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# Gauge-invariant Boltzmann equation and the fluid limit

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## Abstract

This paper investigates the collisionless Boltzmann equation up to second order in the cosmological perturbations. It describes the gauge dependence of the distribution function and the construction of a gauge-invariant distribution function and brightness, and then derives the gauge-invariant fluid limit.

## 1. Introduction

The origin of the large-scale structure is nowadays understood from the gravitational collapse of initial density perturbations, which were produced by amplification of the quantum fluctuations in the inflaton field [1]. The properties of the large-scale structure depend both on the initial conditions at the end of inflation and on the growth of perturbations in a universe filled with non-relativistic matter and radiation. The theory of cosmological perturbations is thus a cornerstone of our understanding of the large-scale structure. The evolution of radiation (photons and neutrinos) needs to be described by a Boltzmann equation [2–4]. Two types of perturbative schemes have extensively been used in the literature in order to describe the evolution of the cosmological perturbations. The first is a 1 + 3 covariant splitting of spacetime [5–7] and the second is a more pedestrian-coordinate-based approach. In the first approach, exact equations on the physical spacetime are derived and perturbative solutions around a background solution are then calculated. In the second approach, an averaging procedure is implicitly assumed and, starting from a background spacetime, perturbation variables satisfying the equations of motion order by order are constructed. In the 1 + 3 approach, the variables used are readily covariant, but the absence of background spacetime can be a problem to simplify the resolution by performing a mode expansion, since the Helmholtz function is in general not defined on the physical spacetime. In the coordinate-based approach, all perturbation variables live on the background spacetime and enjoy the advantages of its highly symmetric properties. However, this extra mathematical structure is at the origin of the gauge issue through the identification mapping that needs to be defined

between the background spacetime and the physical spacetime. Thus, the gauge dependence needs to be understood. An elegant solution is to construct gauge-invariant variables *à la* Bardeen both for the metric perturbation variables [8] and for the distribution function [9, 10]. Since the Boltzmann and Einstein equations are covariant, they can be expressed solely in terms of gauge-invariant variables provided we have a full set at hand. A full comparison of these two formalisms has been performed at first order in [11] and for gravitational waves at second order in [12].

In the coordinate-based approach, the true degrees of freedom identified from the Lagrangian formalism are quantized. They transfer to classical perturbations which inherit a nearly scale invariant power spectrum and Gaussian statistics, when their wavelength stretches outside the horizon, thus providing initial conditions for the standard big-bang model. Conserved quantities [13, 14] enable us to ignore the details of the transition between the inflation and the standard big-bang model (see, however, [15]), and the evolution details only need to be known when the wavelength re-enters the horizon. A first step to extend this procedure in the  $1 + 3$  formalism has been taken in [16], where conserved quantities were defined. As for the degrees of freedom which need to be quantized, the first proposal was made in [17], in order to identify them, but it has not yet been motivated by a Lagrangian formulation.

The properties of the observed cosmic microwave background (CMB) anisotropies have confirmed the validity of the linear perturbation theory around a spatially homogeneous and isotropic universe and have set strong constraints on the origin of structures, as predicted by inflation. It now becomes necessary, with the forthcoming increasing precision of data that may allow us to detect deviation from Gaussianity [18], to study the second-order approximation, in order to discuss the accuracy of these first-order results. These non-Gaussian features are also of first importance, since they can help discriminate between different inflation theories. Indeed, one-field driven inflation leads to very small levels of primordial non-Gaussianity [19], whereas multifield inflation can present significant non-Gaussian features [20, 21]. However, since nonlinear effects also appear through nonlinear evolution, that is from the second-order approximation and beyond of the evolution equations, the study of second-order evolution equations is necessary in order to distinguish between primordial and evolutionary non-Gaussianities (see [22] for a review on non-Gaussianity). Second-order Einstein and Boltzmann equations have been written in the  $1 + 3$  formalism [23, 24], but not integrated numerically, partly because the mode expansion is not defined on the physical spacetime, and this would then require a four-dimensional numerical integration. However, the promising formalism of [25], which builds a bridge between the  $1 + 3$  formalism and the coordinate-based approach, might shed some light on these issues. Similarly, in the coordinate-based approach, the second-order Einstein equations have been written in terms of gauge-invariant variables [26], and a first attempt has been made to write the Boltzmann equation in a given gauge for the different species filling the universe and to solve them analytically [27, 28].

The goal of this paper is to provide the full mathematical framework for handling distribution functions at second order in the coordinate-based approach taking into account the gauge issue. This will clarify the existing literature and point out some existing mistakes. In section 2 we first briefly review the gauge transformations and the procedure to build gauge-invariant variables. We then present in section 3 the transformation properties of the distribution function, and express them up to second order. In section 4 we define the gauge-invariant distribution function and the gauge-invariant brightness up to second order in the particular case of radiation (but this is readily extendable to cold dark matter). We then deduce in section 5, from the Boltzmann equation, the evolution of the gauge-invariant brightness in

its simplest collisionless form, at first and second orders. To finish, in section 6 we express the fluid limit as a consistency check of our results.

## 2. Overview on gauge transformations and gauge-invariant variables

### 2.1. First- and second-order perturbations

We assume that, at lowest order, the universe is well described by a Friedmann–Lemaître spacetime (FL) with flat spatial sections. The most general form of the metric for an almost FL universe is

$$\begin{aligned} ds^2 &= g_{\mu\nu} dx^\mu dx^\nu \\ &= a(\eta)^2 \{ -(1 + 2\Phi) d\eta^2 + 2\omega_i dx^i d\eta + [(1 - 2\Psi)\delta_{ij} + h_{ij}] dx^i dx^j \}, \end{aligned} \quad (1)$$

where  $\eta$  is the conformal time and  $a$  is the scale factor. We perform a scalar–vector–tensor decomposition as

$$\omega_i = \partial_i B + B_i, \quad (2)$$

$$h_{ij} = 2E_{ij} + \partial_i E_j + \partial_j E_i + 2\partial_i \partial_j E, \quad (3)$$

where  $B_i$ ,  $E_i$  and  $E_{ij}$  are transverse ( $\partial^i E_i = \partial^i B_i = \partial^i E_{ij} = 0$ ), and  $E_{ij}$  is traceless ( $E_i^i = 0$ ). There are four scalar degrees of freedom ( $\Phi$ ,  $\Psi$ ,  $B$ ,  $E$ ), four vector degrees of freedom ( $B_i$ ,  $E_i$ ) and two tensor degrees of freedom ( $E_{ij}$ ). Each of these perturbation variables can be split into first- and second-order parts as

$$W = W^{(1)} + \frac{1}{2} W^{(2)}. \quad (4)$$

This expansion scheme will refer, as we shall see, to the way gauge transformations and gauge-invariant (GI) variables are defined. First-order variables are solutions of first-order equations which have been extensively studied (see [29] for a review). Second-order equations will involve purely second-order terms, e.g.  $W^{(2)}$  and terms quadratic in the first-order variables, e.g.  $[W^{(1)}]^2$ . There will thus never be any ambiguity about the order of perturbation variables involved as long as the order of the equation considered is known. Consequently, we will often omit to specify the order superscript when there is no risk of confusion.

At first order, 4 of the 10 metric perturbations are gauge degrees of freedom and the 6 remaining degrees of freedom reduce to 2 scalars, 2 vectors and 2 tensors. The three types of perturbations decouple and can thus be treated separately. As long as no vector source terms are present, which is generally the case when no magnetic field or topological defect is taken into account, the vector modes decay as  $a^{-2}$ . Thus, we can safely discard them and set  $E_i^{(1)} = B_i^{(1)} = 0$ . In the following, we shall not include vector modes for the sake of clarity. We checked that our arguments and derivation can trivially (but at the expense of much lengthy expressions) take them into account.

In the fluid description, we assume that the matter content of the universe can be described by a mixture of fluids. The 4-velocity of each fluid is decomposed as

$$u^\mu = \frac{1}{a} (\delta_0^\mu + v^\mu). \quad (5)$$

The perturbation  $v^\mu$  has only three independent degrees of freedom since  $u^\mu$  must satisfy  $u_\mu u^\mu = -1$ . The spatial components can be decomposed as

$$v^i = \partial^i v + \bar{v}^i, \quad (6)$$

$\bar{v}^i$  being the vector degree of freedom and  $v$  the scalar degree of freedom. The stress–energy tensor of this fluid is of the form

$$T_{\mu\nu} = \rho u_\mu u_\nu + P(g_{\mu\nu} + u_\mu u_\nu), \quad (7)$$

where the density and pressure are expanded as follows:

$$\rho = \bar{\rho} + \delta\rho, \quad P = \bar{P} + \delta P. \quad (8)$$

At the background level, the form of the stress–energy tensor is completely fixed by the symmetry properties of the FL spacetime. However, at the perturbation level, one must consider an anisotropic stress component,  $\pi_{\mu\nu}$  with  $\pi_\mu^\mu = u^\mu \pi_{\mu\nu} = 0$ . The pressure and density of the fluid are related by an equation of state,  $P = \rho/3$ , in the case of radiation.

At first order, the formalism developed by the seminal work of [8] provides a full set of gauge-invariant variables (GIV). Thanks to the general covariance of the equations at hand (Einstein equations, conservation equations, Boltzmann equation), it was shown that it was possible to get first-order equations involving only these gauge-invariant variables. In addition, if these gauge-invariant variables reduce, in a particular gauge, to the perturbation variables that we use in this particular gauge, then the computation of the equation can be simplified. Actually, we only need to compute the equations in this particular gauge, as long as it is completely fixed, and then to promote by identification our perturbation variables to the gauge-invariant variables. Thus, provided we know this full set of gauge-invariant variables, the apparent loss of generality by fixing the gauge in a calculation is in fact just a way to simplify computations. Eventually, we will reinterpret the equations as being satisfied by gauge-invariant variables. The full set of first-order gauge-invariant variables is well known and is reviewed in [29] and [30]. As gauge transformations up to any order were developed, it remained uncertain [31], whether or not a full set of gauge-invariant variables could be built for second and higher orders. This has been recently clarified [26], and the autosimilarity of the transformation rules for different orders can be used as a guide to build the gauge-invariant variables at any order. We present a summary of the ideas presented in [31] about gauge transformations and the construction of gauge-invariant variables [26].

## 2.2. Points identification on manifolds

When working with perturbations, we consider two manifolds: a background manifold,  $\mathcal{M}_0$ , with associated metric  $\bar{g}$ , which in our case is the FL spacetime, and the physical spacetime  $\mathcal{M}_1$  with the metric  $g$ . We consider that the variation of metric boils down to a comparison between tensor fields on distinct manifolds. Thus, in order to give a sense to ‘ $\delta g(P) = g(P) - \bar{g}(\bar{P})$ ’, we need to identify the points  $P$  and  $\bar{P}$  between these two manifolds and also to set up a procedure for comparing tensors. This will also be necessary for the comparison of any tensor field.

One solution to this problem [31] is to consider an embedding (4+1)-dimensional manifold  $\mathcal{N} = \mathcal{M} \times [0, 1]$ , endowed with the trivial differential structure induced, and the projections  $\mathcal{P}_\lambda$  on submanifolds with  $\mathcal{P}_0(\mathcal{N}) = \mathcal{M} \times \{0\} = \mathcal{M}_0$  and  $\mathcal{P}_1(\mathcal{N}) = \mathcal{M} \times \{1\} = \mathcal{M}_1$ . The collection of  $\mathcal{M}_\lambda \equiv \mathcal{P}_\lambda(\mathcal{N})$  is a foliation of  $\mathcal{N}$ , and each element is diffeomorphic to the physical spacetime  $\mathcal{M}_1$  and the background spacetime  $\mathcal{M}_0$ . The gauge choice on this stack of spacetimes is defined as a vector field  $X$  on  $\mathcal{N}$  which satisfies  $X^4 = 1$  (the component along the spacetime slicing  $\mathbb{R}$ ). A vector field defines integral curves that are always tangent to the vector field itself, hence inducing a one-parameter group of diffeomorphisms  $\phi(\lambda, \cdot)$ , also noted  $\phi_\lambda(\cdot)$ , a flow, leading in our case from  $\phi(0, p \in \mathcal{P}_0(\mathcal{N})) = p \in \mathcal{P}_0(\mathcal{N})$  along the integral curves to  $\phi(1, p \in \mathcal{P}_0(\mathcal{N})) = q \in \mathcal{P}_1(\mathcal{N})$ . Due to the never-vanishing last component of  $X$ , the integral curves will always be transverse to the stack of spacetimes and the points lying on

the same integral curve, belonging to distinct spacetimes, will be identified. Additionally, the property  $X^4 = 1$  ensures that  $\phi_{\lambda,X}(\mathcal{P}_0(\mathcal{N})) = \mathcal{P}_\lambda(\mathcal{N})$ , i.e. the flow carries a spacetime slice to another. This points' identification is necessary when comparing tensors, but we already see that the arbitrariness in the choice of a gauge vector field  $X$  should not have a physical meaning, and this is the well-known *gauge freedom*.

### 2.3. Tensors' comparison and perturbations

The induced transport, along the flow, of tensors living on the tangent bundle, is determined by the push-forward  $\phi_{*\lambda}$  and the pull-back  $\phi_\lambda^*$  [32] associated with an element  $\phi_\lambda$  of the group of diffeomorphisms. These two functions encapsulate the transformation properties of the tangent and co-tangent spaces at each point and its image. Indeed, the pull-back can be linked to the local differential properties of the vector field embedded by the Lie derivatives along the vector field in a Taylor-like fashion (see [32] or [31]):

$$\Phi_{X,\lambda}^*(T) = \sum_{k=0}^{k=\infty} \frac{\lambda^k}{k!} \mathcal{L}_X^k T, \quad (9)$$

for any tensor  $T$ .

A remark about coordinate changes is in order here. When the tensor  $T$  is a coordinate  $x^\mu$  (once  $\mu$  is fixed, it is a scalar field), the previous definition reduces to the standard finite coordinates transformation

$$x'^\mu \equiv \Phi_{\lambda,\xi}^*(x_\mu) = x^\mu + \lambda \xi^\mu + \frac{\lambda^2}{2} \xi_{,\nu}^\mu \xi^\nu + \dots \quad (10)$$

This is the standard way of defining an active transformation on the manifold, by transporting a point of coordinates  $x^\mu$  to a point of coordinates  $x'^\mu$ . This transformation, when performed on the coordinate system—considering the coordinates as a grid on the manifold that one would displace according to the active transformation—induces a passive coordinates transformation, if we decide that the new coordinates of a point  $q$  are the coordinates of the point  $p$  such that  $\phi_\lambda(p) = q$ . When considering a transformation induced by a field  $\xi$ , we will refer to the passive coordinates transformation induced by the active transportation of the coordinates system.

The expansion of equation (9) on  $\mathcal{P}_0(\mathcal{N})$  provides a way to compare a tensor field on  $\mathcal{P}_\lambda(\mathcal{N})$  to the corresponding one on the background spacetime  $\mathcal{P}_0(\mathcal{N})$ . The background value being  $T_0 \equiv \mathcal{L}_X^0 T|_{\mathcal{P}_0(\mathcal{N})}$ , we obtain a natural definition for the tensor perturbation

$$\Delta_X T_\lambda \equiv \sum_{k=1}^{k=\infty} \frac{\lambda^k}{k!} \mathcal{L}_X^k T \Big|_{\mathcal{P}_0(\mathcal{N})} = \Phi_{X,\lambda}^*(T) - T_0. \quad (11)$$

The subscript  $X$  reminds the gauge dependence. We can read the  $n$ th order perturbation as

$$\delta_X^{(n)} T \equiv \mathcal{L}_X^n T \Big|_{\mathcal{P}_0(\mathcal{N})}, \quad (12)$$

which is consistent with the expansion of perturbation variables of the physical metric in equation (4), since the physical spacetime is labeled by  $\lambda = 1$ . However, the fact that the intermediate spacetime slices  $\mathcal{P}_\lambda(\mathcal{N})$  are labeled by  $\lambda$  removes the absolute meaning of order-by-order perturbations, as it can be seen from equation (11). The entire structure embedded by  $\mathcal{N}$  is more than just a convenient construction, and this will have important consequences in gauge changes as we will now detail.

#### 2.4. Gauge transformations and gauge invariance

If we consider two gauge choices  $X$  and  $Y$ , a gauge transformation from  $X$  to  $Y$  is defined as the diffeomorphism

$$\phi_{X \rightarrow Y, \lambda} = (\phi_{X, \lambda})^{-1}(\phi_{Y, \lambda}), \quad (13)$$

and it induces a pull-back which carries the tensor  $\Delta_X T_\lambda$ , which is the perturbation in the gauge  $X$ , to  $\Delta_Y T_\lambda$ , which is the perturbation in gauge  $Y$  since

$$\begin{aligned} \phi_{X \rightarrow Y, \lambda}^*(\Delta_X T_\lambda + T_0) &= [(\phi_{X, \lambda})^{-1}(\phi_{Y, \lambda})]^* \phi_{X, \lambda}^*(T) \\ &= \phi_{Y, \lambda}^*(\phi_{X, \lambda}^*)^{-1} \phi_{X, \lambda}^*(T) \\ &= \phi_{Y, \lambda}^*(T) \\ &= \Delta_Y T_\lambda + T_0. \end{aligned} \quad (14)$$

As demonstrated in [31], this family (indexed by  $\lambda$ ) of gauge transformations fails to be a one-parameter group due to the lack of the composition rule. It should be Taylor-expanded using the so-called knight diffeomorphism along a sequence of vector fields  $\xi_i$ . For the three first orders, the expression of this knight diffeomorphism is

$$\begin{aligned} \Delta_Y T_\lambda &= \phi_{X \rightarrow Y, \lambda}^* \Delta_X T_\lambda \\ &= \Delta_X T_\lambda + \lambda \mathcal{L}_{\xi_1} \Delta_X T_\lambda + \frac{\lambda^2}{2!} (\mathcal{L}_{\xi_2} + \mathcal{L}_{\xi_1}^2) \Delta_X T_\lambda + \frac{\lambda^3}{6} (\mathcal{L}_{\xi_3} + 3\mathcal{L}_{\xi_1} \mathcal{L}_{\xi_2} + \mathcal{L}_{\xi_1}^3) \Delta_X T_\lambda. \end{aligned} \quad (15)$$

The vector fields  $\xi_1$ ,  $\xi_2$  and  $\xi_3$  are related to the gauge vector fields  $X$  and  $Y$  by  $\xi_1 = Y - X$ ,  $\xi_2 = [X, Y]$  and  $\xi_3 = [2X - Y, [X, Y]]$ . By substitution of the perturbation by its expression in equation (11), we identify order by order in  $\lambda$ , and obtain the transformation rules for perturbations order by order. The first and second order transformation rules, on which we will focus our attention, are

$$\delta_Y^{(1)} T - \delta_X^{(1)} T = \mathcal{L}_{\xi_1} T_0, \quad \delta_Y^{(2)} T - \delta_X^{(2)} T = 2\mathcal{L}_{\xi_1} \delta_X^{(1)} T + (\mathcal{L}_{\xi_2} + \mathcal{L}_{\xi_1}^2) T_0. \quad (16)$$

The fact that we had to follow  $n$  integral curves of  $n$  distinct vector fields for  $n$ th order perturbations is a characteristic of knight diffeomorphisms. It arises from the fact that, for the whole differential structure of  $\mathcal{N}$  to hold, gauge changes are a more general type of transformation than simple vector-field-induced flows. Consequently, the Taylor-like expansion must be of a more general type. Indeed, for a given gauge change between  $X$  and  $Y$ , the family of gauge changes  $\phi_{X \rightarrow Y, \lambda}$  labeled by  $\lambda$  is not always a group in  $\lambda$ , and this happens for instance if  $[X, Y] \neq 0$  (see [31] for a graphic intuition). Although we could, for a fixed  $\lambda = \lambda_0$ , find  $\xi$  such that equation (16) takes a form like equation (11) up to a given order, for instance by fixing  $\lambda_0 = 1$ , and choosing

$$\xi \equiv \xi_1 + \frac{1}{2} \xi_2 + \frac{1}{3!} (\xi_3 + \frac{3}{2} [\xi_1, \xi_2]), \quad (17)$$

this would mean that intermediate spacetimes are useless, and we would then ask Einstein equations to hold only for  $\mathcal{P}_0(\mathcal{N})$  and  $\mathcal{P}_{\lambda_0}(\mathcal{N})$ . This would lead to equations in the perturbation variables that mix different orders. The resulting solution, for second order and above, would be very difficult to find.

#### 2.5. Perturbed Einstein equations

Instead, we prefer to use this more complicated but cleaner knight diffeomorphism (equation (16)) to change gauge. It keeps the differential structure built on  $\mathcal{N}$  and we additionally demand Einstein equations to be satisfied on each  $\mathcal{P}_\lambda(\mathcal{N})$ . This can be used to differentiate Einstein equations to first order w.r.t  $\lambda$  and take the limit  $\lambda \rightarrow 0$  in order to get a

set of equations that formally take the form  $\mathcal{E}_1[\delta^{(1)}g, \delta^{(1)}T] = 0$ . Once solved for the solutions of the first-order Einstein equation, we can differentiate twice the Einstein equation w.r.t  $\lambda$  and get an equation of the type

$$\mathcal{E}_2[\delta^{(2)}g, \delta^{(2)}T] = S[\delta^{(1)}g, \delta^{(1)}T], \quad (18)$$

where  $S$  stands for a source term quadratic in the first-order variables (see [12] for a concrete example).

We see that the decomposition of perturbation variables in the form given by equation (4) will trigger a similarity between the equations, i.e.  $\mathcal{E}_1$  and  $\mathcal{E}_2$  have the same form. Purely second-order perturbation variables satisfy the same equation as first-order perturbation variables do, but with a source term. With known sources and known solutions to the homogenous equation, the Green function method enables us to solve, at least formally, the second-order equations, and by recursion at any order. To summarize, the Taylor expansion ‘Taylorizes’ the process for solving the equations by dividing tasks among orders, since Einstein equations are satisfied order by order.

## 2.6. Gauge-invariant variables

General covariance, i.e. the fact that physics should not depend on a particular choice of coordinates, is an incentive to work with gauge-invariant quantities. As we note from equation (16), a tensor  $T$  is gauge invariant up to the  $n$ th order if it satisfies  $\mathcal{L}_\xi \delta_X^{(r)} T = 0$  for any vector field  $\xi$  and any  $r \leq n$ , as can be deduced by recursion. A consequence of this strong condition is that a tensor is gauge invariant up to order  $n$  if and only if  $T_0$  and all its perturbations of order lower than  $n$  either vanish or are constant scalars or are combinations of Kronecker deltas with constant coefficients. The Einstein equation is of the form  $G - T = 0$ , and for this reason it is totally gauge invariant. However, we cannot find non-trivial tensorial quantities (that is, different from  $G - T$ ) gauge invariant up to the order we intend to study perturbations, with which we could express the perturbed set of Einstein equations.

Consequently, we will lower our goal and we will build, by combinations of perturbed tensorial quantities, gauge-invariant variables. These combinations will not be the perturbation of an underlying tensor. This method will prove to be very conclusive since a general procedure exists for perturbations around FL. Eventually, we shall identify observables among these gauge-invariant variables and the fact that they are not the perturbation of a tensor will not matter. It has to be emphasized that the transformation rules of these combinations are not intrinsic and cannot be deduced directly from the knight diffeomorphism since they are not tensorial quantities. Instead, we have to form the combination before and after the gauge change in order to deduce their transformation rules.

We now summarize the standard way to build gauge-invariant variables. For simplicity we consider only the scalar part of the gauge transformations, since we will not consider vector modes in the metric and fluid perturbation variables (again, this could be done, but would just obfuscate the explanations). In the following, we split  $\xi_r^\mu$  as

$$\xi_r^0 = T^{(r)}, \quad \xi_r^i = \partial^i L^{(r)}, \quad \text{with } r = 1, 2. \quad (19)$$

## 2.7. First-order gauge-invariant variables

In the subsequent work, we present the transformation rules of perturbed quantities in a simplified notation. Instead of writing  $W_Y^{(r)} = W_X^{(r)} + f(\xi_1, \dots, \xi_r)$ , in order to state that the difference between the expression of the  $r$ th order perturbed variable  $W$  in gauge  $Y$  and in gauge  $X$  is a function  $f$  of the knight-diffeomorphism fields  $\xi_1, \dots, \xi_r$ , we prefer to

write  $W^{(r)} \rightarrow W^{(r)} + f(\xi_1, \dots, \xi_r)$ . We recall that the expressions of the fields  $(\xi_i)_{1 \leq i \leq r}$  necessary for the knight diffeomorphism are expressed in function of the gauge fields  $X$  and  $Y$  (see equation (16)). From the transformation rules (16), we deduce that the first-order perturbations of the metric tensor (1) transform as

$$\Phi^{(1)} \rightarrow \Phi^{(1)} + T^{(1)'} + \mathcal{H}T^{(1)} \quad (20)$$

$$B^{(1)} \rightarrow B^{(1)} - T^{(1)} + L^{(1)'} \quad (21)$$

$$\Psi^{(1)} \rightarrow \Psi^{(1)} - \mathcal{H}T^{(1)} \quad (22)$$

$$E^{(1)} \rightarrow E^{(1)} + L^{(1)} \quad (23)$$

$$E_{ij}^{(1)} \rightarrow E_{ij}^{(1)}, \quad (24)$$

while the scalar quantities related to matter transform as

$$\begin{aligned} \delta^{(1)}\rho &\rightarrow \delta^{(1)}\rho + \bar{\rho}'T^{(1)} \\ \delta^{(1)}P &\rightarrow \delta^{(1)}P + \bar{P}'T^{(1)} \end{aligned} \quad (25)$$

$$\begin{aligned} v^{(1)} &\rightarrow v^{(1)} - L^{(1)'} \\ \delta^{(1)}\pi^{ij} &\rightarrow \delta^{(1)}\pi^{ij}, \end{aligned} \quad (26)$$

where a prime denotes a derivative w.r.t the conformal time  $\eta$  and where  $\mathcal{H} \equiv a'/a$ .

From now on, we shall refer to these first-order transformation rules defined by  $\xi_1$  as  $\mathcal{T}_{\xi_1}(\Phi^{(1)}), \mathcal{T}_{\xi_1}(B^{(1)}), \dots$  or simply  $\mathcal{T}(\Phi^{(1)}), \mathcal{T}(B^{(1)}), \dots$ . For instance  $\mathcal{T}(\Phi^{(1)}) = \Phi^{(1)} + T^{(1)'} + \mathcal{H}T^{(1)}$ .

We first note that the first-order tensorial modes and the first-order anisotropic stress are automatically gauge invariant. For the other perturbation variables, which are not automatically gauge invariant, they are two ways to understand the procedure to build gauge-invariant combinations. The first point of view in building gauge-invariant variables consists in finding a way to get rid of the undesired transformation rule. To do so, we remark that the combinations  $B^{(1)} - E^{(1)'}$  and  $-E^{(1)'}$  transform under a gauge change as  $B^{(1)} - E^{(1)'} \rightarrow B^{(1)} - E^{(1)'} - T^{(1)}$  and  $-E^{(1)'} \rightarrow -E^{(1)'} - L^{(1)}$  respectively. We can use these combinations to add ad hoc compensating terms to  $\Phi^{(1)}$  and  $\Psi^{(1)}$  by defining

$$\hat{\Phi}^{(1)} \equiv \Phi^{(1)} + (B^{(1)} - E^{(1)'})' + \mathcal{H}(B^{(1)} - E^{(1)'}) \quad (27)$$

$$\hat{\Psi}^{(1)} \equiv \Psi^{(1)} - \mathcal{H}(B^{(1)} - E^{(1)'}). \quad (28)$$

$\hat{\Phi}^{(1)}$  and  $\hat{\Psi}^{(1)}$  are now gauge invariant, by construction. This can also be understood, from a second point of view, as a gauge transformation for  $\Phi^{(1)}$  and  $\Psi^{(1)}$  toward the Newtonian gauge (NG) [1], defined by  $\xi_{\rightarrow \text{NG}}^{(1)}$ , decomposed in  $T_{\rightarrow \text{NG}}^{(1)} = B^{(1)} - E^{(1)'}$ ,  $L_{\rightarrow \text{NG}}^{(1)} = -E^{(1)}$ , which transform the perturbation variables as

$$B^{(1)} \rightarrow 0 \quad (29)$$

$$E^{(1)} \rightarrow 0 \quad (30)$$

$$\Phi^{(1)} \rightarrow \hat{\Phi}^{(1)} \equiv \Phi_{\text{NG}}^{(1)} = \Phi^{(1)} + \mathcal{H}(B^{(1)} - E^{(1)'}) + (B^{(1)} - E^{(1)'})' \quad (31)$$

$$\Psi^{(1)} \rightarrow \hat{\Psi}^{(1)} \equiv \Psi_{\text{NG}}^{(1)} = \Psi^{(1)} - \mathcal{H}(B^{(1)} - E^{(1)'}). \quad (32)$$

Similarly, the gauge-invariant variables that would reduce to  $\delta\rho$ ,  $\delta P$  and  $v$  are

$$\begin{aligned}\delta^{(1)}\hat{\rho} &\equiv \delta_{\text{NG}}^{(1)}\rho = \delta^{(1)}\rho + \bar{\rho}'(B^{(1)} - E^{(1)})' \\ \delta^{(1)}\hat{P} &\equiv \delta_{\text{NG}}^{(1)}P = \delta^{(1)}P + \bar{P}'(B^{(1)} - E^{(1)})' \\ \hat{v}^{(1)} &\equiv v_{\text{NG}}^{(1)} = v^{(1)} + E^{(1)'} \\ \hat{\pi}^{ij(1)} &\equiv \delta_{\text{NG}}^{(1)}\pi^{ij} = \delta^{(1)}\pi^{ij}.\end{aligned}\tag{33}$$

Since we have ignored the vector gauge degrees of freedom,  $B^{(1)}$  and  $E^{(1)}$  are the two gauge-variant variables of the metric perturbation while  $\hat{\Phi}^{(1)}$  and  $\hat{\Psi}^{(1)}$  are the gauge-invariant parts. As mentioned before, we then force the gauge-invariant variables in the perturbed metric by replacing  $\Phi^{(1)}$  with  $\hat{\Phi}^{(1)} - \mathcal{H}(B^{(1)} - E^{(1)'}) + (B^{(1)} - E^{(1)'})'$  and applying similar procedures for  $\Psi^{(1)}$ ,  $\delta^{(1)}\rho$  and  $\delta^{(1)}P$ . When developing Einstein equations, we know that general covariance will eventually keep only gauge-invariant terms. Thus, we can either do a full calculation and witness the terms involving the degrees of freedom  $B^{(1)}$  and  $E^{(1)}$  disappear or perform the calculations with  $B^{(1)}$  and  $E^{(1)}$  set to zero and obtain the perturbed Einstein equations only in function of gauge-invariant variables. The latter simplifies the computation, which is useful when going to higher orders. The advantage of the second point of view is that the addition of the compensating terms of the first point of view can be seen as a first-order gauge change toward the Newtonian gauge with  $\xi_{\rightarrow\text{NG}}^{(1)}$  (decomposed as  $T_{\rightarrow\text{NG}}^{(1)}$  and  $L_{\rightarrow\text{NG}}^{(1)}$ ). These enable us to decompose the perturbed metric in a gauge-invariant part and a gauge-variant part as

$$\delta^{(1)}g \equiv \delta^{(1)}\tilde{g} + \mathcal{L}_{-\xi_{\rightarrow\text{NG}}^{(1)}}\tilde{g},\tag{34}$$

as can be seen from the transformation rules under a gauge change characterized by  $\xi_1$

$$\delta^{(1)}\tilde{g} \rightarrow \delta^{(1)}\tilde{g}, \quad -\xi_{\rightarrow\text{NG}}^{(1)} \rightarrow -\xi_{\rightarrow\text{NG}}^{(1)} + \xi_1.\tag{35}$$

This property, which is not general, but happens to hold in the case of cosmological perturbation (i.e. around FL metric) is the key to extending this construction to second order.

It should be noted that this procedure, although achieved by defining gauge-invariant variables which reduce to the perturbation variables in the Newtonian gauge, can be extended to other types of gauge-invariant variables which reduce to perturbation variables in another gauge. For instance, we can use the transformation properties of  $\Psi^{(1)}$  and  $E^{(1)}$  to add the compensating terms to  $\Phi^{(1)}$ ,  $B^{(1)}$  and other variables. The transformation rules  $\Psi^{(1)}/\mathcal{H} \rightarrow \Psi^{(1)}/\mathcal{H} - T^{(1)}$  and  $-E^{(1)} \rightarrow -E^{(1)} - L^{(1)}$  make it straightforward to build these compensating terms. We need to define  $\xi_{\rightarrow\text{FG}}^{(1)}$  decomposed with  $T_{\rightarrow\text{FG}}^{(1)} = \Psi^{(1)}/\mathcal{H}$  and  $L_{\rightarrow\text{FG}}^{(1)} = -E^{(1)}$ . The gauge-invariant variables defined with this procedure reduce to the perturbation variables in the flat gauge ( $E^{(1)} = 0$ ,  $\Psi^{(1)} = 0$ ), and are

$$\tilde{B} \equiv B_{\text{FG}} = B^{(1)} - \frac{\Psi^{(1)}}{\mathcal{H}} - E^{(1)'}, \quad \tilde{\Phi}^{(1)} \equiv \Phi_{\text{FG}}^{(1)} = \Phi^{(1)} + \Psi^{(1)} + \left(\frac{\Psi^{(1)}}{\mathcal{H}}\right)'. \tag{36}$$

## 2.8. Second-order gauge-invariant variables

For second-order perturbations, equation (16) gives the following transformation rules:

$$\begin{aligned}\Phi^{(2)} &\rightarrow \Phi^{(2)} + T'^{(2)} + \mathcal{H}T^{(2)} + S_\Phi \\ B^{(2)} &\rightarrow B^{(2)} - T^{(2)} + L'^{(2)} + S_B \\ \Psi^{(2)} &\rightarrow \Psi^{(2)} - \mathcal{H}T^{(2)} + S_\Psi \\ E^{(2)} &\rightarrow E^{(2)} + L^{(2)} + S_E\end{aligned}$$

$$\begin{aligned}
E_{ij}^{(2)} &\rightarrow E_{ij}^{(2)} + S_{Eij} \\
\delta^{(2)}\rho &\rightarrow \delta^{(2)}\rho + \bar{\rho}'T^{(2)} + S_\rho \\
\delta^{(2)}P &\rightarrow \delta^{(2)}P + \bar{P}'T^{(2)} + S_P \\
v^{(2)} &\rightarrow v^{(2)} - L^{(2)'} + S_v \\
\pi^{ij(2)} &\rightarrow \pi^{ij(2)} + 2T^{(1)}(\pi^{ij(1)})' + 2\partial^k L^{(1)}\partial_k \pi^{ij(1)} - 2\pi^{ik(1)}\partial_k \partial^j L^{(1)} - 2\pi^{jk(1)}\partial_k \partial^i L^{(1)},
\end{aligned} \tag{37}$$

where the source terms are quadratic in the first-order variables  $T^{(1)}$ ,  $L^{(1)}$ ,  $\Phi^{(1)}$ ,  $\Psi^{(1)}$ . We collect the expressions of these terms in appendix A. In the rest of this paper, we shall refer to these second-order transformation rules associated with  $(\xi) \equiv (\xi_1, \xi_2)$  as  $\mathcal{T}_{(\xi)}(\Phi^{(2)})$ ,  $\mathcal{T}_{(\xi)}(B^{(2)})$ ,  $\dots$  or simply  $\mathcal{T}(\Phi^{(2)})$ ,  $\mathcal{T}(B^{(2)})$ ,  $\dots$ . These transformation rules are much more complicated than their first-order counterparts. However, the combination defined by  $F \equiv \delta^{(2)}g + 2\mathcal{L}_{\xi \rightarrow \text{NG}}^{(1)}\delta^{(1)}g + \mathcal{L}_{\xi \rightarrow \text{NG}}^2 \bar{g}$  enjoys the simple transformation rule  $F \rightarrow F + \mathcal{L}_{\xi_2 + [\xi \rightarrow \text{NG}, \xi_1]} \bar{g}$  under a gauge change defined by  $\xi_2$  and  $\xi_1$  (see [26]). As a result, its transformation rule mimics that of first-order perturbations under a gauge change. This means that if we decompose  $F$  in the same way as we did for the metric with

$$\begin{aligned}
\Phi_F &\equiv \Phi^{(2)} + S_\Phi(\xi_{\rightarrow \text{NG}}^{(1)}) \\
\Psi_F &\equiv \Psi^{(2)} + S_\Psi(\xi_{\rightarrow \text{NG}}^{(1)}) \\
B_F &\equiv B^{(2)} + S_B(\xi_{\rightarrow \text{NG}}^{(1)}) \\
E_F &\equiv E^{(2)} + S_E(\xi_{\rightarrow \text{NG}}^{(1)}) \\
E_{Fij} &\equiv E_{ij}^{(2)} + S_{Eij}(\xi_{\rightarrow \text{NG}}^{(1)}),
\end{aligned} \tag{38}$$

then the transformation rules for these quantities will be similar to those of equation (29), but with the vector  $\xi_2 + [\xi \rightarrow \text{NG}, \xi_1]$  instead of  $\xi_1$ . Consequently, we shall use the same combinations in order to construct gauge-invariant variables which are

$$\begin{aligned}
\hat{\Phi}^{(2)} &\equiv \Phi_F + (B_F - E_F)' + \mathcal{H}(B_F - E_F) \\
\hat{\Psi}^{(2)} &\equiv \Psi_F - \mathcal{H}(B_F - E_F) \\
\hat{E}_{ij}^{(2)} &\equiv E_{Fij}.
\end{aligned} \tag{39}$$

As for the first order, this addition of compensating terms can be understood, from the second point of view, as defining the gauge-invariant variables as the perturbation variables in a given gauge. In our case, it is the Newtonian gauge since it transforms  $B$  and  $E$  into a null value. This transformation is defined by  $\xi_{\rightarrow \text{NG}}^{(2)}$  that we decompose in

$$\begin{aligned}
T_{\rightarrow \text{NG}}^{(2)} &= B^{(2)} - E'^{(2)} + S_B^{(2)}(\xi_{\rightarrow \text{NG}}^{(1)}) - S_E'^{(2)}(\xi_{\rightarrow \text{NG}}^{(1)}) \\
L_{\rightarrow \text{NG}}^{(2)} &= -E^{(2)} - S_E^{(2)}(\xi_{\rightarrow \text{NG}}^{(1)}).
\end{aligned} \tag{40}$$

The second-order gauge-invariant variables can be seen as

$$\begin{aligned}
\hat{\Phi}^{(2)} &\equiv \delta_{\text{NG}}^{(2)}\Phi \\
m_S \hat{\Psi}^{(2)} &\equiv \delta_{\text{NG}}^{(2)}\Psi
\end{aligned} \tag{41}$$

$$\begin{aligned}
\hat{E}_{ij}^{(2)} &\equiv \delta_{\text{NG}}^{(2)}E_{ij} \\
\delta^{(2)}\hat{\rho} &\equiv \delta_{\text{NG}}^{(2)}\rho \\
\delta^{(2)}\hat{P} &\equiv \delta_{\text{NG}}^{(2)}P \\
\hat{v}^{(2)} &\equiv v_{\text{NG}}^{(2)} \\
\hat{\pi}^{ij(2)} &\equiv \delta_{\text{NG}}^{(2)}\pi^{ij},
\end{aligned} \tag{42}$$

where the index NG indicates that we transformed the quantity with formula (16), with the vector fields  $\xi_{\rightarrow \text{NG}}^{(1)}$  and  $\xi_{\rightarrow \text{NG}}^{(2)}$  defined above. This means that we have split the second-order metric according to

$$\delta^{(2)}g = \delta^{(2)}\tilde{g} + \mathcal{L}_{-\xi_{\rightarrow \text{NG}}^{(2)}}\bar{g} + 2\mathcal{L}_{-\xi_{\rightarrow \text{NG}}^{(1)}}\delta^{(1)}g - \mathcal{L}_{-\xi_{\rightarrow \text{NG}}^{(1)}}^2\bar{g}, \quad (43)$$

where  $\delta^{(2)}\tilde{g}$  is the gauge-invariant part and  $-\xi_{\rightarrow \text{NG}}^{(2)}$  is the gauge-variant part, as can be seen from the transformation rules under a gauge change characterized by  $(\xi_1, \xi_2)$ :

$$\delta^{(2)}\tilde{g} \rightarrow \delta^{(2)}\tilde{g}, \quad -\xi_{\rightarrow \text{NG}}^{(2)} \rightarrow -\xi_{\rightarrow \text{NG}}^{(2)} + \xi_2 + [\xi_{\rightarrow \text{NG}}^{(1)}, \xi_1]. \quad (44)$$

As for the first order, we can choose other types of combinations, for instance those which are equivalent to setting the gauge as being flat, by using this procedure. In this case, the vector field  $\xi_{\rightarrow \text{FG}}^{(2)}$  is decomposed in

$$T_{\rightarrow \text{FG}}^{(2)} = \frac{\Psi^{(2)}}{\mathcal{H}} + \frac{1}{\mathcal{H}}S_{\Psi}^{(2)}(\xi_{\rightarrow \text{FG}}^{(1)}), \quad L_{\rightarrow \text{FG}}^{(2)} = -E^{(2)} - S_E^{(2)}(\xi_{\rightarrow \text{FG}}^{(1)}). \quad (45)$$

It should also be mentioned that the existence of an inverse Laplacian  $\Delta^{-1}$  of the background spacetime, i.e. a corresponding Green function with boundary conditions, is required for the entirety of this procedure. In other words, when working in Fourier space, all our conclusions will be valid only for modes which do not belong to the kernel of  $\Delta$ .

### 3. Gauge transformation of the distribution function

#### 3.1. Pre-Riemannian distribution function

So far, we have set up the mathematical framework to identify points between the background spacetime and the perturbed spacetimes through a gauge field  $X$ . This enabled us to define the perturbation of tensors and to calculate their transformation properties under a gauge transformation. However, this only allows us to perform a fluid treatment of the radiation. In the statistical description for a set of particles, we assume that each particle has a given impulsion  $p^\mu$  and is located at a given position [33]. The equations then have to describe the phase-space distribution of the particles. If the number of particles is high enough, we can define a probability density, the distribution function, of finding a particle in an infinitesimal volume of the phase space. Now, let us focus our attention on this distribution function. The distribution function is a function of the point considered (i.e. its coordinates  $x^\mu$ ) and also a function of the tangent space at this point whose coordinate we label by  $p^\nu\partial_\nu$ . There is no special reason for this function to be linear in  $p^\nu\partial_\nu$ , but we can expand it, without any loss of generality, in power series of tensors according to

$$f(x^\nu, p^\nu) = \sum_k \mathcal{F}_{\mu_1 \dots \mu_k}(x^\nu) p^{\mu_1} \dots p^{\mu_k}. \quad (46)$$

The distribution function is then decomposed as the sum of all the multipoles  $\mathcal{F}_{\mu_1 \dots \mu_k}$  evaluated in a particular point of the tangent space. From the previous section, we know the transformation rules for these tensorial quantities; thus  $f$  transforms according to

$$\mathcal{T}_{(\xi)}[f(x^\nu, p^\nu)] \equiv \sum_k \mathcal{T}_{(\xi)}[\mathcal{F}_{\mu_1 \dots \mu_k}(x^\nu)] p^{\mu_1} \dots p^{\mu_k}, \quad (47)$$

where  $\mathcal{T}_{(\xi)}$  refers to the knight diffeomorphism with the set of vectors  $(\xi_1, \xi_2, \dots)$ .

As we do not necessarily want to refer explicitly to the decomposition in multipoles, we use the fact that for any vector  $\xi = \xi^\mu\partial_\mu$ , which defines a flow on the background spacetime

$\mathcal{P}_0(\mathcal{N})$ , we can define an induced flow (a natural lift) on the vector tangent bundle  $T\mathcal{P}_0(\mathcal{N})$  directed by the vector field  $T\xi = [\xi^\mu \partial_\mu, p^\nu (\partial_\nu \xi^\mu) \frac{\partial}{\partial p^\mu}]$ . This implies the useful property

$$\mathcal{L}_\xi(\mathcal{F}_{\mu_1 \dots \mu_p}) p^{\mu_1} \dots p^{\mu_p} = \mathcal{L}_{T\xi}(\mathcal{F}_{\mu_1 \dots \mu_p}) p^{\mu_1} \dots p^{\mu_p}. \quad (48)$$

With this definition, we can rewrite the transformation rule for  $f$  as

$$\mathcal{T}_{(\xi)}[f(x^\nu, p^\nu)] = \mathcal{T}_{(T\xi)}[f(x^\nu, p^\nu)], \quad (49)$$

where now  $\mathcal{T}_{(T\xi)}$  refers to the knight diffeomorphism with the set of vectors  $(T\xi_1, T\xi_2, \dots)$ .

The evolution of the distribution function is dictated by the Boltzmann equation  $\frac{df}{d\eta} = C[f]$ , where the rhs is the collision term which encodes the local physics. This collision term can be easily expressed in the local Minkowskian frame defined by a tetrad field  $e_a$ , from known particles physics. For this reason, the framework developed to define gauge transformations for a general manifold has to be extended to the case of Riemannian manifold. Instead of using the coordinates basis  $\partial_\mu$  to express a vector of tangent space as  $V = p^\mu \partial_\mu$ , we prefer to use the tetrads basis  $e_a$  and write  $V = \pi^a e_a$ . In terms of coordinates, this means that the distribution function is a function of  $x^\mu$  and  $\pi^a$ . When expressing the physics with the tetrad fields, the metric is not just one of the many tensors of the theory whose properties under a gauge transformation we need to know, but rather a central feature of the manifold, since it determines the tetrads (up to a Lorentz transformation) required to express the distribution function. As the metric is a tensor, and as the tetrads are defined according to the metric, the extension is inherited from the previous section.

### 3.2. Tetrads

**3.2.1. Definitions.** On each slice  $\mathcal{P}_\lambda(\mathcal{N})$ , we should have four vector fields<sup>1</sup> (and their associated 1-form fields) labeled by  $a = 0, 1, 2, 3$ , which satisfy the normalization conditions

$$e_a^\mu e_b^\nu g_{\mu\nu} = \eta_{ab}, \quad e_\mu^a e_\nu^b g^{\mu\nu} = \eta^{ab}. \quad (50)$$

With these notations, the indices  $a, b, c \dots$  are raised and lowered with  $\eta_{ab}$ .

With the formalism developed for tensors, we carry this tetrad field onto the background spacetime using a gauge field  $X$  with

$$e_{a,X}^\mu \equiv \phi_{\lambda,X}^*(e_a) = \sum_{k=0}^{k=\infty} \frac{\lambda^k}{k!} \mathcal{L}_X^k e_a \quad (51)$$

$$\delta_X^{(n)} e_a \equiv \mathcal{L}_X^n e_a \Big|_{\mathcal{P}_0(\mathcal{N})}, \quad \bar{e}_a \equiv \delta_X^{(0)} e_a,$$

and similar formulae for  $e^a$ .

As  $\bar{e}_a$  is a basis of the tangent space on the background spacetime (and  $\bar{e}^a$  a basis of its dual space),  $e_{a,X}^\mu$  and  $e_{\mu,X}^a$  can be expressed in the generic form

$$e_{a,X} = R_{a,X}^b \bar{e}_b, \quad e_X^b = \bar{e}^a S_{a,X}^b, \quad R_{a,X}^c S_{c,X}^b = S_{a,X}^c R_{c,X}^b = \delta_a^b, \quad (52)$$

where

$$R_{ab,X} \equiv \sum_k \frac{\lambda^k}{k!} R_{ab,X}^{(k)} \quad S_{ab,X} \equiv \sum_k \frac{\lambda^k}{k!} S_{ab,X}^{(k)}. \quad (53)$$

Order by order, this reads

$$\delta_X^{(n)} e_a = R_{a,X}^{(n)b} \bar{e}_b, \quad \delta_X^{(n)} e^b = \bar{e}^a S_{a,X}^{(n)b}. \quad (54)$$

<sup>1</sup> The fifth direction which arises from the extension of the manifold from  $\mathcal{M}$  to  $\mathcal{N}$  is ignored as the component of any tensor is required to vanish in this direction. We thus consider the tangent space at each point of  $\mathcal{N}$  as being four dimensional.

3.2.2. *Normalization condition.* Tetrads are four vector fields which satisfy equation (50) and are thus related to the metric. Consequently, the perturbations of the tetrad defined above are partly related to the perturbations of the metric. When pulled back to the background spacetime, equation (50) implies

$$\begin{aligned}\phi_{\lambda,X}^*(\eta_{ab}) &= \eta_{ab} = \phi_{\lambda,X}^*(e_a^\mu e_b^\nu g_{\mu\nu}) \\ &= \phi_{\lambda,X}^*(e_a^\mu) \phi_{\lambda,X}^*(e_b^\nu) \phi_{\lambda,X}^*(g_{\mu\nu}).\end{aligned}\quad (55)$$

Identifying order by order (in terms of  $\lambda$ ), we get in particular for the first and second orders

$$\bar{e}_b \cdot \delta_X^{(1)} e_a + \bar{e}_a \cdot \delta_X^{(1)} e_b + \delta_X^{(1)} g(\bar{e}_a, \bar{e}_b) = 0 \quad (56)$$

$$\bar{e}_b \cdot \delta_X^{(2)} e_a + \bar{e}_a \cdot \delta_X^{(2)} e_b + \delta_X^{(2)} g(\bar{e}_a, \bar{e}_b) + \delta_X^{(1)} e_b \cdot \delta_X^{(1)} e_a + \delta_X^{(1)} g(\delta_X^{(1)} e_a, \bar{e}_b) + \delta_X^{(1)} g(\bar{e}_a, \delta_X^{(1)} e_b) = 0,$$

where a dot product stands for  $\bar{g}(-, -)$ . From the constraints (56), we can determine the symmetric part of  $R_{ab}^{(n)}$  as

$$R_{(ab),X}^{(1)} = -\frac{1}{2} \delta_X^{(1)} g(\bar{e}_a, \bar{e}_b) \quad (57)$$

$$R_{(ab),X}^{(2)} = -\frac{1}{2} \delta_X^{(2)} g(\bar{e}_a, \bar{e}_b) - \delta_X^{(1)} g(R_{ac,X}^{(1)} \bar{e}^c, \bar{e}_b) - \delta_X^{(1)} g(\bar{e}_a, R_{bc,X}^{(1)} \bar{e}^c) - R_{a,X}^{(1)c} R_{bc,X}^{(1)}, \quad (58)$$

which are related to the components of the inverse by

$$S_{ab,X}^{(1)} = -R_{ab,X}^{(1)} \quad (59)$$

$$S_{ab,X}^{(2)} = -R_{ab,X}^{(2)} + 2R_{a,X}^{(1)c} R_{cb,X}^{(1)}. \quad (60)$$

The antisymmetric part,  $R_{[ab],X}$ , still remains to be chosen as it corresponds to the Lorentz transformation freedom (boost and rotation), which is allowed by definition (50). A first and easy choice would be  $R_{[ab],X}^{(n)} = 0$  for any  $n$ . However, as mentioned above, we eventually want to decompose a vector  $p^\mu \partial_\mu$  on tangent space as

$$p^\mu \partial_\mu = \pi^a e_a = \pi^a e_a^\mu \partial_\mu, \quad (61)$$

and identify  $\pi^0$  with the energy and  $\pi^i$  with the momentum (although conserved quantities are generally ill defined in general relativity, energy and momentum can be defined when performing perturbations around a maximally symmetric background [35] as is the case here). When working with coordinates, we want to express physical quantities, as measured by comoving observers, i.e. observers of constant spatial coordinates, whose motion is defined by the 1-form  $(d\eta)_\mu$  [36]. We thus require  $(e^0)_\mu \sim (d\eta)_\mu$ , which is equivalent to choosing  $S_{a_i,0,X}^{(n)} = 0$  for any  $n$ , where  $a_i = 1, 2, 3$ . This choice allows us to fix the boost in  $S^{(n)}$  by imposing the condition  $S_{[a_i,0],X}^{(n)} = -S_{[0a_i],X}^{(n)} = -S_{(a_i,0),X}^{(n)}$ . As equation (52) implies that for any  $n$

$$S_{ac}^{(n)} + R_{ac}^{(n)} + \prod_{\substack{\{p+q=n, \\ p \geq 1, q \geq 1\}}} S_a^{(p)b} R_{bc}^{(q)} \frac{n!}{p!q!} = 0, \quad (62)$$

it can be checked by recursion that this implies

$$R_{[a_i,0],X}^{(n)} = -R_{[0a_i],X}^{(n)} = -R_{(a_i,0),X}^{(n)}. \quad (63)$$

We also fix the rotation by requiring  $S_{[a_i a_j],X}^{(n)} = 0$ , and it can be checked similarly, by recursion on equation (62), that this implies  $R_{[a_i a_j],X}^{(n)} = 0$ .

### 3.3. Gauge transformation of tetrads

Under a gauge transformation, we can deduce the transformation properties of the tetrad from those of the perturbed metric. For simplicity, we restrict to scalar and tensor perturbations, but this is completely general and can be easily extended to include vectors. In the FL case, we use a natural background tetrad associated with Cartesian coordinates  $\bar{e}_0 = (\partial_\eta)/a$ ,  $\bar{e}_{b_i} = (\partial_i)/a$ , in order to evaluate equation (57). The notation  $b_i$  refers to Lorentz (SO(1,3)) indices running from 1 to 3, whereas  $i$  is a coordinate index running from 1 to 3. When uselessly obfuscating the explanation, we will not make the distinction and change  $b_i$  for  $i$ . We report the detailed expressions for the transformation of the tetrads for the first and second orders in appendix B.

## 4. Distribution function

Now that the transformation properties of the tetrads are known, we turn to the general transformation of a distribution function  $f(x^\mu, \pi^a)$ .

### 4.1. Multipolar expansion

Any function  $f(x^\mu, \pi^a)$  can be expanded in symmetric trace-free multipoles as [37]

$$f(x^\mu, \pi^a) = \sum_p F_p(x^\mu, \pi^a) \quad (64)$$

with

$$\begin{aligned} F_p(x^\nu, \pi^a) &\equiv \mathcal{F}_{\mu_1 \dots \mu_p}(x^\nu) p^{\mu_1} \dots p^{\mu_p} \\ &= [\mathcal{F}_{\mu_1 \dots \mu_p}(x^\nu) e_{a_1}^{\mu_1} \dots e_{a_p}^{\mu_p}] \pi^{a_1} \dots \pi^{a_p} \\ &\equiv F_{a_1 \dots a_p}(x^\nu) \pi^{a_1} \dots \pi^{a_p}. \end{aligned} \quad (65)$$

We do not need any additional identification procedure for the tangent spaces through a gauge field, in order to identify points of the tangent space of the slices  $T\mathcal{P}_\lambda(\mathcal{N})$ . Indeed, once the metric and a gauge field  $X$  are chosen, there exists a natural identification with the tetrad fields. First, and as mentioned before, we identify the points of  $\mathcal{N}$  which lie on the same integral curves of  $X$ , that is, we identify a point  $P \in \mathcal{P}_0(\mathcal{N})$  and  $\Phi_{\lambda, X}(P) \in \mathcal{P}_\lambda(\mathcal{N})$ . Then, we identify vectors of their respective tangent spaces, if the coordinates of these vectors in their respective local tetrad frames  $\bar{e}_a$  and  $e_a$  are the same. To be short, we identify  $\pi^a e_a$  and  $\pi^a \bar{e}_a$ . As a consequence, for any given set  $\{a_1, \dots, a_p\}$ , the function  $F_{a_1 \dots a_p}(x^\nu)$  is a scalar field.  $F_{a_1 \dots a_p}(x^\nu)$  is then pulled back on the background spacetime using the gauge field  $X$ , and we define in this way perturbations

$$\Phi_{\lambda, X}^*[F_{a_1 \dots a_p}(x^\nu)] \equiv F_{X, a_1 \dots a_p}(x^\nu) \equiv \sum_\lambda \frac{\lambda^n}{n!} \delta_X^{(n)} F_{a_1 \dots a_p}(x^\nu) \quad (66)$$

and

$$F_{p, X}(x^\nu, \pi^a) \equiv F_{X, a_1 \dots a_p}(x^\nu) \pi^{a_1} \dots \pi^{a_p}. \quad (67)$$

This perturbation scheme induces a perturbation procedure for the distribution function  $f$  as

$$\begin{aligned} f_X(x^\nu, \pi^a) &\equiv \sum_n \frac{\lambda^n}{n!} \delta_X^{(n)} f(x^\nu, \pi^a), \\ \delta_X^{(n)} f(x^\mu, \pi^a) &\equiv \sum_p \delta_X^{(n)} F_{a_1 \dots a_p}(x^\nu) \pi^{a_1} \dots \pi^{a_p}. \end{aligned} \quad (68)$$

It is essential to stress that  $\pi^a$  is not a perturbed quantity; it is a coordinate of the locally Minkowskian tangent space. However, the tetrad field allows us to see  $p^\mu$  as a perturbed vector since  $p^\mu(\pi^a) = e_a^\mu \pi^a$ . In other words, for a given  $\pi^a$ , there is an associated vector whose order-by-order perturbation in a given gauge  $X$  is given by  $p_X^{\mu(n)} \equiv e_{a,X}^{\mu(n)} \pi^a$ .

#### 4.2. Gauge transformation: general case

We can deduce the transformation rule under a gauge change directly on the form (65), pulled back to the background spacetime,

$$\mathcal{T}[f_X(x^\nu, \pi^a)] \equiv \sum_p \mathcal{T}[\mathcal{F}_{X,\mu_1 \dots \mu_p}(x^\nu)] \mathcal{T}(e_{a_1, X}^{\mu_1}) \dots \mathcal{T}(e_{a_p, X}^{\mu_p}) \pi^{a_1} \dots \pi^{a_p}. \quad (69)$$

The first factor in this expression is tensorial. Exactly as for the pre-Riemannian case, its transformation rule is dictated by the knight diffeomorphism, whereas we get the transformation rules of the tetrads from equations (B.2) and (B.4). As we do not necessarily want to refer explicitly to the multipole expansion, the first factor is rewritten by considering  $f$  as a function of  $p^\mu$  using  $\pi^a = e_{v,X}^a p^\nu$ , and applying equation (49). We then have to consider the resulting distribution function as a function of  $\pi^a$ , knowing that the inversion is now given by  $p^\mu(\pi^a) = \mathcal{T}(e_a^\mu) \pi^a$ . This will account for  $\mathcal{T}(e_{a_1, X}^{\mu_1})$  in equation (69). In a compact form, it reads

$$\mathcal{T}[f_X(x^\nu, \pi^a)] = \mathcal{T}_{(\mathcal{T}\xi)} \{ f_X[x^\nu, e_\mu^a p^\mu] \} \Big|_{p^\mu = \mathcal{T}(e_b^\mu) \pi^b}. \quad (70)$$

To obtain an order-by-order formula, we explicitly define these three steps using a Taylor expansion. First, we use that

$$f_X(x^\nu, \pi^a) = \left[ \exp \left( \bar{e}_\mu^b p^\mu S_{b,X}^a \frac{\partial}{\partial \pi^a} \right) f_X \right] (x^\nu, \bar{e}_\mu^b p^\mu) \equiv g_X(x^\nu, p^\mu), \quad (71)$$

in order to consider  $f$  as a function of  $p^\mu$ . We then Taylor-expand back the result of the knight diffeomorphism in order to read the result as a function of  $\pi^a$ ,

$$\mathcal{T}[f_X(x^\nu, \pi^a)] = \left[ \exp \left( \bar{e}_b^\mu \pi^a \mathcal{T}(R_{a,X}^b) \frac{\partial}{\partial p^\mu} \right) \mathcal{T}_{(\mathcal{T}\xi)}(g_X) \right] (x^\nu, \bar{e}_a^\mu \pi^a). \quad (72)$$

The derivatives in the previous expressions have to be ordered on the right in each term of the expansion in power series of the exponential. When identifying order by order, we need to take into account the expansion in  $R_{ab}$  and  $S_{ab}$ , in the exponentials and also in the knight diffeomorphism.

We have provided the general transformation rules for the distribution function and we will now specify the transformation properties of the first- and second-order distribution functions.

#### 4.3. The mass shell

The transformation properties of  $\delta_X^{(n)} e_a^\mu$  have been chosen so that, in the special case of  $f \equiv g_{\mu\nu} p^\mu p^\nu = g_{\mu\nu} e_a^\mu e_b^\nu \pi^a \pi^b = \pi_a \pi^a$ , it remains unchanged under a gauge transformation, i.e.  $\mathcal{T}(\pi^a \pi_a) = \pi^a \pi_a$ . Since the tetrads must satisfy equation (50), then  $\delta_X^{(n)} f = 0$  for  $n \geq 1$ , and it implies this property trivially. As a consequence, any function of the form  $\delta(\pi_a \pi^a - m^2) f(x^\mu, \pi^a)$  transforms as  $\delta(\pi_a \pi^a - m^2) \mathcal{T}[f(x^\mu, \pi^a)]$ , where  $m^2$  is the mass of the particles described by the distribution function. In other words, the transformation of the distribution function remains on the mass shell, as has already been mentioned in [9]. We will make use of this property when computing the transformation rules of the distribution function.

## 5. Application to the perturbation of the Boltzmann equation for radiation

The formalism developed in the previous section is general. We will now apply it to the particular FL case, and from now on we will also focus on the radiation case, that is the case where  $m^2 = 0$ . For the first and the second orders, we will present the transformation rules of the distribution function for radiation and build a gauge-invariant distribution function as well as a gauge-invariant brightness. We will then write the evolution equation of this gauge-invariant brightness in the case where the photon travels freely through spacetime without being affected by diffusion processes. This is obtained using the collisionless Boltzmann equation

$$\frac{df}{d\eta} = \frac{\partial f}{\partial \eta} + \frac{\partial f}{\partial x^i} \frac{\partial x^i}{\partial \eta} + \frac{\partial f}{\partial \pi^0} \frac{\partial \pi^0}{\partial \eta} + \frac{\partial f}{\partial n^i} \frac{\partial n^i}{\partial \eta} = 0, \quad (73)$$

where  $n^i \equiv \pi^i/\pi^0$ , from which we will extract the background, the first- and the second-order equations after having pulled it back to the background spacetime. In order to do so, we need to know  $\frac{\partial \pi^0}{\partial \eta}$  and  $\frac{\partial \pi^i}{\partial \eta}$ . By considering  $p^\mu$  as a perturbed vector, as mentioned in section (4.1), these can be expressed from the geodesic equation

$$p^0 \frac{dp^\mu}{d\eta} = -\Gamma_{\nu\sigma}^\mu p^\nu p^\sigma \quad (74)$$

that we pull back to the background spacetime in order to extract order-by-order equations. Similarly,  $\frac{\partial x^i}{\partial \eta}$  is given by the order-by-order expressions of  $p^0 \frac{\partial x^i}{\partial \eta} = p^i$ , when pulled back to the background spacetime.

At the background level, the space is homogeneous and isotropic. Consequently, the distribution function depends neither on the direction  $n^i$  of the photon nor on the position in space  $x^i$ . It only depends on  $\pi^0$  and  $\eta$ , which implies that  $\frac{\partial \bar{f}}{\partial n^i} = \frac{\partial \bar{f}}{\partial x^i} = 0$ . Since the background geodesic deviation equation implies  $\frac{\partial \pi^0}{\partial \eta} = -\mathcal{H}\pi^0$ , the collisionless Boltzmann equation reads at the background level

$$\left. \frac{\partial \bar{f}}{\partial \eta} \right|_\pi - \mathcal{H}\pi^0 \frac{\partial \bar{f}}{\partial \pi^0} = 0. \quad (75)$$

### 5.1. Gauge transformation at first order

In order to better understand the seemingly heavy but powerful formalism of section 4.2, let us apply it to the first-order gauge transformation of the photon distribution function  $f$  in the Boltzmann equation. In this case, equation (70) for  $\xi_1 = (T, L)$  leads to

$$\mathcal{T}[\delta(\pi_c \pi^c) \delta_X^{(1)} f] = \delta(\pi_c \pi^c) \left\{ \mathcal{L}_{T\xi_1}[\bar{f}(x^\nu, ap^\mu)] + [\mathcal{T}(R_{a,X}^{(1)b}) + S_{a,X}^{(1)b}] \pi^a \frac{\partial \bar{f}}{\partial \pi^b} \right\}. \quad (76)$$

The expressions of  $R_{a,X}^b$  and  $S_{a,X}^b$ , and their transformation rules for the FL case, are given in appendix B. Using the fact that  $\bar{f}$  is only a function of  $\pi^0$  due to the term  $\delta(\pi_c \pi^c)$ ,

$$\mathcal{L}_{T\xi_1}[\bar{f}(x^\nu, ap^\mu)] = T \frac{\partial}{\partial \eta} \Big|_p \bar{f}(x^\nu, ap^\mu) + \frac{\partial \bar{f}}{\partial \pi^0} \pi^0 (T' + n^i \partial_i T) \quad (77)$$

$$[\mathcal{T}(R_{0,X}^{(1)0}) + S_{0,X}^{(1)0}] \pi^0 \frac{\partial \bar{f}}{\partial \pi^0} = -\frac{\partial \bar{f}}{\partial \pi^0} \pi^0 (T' + \mathcal{H}T). \quad (78)$$

Note that there is no term involving  $[\mathcal{T}(R_{i,X}^{(1)0}) + S_{i,X}^{(1)0}] \pi^i \frac{\partial \bar{f}}{\partial \pi^0}$ , thanks to the prescription in the choice of the tetrad in section 3.2.2.

We then express the derivatives as

$$\left. \frac{\partial \bar{f}(x^\nu, ap^\mu)}{\partial \eta} \right|_p = \left. \frac{\partial \bar{f}}{\partial \eta} \right|_\pi + \frac{\partial \bar{f}}{\partial \pi^0} \mathcal{H} \pi^0. \quad (79)$$

Putting all the pieces together, we finally get that

$$\begin{aligned} \mathcal{T}[\delta(\pi^c \pi_c) \delta_X^{(1)} f] &= \delta(\pi^c \pi_c) \left( \frac{\partial \bar{f}}{\partial \pi^0} \pi^0 n^i \partial_i T + T \left. \frac{\partial \bar{f}}{\partial \eta} \right|_\pi \right) \\ &= \delta(\pi^c \pi_c) \frac{\partial \bar{f}}{\partial \pi^0} \pi^0 (\mathcal{H} T + n^i \partial_i T), \end{aligned} \quad (80)$$

where in the last step we have made use of the background Boltzmann equation (75).

It can be checked that by considering  $f$  as a function of  $\sqrt{\pi^i \pi_i}$  instead of  $\pi^0$ , as allowed by the factor  $\delta(\pi^c \pi_c)$ , we recover the same result as performed in [9]. However this is slightly more intricate, as it now apparently depends on the three variables  $\pi^i$  which are in fact not independent at the background level.

Although the mathematical framework can seem to be heavy, we did not need to define an extension of the distribution function outside the mass shell nor a gauge transformation field parallel to the mass shell as in [9]. We have first built the distribution function using the tetrad field (it is a function of  $\pi^a$  and not an explicit function of  $p^\mu$ ). Then, as explained in section 4.3, the normalization condition (50), when expressed at each order in equations (56), ensures that it remains on the mass shell during a gauge transformation that we perform using the rules derived for tensors.

### 5.2. First-order gauge-invariant distribution function for radiation

Now that transformation properties of the first-order distribution function are known, we can use the results of section 2 to define a gauge-invariant distribution function by

$$\begin{aligned} \hat{f}^{(1)} &\equiv \delta_{\text{NG}}^{(1)} f = \delta_X^{(1)} f + \mathcal{T}_{\xi \rightarrow \text{NG}}^{(1)} (\delta_X^{(1)} f) \\ &= \delta_X^{(1)} f + \frac{\partial \bar{f}}{\partial \pi^0} \pi^0 [\mathcal{H}(B^{(1)} - E^{(1)'}) + n^i \partial_i (B^{(1)} - E^{(1)'})]. \end{aligned} \quad (81)$$

As for tensorial quantities, we can choose for instance  $\xi_{\rightarrow \text{FG}}^{(1)}$  in the above expression, in order to define another gauge-invariant distribution function. Its expression is given by

$$\begin{aligned} \tilde{f}^{(1)} &\equiv \delta_{\text{FG}}^{(1)} f = \delta_X^{(1)} f + \mathcal{T}_{\xi \rightarrow \text{FG}}^{(1)} (\delta_X^{(1)} f) \\ &= \delta_X^{(1)} f + \frac{\partial \bar{f}}{\partial \pi^0} \pi^0 \left[ \Psi^{(1)} + \frac{n^i \partial_i \Psi^{(1)}}{\mathcal{H}} \right]. \end{aligned} \quad (82)$$

These two first-order gauge-invariant distribution functions are related by

$$\tilde{f}^{(1)} - \hat{f}^{(1)} = \frac{\partial \bar{f}}{\partial \pi^0} \pi^0 \left[ \hat{\Psi}^{(1)} + \frac{n^i \partial_i \hat{\Psi}^{(1)}}{\mathcal{H}} \right]. \quad (83)$$

It is worth remarking that in the previous literature [9], another gauge-invariant distribution is defined, namely

$$\begin{aligned} F^{(1)} &\equiv \delta_X^{(1)} f + \frac{\partial \bar{f}}{\partial \pi^0} \pi^0 [\Psi^{(1)} + n^i \partial_i (B^{(1)} - E^{(1)'})] \\ &= \hat{f}^{(1)} + \frac{\partial \bar{f}}{\partial \pi^0} \pi^0 \hat{\Psi}^{(1)}. \end{aligned} \quad (84)$$

Though it cannot be interpreted as the perturbation of the distribution function in a given gauge since it mixes  $\xi \rightarrow_{\text{NG}}$  and  $\xi \rightarrow_{\text{FG}}$ , this is a better variable to highlight the conformal invariance of the photons propagation and to compare with the null cone integration method [34].

This first-order analysis illustrates the power of this formalism which can be generalized to higher orders in perturbations.

### 5.3. First-order collisionless Boltzmann equation for radiation

Integrating the gauge-invariant distribution function of radiation over  $\pi^0$ , we define the gauge-invariant brightness, which is the energy perturbation per unit solid angle in a given direction

$$\hat{\mathcal{I}}^{(1)}(x^\mu, n^i) \equiv 4\pi \int \hat{f}^{(1)}(x^\mu, \pi^0, n^i) (\pi^0)^3 d\pi^0. \quad (85)$$

We choose the normalization of the background distribution function such that the background brightness reduces to the energy density (see section 6 for the fluid approximation)

$$\bar{\mathcal{I}}(\eta) \equiv 4\pi \int \bar{f}(\eta, \pi^0) (\pi^0)^3 d\pi^0 = \bar{\rho}. \quad (86)$$

We can associate gauge-invariant symmetric trace-free moments,  $\hat{\mathcal{F}}_{i_1 \dots i_n}$ , to this brightness by using the decomposition

$$\hat{\mathcal{I}}^{(1)}(x^\mu, n^i) \equiv \sum_p \hat{\mathcal{F}}_{i_1 \dots i_p}^{(1)}(x^\mu) n^{i_1} \dots n^{i_p}. \quad (87)$$

With these definitions, the integral  $\int (\pi^0)^3 d\pi^0$  on the first-order Boltzmann equation leads to the evolution equation for  $\hat{\mathcal{I}}^{(1)}$  [38]:

$$\left( \frac{\partial}{\partial \eta} + n^i \partial_i \right) \frac{\hat{\mathcal{I}}^{(1)}}{4} + \mathcal{H} \hat{\mathcal{I}}^{(1)} + (n^i \partial_i \hat{\Phi}^{(1)} - \hat{\Psi}^{(1)'}) \bar{\mathcal{I}} = 0, \quad (88)$$

where we have ignored the tensor terms for simplicity. Similarly, a gauge-invariant brightness  $\tilde{\mathcal{I}}^{(1)}$ , associated with  $\tilde{f}^{(1)}$ , and a gauge-invariant brightness  $M^{(1)}$ , [9] associated with  $\mathcal{F}^{(1)}$ , can be defined. They are related to  $\hat{\mathcal{I}}^{(1)}$  by

$$\tilde{\mathcal{I}}^{(1)} = \hat{\mathcal{I}}^{(1)} - 4\bar{\mathcal{I}} \left( \hat{\Psi}^{(1)} + \frac{n^i \partial_i \hat{\Psi}^{(1)}}{\mathcal{H}} \right) \quad M^{(1)} = \hat{\mathcal{I}}^{(1)} - 4\bar{\mathcal{I}} \hat{\Psi}^{(1)}. \quad (89)$$

### 5.4. Gauge transformation at second order

At second order, the general gauge transformation of the distribution function (70) for  $(\xi) = (\xi_1, \xi_2)$ ,  $(T\xi) = (T\xi_1, T\xi_2)$  is given in detail in appendix C. After simplifications, it reads

$$\begin{aligned} \mathcal{T} \left( \delta_X^{(2)} f \right) &= \frac{\partial \tilde{f}}{\partial \eta} (T^{(2)} + T T' + \partial_i T \partial^i L) \\ &+ \frac{\partial \tilde{f}}{\partial \pi^0} \pi^0 \{ n^i \partial_i T^{(2)} - 2n^j [(\partial_i \partial_j E + E_{ij} + \partial_i \partial_j L) \partial^i T - \Psi \partial_j T] \\ &+ \partial_i T \partial^i T + (T n^i \partial_i T)' + n^i \partial_i (\partial^j L \partial_j T) + 2\Phi n^i \partial_i T \} \\ &+ \frac{\partial^2 \tilde{f}}{\partial (\pi^0)^2} (\pi^0)^2 (n^i \partial_i T n^j \partial_j T) + 2 \frac{\partial^2 \tilde{f}}{\partial \eta \partial \pi^0} T n^i \partial_i T + \frac{\partial^2 \tilde{f}}{\partial \eta^2} T^2 \\ &+ 2 \frac{\partial \delta_X^{(1)} f}{\partial \pi^0} \pi^0 n^j \partial_j T + 2 \frac{\partial \delta_X^{(1)} f}{\partial \pi^i} \pi^0 \partial^i T + 2 \partial^i L \partial_i \delta_X^{(1)} f + 2T \frac{\partial \delta_X^{(1)} f}{\partial \eta}. \end{aligned} \quad (90)$$

This is a cornerstone expression in our study of the second-order distribution function. As for the fluid quantities, knowing the transformation rules under a second-order gauge change is enough to define a second-order gauge-invariant distribution function which is required to write the second-order Boltzmann equation only in terms of gauge-invariant variables. As for tensors, several gauge-invariant distribution functions can be defined, and this relation is also required to express how the different gauge-invariant distribution functions are related.

### 5.5. Second-order gauge-invariant distribution function for radiation

Again, we can use the results of section 2.8 to define a gauge-invariant distribution function as

$$\hat{f}^{(2)} \equiv \delta_{\text{NG}}^{(2)} f = \delta_X^{(2)} f + \mathcal{T}_{(\xi \rightarrow \text{NG}, \xi \rightarrow \text{NG})}^{(1)} [\delta_X^{(2)} f]. \quad (91)$$

As for tensorial quantities, we can choose for instance  $(\xi \rightarrow \text{FG}, \xi \rightarrow \text{FG})^{(1)}$ , in order to build another second-order gauge-invariant distribution function

$$\tilde{f}^{(2)} \equiv \delta_{\text{FG}}^{(2)} f = \delta_X^{(2)} f + \mathcal{T}_{(\xi \rightarrow \text{FG}, \xi \rightarrow \text{FG})}^{(1)} [\delta_X^{(2)} f]. \quad (92)$$

The difference between these two gauge-invariant distribution functions is also gauge invariant and is consequently expressed only in terms of gauge-invariant quantities. For the sake of completeness, we give the form of the relation between these two gauge-invariant distribution functions,

$$\begin{aligned} \tilde{f}^{(2)} - \hat{f}^{(2)} = & \frac{1}{\mathcal{H}^2} \frac{\partial \tilde{f}}{\partial \pi^0} \pi^0 \left\{ n^k \partial_k [\mathcal{H} \hat{\Psi}^{(2)} + \hat{\Psi}^{(1)} \hat{\Psi}^{(1)'} + 2\mathcal{H} \hat{\Psi}^{(1)2}] + \partial_i \hat{\Psi}^{(1)} \partial^i \hat{\Psi}^{(1)} \right. \\ & + n^k \partial_k \left[ -\frac{\Delta^{-1}}{2\mathcal{H}} (\Delta \hat{\Psi}^{(1)})^2 + \frac{\Delta^{-1}}{2\mathcal{H}} (\partial^i \partial_j \hat{\Psi}^{(1)} \partial^j \partial_i \hat{\Psi}^{(1)}) \right] \\ & \left. - 2\mathcal{H} n^j E_{ij} \partial^i \hat{\Psi}^{(1)} + (\hat{\Psi}^{(1)} n^i \partial_i \hat{\Psi}^{(1)})' + 2\mathcal{H} (\hat{\Phi}^{(1)} + \hat{\Psi}^{(1)}) n^i \partial_i \hat{\Psi}^{(1)} \right\} \\ & + \frac{1}{\mathcal{H}^2} \frac{\partial^2 \tilde{f}}{\partial (\pi^0)^2} (\pi^0)^2 [n^i \partial_i \hat{\Psi}^{(1)} n^j \partial_j \hat{\Psi}^{(1)}] + \frac{2}{\mathcal{H}^2} \frac{\partial^2 \tilde{f}}{\partial \eta \partial \pi^0} \hat{\Psi}^{(1)} n^i \partial_i \hat{\Psi}^{(1)} \\ & + \frac{1}{\mathcal{H}^2} \frac{\partial \tilde{f}}{\partial \eta} \left[ \mathcal{H} \hat{\Psi}^{(2)} + \hat{\Psi}^{(1)} \hat{\Psi}^{(1)'} + 2\mathcal{H} \hat{\Psi}^{(1)2} + \mathcal{H} \hat{\Psi}^{(1)} \left( \frac{\hat{\Psi}^{(1)}}{\mathcal{H}} \right)' \right. \\ & \left. - \frac{\Delta^{-1}}{2\mathcal{H}} (\Delta \hat{\Psi}^{(1)})^2 + \frac{\Delta^{-1}}{2\mathcal{H}} (\partial^i \partial_j \hat{\Psi}^{(1)} \partial^j \partial_i \hat{\Psi}^{(1)}) \right] + \frac{2}{\mathcal{H}} \frac{\partial \hat{f}^{(1)}}{\partial \pi^0} \pi^0 n^j \partial_j \hat{\Psi}^{(1)} \\ & + \frac{2}{\mathcal{H}} \frac{\partial \hat{f}^{(1)}}{\partial \pi^i} \pi^0 \partial^i \hat{\Psi}^{(1)} + 2 \frac{\hat{\Psi}^{(1)}}{\mathcal{H}} \frac{\partial \hat{f}^{(1)}}{\partial \eta} + \frac{1}{\mathcal{H}^2} \frac{\partial^2 \tilde{f}}{\partial \eta^2} (\hat{\Psi}^{(1)})^2. \end{aligned} \quad (93)$$

This clearly demonstrates the power of our formalism since, in contrast to the first order, this relation cannot be guessed intuitively. Also note that this is non-local as it is generally the case for second-order gauge-invariant quantities.

### 5.6. The second-order gauge-invariant collisionless Boltzmann equation for radiation

Similarly to the first-order case, we define the second-order brightness as

$$\hat{\mathcal{I}}^{(2)}(x^\mu, n^i) \equiv 4\pi \int \hat{f}^{(2)}(x^\mu, \pi^0, n^i) (\pi^0)^3 d\pi^0. \quad (94)$$

We also define the second-order gauge-invariant moments associated with this gauge-invariant brightness by the second-order version of equation (87). The derivation of the collisionless

Boltzmann equation in the Newtonian gauge is detailed in [27, 28]. Once the integral  $\int (\pi^0)^3 d\pi^0$  is performed, it leads to an evolution equation for the brightness. As this is a scalar equation, it is gauge invariant and can be expressed only in terms of the gauge-invariant quantities that we have defined and which reduce to the perturbation variables in the Newtonian gauge. Explicitly, it reads

$$\begin{aligned} \left( \frac{\partial}{\partial \eta} + n^i \partial_i \right) \frac{\hat{\mathcal{I}}^{(2)}}{4} + \mathcal{H} \hat{\mathcal{I}}^{(2)} + \bar{\mathcal{I}} n^i \partial_i \hat{\Phi}^{(2)} + 2\bar{\mathcal{I}} (\Psi^{(1)} - \Phi^{(1)}) n^i \partial_i \Phi^{(1)} \\ + \frac{1}{2} [\partial_j (\hat{\Phi}^{(1)} + \hat{\Psi}^{(1)}) n^i n^j - \partial^i (\hat{\Phi}^{(1)} + \hat{\Psi}^{(1)})] \frac{\partial \hat{\mathcal{I}}^{(1)}}{\partial n^i} \\ - 2\hat{\mathcal{I}}^{(1)} (\hat{\Psi}^{(1)'} - n^j \partial_j \hat{\Phi}^{(1)}) - \bar{\mathcal{I}} (\hat{\Psi}^{(2)'} + 4\hat{\Psi}^{(1)} \hat{\Psi}^{(1)'}) \\ + \frac{1}{2} (\hat{\Phi}^{(1)} + \hat{\Psi}^{(1)}) n^i \partial_i \hat{\mathcal{I}}^{(1)} = 0. \end{aligned} \quad (95)$$

Up to this stage, we agree with the expressions of [27, 28].

## 6. Fluid approximation

If we want to recover the transformation rule and the gauge-invariant variables for the energy density, the pressure and the velocity of radiation, we need to define a stress–energy tensor from the distribution function of radiation. We already know from special relativity how to define such a tensor. We generalize it by using the local Minkowskian frame

$$\begin{aligned} T^{ab}(x^\mu) &= \int d\pi^0 d^3 \pi^i \delta(\pi^c \pi_c) f(x^\mu, \pi^d) \pi^a \pi^b \\ &= \int (\pi^0)^3 f(x^\mu, \pi^d) n^a n^b d\pi^0 d^2 n^i, \end{aligned} \quad (96)$$

where  $n^a \equiv n^i = \pi^i / \pi^0$ , if  $a = 1, 2, 3$  and  $n^a = 1$  if  $a = 0$ . In order to evaluate the stress–energy tensor, we have performed one of the integrals which removes the Dirac contribution  $\delta(\pi^a \pi_a)$ :

$$\int \delta(\pi^a \pi_a) G(x, \pi^a) d\pi^0 d^3 \pi^i = \int G(x, \pi^0, n^i) \pi^0 d\pi^0 d^2 n^i. \quad (97)$$

Several useful relations for handling integrals of the background distribution function are reported in appendix D. If we are dealing with several species, we can still define a stress–energy tensor for each component, as long as we are dealing with weakly interacting gases. This is the standard kinetic approach in which the interaction between two species is encoded in the collision term of the Boltzmann equation [2].

We define  $\rho$ ,  $P$ , the velocity  $U^a$  and the anisotropic stress  $\Pi^{ab}$ ,

$$T^{ab} = \rho U^a U^b + P \perp^{ab} + \Pi^{ab}, \quad (98)$$

with  $\perp^{ab} \equiv \eta^{ab} + U^a U^b$ , and the properties  $U^a U_a = -1$ ,  $\Pi^{ab} \perp_{ab} = 0$ ,  $U_a \Pi^{ab} = 0$ . However, fluid quantities are usually expressed using the canonical basis associated with coordinates  $\partial_\mu$  and not the tetrad field. We thus define  $u^\mu = U^a e_a^\mu$  as the coordinates of the velocity in this canonical basis, and we decompose it as in equation (5). Similarly, we define the anisotropic stress expressed in the canonical basis by  $\pi^{\mu\nu} = e_a^\mu e_b^\nu \Pi^{ab}$ . Some confusion can arise from the fact that physicists often design a vector by its coordinates. With this symbolic convention,  $U^a$  and  $u^\mu$  are mathematically the same vector, but expressed in a different basis since  $U^a e_a = u^\mu \partial_\mu$ . The relations between  $U^a$  and  $u^\mu$  up to second order are

$$\bar{U}^0 = a\bar{u}^0 = 1 \quad \bar{U}^i = a\bar{u}^i = 0, \quad (99)$$

and

$$\begin{aligned}\delta_X^{(1)} U^0 &= 0 \\ \delta_X^{(1)} U^i &= \partial^i (v^{(1)} + B^{(1)}),\end{aligned}\tag{100}$$

$$\begin{aligned}\delta_X^{(2)} U^0 &= \partial_i (v + B) \partial^i (v + B) \\ \delta_X^{(2)} U^i &= \partial^i (v^{(2)} + B^{(2)}) - 2\Phi \partial^i B + 2\Psi \partial^i (B - v) + 2\partial^j (v - B) (\partial^i \partial_j E + E_j^i).\end{aligned}\tag{101}$$

Similarly, the relations between the spatial components of  $\pi^{\mu\nu}$  and  $\Pi^{ab}$  are

$$\begin{aligned}\delta_X^{(1)} \pi^{ij} &= \frac{1}{a^2} \delta_X^{(1)} \Pi^{ij} \\ \delta_X^{(2)} \pi^{ij} &= \frac{1}{a^2} [\delta_X^{(2)} \Pi^{ij} + 2\delta_X^{(1)} \Pi^{ik} (\Psi^{(1)} \delta_k^j - \partial_k \partial^j E^{(1)} - E_k^{(1)j}) \\ &\quad + 2\delta_X^{(1)} \Pi^{jk} (\Psi^{(1)} \delta_k^i - \partial_k \partial^i E^{(1)} - E_k^{(1)i})].\end{aligned}\tag{102}$$

The fluid quantities can be extracted from equation (98) as follows:

$$\rho = T^{ab} U_a U_b,\tag{103}$$

$$3P = T^{ab} \perp_{ab},\tag{104}$$

$$\Pi_{ab} = T^{cd} (\perp_{ca} \perp_{db} - \frac{1}{3} \perp_{cd} \perp_{ab}),\tag{105}$$

$$(\rho + P) U^0 U^i = T^{0i}.\tag{106}$$

It is easy to see that the factor  $\delta(\pi_a \pi^a)$  in the integral of definition (96) of the stress–energy tensor implies that  $P = \rho/3$ .

The system of definitions (103)–(106) determines the fluid quantities. Indeed, these quantities can now be calculated iteratively at any order once equation (96) is pulled back to the background spacetime. Since  $\bar{U}^0 = 1$  and  $\bar{U}^i = 0$ ,  $\bar{\rho} \bar{P}$  and  $\bar{\Pi}^{ab}$  are given by

$$\bar{\rho} = 3\bar{P} = \bar{T}^{00} \bar{U}_0 \bar{U}_0, \quad \bar{\Pi}^{ab} = 0,\tag{107}$$

as expected from the background symmetries for a fluid of radiation. Then, since  $U^0 = \sqrt{U^i U_i + 1}$ , and using equation (106), we can determine the first-order expression of the velocity

$$\delta_X^{(1)} U^0 = 0 \quad \delta_X^{(1)} U^i = \frac{3}{4\bar{\rho}} \delta_X^{(1)} T^{0i}.\tag{108}$$

Repeating this procedure, we obtain from equations (103)–(106)

$$\begin{aligned}\delta_X^{(1)} \rho &= 3\delta_X^{(1)} P = \delta_X^{(1)} T^{00} \bar{U}_0 \bar{U}_0 \\ \delta_X^{(1)} \Pi^{ij} &= \delta_X^{(1)} T^{ij} - \frac{\delta^{ij}}{3} \delta_X^{(1)} T_k^k,\end{aligned}\tag{109}$$

and the condition  $U_a \Pi^{ab} = 0$  implies

$$\begin{aligned}\delta_X^{(1)} \Pi^{i0} &= \delta_X^{(1)} \Pi^{00} = 0 \\ \delta_X^{(2)} \Pi^{00} &= 0 \\ \delta_X^{(2)} \Pi^{0i} &= 2\delta_X^{(1)} \Pi^{ij} \delta_X^{(1)} U_j.\end{aligned}\tag{110}$$

Again, using equation (106), we determine the second-order perturbation of the velocity

$$\delta_X^{(2)} U^0 = \delta_X^{(1)} U^i \delta_X^{(1)} U_i \quad (111)$$

$$\delta_X^{(2)} U^i = \frac{3}{4\bar{\rho}} (\delta_X^{(2)} T^{0i} - \delta_X^{(2)} \Pi^{0j}) - 2 \frac{\delta_X^{(1)} \rho}{\bar{\rho}} \delta_X^{(1)} U^i. \quad (112)$$

Iterating, we obtain from equations (103)–(106)

$$\begin{aligned} \delta_X^{(2)} \rho &= 3\delta_X^{(2)} P = \delta_X^{(2)} T^{00} \bar{U}_0 \bar{U}_0 + 2\bar{T}^{00} \bar{U}_0 \delta_X^{(2)} U_0 \\ \delta_X^{(2)} \Pi^{ij} &= \delta_X^{(2)} T^{ij} - \frac{\delta^{ij}}{3} \delta_X^{(2)} T_k^k - \frac{8}{3} \bar{\rho} \left( \delta_X^{(1)} U^i \delta_X^{(1)} U^j - \frac{\delta^{ij}}{3} \delta_X^{(1)} U^k \delta_X^{(1)} U_k \right). \end{aligned} \quad (113)$$

This shows that, by iterating this procedure, the fluid quantities can be determined up to order  $n$  if  $f$ , that is  $T^{ab}$ , is known up to order  $n$ . This means that, by knowing the transformation rule of  $f$  under a gauge transformation, we can deduce the transformation rules of the fluid quantities built out of it ( $\rho$ ,  $P$ ,  $U^a$ ,  $\Pi^{ab}$ ). Eventually, we are interested in their expressions in the canonical basis in order to compare with the results of section 2, and we need to use equations (99)–(101) and (102).

### 6.1. First-order fluid quantities transformation

At first order, from relations (109) and (100), and the transformation rule for  $f$ , equation (80), we deduce after some algebra that  $\delta_X^{(1)} \rho$  transforms as in equation (25). Similarly, from equation (108), relation (100) and the transformation rule for  $f$ , equation (80), we deduce that  $v^{(1)}$  transforms as in equation (25). By the same method, we easily recover that  $\delta^{(1)} \pi^{ij}$  is gauge invariant.

### 6.2. First-order fluid equations

In order to recover the gauge-invariant conservation equation and the Euler equation of the fluid approximation at first order, we define the first-order gauge-invariant stress–energy tensor by

$$\hat{T}^{ab(1)}(x^\mu) \equiv \int (\pi^0)^3 \hat{f}^{(1)}(x^\mu, \pi^c) n^a n^b d\pi^0 d^2\Omega = \int \hat{\mathcal{T}}^{(1)} n^a n^b \frac{d^2\Omega}{4\pi}, \quad (114)$$

and its associated first-order gauge-invariant fluid quantities,  $\hat{\rho}^{(1)}$ ,  $\hat{P}^{(1)}$ ,  $\hat{v}^{(1)}$  and  $\hat{\pi}^{ij(1)}$ , built from the same types of relation as in the set of equations (103)–(106) and expressed in the canonical basis with equations (100) and (102). Because of the comparison performed in the previous section, these quantities match those defined in equation (33), and this justifies the fact that we use the same notation. We need the useful relations between the first moments and the fluid quantities

$$\hat{\mathcal{F}}^{(1)} = \int \hat{\mathcal{T}}^{(1)} \frac{d\Omega}{4\pi} = \delta^{(1)} \hat{\rho}, \quad (115)$$

$$\hat{\mathcal{F}}^{i(1)} = \int \hat{\mathcal{T}}^{(1)} n^i \frac{d\Omega}{4\pi} = \frac{4}{3} \bar{\rho} \partial^i \hat{v}^{(1)}, \quad (116)$$

$$\hat{\mathcal{F}}^{ij(1)} = \int \hat{\mathcal{T}}^{(1)} \left( n^i n^j - \frac{\delta^{ij}}{3} \right) \frac{d\Omega}{4\pi} = \hat{\Pi}^{ij(1)}. \quad (117)$$

Performing  $\int d\Omega$  on the brightness evolution equation (88), we recover the first-order conservation equation. However, performing  $\int n^i d\Omega$ , we recover the first-order Euler equation

as expressed in appendix E, *only if* we neglect the first-order anisotropic pressure. This comes from the fact that the statistical description of radiation leads to an infinite hierarchy of equations coupling moments of orders  $p - 1$ ,  $p$  and  $p + 1$  [39], whereas the fluid description keeps only the equations involving the monopole and the dipole.

### 6.3. Second-order fluid quantities transformation

In order to establish the second-order comparison with the fluid description, we need to know how to perform an integral involving  $\delta_X^{(1)} f$ , for instance on  $2 \frac{\partial \delta_X^{(1)} f}{\partial \pi^0} \pi^0 n^j \partial_j T + 2 \frac{\partial \delta_X^{(1)} f}{\partial \pi^i} \pi^0 \partial^i T$ . We will thus make use of the multipolar expansion

$$\delta_X^{(1)} f = \bar{f} \frac{\delta_X^{(1)} \rho}{\bar{\rho}} + 4 \bar{f} \partial_i (v^{(1)} + B^{(1)}) n^i + \frac{15 \bar{f}}{2 \bar{\rho}} \delta_X^{(1)} \Pi_{ij} n^i n^j + \dots, \quad (118)$$

from which it can be checked that we recover the correct fluid quantities when used to compute  $\delta_X^{(1)} T^{ab}$  in equation (96).

Using the same method as for the first order, with relations (113) and (101), and the transformation rule for the second-order distribution function, equation (90), we deduce that  $\delta^{(2)} \rho$  transforms as in equation (37). Additionally, from relations (111), (112), (101) and (110), we deduce that  $v^{(2)}$  transforms as in equation (37).

We also note that from definition (113), relations (111), (112) and the transformation rule for  $f$ , equation (90), we deduce that  $\delta^{(2)} \Pi^{ij}$  transforms according to

$$\delta^{(2)} \Pi^{ij} \rightarrow \delta^{(2)} \Pi^{ij} + 2T (\delta^{(1)} \Pi^{ij})' + 2 \partial^k L \partial_k (\delta^{(1)} \Pi^{ij}). \quad (119)$$

When expressed in the canonical basis ( $\pi^{\mu\nu} \equiv e_a^\mu e_b^\nu \Pi^{ab}$ ), we recover exactly the transformation rule of the anisotropic stress given in equation (37).

This is one of the major results of this paper. We recover the perfect fluid transformation rules for the energy density, the pressure, the velocity and the anisotropic stress given in [40] up to second order, when starting from the statistical description.

### 6.4. Second-order fluid equations

In order to recover the gauge-invariant conservation equation and the Euler equation of the fluid approximation at the second order, we follow the same procedure as for the first-order case. We thus define the second-order gauge-invariant stress-energy tensor by

$$\hat{T}^{ab(2)}(x^\mu) \equiv \int (\pi^0)^3 \hat{f}^{(2)}(x^\mu, \pi) n^a n^b d\pi^0 d^2\Omega = \int \hat{\mathcal{T}}^{(2)} n^a n^b \frac{d^2\Omega}{4\pi}, \quad (120)$$

and its associated second-order gauge-invariant fluid quantities,  $\hat{\rho}^{(2)} \hat{P}^{(2)} \hat{v}^{(2)}$  and  $\hat{\pi}^{ij(2)}$ , built from the same types of relations as in the set of equations (103)–(106) and expressed in the canonical basis with equations (101) and (102). Because of the comparison performed in the previous section, these quantities match those defined in equation (42), thus justifying the fact that we use the same notation.

In order to recover the conservation and Euler equations of the fluid approximation we perform the integral  $\int \frac{d\Omega}{4\pi}$  and  $\int \frac{d\Omega}{4\pi} n^i$  on this equation. However, at the second order this has to be done with care since the link between the second-order gauge-invariant brightness and the second-order fluid quantities is given by

$$\hat{\mathcal{F}}^{(2)} = \int \hat{\mathcal{T}}^{(2)} \frac{d\Omega}{4\pi} = \delta^{(2)} \hat{\rho} + \frac{8}{3} \bar{\rho} \partial_i \hat{v}^{(1)} \partial^i \hat{v}^{(1)}, \quad (121)$$

$$\hat{\mathcal{F}}^{i(2)} = \int \hat{\mathcal{L}}^{(2)} n^i \frac{d\Omega}{4\pi} = \frac{4}{3} \bar{\rho} (\partial^i \hat{v}^{(2)} - 2\hat{\Psi}^{(1)} \partial^i \hat{v}^{(1)}) + \frac{8}{3} \delta^{(1)} \hat{\rho} \partial^i \hat{v}^{(1)}, \quad (122)$$

$$\begin{aligned} \hat{\mathcal{F}}^{ij(2)} &= \int \hat{\mathcal{L}}^{(2)} \left( n^i n^j - \frac{\delta^{ij}}{3} \right) \frac{d\Omega}{4\pi} \\ &= \hat{\Pi}^{ij(2)} + \frac{8}{3} \bar{\rho} \left[ \partial_i \hat{v}^{(1)} \partial_j \hat{v}^{(1)} - \frac{\delta_{ij}}{3} (\partial_k \hat{v}^{(1)} \partial^k \hat{v}^{(1)}) \right]. \end{aligned} \quad (123)$$

This clearly differs from expressions (5.10) and (6.33) of [27] where the term quadratic in  $v$  in  $\hat{\mathcal{F}}^{(2)}$ , the term quadratic in  $\Psi$  and  $v$  in  $\hat{\mathcal{F}}^{i(2)}$  are not there. The difference in the energy density perturbation as extracted from  $\hat{\mathcal{F}}^{(2)}$  comes from the fact that the fractional energy density  $\Delta^{(2)}$  for the radiation is defined as seen by the observer of velocity  $e_\mu^0 \sim (d\eta)_\mu$ , whereas we define it in the fluid frame. The fractional energy density that they define is related to our quantities by  $\bar{\rho} \Delta^{(2)} = \delta_{\text{NG}}^{(2)} T^{00} \bar{U}_0 \bar{U}_0$ . The difference in the expressions for the fractional energy density can be traced using equations (113) with equation (111). However, this is only a matter of definition and it is consistent with equation (7.2) of [27]. Implicitly, the authors of [27] do also use a tetrad basis in their section 3 in order to identify coordinates of the tangent space between the background and the perturbed spacetime, in the same way as explained below equation (65). Their  $p$  is equal to our  $\pi^0$ , and the unit vectors  $n^i$  match when restricting to the Newtonian gauge. Equations (3.6) and (3.7) of [27] are equivalent to equation (61) when expressed in the Newtonian gauge with the use of equations (52), (B.1) and (B.3). As for the difference in the velocity perturbation as defined from  $\hat{\mathcal{F}}^{i(2)}$ , it comes from the fact that their definition for  $v_\gamma^{i(2)}$  has to be interpreted in the tetrad basis, and therefore it matches  $\delta_{\text{NG}}^{(2)} U^i$ . However, the difference between the tetrad basis and the canonical basis is not computed as in equation (112), and it explains the discrepancy. This can also be checked on the second-order extraction of equation (7.3) in [27]. Indeed, there is an extra term quadratic in  $\Psi$  and  $v^i$  when compared to equation (2.15) of [41], as a trace of the difference between our perturbed velocity, which matches the definition in the canonical basis usually given by equations (5) and (4), and their perturbed velocity. However, the equations involving  $v_\gamma^{i(2)}$  in [27, 28] such as equation (4.6) are consistent with this difference, though the physical interpretation  $v_\gamma^{i(2)}$  as being the perturbed velocity of photons in the canonical basis is not correct.

The computation of a term like  $\frac{\partial \hat{\mathcal{F}}^{(1)}}{\partial n^i}$  is easily performed using the multipolar expansion

$$\hat{\mathcal{F}}^{(1)} = \bar{f} \frac{\delta^{(1)} \hat{\rho}}{\bar{\rho}} + 4 \bar{f} \partial_i \hat{v}^{(1)} n^i + \frac{15 \bar{f}}{2 \bar{\rho}} \hat{\Pi}_{ij}^{(1)} n^i n^j + \dots \quad (124)$$

Applying this method, we recover the second-order conservation equation detailed in appendix E. As for the Euler equation, we recover it at second order only if we neglect the anisotropic stress up to second order (beware that the anisotropic stress is different from the second moment of the distribution as can be seen in equation (123)) and use the first-order Euler equation.

This is also a major result of this paper. We recover the fluid gauge-invariant equations up to second order, only if we can neglect the anisotropic stress up to second order. It remains to be shown that this is extended up to any order, as we expect.

Let us also stress that in [28], the term  $\frac{\partial \hat{\mathcal{F}}^{(1)}}{\partial n^i}$  is evaluated using  $\frac{\partial \hat{\mathcal{F}}^{(1)}}{\partial n^i} = \frac{\partial \hat{f}}{\partial x^j} \frac{\partial x^j}{\partial n^i}$ , in order to derive equations (4.1) and (4.6). However, this is not correct since  $\hat{f}$  is a function of the independent variables  $\eta, x^i, \pi^0, n^i$ . Even though they are related on a photon geodesic, they are independent in the analytic expression of  $\hat{f}$ . Additionally this method is not fruitful because  $\frac{\partial x^j}{\partial n^i} \neq \delta_i^j (\eta - \eta_i)$ , since  $n^i$  does not parametrize a photon geodesic.

Consequently, the subsequent analytic expressions of this reference solving the conservation and Euler equation are not correct (for both radiation and cold dark matter) though the Boltzmann equation is correct. This can also be seen directly from the fact that these equations do not match fluid approximation equations of appendix E. Once corrected for this mistake and taking into account the differences mentioned before, we can check that the collisionless part of the conservation and Euler equations (4.1) and (4.6) of [27] match our equations.

### 6.5. Validity of the fluid approximation in the literature

In this paper, we have considered so far the fluid approximation as a theoretical framework in which we restrict the description of a species to its energy density and its velocity. The computations involved for the distribution function at second order were rather long, and it was used as a consistency check for the gauge transformation rules and the collisionless Boltzmann equation. Since the fluid approximation is built out of the kinetic theory, it was indeed expected that all the conclusions made in this statistical description could find their fluid approximation counterpart.

It is now necessary to determine under which conditions this can be done, that is when the anisotropic stress can be neglected. This requires us to work on the physics of coupled species, baryons and photons, in the cosmological context. The collision term as well as its physical implications has been studied in [28], and it is very likely that the extraction of its quadrupole in equation (4.18) is not affected by the previous considerations. Indeed, in the tight coupling limit (which requires only the collision term) for a system of photons and electrons highly coupled through the Compton diffusion, the authors of [28] find that the quadrupole satisfies

$$\hat{\mathcal{F}}^{ij(2)} \simeq \frac{8}{3}\bar{\rho} \left[ \partial^i \hat{v}^{(1)} \partial^j \hat{v}^{(1)} - \frac{\delta^{ij}}{3} (\partial_k \hat{v}^{(1)} \partial^k \hat{v}^{(1)}) \right]. \quad (125)$$

This result is necessary to determine in which case the fluid approximation can be used. Comparing it with equation (123), we immediately see that the physical interpretation of this result is that the second-order anisotropic stress of radiation  $\hat{\Pi}^{ij(2)}$  is suppressed in the tight coupling limit. As a consequence, the fluid approximation can be used in the tight coupling limit also at second order in perturbations.

## 7. Conclusion

In this paper, we have performed a general investigation of the gauge invariance of the distribution function. This allows us to recover very easily the standard results at the first order and to extend them at the second order. We derived the fluid approximation at first and second orders. This required us to carefully define the stress–energy tensor in the local Minkowskian frame. At the second order, our results differ from those previously derived in the literature [27, 28]. We have tackled down the origin of the differences and shown that it was lying in an incorrect identification between the tetrad and the canonical basis. Our analysis, restricted to the collisionless case, puts the second-order Boltzmann equation, needed if we intend to study non-Gaussianities in the CMB, on firm ground.

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## Appendix A. Source terms in second-order transformations

The perturbation variables in the decomposition (1) are extracted as follows:

$$\begin{aligned}
\Phi &= -\frac{1}{2a^2} \delta g_{00}, \\
\Psi &= -\frac{1}{4a^2} (\delta^{ij} - \Delta^{-1} \partial^i \partial^j) \delta g_{ij}, \\
B &= \frac{1}{a^2} \Delta^{-1} \partial^i \delta g_{0i}, \\
E &= \frac{1}{4a^2} (\Delta \Delta)^{-1} (3\partial^i \partial^j - \Delta \delta^{ij}) \delta g_{ij}, \\
E_{pq} &= \frac{1}{2a^2} (\delta_p^r - \Delta^{-1} \partial_p \partial^r) (\delta_q^s - \Delta^{-1} \partial_q \partial^s) \left( \delta_r^i \delta_s^j - \frac{1}{3} \delta_{rs} \delta^{ij} \right) \delta g_{ij}.
\end{aligned} \tag{A.1}$$

Using this method we can read the source terms defined in equation (37), which are quadratic in the gauge change variables  $T$ ,  $L$  and the perturbation variables  $\Phi$ ,  $\Psi$ ,  $B$ ,  $E$ ,  $E_{ij}$ , in equation (16):

$$\begin{aligned}
S_\Phi &= T(T'' + 5\mathcal{H}T' + (\mathcal{H}' + 2\mathcal{H}^2)T + 4\mathcal{H}\Phi + 2\Phi') + T'(2T' + 4\Phi) + \partial_i L \partial^i (T' + \mathcal{H}T + 2\Phi) \\
&\quad + \partial_i L' \partial^i (T - 2B - L'),
\end{aligned} \tag{A.2}$$

$$\begin{aligned}
S_\Psi &= -T(\mathcal{H}T' + (\mathcal{H}' + 2\mathcal{H}^2)T - 2\Psi' - 4\mathcal{H}\Psi) - \partial_i (\mathcal{H}T - 2\Psi) \partial^i L - \frac{1}{2} (\delta^{ij} - \Delta^{-1} \partial^i \partial^j) \\
&\quad \times \left[ \partial_j (2B + L' - T) \partial_i T + \partial_i \partial^k L (2\partial_k \partial_j L + 2\partial_k \partial_j E + 2E_{kj} + (2\mathcal{H}T - 2\Psi) \delta_{kj}) \right. \\
&\quad + T \partial_i \partial_j (L' + 2\mathcal{H}L) + T(2E'_{ij} + 2\partial_i \partial_j E' + 4\mathcal{H}E_{ij} + 4\mathcal{H}\partial_i \partial_j E) \\
&\quad \left. + \partial^k L \partial_k (\partial_i \partial_j L + 2E_{ij} + 2\partial_i \partial_j E) \right].
\end{aligned} \tag{A.3}$$

$S_\Psi$  is slightly different from [13, 44] since, in these works, the extraction of metric perturbation variables is not performed according to equation (A.1). However, this mistake does not matter for their study that focused on the long wavelength limit

$$\begin{aligned}
S_B &= \Delta^{-1} \partial^i \{ T' \partial_i (2B + L' - T) + \partial^j L' [2\partial_i \partial_j L + 2(\mathcal{H}T - 2\Psi) \delta_{ij} + 4(E_{ij} + \partial_i \partial_j E)] \\
&\quad + \partial^j \partial_i L \partial_j (2B + L' - T) + \partial^j L \partial_j \partial_i (2B + L' - T) \\
&\quad + \partial_i T (-4\Phi - 2T' - 2\mathcal{H}T) + T \partial_i (2B + L' - T') \\
&\quad + 2\mathcal{H}T \partial_i (2B + L' - T) \},
\end{aligned} \tag{A.4}$$

$$\begin{aligned}
S_E &= (\Delta \Delta)^{-1} \left( \frac{3}{2} \partial^i \partial^j - \frac{1}{2} \Delta \delta^{ij} \right) \{ \partial_j (2B + L' - T) \partial_i T \\
&\quad + \partial_i \partial^k L [2\partial_k \partial_j L + 2\partial_k \partial_j E + 2E_{kj} + (2\mathcal{H}T - 2\Psi) \delta_{kj}] \\
&\quad + T \partial_i \partial_j (L' + 2\mathcal{H}L) + T(2E'_{ij} + 2\partial_i \partial_j E' + 4\mathcal{H}E_{ij} + 4\mathcal{H}\partial_i \partial_j E) \\
&\quad + \partial^k L \partial_k (\partial_i \partial_j L + 2E_{ij} + 2\partial_i \partial_j E) \},
\end{aligned} \tag{A.5}$$

$$\begin{aligned}
S_{E pq} = & (\delta_p^r - \Delta^{-1} \partial_p \partial^r) (\delta_q^s - \Delta^{-1} \partial_q \partial^s) \left( \delta_r^i \delta_s^j - \frac{\delta_{rs}}{3} \delta^{ij} \right) \\
& \times \{ \partial_i \partial^k L [2 \partial_k \partial_j L + 2 \partial_k \partial_j E + 2 E_{kj} + (2 \mathcal{H} T - 2 \Psi) \delta_{kj}] \\
& + T \partial_i \partial_j (L' + 2 \mathcal{H} L) + \partial_j (2 B + L' - T) \partial_i T \\
& + T (2 E'_{ij} + 2 \partial_i \partial_j E' + 4 \mathcal{H} E_{ij} + 4 \mathcal{H} \partial_i \partial_j E) \\
& + \partial^k L \partial_k (\partial_i \partial_j L + 2 E_{ij} + 2 \partial_i \partial_j E) \}, \tag{A.6}
\end{aligned}$$

$$S_\rho = T(\bar{\rho}'' T + \bar{\rho}' T' + 2 \delta \rho') + \partial^i L \partial_i (2 \delta \rho + \bar{\rho}' T), \tag{A.7}$$

$$S_P = T(\bar{P}'' T + \bar{P}' T' + 2 \delta P') + \partial^i L \partial_i (2 \delta P + \bar{P}' T), \tag{A.8}$$

$$\begin{aligned}
S_v = & \Delta^{-1} \partial_i [\mathcal{H} T \partial^i (L' - 2v) + T \partial^i (2v' - L'')] \\
& + L^j \partial_j \partial^i (2v - L') + \partial^i L' (\mathcal{H} T + T' + 2\Phi) + \partial^j (L' - 2v) \partial_j \partial^i L. \tag{A.9}
\end{aligned}$$

## Appendix B. Transformation rules of the tetrad fields

$R_{ab}$  and  $S_{ab}$  are defined in equation (52). The perturbation variables of the metric are defined in equation (1).

### B.1. First order

$$\begin{aligned}
R_{00,X}^{(1)} &= -S_{00,X}^{(1)} = \Phi^{(1)} \\
R_{0a_i,X}^{(1)} &= -S_{0a_i,X}^{(1)} = -\partial_{a_i} B^{(1)} \\
R_{a_i 0,X}^{(1)} &= -S_{a_i 0,X}^{(1)} = 0 \\
R_{a_i a_k,X}^{(1)} &= -S_{a_i a_k,X}^{(1)} = \Psi^{(1)} \delta_{a_i a_k} - \partial_{a_k} \partial_{a_i} E^{(1)} - E_{a_i a_k}^{(1)}. \tag{B.1}
\end{aligned}$$

We can read directly from these expressions the transformation rules for the tetrad

$$\begin{aligned}
\delta_Y^{(1)} e_0^\mu &= \mathcal{T}(\delta_X^{(1)} e_0^\mu) = -\mathcal{T}(\Phi^{(1)}) \bar{e}_0^\mu - \bar{e}_{a_i}^\mu \partial^{a_i} \mathcal{T}(B^{(1)}) \\
\delta_Y^{(1)} e_{a_i}^\mu &= \mathcal{T}(\delta_X^{(1)} e_{a_i}^\mu) = \mathcal{T}(\Psi^{(1)}) \bar{e}_{a_i}^\mu - \bar{e}_{a_k}^\mu \partial^{a_k} \partial_{a_i} \mathcal{T}(E^{(1)}). \tag{B.2}
\end{aligned}$$

### B.2. Second order

$$\begin{aligned}
R_{00,X}^{(2)} &= \Phi^{(2)} - 3\Phi^2 + \partial_i B \partial^i B \\
R_{0a_i,X}^{(2)} &= -\partial_{a_i} B^{(2)} + (2\Phi - 4\Psi) \partial_{a_i} B + 4\partial^{a_j} B (\partial_{a_i} \partial_{a_j} E + E_{a_i a_j}) \\
R_{a_i 0,X}^{(2)} &= -S_{a_i 0,X}^{(2)} = 0 \\
R_{a_i a_k,X}^{(2)} &= -S_{a_i a_k,X}^{(2)} \\
&= \Psi^{(2)} \delta_{a_i a_k} - (\partial_{a_k} \partial_{a_i} E^{(2)} + E_{a_k a_i}^{(2)}) + \Psi^2 \delta_{a_i a_k} \\
&\quad + 3(\partial_{a_i} \partial^{a_i} E + E_{a_i}^{a_i}) (\partial_{a_i} \partial_{a_k} E + E_{a_i a_k}) \\
&\quad - 6\Psi (\partial_{a_i} \partial_{a_k} E + E_{a_i a_k}) \\
-S_{00,X}^{(2)} &= \Phi^{(2)} - \Phi^2 + \partial_i B \partial^i B \\
-S_{0a_i,X}^{(2)} &= -\partial_{a_i} B^{(2)} - 2\Psi \partial_{a_i} B + 2\partial^{a_j} B (\partial_{a_i} \partial_{a_j} E + E_{a_i a_j}). \tag{B.3}
\end{aligned}$$

In these formulae, we have omitted the first-order superscript as there is no possible confusion. In the following, we will also omit the first-order superscript. The transformation rules for the

tetrads can be read, as we did for the first-order case:

$$\begin{aligned}
\mathcal{T}(\delta_X^{(2)} e_0^\mu) &= -[\mathcal{T}(\Phi^{(2)}) - 3\mathcal{T}(\Phi)^2 + \partial_i \mathcal{T}(B) \partial^i \mathcal{T}(B)] \bar{e}_0^\mu \\
&\quad + \{-\partial^{a_i} \mathcal{T}(B^{(2)}) + [2\mathcal{T}(\Phi) - 4\mathcal{T}(\Psi)] \partial^{a_i} \mathcal{T}(B) \\
&\quad + 4\partial^{a_j} \mathcal{T}(B) [\partial^{a_i} \partial_{a_j} \mathcal{T}(E) + E_{a_j}^{a_i}]\} \bar{e}_{a_i}^\mu \\
\mathcal{T}(\delta_X^{(2)} e_{a_i}^\mu) &= [\mathcal{T}(\Psi^{(2)}) + \mathcal{T}(\Psi)^2] \bar{e}_{a_i}^\mu \\
&\quad + \{-\partial^{a_k} \partial_{a_i} \mathcal{T}(E^{(2)}) + 3[\partial_{a_i} \partial^{a_j} \mathcal{T}(E) + E_{a_i}^{a_j}][\partial^{a_k} \partial_{a_j} \mathcal{T}(E) + E_{a_j}^{a_k}] \\
&\quad - 6\mathcal{T}(\Psi) [\partial^{a_k} \partial_{a_i} \mathcal{T}(E) + E_{a_i}^{a_k}]\} \bar{e}_{a_k}^\mu.
\end{aligned} \tag{B.4}$$

### Appendix C. Transformation of $\delta^{(2)} f$

$$\begin{aligned}
\mathcal{T}(\delta_X^{(2)} f) &= \left\{ (\mathcal{L}_{T\xi_2} + \mathcal{L}_{T\xi_1}^2) [\bar{f}(x^\nu, ap^\mu)] + 2\mathcal{L}_{T\xi_1} [\delta_X^{(1)} f(x^\nu, ap^\mu)] \right. \\
&\quad + [\mathcal{T}(R_{a,X}^{(2)c}) + S_{a,X}^{(2)c} + 2S_{a,X}^{(1)d} \mathcal{T}(R_{d,X}^{(1)c})] \pi^a \frac{\partial \bar{f}}{\partial \pi^c} \\
&\quad + [\mathcal{T}(R_{a,X}^{(1)b}) \mathcal{T}(R_{c,X}^{(1)d}) + S_{a,X}^{(1)b} S_{c,X}^{(1)d} + 2S_{a,X}^{(1)b} \mathcal{T}(R_{c,X}^{(1)d})] \pi^a \pi^c \frac{\partial^2 \bar{f}}{\partial \pi^b \partial \pi^d} \\
&\quad + 2\mathcal{T}(R_{a,X}^{(1)b}) \pi^a \frac{\partial}{\partial \pi^b} \mathcal{L}_{T\xi_1} [\bar{f}(x^\nu, ap^\mu)] + 2\mathcal{L}_{T\xi_1} \left[ S_{a,X}^{(1)b} \pi^a \frac{\partial}{\partial \pi^b} \bar{f}(x^\nu, ap^\mu) \right] \\
&\quad \left. + 2[\mathcal{T}(R_{a,X}^{(1)b}) + S_{a,X}^{(1)b}] \pi^a \frac{\partial}{\partial \pi^b} \delta_X^{(1)} f(x^\nu, ap^\mu) \right\}.
\end{aligned} \tag{C.1}$$

These individual terms are explicitly given by

$$\begin{aligned}
&[\mathcal{T}(R_{0,X}^{(2)0}) + S_{0,X}^{(2)0} + 2S_{0,X}^{(1)0} \mathcal{T}(R_{0,X}^{(1)0})] \pi^0 \frac{\partial \bar{f}}{\partial \pi^0} \\
&= [-(T^{(2)'}) + \mathcal{H}T^{(2)} + S_\Phi(T, L)] + 4\Phi(T' + \mathcal{H}T) + 3(T' + \mathcal{H}T)^2 \\
&\quad - 2\partial_i B \partial^i (-T + L') - \partial_i (-T + L') \partial^i (-T + L') \pi^0 \frac{\partial \bar{f}}{\partial \pi^0},
\end{aligned} \tag{C.2}$$

$$[\mathcal{T}(R_{0,X}^{(1)0}) \mathcal{T}(R_{0,X}^{(1)0}) + S_{0,X}^{(1)0} S_{0,X}^{(1)0} + 2S_{0,X}^{(1)0} \mathcal{T}(R_{0,X}^{(1)0})] \pi^0 \pi^0 \frac{\partial^2 \bar{f}}{\partial \pi^0 \partial \pi^0} = \frac{\partial^2 \bar{f}}{\partial (\pi^0)^2} (\pi^0)^2 (T' + \mathcal{H}T)^2, \tag{C.3}$$

$$\begin{aligned}
&2[\mathcal{T}(R_{a,X}^{(1)b}) + S_{a,X}^{(1)b}] \pi^a \frac{\partial}{\partial \pi^b} \delta_X^{(1)} f(x^\nu, ap^\mu) \\
&= -2 \frac{\partial \delta_X^{(1)} f}{\partial \pi^0} \pi^0 (T' + \mathcal{H}T) - 2 \frac{\partial \delta_X^{(1)} f}{\partial \pi^i} \pi^0 (-\partial^i T + \partial^i L') \\
&\quad - 2 \frac{\partial \delta_X^{(1)} f}{\partial \pi^i} (\pi^j \partial^i \partial_j L + \mathcal{H} \pi^i T),
\end{aligned} \tag{C.4}$$

$$\begin{aligned}
&2\mathcal{T}(R_{a,X}^{(1)b}) \pi^a \frac{\partial}{\partial \pi^b} \mathcal{L}_{T\xi_1} [\bar{f}(x^\nu, ap^\mu)] \\
&= -2 \left[ \frac{\partial^2 \bar{f}}{\partial (\pi^0)^2} (\pi^0)^2 + \frac{\partial \bar{f}}{\partial \pi^0} \pi^0 \right] (T' + \mathcal{H}T) (\Phi + T' + \mathcal{H}T)
\end{aligned}$$

$$\begin{aligned}
& -2 \frac{\partial^2 \bar{f}}{\partial (\pi^0)^2} (\pi^0)^2 (n^i \partial_i T) (\Phi + T' + \mathcal{H}T) - 2T \frac{\partial^2 \bar{f}}{\partial \eta \partial \pi^0} (\Phi + T' + \mathcal{H}T) \\
& - 2 \frac{\partial \bar{f}}{\partial \pi^0} \pi^0 (B^i - \partial^i T + \partial^i L') \partial_i T \\
& - 2 \frac{\partial \bar{f}}{\partial \pi^0} \pi^0 [n^j \partial^i \partial_j (E + L) + n^j E_j^i + n^i (-\Psi + \mathcal{H}T)] \partial_i T, \tag{C.5}
\end{aligned}$$

$$\begin{aligned}
2\mathcal{L}_{T\xi_1} [S_{c,X}^{(1)b} a\pi^c \frac{\partial}{\partial \pi^b} \bar{f}(x^\nu, a\pi^\mu)] &= 2 \frac{\partial^2 \bar{f}}{\partial \eta \partial \pi^0} \pi^0 \Phi T + 2 \frac{\partial \bar{f}}{\partial \pi^0} \pi^0 (\Phi' T + \partial_i \Phi \partial^i L) \\
&+ 2 \left[ \frac{\partial^2 \bar{f}}{\partial (\pi^0)^2} (\pi^0)^2 + \frac{\partial \bar{f}}{\partial \pi^0} \pi^0 \right] \Phi (T' + \mathcal{H}T + n^i \partial_i T), \tag{C.6}
\end{aligned}$$

$$\begin{aligned}
(\mathcal{L}_{T\xi_2} + \mathcal{L}_{T\xi_1}^2) [\bar{f}(x^\nu, a\pi^\mu)] &= T^{(2)} \frac{\partial \bar{f}}{\partial \eta} + \frac{\partial \bar{f}}{\partial \pi^0} \pi^0 (T^{(2)'} + \mathcal{H}T^{(2)} + n^i \partial_i T^{(2)}) \\
&+ \frac{\partial^2 \bar{f}}{\partial \eta^2} T^2 + \frac{\partial \bar{f}}{\partial \eta} (TT' + \partial_i T \partial^i L) + \frac{\partial^2 \bar{f}}{\partial \pi^0 \partial \eta} \pi^0 2T (T' + \mathcal{H}T + n^i \partial_i T) \\
&+ \frac{\partial^2 \bar{f}}{\partial (\pi^0)^2} (\pi^0)^2 [2n^i \partial_i T (\mathcal{H}T + T') + (n^i \partial_i T) (n^j \partial_j T) + \mathcal{H}^2 T^2 + 2\mathcal{H}TT' + (T')^2] \\
&+ \frac{\partial \bar{f}}{\partial \pi^0} \pi^0 [TT'' + \mathcal{H}'T^2 + 3\mathcal{H}TT' + Tn^i \partial_i T' + T'n^i \partial_i T + \mathcal{H}^2 T^2 + 2\mathcal{H}Tn^i \partial_i T \\
&+ \partial_j T' \partial^j L + \partial_j T \partial^j L' + \mathcal{H} \partial_j T \partial^j L + \partial^j L n^i \partial_i \partial_j T + \partial^j T n^i \partial_i \partial_j L + (T')^2], \tag{C.7}
\end{aligned}$$

$$\begin{aligned}
2\mathcal{L}_{T\xi_1} [\delta_X^{(1)} f(x^\nu, a\pi^\mu)] &= 2(\pi^0 T' + \pi^j \partial_j T) \frac{\partial \delta_X^{(1)} f}{\partial \pi^0} + 2(\pi^0 \partial^i L' + \pi^j \partial^i \partial_j L) \frac{\partial \delta_X^{(1)} f}{\partial \pi^i} \\
&+ 2\partial^i L \frac{\partial \delta_X^{(1)} f}{\partial x^i} + 2T \left( \frac{\partial \delta_X^{(1)} f}{\partial \eta} + \frac{\partial \delta_X^{(1)} f}{\partial \pi^0} \mathcal{H}\pi^0 + \frac{\partial \delta_X^{(1)} f}{\partial \pi^i} \mathcal{H}\pi^i \right). \tag{C.8}
\end{aligned}$$

In the above formulae, we have omitted to write the fact that the derivatives with respect to  $\eta$  or  $x^i$  are taken at fixed  $\pi^a$ .

#### Appendix D. Integral relations necessary to derive the fluid limit

The integrations on angular directions can be handled with the general formulae (see [43])

$$\int n^{i_1} \dots n^{i_k} \frac{d^2 \Omega}{4\pi} = 0 \quad \text{if } k = 2p + 1 \tag{D.1}$$

$$\int n^{i_1} \dots n^{i_k} \frac{d^2 \Omega}{4\pi} = \frac{1}{k+1} (\delta^{(i_1 i_2} \dots \delta^{i_{(k-1) i_k})}) \quad \text{if } k = 2p. \tag{D.2}$$

By successive integration by parts, we also obtain the following useful results:

$$\begin{aligned}
& \int \bar{f}(x^\mu, \pi^0) (\pi^0)^3 d\pi^0 d^2 \Omega = \bar{\rho}(x^\mu), \\
& \int \frac{\partial \bar{f}(x^\mu, \pi^0)}{\partial \pi^0} (\pi^0)^4 d\pi^0 d^2 \Omega = -4\bar{\rho}(x^\mu), \\
& \int \frac{\partial^2 \bar{f}(x^\mu, \pi^0)}{\partial^2 \pi^0} (\pi^0)^5 d\pi^0 d^2 \Omega = 20\bar{\rho}(x^\mu). \tag{D.3}
\end{aligned}$$

## Appendix E. The fluid limit for radiation

As explained in section 2.1, second-order quantities involve either purely second-order perturbation variables or terms quadratic in first-order perturbation variables. As long as the order of the quantity is known, we can omit the order superscript in order to simplify notations.

### E.1. Geometric quantities

In the Newtonian gauge, ignoring vector perturbations for simplicity, the non-vanishing Christoffel symbols associated with the metric (1) are for the background

$${}^{(0)}\Gamma_{00}^0 = \mathcal{H}, \quad {}^{(0)}\Gamma_{jk}^0 = \mathcal{H}\delta_{jk}, \quad {}^{(0)}\Gamma_{0j}^i = \mathcal{H}\delta_j^i. \quad (\text{E.1})$$

At the first order, we get

$${}^{(1)}\Gamma_{00}^0 = \Phi', \quad {}^{(1)}\Gamma_{0j}^0 = \partial_j \Phi, \quad {}^{(1)}\Gamma_{00}^i = \partial^i \Phi, \quad (\text{E.2})$$

$${}^{(1)}\Gamma_{jk}^0 = 2\mathcal{H}E_{jk} + E'_{jk} - (2\mathcal{H}\Phi + \Psi' + 2\mathcal{H}\Psi)\delta_{jk}, \quad (\text{E.3})$$

$${}^{(1)}\Gamma_{0j}^i = E'^i{}_j - \Psi'\delta_j^i, \quad (\text{E.4})$$

$${}^{(1)}\Gamma_{jk}^i = 2\partial_{(k}[E_{j)}^i - \Psi\delta_{jk}^i] - \partial^i(E_{jk} - \Psi\delta_{jk}), \quad (\text{E.5})$$

where  $A_{(ij)} \equiv (A_{ij} + A_{ji})/2$ . At the second order, we obtain

$${}^{(2)}\Gamma_{00}^0 = \Phi' - 4\Phi\Phi', \quad {}^{(2)}\Gamma_{0j}^0 = \partial_j \Phi - 4\Phi\partial_j \Phi, \quad (\text{E.6})$$

$${}^{(2)}\Gamma_{00}^i = \partial^i \Phi - 4E^{ij}\partial_j \Phi + 4\Psi\partial^i \Phi, \quad (\text{E.7})$$

$${}^{(2)}\Gamma_{jk}^0 = [-2\mathcal{H}\Psi - \Psi' + 4\Phi\Psi' - 2\mathcal{H}\Phi + 8\mathcal{H}\Phi(\Phi + \Psi)]\delta_{jk} \\ + 2\mathcal{H}E_{jk} - 8\Phi\mathcal{H}E_{jk} + E'_{jk} - 4\Phi E'_{jk}, \quad (\text{E.8})$$

$${}^{(2)}\Gamma_{0j}^i = E_j^i + 4\Psi E_j^i - \Psi'\delta_j^i - 4\Psi\Psi'\delta_j^i - 4E^{ik}E'_{kj} + 4\Psi E_j^{i'}, \quad (\text{E.9})$$

$${}^{(2)}\Gamma_{jk}^i = 2\partial_{(k}[E_{j)}^i - \Psi\delta_{jk}^i] - \partial^i(E_{jk} - \Psi\delta_{jk}) \\ + 4(E^{il} - \Psi\delta^{il})\{[\partial_l(E_{jk} - \Psi\delta_{jk}) + \partial_j(E_k^i - \Psi\delta_k^i)] \\ - [l \rightarrow kk \rightarrow jj \rightarrow l] - [l \rightarrow jk \rightarrow lj \rightarrow k]\}. \quad (\text{E.10})$$

### E.2. The radiation fluid equations

The conservation equation  $\nabla_\mu T^{\mu\nu}$  for the stress–energy tensor (7) with a radiation equation of state  $P = \rho/3$  and where we assume  $\pi^{\mu\nu} = 0$  are the conservation equation

$$(\delta^{(1)}\rho)' + 4\mathcal{H}\delta^{(1)}\rho + \frac{4}{3}\bar{\rho}(\Delta v^{(1)} - 3\Psi^{(1)'}) = 0, \\ (\delta^{(2)}\rho)' + 4\mathcal{H}\delta^{(2)}\rho + \frac{4}{3}\bar{\rho}(\Delta v^{(2)} - 3\Psi^{(2)'}) = S_c, \quad (\text{E.11})$$

and the Euler equation

$$v^{(1)'} + \Phi^{(1)} + \frac{\delta^{(1)}\rho}{4\bar{\rho}} = 0, \quad v^{(2)'} + \Phi^{(2)} + \frac{\delta^{(2)}\rho}{4\bar{\rho}} = S_e, \quad (\text{E.12})$$

where the source terms in the second-order equations, which are quadratic in the first-order perturbation variables, are given by

$$S_c = \frac{8}{3} \{ \delta \rho \Psi' + 6 \bar{\rho} \Psi \Psi' - (\Phi + \delta) \bar{\rho} \Delta v + \partial_i v [ -\partial^i \delta \rho - 2 \bar{\rho} \partial^i v' - 2 \bar{\rho} \partial^i \Phi + 3 \bar{\rho} \partial^i \Psi ] \}, \quad (\text{E.13})$$

$$\partial_i S_e = -2 \left( \frac{\delta \rho}{\bar{\rho}} \partial_i v \right)' + 10 \Psi' \partial_i v + 4 \Psi \partial_i v' - 2 \partial_j (\partial^j v \partial_i v) + 2 \Phi \partial_i v' - 2 \frac{\delta \rho}{\bar{\rho}} \partial_i \Phi + 4 \Phi \partial_i \Phi. \quad (\text{E.14})$$

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