The tight-coupling approximation for baryon acoustic oscillations

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(Dated: December 2, 2010)

The tight-coupling approximation (TCA) used to describe the early dynamics of the baryons-photons system is systematically built to higher orders in the inverse of the interaction rate. This expansion can be either used to grasp the physical effects by deriving simple analytic solutions or to obtain a form of the system which is stable numerically at early times. In linear cosmological perturbations, we estimate numerically its precision, and we discuss the implications for the baryons acoustic oscillations. The TCA can be extended to the second order cosmological perturbations, and in particular we recover that vorticity is not generated at lowest order of this expansion.

PACS numbers: 98.80.-k

The baryon acoustic oscillations (BAO) generated when baryons and photons were highly coupled have now been observed in the cosmic microwave background [1, 2] and in the large scale structures [3]. The shape and amplitude of these oscillations cannot be obtained analytically and one resorts either to a numerical resolution or to a less accurate WKB approximation on the full system of dynamical equations for the cosmic fluids and the metric. On the one hand the numerical integration has stability issues due to strong restoring forces, and on the other hand the precision of the WKB approximation is limited. The tight-coupling approximation, which is an expansion in the interaction rate [see Eq. (17) below], is a way to avoid these two issues [4, 5]. The orders of the approximation are denoted by TCA-n, and one can always consider the equations up to TCA-n with n sufficiently large to ensure the required accuracy, in the range of time for which the expansion is converging. In this paper, we first recast the baryons-photons dynamics into a total fluid system, and we then present the TCA expansion. We also estimate numerically the convergence before recombination. A closed form at TCA-1 for photons density perturbations is then derived from which the main features of the dynamics are deduced. At second order in cosmological perturbations, the TCA-0 is sufficient at early times given the precision required for all practical purposes. An exception arises for vorticity since it is not generated for a single perfect fluid, and we recover the known result that it is not generated below TCA-1 [6, 7]. Our analysis is of interest for high precision BAO computations but also sheds some light on the fluid approximation of the baryons-photons system in general relativity and cosmology by offering another method for solving cosmological perturbations.

Perturbations

Geometry: We perturb the metric in the conformal Newtonian gauge according to

$$\mathrm{d}s^2 = a^2\left[(1-2\Psi)\delta_{ij}\mathrm{d}x^i\mathrm{d}x^j - (1+2\Phi)\mathrm{d}\eta^2 + 2S_i\mathrm{d}x^i\mathrm{d}\eta\right]$$

where a is the scale factor and η the conformal time. S_i is a vector type peturbation ($\partial_i S^i = 0$) considered only at second order in perturbations since vector perturbations decay at first order if not sourced. We also do not consider the tensor perturbations. A prime denotes a derivative with respect to η, and $\mathcal{H} \equiv a'/a$.

Baryons-photons system: The stress energy tensor of a species labelled by s is decomposed as

$$T^{\mu\nu}_{\mathrm{s}} = (\rho_{\mathrm{s}} + P_{\mathrm{s}})u^\mu_{\mathrm{s}}u^\nu_{\mathrm{s}} + P_{\mathrm{s}}g^{\mu\nu} + P_{\mathrm{s}}\Pi^{\mu\nu}_{\mathrm{s}}\,. \qquad (1)$$

The energy density is perturbed according to $\rho_{\mathrm{s}} = \bar\rho_{\mathrm{s}}(1+\delta_{\mathrm{s}})$, velocities are perturbed according to $u^i_{\mathrm{s}} = 1/a[(1+\Psi)v^i_{\mathrm{s}} - S^i]$ such that v^i_{s} matches the tetrad components of the velocity [8]. We assume that baryons have no anisotropic stress ($\Pi^{\mathrm{b}}_{ij} = 0$), have equation of state $w_{\mathrm{b}} \equiv \bar P_{\mathrm{b}}/\bar\rho_{\mathrm{b}} = 0$, whereas photons have equation of state $w_\gamma = 1/3$. The adiabatic speed of sound $c^2_{\mathrm{s}} \equiv P'_{\mathrm{s}}/\rho'_{\mathrm{s}}$ is constant for these two fluids and satisfies $c^2_{\gamma/\mathrm{b}} = w_{\gamma/\mathrm{b}}$. In general for a perfect fluid, if w_{s} is not constant, $c^2_{\mathrm{s}} \neq w_{\mathrm{s}}$ and $w'_{\mathrm{s}} = -3\mathcal{H}(c^2_{\mathrm{s}} - w_{\mathrm{s}})(1+w_{\mathrm{s}})$. For photons, free-streaming produces an anisotropic stress Π^γ_{ij} and in general we have to consider the moments of the temperature $\Theta_{i_1\dots i_\ell}$, and also the electric and magnetic type multipoles ($E_{i_1\dots i_\ell}$ and $B_{i_1\dots i_\ell}$) to describe linear polarization. The lowest multipoles are related to the fluid quantities up to second order by [8]

$$\Theta_i = \frac{\rho_\gamma v^\gamma_i}{\bar\rho_\gamma}\,, \quad \Theta_{ij} = \frac{5}{2}\left(\frac{\rho_\gamma\Pi^\gamma_{ij}}{4\bar\rho_\gamma} + v^\gamma_{\langle i}v^\gamma_{j\rangle}\right)\,. \qquad (2)$$

At first order, there are only scalar perturbations, the magnetic type multipoles vanish and the temperature scalar multipoles are defined by $\Theta_{i_1\dots i_\ell} \equiv (-1)^\ell(2\ell-1)!/\ell!\partial_{\langle i_1}\dots\partial_{i_\ell\rangle}\hat\Theta_\ell$ (and similarly for electric multipoles) where $\langle\dots\rangle$ means symmetric traceless part of the indices. In Fourier space, for a given mode $\mathbf{k}$ with $k \equiv |\mathbf{k}|$, we define dimensionless multipoles from the previous ones by $\Theta_\ell \equiv k^{-\ell}\hat\Theta_\ell$, and similarly for electric multipoles. For velocities, we use instead $v^{\mathrm{s}}_i \equiv \partial_i\hat v^{\mathrm{s}}$ and $v_{\mathrm{s}} = \hat v_{\mathrm{s}}/k$, that is $v_\gamma = -\Theta_1$.

Baryons-photons system and tight-coupling

Fluid equations: In Fourier space, the conservation and the Euler equations at linear order are, with $\mathrm{s} = \mathrm{b}, \gamma$

$$\mathrm{C_s} \equiv \left(\frac{\delta_\mathrm{s}}{1+w_\mathrm{s}}\right)' - k v_\mathrm{s} - 3\Psi' = 0 \tag{3}$$

$$\mathrm{E_s} \equiv v_\mathrm{s}' + (1-3c_\mathrm{s}^2)\mathcal{H} v_\mathrm{s} + k\Phi + \frac{k c_\mathrm{s}^2}{1+w_\mathrm{s}}\delta_\mathrm{s} - \frac{2k}{5}\Theta_2^\mathrm{s} = \mathcal{C}_\mathrm{s}.$$

The collision terms for baryons and photons are obtained from the Boltzmann equation [5] and are given by

$$\mathcal{C}_\gamma \equiv \tau'(v_\mathrm{b} - v_\gamma)\,, \quad \mathcal{C}_\mathrm{b} \equiv -\mathcal{C}_\gamma / R \tag{4}$$

with $R \equiv 3\bar\rho_\mathrm{b}/(4\bar\rho_\gamma)$, and the interaction rate is $\tau' \equiv a n_\mathrm{e} \sigma_\mathrm{T}$ where n_e is the number density of free electrons and σ_T the Thomson cross-section.

The plasma of photons and baryons is an unperfect fluid (labelled by pl) whose energy density and velocity are given by

$$\rho_\mathrm{pl} \equiv \sum_{\mathrm{s}=\mathrm{b},\gamma} \rho_\mathrm{s}\,, \quad (\rho_\mathrm{pl} + P_\mathrm{pl}) u^\mu_\mathrm{pl} \equiv \sum_{\mathrm{s}=\mathrm{b},\gamma} (\rho_\mathrm{s} + P_\mathrm{s}) u^\mu_\mathrm{s}. \tag{5}$$

We can infer easily the equation of state of the plasma, and by considering its time evolution we find its adiabatic speed of sound. They read

$$w_\mathrm{pl} = (3+4R)^{-1}\,, \quad c^2_\mathrm{pl} = [3(1+R)]^{-1}\,. \tag{6}$$

If we define a reduced energy density contrast by $(1+w_\mathrm{s})\Delta_\mathrm{s} \equiv (1+w_\mathrm{pl})\delta_\mathrm{s}$, we obtain from Eqs. (5) the very simple first order relations

$$\delta_\mathrm{pl} = \mathrm{Bar}(\Delta_\mathrm{b}, \Delta_\gamma), \quad v_\mathrm{pl} = \mathrm{Bar}(v_\mathrm{b}, v_\gamma)\,, \tag{7}$$

where we defined $\mathrm{Bar}(X_\mathrm{b}, X_\gamma) \equiv (R X_\mathrm{b} + X_\gamma)/(1+R)$. The conservation equations are then rewritten in a compact form as

$$\mathrm{C_s} = \left(\frac{\Delta_\mathrm{s}}{1+w_\mathrm{pl}}\right)' - k v_\mathrm{s} - 3\Psi' = 0\,. \tag{8}$$

In order to fully characterize this two fluids system we also need to describe the differences between them by the entropy perturbation $S \equiv 3[(\rho_\mathrm{b}/\bar\rho_\mathrm{b})^{1/3} - (\rho_\gamma/\bar\rho_\gamma)^{1/4}]$, and the velocity slip $V^\mu \equiv u^\mu_\mathrm{b} - u^\mu_\gamma$. Both quantities vanish for adiabatic initial conditions at any order in perturbations. We obtain at linear order

$$(1+w_\mathrm{pl}) S = \Delta_\mathrm{b} - \Delta_\gamma\,, \qquad V = v_\mathrm{b} - v_\gamma\,. \tag{9}$$

In the rest of this paper, we abbreviate w_pl to w and c^2_pl to c^2. The photons-baryons system is either described with the intrinsic fluid variables $(\delta_\gamma, \delta_\mathrm{b}, v_\mathrm{b}, v_\gamma)$ or with the total fluid variables $(\delta_\mathrm{pl}, v_\mathrm{pl}, S, V)$ but the latter choice of variables is better suited for the TCA.

Tight-coupling expansion: From the definition (7), considering $\mathrm{Bar}(\mathrm{C_b}, \mathrm{C}_\gamma)$ and $\mathrm{Bar}(\mathrm{E_b}, \mathrm{E}_\gamma)$ with Eqs. (8) and (4) leads to the plasma equations

$$\mathrm{C_{pl}} = \mathcal{H}\frac{RS}{(1+R)^2}\,, \tag{10}$$

$$\mathrm{E_{pl}} = k\left[\frac{RS}{3(1+R)^2} + \frac{1}{(1+R)}\frac{2\Theta_2}{5}\right]. \tag{11}$$

Taking the differences $\mathrm{E_b} - \mathrm{E}_\gamma$ and $\mathrm{C_b} - \mathrm{C}_\gamma$, we also obtain the entropy and velocity slip equations

$$S' - kV = 0\,, \tag{12}$$

$$-(1+R)/R\tau' V = \tag{13}$$

$$\left[V' + \mathcal{H}(v_\mathrm{pl} + 3c^2 V) + k\left(\frac{2}{5}\Theta_2 - \frac{\delta_\mathrm{pl}}{3(1+w)} + Rc^2 S\right)\right].$$

The dynamics of the quadrupole Θ_2 is inferred from the Boltzmann hierarchy [8]. Combined with the dynamics of E_2 we get

$$\tau'\Theta_2 = 2E_2' - \frac{4\Theta_2'}{3} + k\left(-\frac{8v_\gamma}{9} - \frac{4\Theta_3}{7} + \frac{10E_3}{21}\right) \tag{14}$$

where we must then use that $v_\gamma = v_\mathrm{pl} - RV/(1+R)$. The same method leads to

$$\tau' E_2 = -3E_2' + \frac{\Theta_2'}{3} + k\left(\frac{2v_\gamma}{9} + \frac{\Theta_3}{7} + \frac{5E_3}{7}\right). \tag{15}$$

For higher order moments, the Boltzmann hierarchy leads directly to

$$\tau' E_\ell = -E_\ell' + k\left[\frac{(\ell-1)(\ell+3)}{(\ell+1)(2\ell+3)}E_{\ell+1} - \frac{\ell}{2\ell-1}E_{\ell-1}\right]$$

$$\tau'\Theta_\ell = -\Theta_\ell' + k\left(\frac{\ell+1}{2\ell+3}\Theta_{\ell+1} - \frac{\ell}{2\ell-1}\Theta_{\ell-1}\right). \tag{16}$$

When combined with the Einstein equations to determine the perturbations of the metric, these equations completely determine the system. However, at early times $\tau' \propto a^{-2}$ and the restoring forces can be huge and require very small numerical steps. The TCA consists in not solving the dynamical equations for $(V, \Theta_{\ell\geq 2}, E_{\ell\geq 2})$ but to use instead the expressions (13-16) multiplied by $1/\tau'$ to obtain expressions for these variables as constraints which are functions of plasma variables but also functions of themselves. To obtain a closed system out of this infinite recursion, all variables are expanded in powers of the TCA parameter ϵ in the form

$$X = \sum_{p=0}^{\infty} \epsilon^p X^{(p)} \qquad \text{with} \qquad \epsilon \equiv \frac{\mathcal{H}}{\tau'} \tag{17}$$

and we use that $V^{(0)} = 0$ and $\Theta^{(n)}_{\ell\geq 2} = E^{(n)}_{\ell\geq 2} = 0$ for $n \leq (\ell - 2)$. The expansion (17) can be obtained by replacing recursively the constraints in themselves, and getting rid of the time derivative by using the plasma and entropy equations. We stop at the desired TCA order, gaining one order at each recursion, and then the constraints are replaced in the plasma and entropy equations. We obtain finally a *closed* and *first order* differential system up to the desired TCA order. More precisely, the result is stated as

(*): Defining the dynamical and constrained variables $\vec{Y} \equiv (v_{\rm pl}, \delta_{\rm pl}, S, \Phi, \Psi)$ and $\vec{Z} \equiv (V, \Theta_2, E_2, \Theta_3, E_3, \dots)$, then to solve for their dynamics there exists two sets of matrices $\mathbf{M}_p$ and $\mathbf{N}_p$, the coefficients of which are sums of products of the $(k/\mathcal{H})^q$, the $\mathcal{H}^{-(q+1)}\mathrm{d}^q H/\mathrm{d}\eta^q$, and some functions of R (this type of matrix is called *generic* in the following), such that the equations to solve are

$$\vec{Y}' = \mathcal{H}\sum_{p=0}^{\infty}\epsilon^p \mathbf{M}_p.\vec{Y}, \quad \vec{Z} = \sum_{p=0}^{\infty}\epsilon^p \mathbf{N}_p.\vec{Y} \tag{18}$$

with $\mathbf{N}_0 = 0$. This implies in particular that there exists generic matrices $\mathbf{K}_p$ such that $\vec{Z}' = \mathcal{H}\sum_{p=1}^{\infty}\epsilon^p\mathbf{K}_p.\vec{Y}$ [obtained from $\mathbf{K}_p \equiv (\epsilon^p\mathbf{N}_p)'/(\mathcal{H}\epsilon^p) + \sum_{r=1}^{p}\mathbf{N}_r.\mathbf{M}_{p-r}$]. If we define the variables *up to TCA-n* by $X^{<n} \equiv \sum_{p=0}^{n}\epsilon^p X^{(p)}$, then $\vec{Y}^{<n}$ and $\vec{Z}^{<n}$ are obtained from a truncation of the expansion (18).

In order to show (*) we will determine these matrices recursively. Hereafter $\mathbf{P}$, $\mathbf{Q}$, $\mathbf{U}$, $\mathbf{W}$, $\mathbf{B}$, and $\mathbf{C}$ are generic matrices. First, the dynamics of Ψ from Einstein equations is of the form $\Psi' = \mathcal{H}\mathbf{P}.\vec{Y}$, but Φ is found from a constraint $\Phi - \Psi \propto \Theta_2$ and its dynamics depends on Θ_2' [5]. Using this, the plasma-entropy equations are of the form $\vec{Y}' = \mathcal{H}[\mathbf{Q}.\vec{Y} + \mathbf{U}.\vec{Z} + \mathbf{W}.\vec{Z}'/\mathcal{H}]$, the dependence in $\vec{Z}'$ being just on $\Theta_2^{\gamma\prime}$. The $\mathbf{M}_p$ are not independent since $\mathbf{M}_0 = \mathbf{Q}$ and $\mathbf{M}_p = \mathbf{U}.\mathbf{N}_p + \mathbf{W}.\mathbf{K}_p$ if $p > 0$. Hence, the $\mathbf{K}_p$ and $\mathbf{M}_p$ are deduced from the $\mathbf{N}_p$. For $p = 0$, $\mathbf{M}_0 = \mathbf{Q}$ and $\mathbf{N}_0 = \mathbf{K}_0 = 0$. At TCA-0, it is as if there was a single perfect fluid. Now, if the matrices are known up to a given $p \geq 0$, that is if we know $\vec{Y}'^{<p}$ and $\vec{Z}^{<p}$ (and thus $\vec{Z}'^{<p}$), then we can find $\mathbf{K}_{p+1}$, and using that the constraints for $\vec{Z}$ is of the form

$$\vec{Z} = \epsilon\left[\mathbf{A}.\vec{Z}'/\mathcal{H} + \mathbf{B}.\vec{Z} + \mathbf{C}.\vec{Y}\right] \tag{19}$$

where $\mathbf{A}$ is a constant matrix, we replace $\vec{Y}'^{<p}$, $\vec{Z}^{<p}$ and $\vec{Z}'^{<p}$ in the right hand side. We then get $\mathbf{N}_{p+1}$ from

$$\mathbf{N}_{p+1} = \mathbf{A}.\mathbf{K}_p + \mathbf{B}.\mathbf{N}_p \tag{20}$$

for $p > 1$ and $\mathbf{N}_1 = \mathbf{C}$ otherwise, from which we find $\mathbf{M}_{p+1}$. At early times $(\tau')' = -2\mathcal{H}\tau'$, and $R' = \mathcal{H}R$, and this is why matrices are of generic type. For a realistic case, neutrinos an cold dark matter should be considered and added to $\vec{Y}$. Note also that when recombination starts, τ' is no more scaling like $\propto a^{-2}$ and one must use its complete expression. However, $(\tau')'$ factors appear only at TCA-2. The public code CAMB [9] uses the TCA-1 with intrinsic fluid variables instead of total fluid variables, and such factors appear already at that order. With this algorithm, we can find the equations up to a given TCA-n if one uses abstract calculus to perform the recursion. Our approach clarifies the method adopted in Ref. [10] where (*) is implicitely used up to TCA-2. At early times, it can be used to avoid instablities due to the high interaction rate. Using higher orders of the TCA enables to improve the accuracy, given that up to TCA-n th precision is roughly of order $(k/\mathcal{H}\epsilon)^n = (k/\tau')^n$ for large modes and ϵ^n otherwise. In Fig. 1 we plot the error between the full numerical integration and the successive orders of the TCA for v_γ.

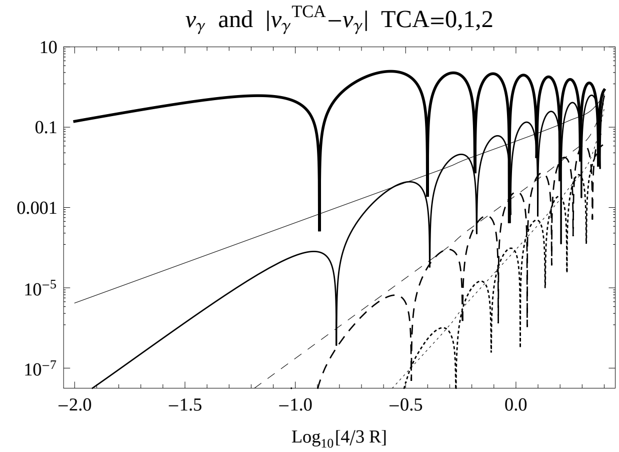

FIG. 1: Top thick line: exact v_γ with $k = 0.2\mathrm{Mpc}^{-1}$. The differences with the solutions up to TCA-$0, 1, 2$ are in continuous, dashed and dotted lines. The orders expected $(k/\tau')^{1,2,3}$, are depicted in thin lines of the same type.

Baryon acoustic oscillations: Qualitative features of the BAO can be inferred by computing the evolution of δ_γ at TCA-1. Considering the combination $\mathrm{C}_\gamma' + k\mathrm{E}_\gamma$ and expressing Θ_2 up to TCA-1 and V up to TCA-2 in order to obtain V/ϵ up to TCA-1, and further ignoring the variations of the potentials, we obtain

$$\delta_\gamma''\left[1 - \frac{\epsilon R^2}{(1+R)^2}\right] + k^2c^2\delta_\gamma\left[1 + \frac{\epsilon R^2}{1+R}\left(\frac{2+R}{1+R} - r\right)\right]$$
$$+\delta_\gamma'\left[\frac{\mathcal{H}R}{1+R} + \epsilon\frac{k^2}{\mathcal{H}}c^2\left(\frac{16}{15} + \frac{R^2}{1+R}\right) + \dots\right] = -\frac{4}{3}k^2\Phi\,,$$

where $r \equiv (\ln\tau')'/\mathcal{H}$ and the dots in the friction term represent terms which are suppressed by a factor $(\mathcal{H}/k)^2$ with respect to the dominant TCA-1 term. We thus recover that up to TCA-1, the viscosity is damping the oscillations [11], but we also find that the pseudo-period of oscillations, and thus the sound horizon, is slightly modified by the TCA-1 terms. However for large modes, $k/\tau' > 1$ before $\epsilon > 1$ and the effect of viscosity is expected to be dominant over the modified pseudo-period variation. This approach is of limited precision, since the TCA cannot be trusted around recombination, and only the full numerical integration makes sense.

Second order cosmological perturbations: vorticity

At second order, the same procedure can be followed. Given the precision that we need to reach, the TCA-0 is sufficient, and the equations are those of a perfect fluid. They can be found e.g. in Ref. [12] (with a different definition of the velocity perturbation). There is however a case where the TCA-0 is not sufficient. Indeed, vorticity is not generated for a perfect fluid [13] and thus not at TCA-0 [6]. One needs then to consider the equations up to TCA-1 at least in order to describe the generation of vorticity. The Euler equation for a perfect fluid reads

$$\begin{aligned}\mathrm{E}^{\mathrm{s}}_i &\equiv v^{\mathrm{s}\prime}_i + (1-3c_{\mathrm{s}}^2)\mathcal{H}v^{\mathrm{s}}_i + \frac{c_{\mathrm{s}}^2}{1+w_{\mathrm{s}}}\partial_i\delta_{\mathrm{s}} + \partial_i\Phi - 4\Psi' v^{\mathrm{s}}_i \\ &+ \frac{1+c_{\mathrm{s}}^2}{1+w_{\mathrm{s}}}\left[(\delta_{\mathrm{s}}v^{\mathrm{s}}_i)' + \mathcal{H}(1-3w_{\mathrm{s}})\delta_{\mathrm{s}}v^{\mathrm{s}}_i + \delta_{\mathrm{s}}\partial_i\Phi\right] \\ &+ \partial_j(v^{\mathrm{s}}_i v^j_{\mathrm{s}}) - (\Phi+\Psi)\left[v^{\mathrm{s}\prime}_i + \mathcal{H}(1-3c_{\mathrm{s}}^2)v^{\mathrm{s}}_i\right] - \partial_i(\Phi^2) \\ &+ \Psi\left[v^{\mathrm{s}\prime}_i + (1-3c_{\mathrm{s}}^2)\mathcal{H}v^{\mathrm{s}}_i + \frac{c_{\mathrm{s}}^2}{1+w_{\mathrm{s}}}\partial_i\delta_{\mathrm{s}} + \partial_i\Phi\right] \\ &+ \frac{(c_{\mathrm{s}}^2)'}{1+w_{\mathrm{s}}}\delta_{\mathrm{s}}u^{\mathrm{s}}_i - \frac{(c_{\mathrm{s}}^2)'}{6\mathcal{H}(1+w_{\mathrm{s}})^2}\partial_i(\delta_{\mathrm{s}}^2) = \mathcal{C}^{v_{\mathrm{s}}}_i\,. \qquad (21)\end{aligned}$$

The vorticity is defined by $\omega^{\mathrm{s}}_{\mu\nu} \equiv \perp^{\mathrm{s}}{}_\mu^{\ \alpha}\perp^{\mathrm{s}}{}_\nu^{\ \beta}\nabla_{[\alpha}u^{\mathrm{s}}_{\beta]}$ where $X_{[\mu\nu]} \equiv \frac{1}{2}(X_{\mu\nu} - X_{\nu\mu})$, and the projector is given by $\perp^{\mathrm{s}}{}_\mu^{\ \nu} \equiv \delta^\nu_\mu + u^{\mathrm{s}}_\mu u^\nu_{\mathrm{s}}$. There is no vorticity a first order since we have discarded the vector modes, but at second order its expression is

$$a\omega^{\mathrm{s}}_{ij} = \partial_{[i}v^{\mathrm{s}}_{j]} + v^{\mathrm{s}}_{[i}\partial_{j]}(\Psi+\Phi) + v^{\mathrm{s}}_{[i}v^{\mathrm{s}\ \prime}_{j]}\,. \qquad (22)$$

For a collisionless perfect fluid, that is if $\mathcal{C}^{v_{\mathrm{s}}}_i = 0$, then from the Euler equation, we can show that the evolution of vorticity is dictated by

$$\Omega^{\mathrm{s}}_{ij} \equiv \omega^{\mathrm{s}\ \prime}_{ij} + (2-3c_{\mathrm{s}}^2)\mathcal{H}\omega^{\mathrm{s}}_{ij} = 0 \qquad (23)$$

which implies that $[\bar{\rho}_{\mathrm{s}}(1+w_{\mathrm{s}})a^5\omega^{\mathrm{s}}_{ij}]$ is conserved. At second order, the velocity of the plasma is given by

$$\mathrm{Bar}(1+\delta_{\mathrm{b}}, 1+\delta_\gamma)u^\mu_{\mathrm{pl}} \equiv \mathrm{Bar}[(1+\delta_{\mathrm{b}})u^\mu_{\mathrm{b}}, (1+\delta_\gamma)u^\mu_\gamma]. \quad (24)$$

Using this, we find as expected that at TCA-0 the baryons-photons plasma behaves like a perfect fluid, since $\mathrm{Bar}(\mathrm{E}^{\mathrm{b}(0)}_i, \mathrm{E}^{\gamma(0)}_i) = \mathrm{E}^{\mathrm{pl}}_i$, and the plasma vorticity satisfies $\Omega^{\mathrm{pl}(0)}_{ij} = 0$. However in full generality

$$\Omega^{\mathrm{pl}}_{ij} + \Omega^{\Pi}_{ij} + \frac{R}{(1+R)^2}\left[\frac{\partial_{[i}\delta^{\mathrm{com}}_{\mathrm{pl}}\partial_{j]}S}{3(1+w_{\mathrm{pl}})} + \partial_{[i}(\partial_k V^k)\partial_{j]}V\right] = 0$$

with $\delta^{\mathrm{com}}_{\mathrm{pl}} \equiv \delta_{\mathrm{pl}} - 3\mathcal{H}(1+w_{\mathrm{pl}})v_{\mathrm{pl}}$, and where $\Omega^{\Pi}_{ij} = 0$ if for photons we neglect the anisotropic stress and use a perfect fluid description. This equation without the last term (which is related to the quadrupole generated by the mixing of the fluids) matches the expression of Ref. [14], given that the non-adiabatic pressure perturbation is

$$\delta P_{\mathrm{nad}} \equiv c_\gamma^2\bar{\rho}_\gamma\delta_\gamma - c_{\mathrm{pl}}^2\bar{\rho}_{\mathrm{pl}}\delta_{\mathrm{pl}} = -\bar{\rho}_{\mathrm{pl}}\frac{R(1+w_{\mathrm{pl}})}{3(1+R)^2}S\,. \quad (25)$$

However we do need to consider the anisotropic stress at TCA-($n \geq 1$). For completeness, its contribution is

$$\begin{aligned}\Omega^{\Pi}_{ij} \equiv \frac{3c^2}{4}\Bigg[&-\frac{1+c^2}{1+w}\partial_{[i}\delta_{\mathrm{pl}}\partial^k\Pi^\gamma_{j]k} + \frac{v^{\mathrm{pl}}_{[i}(c^2\partial^k\Pi^\gamma_{j]k})'}{c^2} + \\ &\frac{R\partial_{[i}S\partial^k\Pi^\gamma_{j]k}}{3(1+R)^2} + \partial_{[i}\partial^k[(1+\delta_\gamma)\Pi^\gamma_{j]k}] + \partial_{[i}\partial^k(\Phi-3\Psi)\Pi^\gamma_{j]k}\Bigg]\end{aligned}$$

In this paper, we have formulated the TCA in the total fluid variables since it can then be extended easily up to any order through a recursions on the equations. Higher orders in the TCA can be used to speed up and refine linear Boltzmann codes [9, 10]. At second order, a TCA-0 solution is generally sufficient, e.g. for the computation of non-Gaussianity generated by non-linear effects [15]. However, the vorticity in the baryons-photons fluid is generated at least at TCA-1, not only from gradients of non-adiabatic pressure perturbations (see Refs. [6, 14]), but also from gradients of the anisotropic stress of photons. This becomes relevant for the numerical estimation of the seed magnetic field created by vortical currents [16] since it is related to the existence of vorticity.

Acknowledgements: C. P. is supported by STFC (UK) grant ST/H002774/1 and thanks J.-P. Uzan and R. Maartens for discussions on the topic.

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