} \mathcal{T}_S(\overset{g}{\nabla}_\mu V^\nu) &= \overset{g}{\nabla}_\mu \mathcal{T}_S(V^\nu) + \Gamma[\{\boldsymbol{g}\}, S]^\nu_{\mu\alpha} \mathcal{T}_S(V^\alpha) && (20)\\ \mathcal{T}_S(\overset{g}{\nabla}_\mu \omega_\nu) &= \overset{g}{\nabla}_\mu \mathcal{T}_S(\omega_\nu) - \Gamma[\{\boldsymbol{g}\}, S]^\alpha_{\mu\nu} \mathcal{T}_S(\omega_\alpha) && (21) \end{aligned}$$

with obvious generalization on more complicated tensors.

To sum it up, a covariant derivative does not transform homogeneously, and in a conformal transformation Christoffel tensors appear. But these Christoffels, if we were to apply a subsequent conformal transformation, transform homogeneously with the weight (19), ensuring that conformal transformations are the action of an Abelian group when applied on covariant derivatives of primary tensors.

1.6. Weyl rescaling of curvature

The Riemann curvature tensor for Levi-Civita connections is defined from the covariant derivative, as

$$(\overset{g}{\nabla}_\alpha \overset{g}{\nabla}_\beta - \overset{g}{\nabla}_\beta \overset{g}{\nabla}_\alpha) V^\mu = -{}^g R_{\alpha\beta\nu}{}^\mu V^\nu \tag{22}$$

and we must deduce its transformation compatible with (10) and (17). We get that the transformation rule of a Riemann tensor is defined to be the Riemann tensor of the transformed metric and connection, hence

$$\mathcal{T}_S({}^g R_{\alpha\beta\mu}{}^\nu) \equiv {}^{\mathcal{T}_S(g)} R_{\alpha\beta\mu}{}^\nu \,. \tag{23}$$

However it is not homogeneous and its explicit transformation is not straightforward. We first easily find

$$\mathcal{T}_S({}^g R_{\alpha\beta\mu}{}^\nu) - {}^g R_{\alpha\beta\mu}{}^\nu = -2\overset{g}{\nabla}_{[\alpha} \Gamma[\{\boldsymbol{g}\}, S]^\nu_{\beta]\mu} + 2\Gamma[\{\boldsymbol{g}\}, S]^\lambda_{\mu[\alpha} \Gamma[\{\boldsymbol{g}\}, S]^\nu_{\beta]\lambda} \,. \tag{24}$$

It is then more painful to replace expressions (5) and we get

$$\mathcal{T}_S({}^g R_{\alpha\beta\mu}{}^\nu) = {}^g R_{\alpha\beta\mu}{}^\nu + U[g,S]_{\alpha\beta\mu}{}^\nu \,, \tag{25}$$

where the last tensor depends on the metric (and not just on its class of equivalence, an important difference with Christoffels), through the derivative of the Christoffel which is $\overset{g}{\nabla}$. It can be expressed as (see D.7 in [2])

$$\begin{aligned} U[g,S]_{\alpha\beta\mu}{}^\nu &= 2\delta^\nu_{[\alpha} \overset{g}{\nabla}_{\beta]} \overset{g}{\nabla}_\mu \ln S - 2g^{\nu\lambda} g_{\mu[\alpha} \overset{g}{\nabla}_{\beta]} \overset{g}{\nabla}_\lambda \ln S && (26)\\ &+ 2(\overset{g}{\nabla}_{[\alpha} \ln S)\delta^\nu_{\beta]} \overset{g}{\nabla}_\mu \ln S - 2(\overset{g}{\nabla}_{[\alpha} \ln S) g_{\beta]\mu} g^{\nu\lambda} \overset{g}{\nabla}_\lambda \ln S - 2g_{\mu[\alpha} \delta^\nu_{\beta]} g^{\lambda\sigma} (\overset{g}{\nabla}_\lambda \ln S)(\overset{g}{\nabla}_\sigma \ln S) \,. \end{aligned}$$

It can be checked after a painful computation (that can be performed with `xConf!`) that

$$U[g,S]_{[\alpha\beta\mu]}{}^\nu = 0 \tag{27}$$

to maintain the first Bianchi identity. This tensor transforms non-homogeneously as

$$\mathcal{T}_{S_2}\left(U[g,S_1]_{\alpha\beta\mu}{}^\nu\right) = U[\mathcal{T}_{S_2}(g), S_1]_{\alpha\beta\mu}{}^\nu \tag{28}$$

and the composition rule is

$$U[g, S_1 S_2]_{\alpha\beta\mu}{}^\nu = U[g,S_2]_{\alpha\beta\mu}{}^\nu + U[\mathcal{T}_{S_2}(g), S_1]_{\alpha\beta\mu}{}^\nu = U[g,S_1]_{\alpha\beta\mu}{}^\nu + U[\mathcal{T}_{S_1}(g), S_2]_{\alpha\beta\mu}{}^\nu \,, \tag{29}$$

which is to be compared to the simpler (7). This is nonetheless what is expected from the composition of two transformations as it ensures that

$$\mathcal{T}_{S_2} \circ \mathcal{T}_{S_1}({}^{g}R_{\alpha\beta\mu}{}^{\nu}) = \mathcal{T}_{S_1 S_2}({}^{g}R_{\alpha\beta\mu}{}^{\nu}) \equiv {}^{\mathcal{T}_{S_1 S_2}(g)}R_{\alpha\beta\mu}{}^{\nu} \,. \tag{30}$$

The transformation of the Ricci tensor follows from contraction of indices and is

$$\mathcal{T}_S({}^{g}R_{\alpha\mu}) = {}^{g}R_{\alpha\mu} + U[g,S]_{\alpha\beta\mu}{}^{\beta} \tag{31}$$

where

$$\begin{aligned} U[g,S]_{\alpha\beta\mu}{}^{\beta} &= -(n-2)\overset{g}{\nabla}_\alpha \overset{g}{\nabla}_\mu \ln S - g_{\alpha\mu} g^{\beta\nu} \overset{g}{\nabla}_\beta \overset{g}{\nabla}_\nu \ln S \\ &\quad +(n-2)(\overset{g}{\nabla}_\alpha \ln S)(\overset{g}{\nabla}_\mu \ln S) - (n-2) g_{\alpha\mu} g^{\beta\nu} (\overset{g}{\nabla}_\beta \ln S)(\overset{g}{\nabla}_\nu \ln S) \,. \end{aligned} \tag{32}$$

Finally, the Ricci scalar transforms as

$$\begin{aligned} \mathcal{T}_S({}^{g}R) &\equiv \mathcal{T}_S({}^{g}R_{\alpha\beta}) \mathcal{T}_S(g^{\alpha\beta}) \\ &= S^{-2}\left[{}^{g}R - 2(n-1) g^{\alpha\beta} \overset{g}{\nabla}_\alpha \overset{g}{\nabla}_\beta \ln S - (n-2)(n-1) g^{\alpha\beta} (\overset{g}{\nabla}_\alpha \ln S)(\overset{g}{\nabla}_\beta \ln S)\right] . \end{aligned} \tag{33}$$

2. Using `xConf`

2.1. Preferred conformal frame

Since all conformally related metrics belong to a class of equivalence, it is convenient to choose one as a reference with respect to which the other ones are expressed. If we choose a metric $g_{\mu\nu}$ to be the reference metric, also called reference frame, then for all tensors we also decide that they are associated to this conformal frame, that is for all tensors

$$\boldsymbol{T} = \mathcal{F}_g[\boldsymbol{T}] \,. \tag{34}$$

A tensor in a different frame $\mathcal{T}_S g$ is just the result of the conformal transformation from the reference frame g to the frame $\mathcal{T}_S g$, that is

$$\mathcal{F}_{\mathcal{T}_S g}[\boldsymbol{T}] \equiv \mathcal{T}_S\left(\mathcal{F}_g[\boldsymbol{T}]\right) = \mathcal{T}_S(\boldsymbol{T}) \,. \tag{35}$$

We also ask

$$\mathcal{F}_{\mathcal{T}_{S_2} g}[\mathcal{F}_{\mathcal{T}_{S_1} g}[\boldsymbol{T}]] = \mathcal{F}_{\mathcal{T}_{S_2} g}[\boldsymbol{T}] \tag{36}$$

hence $\mathcal{F}$ does not act as a group. It is simply a function which selects the desired element in the class of equivalence $\{\boldsymbol{T}\}$ of the Weyl rescalings. We can move from one choice to another choice thanks to

$$\mathcal{F}_{g_2}[\boldsymbol{T}] = \mathcal{T}_{S_{12}}(\mathcal{F}_{g_1}[\boldsymbol{T}]) \tag{37}$$

where $g_2 = \mathcal{T}_{S_{12}}(g_1)$.

The implementation in `xConf` is based on the idea that the first metric defined is the reference metric, and all tensors are first defined in this conformal frame. We now review how the frame choice function $\mathcal{F}$, and the frame transformation function $\mathcal{T}$, are implemented.

2.2. Defining conformally related metrics

We first load the package, which itself loads `xTensor`.

In[1] := `<<xAct‘xConf‘`

(Version and copyright messages)

We then define the four-dimensional manifold `M` with abstract indices $\{\alpha, \beta, \mu, \nu, \lambda\}$:

In[2] := `DefManifold[ M, 4,` $\{\alpha, \beta, \mu, \nu, \lambda\}$ `];`

and the ambient metric `g` of negative signature, along with its associated covariant derivative `CD`:

In[3] := `DefMetric[ -1, g[-`α`,-`β`], CD, {";","`∇`"}, PrintAs->"g" ];`

So far this is not specific to `xConf`. We then use the first function provided in `xConf`, which is an extension of the capabilities of `DefMetric` to define a metric conformally related to the previous one. The scaling function name is chosen to to be S.

In[4] := `DefConformalMetric[g, S, PrintAs -> "G", ConformalMetricName -> "G",`
`SymbolOfCovD -> {":",`$\overset{G}{\nabla}$`}];`

It is possible to customize the name of the conformally related metric and the color and the associate frame. We know have two metrics as can be checked.

In[5] := `$Metrics`

Out[5] := $\{\mathrm{g}, \mathrm{G}\}$

The conformal relations between them are stored.

In[6] := `ConformalRules[G, g]`

Out[6] := $\{\ [\mathrm{G}]^{\underline{\alpha\beta}} \to \mathrm{g}^{\alpha\beta}\mathrm{S}^2\ ,\ \mathrm{i}[\mathrm{G}]^{\underline{\alpha\beta}} \to \frac{\mathrm{g}^{\alpha\beta}}{\mathrm{S}^2},\ [\mathrm{g}\tilde{\tilde{\mathrm{S}}}^2] \to \tilde{\tilde{\mathrm{g}}}\mathrm{S}^8\}$

The last rule is the relation between the determinants of the metric. If several conformally related metrics are defined, then all possible conformal relations between them are stored automatically.

Finally let us define a vector in order to perform simple examples.

In[7] := `DefTensor[k[-`α`], M, PrintAs -> "k"]`

We check that the default naked weight for this vector vanishes.

In[8] := `ConformalWeight[k]`

Out[8] := 0

2.3. A formal representation of the conformal transformation

We can consider this vector in the frames associated with $g_{\mu\nu}$ or $G_{\mu\nu}$ thanks to the `xTensor` inert head `Conformalframe`. That is $\mathcal{F}_g[k_\alpha]$ and $\mathcal{F}_G[k_\alpha]$ are respectively :

In[9] := `ConformalFrame[g][k[-`α`]]`

Out[9] := ${}^{g}[\mathrm{k}_\alpha]$

In[10] := `ConformalFrame[G][k[-`α`]]`

Out[10] := ${}^{\mathrm{G}}[\mathrm{k}_\alpha]$

Since a conformal frame is also viewed as the result of a Weyl rescaling from the reference frame, the effect on sums and products (9) and (10) is also applied automatically, and when applied on a derivative in a given frame, the relation (17) is also used.

In[11] := `ConformalFrame[G][CD[-`β`]@k[-`α`]]`

Out[11] := $\overset{\mathrm{G}}{\nabla}_\beta {}^{\mathrm{G}}[\mathrm{k}_\alpha]$

In practice the option `$FormatConformal = "Color"` allows to visualize frames with colored brackets, with each color associated to a metric of the class of equivalence.

2.4. Passive transformations

To avoid any confusion, which relies on knowing which metric is used to lower and raise indices (which is by default the first metric defined in `xAct`), it is healthy to always write the metrics instead of hiding them in moved indices. This is achieved in `xAct` by the use of `SeparateMetric` and we use it to precondition expressions. Let us first define a rule to use $\ln(S)$ instead of S.

In[12] := `DefTensor[LogS[], M, PrintAs -> "ln(S)"];`

In[13] := `RuleLogS = {cd_?CovDQ[i_][S[]] :> S[] cd[i][LogS[]]};`

The function `ToMetric` allows to express a given expression in terms of tensors conformally transformed in a target frame, using (35). It is not the transformation of an expression, but rather expresses the same quantity with different tensors, hence it can be viewed as a passive transformation.

In[14] := `S[]CD[-μ][k[-β]]`

Out[14] := $\mathrm{S}\nabla_\mu \mathrm{k}_\beta$

In[15] := `ToMetric[G][%];`

In[16] := `Expand[(% // NoScalar)/.  RuleLogS]`

Out[16] := $\overset{\mathrm{G}}{\nabla}_\mu{}^{\mathrm{G}}[\mathrm{k}_\beta] - \mathrm{G}_{\beta\mu}\mathrm{i}\mathrm{G}^{\alpha\lambda}\left(\overset{\mathrm{G}}{\nabla}_\lambda \ln(\mathrm{S})\right){}^{\mathrm{G}}[\mathrm{k}_\alpha] + \left(\overset{\mathrm{G}}{\nabla}_\beta \ln(\mathrm{S})\right){}^{\mathrm{G}}[\mathrm{k}_\mu]$

The function `ToMetric` uses (37) to relate tensors in different frames to the tensors in the target frame. The transformation appearing in this relation is obtained explicitely with (11) for primary tensors, (5) for changing covariant derivatives, and (25) for curvature tensors, after recursive usage of (9) and (10).

2.5. Active transformations

Active transformations correspond to the action of $\mathcal{T}_S$ implemented via the function `ConformalTransformation`. The scale factor of the transformation is specified by setting a pair of metric $(g, \mathcal{T}_S g)$. This pair is given as a first pair of arguments. For instance we obtain $\mathcal{T}_S(k_\mu)$ and $\mathcal{T}_S(k^\mu)$ via

In[17] := `ConformalTransformation[g, G][k[-α]]`

Out[17] := $\mathrm{k}_\alpha \mathrm{S}$

In[18] := `ConformalTransformation[g, G][k[α]]`

Out[18] := $\frac{g^{\alpha\beta}\mathrm{k}_\beta}{\mathrm{S}}$

In[19] := `ContractMetric[%]`

Out[19] := $\frac{\mathrm{k}^\alpha}{S}$

Furthermore we can optionally specify that we want all tensors appearing in the result to be in a given frame. This corresponds to the application of `ToMetric` to the result.

In[20] := `ConformalTransformation[g, G, FinalFrame->G][CD[-α][k[-β]]]`

Out[20] := $\overset{\mathrm{G}}{\nabla}_\alpha{}^{\mathrm{G}}[\mathrm{k}_\beta]$

Obviously, we recover that if we perform a conformal transformation from an initial frame to a target frame on an expression where all tensors are expressed the initial frame, and ask to read the result with tensors in the target frame, all quantities are only replaced with their $\mathcal{F}_{\rm target}[.]$ versions, and the transformation is a purely formal tautology which states that a transformed expression is the expression of the transformed tensorial constituents.

Internally, the function `ConformalTransformation[g1,g2]` first calls `ToMetric[g1]`, so that all tensors appearing are expressed in the desired initial frame. It then applies the head `ConformalFrame[g2]` which formally replaces all $\mathcal{F}_{g_1}$ by $\mathcal{F}_{g_2}$ hence performing in an abstract manner the transformation via the rule (36). Finally it calls `ToMetric[gtarget]` where the target frame, which is by default the initial frame, can be specified with the option `FinalFrame`. Although `ConformalTransformation` is the most important function, it is also has the shortest implementation.

3. Example

Let us show that the massless Klein-Gordon equation does not transform like a primary tensor.

In[21] := `DefTensor[`ϕ`[], M];`

In[22] := `expr=CD[-`α`][CD[`α`][`ϕ`[]]];`

By default the naked weight of the field is 0.

In[23] := `ConformalWeight[`ϕ`]`

Out[23] := 0

Let us check the transformation rule of the Klein-Gordon equation.

In[24] := `ConformalTransformation[g, G][expr]// NoScalar`

Out[24] := $\frac{2\mathrm{g}^{\alpha\beta}\nabla_\alpha \mathrm{S}\nabla_\beta\phi}{\mathrm{S}^3}+\frac{\mathrm{g}^{\alpha\beta}\nabla_\beta\nabla_\alpha\phi}{S^2}$

We notice that it is not an homogeneous transformation due to the first term. Now let us change the conformal weight and add a coupling to curvature.

In[25] := `ConformalWeight[`ϕ`] ^= -1;`

In[26] := `expr2=expr-1/6 `ϕ`[] RicciScalarCD[]`

The result transforms like a primary tensor

In[27] := `ConformalTransformation[g, G][expr2]// NoScalar`

Out[27] := $-\frac{\mathrm{R}[\nabla]\phi}{6S^3}+\frac{\mathrm{g}^{\alpha\beta}\nabla_\beta\nabla_\alpha\phi}{S^3}$

More complicated examples are provided in a folder provided with `xConf`.

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[2] R. M. Wald, General Relativity, 1984.