

Theory of cosmological perturbations in an anisotropic universe

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Abstract. This paper describes the theory of cosmological perturbations around a homogeneous and anisotropic universe of the Bianchi I type. Starting from a general parametrization of the perturbed spacetime *à la Bardeen*, a complete set of gauge invariant variables is constructed. Three physical degrees of freedom are identified and it is shown that, in the case where matter is described by a scalar field, they generalize the Mukhanov–Sasaki variables. In order to show that they are canonical variables, the action for the cosmological perturbations at second order is derived. Two major physical imprints of the primordial anisotropy are identified: (1) a scalar–tensor ‘seesaw’ mechanism arising from the fact that scalar, vector and tensor modes do not decouple and (2) an explicit dependence of the statistical properties of the density perturbations and gravity waves on the wavevector instead of its norm. This analysis extends, but also sheds some light on, the quantization procedure that was developed under the assumption of a Friedmann–Lemaître background spacetime, and allows one to investigate the robustness of the predictions of the standard inflationary scenario with respect to the hypothesis on the symmetries of the background spacetime. These effects of a primordial anisotropy may be related to some anomalies of the cosmic microwave background anisotropies on large angular scales.

Keywords: cosmological perturbation theory, inflation, quantum field theory on curved space, physics of the early universe

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Inflation [1, 2] (see [3] for a recent review of its status and links with high energy physics) is now a cornerstone of the standard cosmological model. Besides solving the standard problems of the big-bang model (homogeneity, horizon, isotropy, flatness,...), it provides a scenario for the origin of the large scale structure of the universe. In its simplest form, inflation has very definite predictions: the existence of adiabatic initial scalar perturbations and gravitational waves, both with Gaussian statistics and an almost scale invariant power spectrum [4, 5]. Other variants, which in general involve more fields, allow e.g. for isocurvature perturbations [7], non-Gaussianity [8], and modulated fluctuations [9]. All these features let us hope that future data will allow a better understanding of the details (and physics) of this primordial phase.

The predictions of inflation are in agreement with most cosmological data and in particular those of the cosmic microwave background (CMB) by the WMAP satellite [6]. The origin of the density perturbations is related to the amplification of vacuum quantum fluctuations of a scalar field during inflation. In particular, the identification of the degrees of freedom that should be quantized (known as the Mukhanov–Sasaki variables [10]), has been performed assuming a Friedmann–Lemaître background spacetime [5]. This means that homogeneity and isotropy (and even flatness) are in fact assumed from the start of the computation. In the standard lore, one assumes that inflation lasts long enough so that all classical inhomogeneities (mainly spatial curvature and shear) have decayed so that it is perfectly justified to start with a flat Friedmann–Lemaître background spacetime when dealing with the computation of the primordial power spectra for the cosmologically observable modes. This is backed up by the ideas of chaotic inflation and eternal inflation [3]. Note however that a (even small) deviation from flatness [11] or isotropy [12] may have an impact on the dynamics of inflation. It would however be more satisfactory to start from an arbitrary spacetime and understand (1) under which conditions it can be driven toward a Friedmann–Lemaître spacetime during inflation and (2) what are the effects on the evolution and quantization of the perturbations.

The first issue has been addressed by considering the onset of inflation in inhomogeneous and spherically symmetric universes, both numerically in [13] and semi-analytically in [14]. The isotropization of the universe was also investigated by considering the evolution of four-dimensional Bianchi spacetimes [15]–[17] and even Bianchi braneworld [18]. No study has focused on the second issue, i.e. the evolution and the quantization of perturbations during a non-Friedmannian inflationary stage, even though the quantization of test fields and particle production in anisotropic spacetime has been considered [19]. Such an analysis would shed some light on the specificity of the standard quantization procedure which assumes a flat Friedmannian background (see however [20]).

From an observational perspective, a debate concerning possible anomalies on large angular scales in the WMAP has recently driven a lot of activity. Among these anomalies, we count the lack of power in the lowest multipoles, the alignment of the lowest multipoles, and an asymmetry between the two hemispheres (see e.g. [21]). The last two, which point toward a departure from the expected statistical isotropy of the CMB temperature

field, appear much stronger. Various explanations for these anomalies, besides an understood systematic effect that may be related to foreground (see e.g. [22]), have been proposed (such as e.g. the imprint of the topology of space [23, 24] the breakdown of local isotropy due to multiple scalar fields [25] or the existence of a primordial preferred direction [26, 27]).

The broken statistical isotropy of the temperature fluctuations may also be related to a violation of local isotropy, and thus from a departure from the Friedmann–Lemaître symmetries. This can arise either from a late time evolution of the universe (see e.g. [28]–[30] in which it is argued that the subtraction of a Bianchi VII_h leaves a statistically isotropic CMB sky) or from the primordial dynamics which would have imprinted the broken statistical isotropy in the initial conditions. The latter has recently been advocated on the basis of a cylindrically symmetric Bianchi I inflationary model [26]. In these models, the shear decays as the inverse of the third power of the scale factor so that it can play a significant role only in the early stage of the inflationary period. Isotropy is asymptotically reached during inflation and the whole subsequent cosmological evolution can be approximated by a Friedmann–Lemaître universe. It follows (1) that the anisotropy is only imprinted in the largest wavelengths and (2) that the constraints on the shear of the observable universe from the isotropy of the CMB [31]–[34] or big-bang nucleosynthesis [35] are satisfied.

The primordial Bianchi I phase modifies the evolution of the modes (in particular gravity wave and scalar perturbations shall be coupled through the shear) and initial conditions (and thus the quantization procedure) has to be performed in a consistent way with the symmetries of the background spacetime during inflation (see however [26] for a proposal in a locally rotational invariant and homogeneous spacetime of the Kantowski–Sachs family).

In this paper, we investigate the general theory of gauge invariant perturbations about a Bianchi I background spacetime during inflation. Bianchi universes are spatially homogeneous spacetimes and are thus of first importance in cosmology since they express mathematically the cosmological principle. The study of perturbations in Bianchi I was roughed out in [36] where the Bardeen formalism was used (see also [37] and [38] for the case of higher dimensional Kaluza–Klein models). A similar work was undertaken in the 1 + 3 covariant formalism [39] but the identification of gravitational waves and the quantization procedure was not addressed (see [40] for the generalization of the Mukhanov–Sasaki variables in this formalism).

Thus, starting from a general parametrization of the perturbed spacetime *à la Bardeen* [41], we will define in section 2, a scalar–vector–tensor decomposition and construct gauge invariant variables. Contrary to the Friedmann–Lemaître case, these three types of perturbations will be coupled through the shear. In section 3, we derive the perturbation equations. We then show in section 4 that they can be reduced to a set of coupled reduced equations with a mixing between scalar and tensor modes; special care will be taken to vector modes. This work will allow us to generalize the Mukhanov–Sasaki variables and paves the way to the study of the cosmological signatures of a primordial anisotropy [49].

1. Cosmological dynamics of Bianchi I universes

Bianchi spacetimes enjoy a group of isometries simply transitive on spacelike hypersurfaces (see [42]–[44] for a mathematical expositions on Bianchi spacetimes). Thus, they are

homogeneous. It follows that the cosmic time t is the only essential dynamical coordinate and Einstein equations will reduce to ordinary differential equations. The dimension of their group of isotropy [42], that is the group of isometries leaving a given point fixed, is $q = 0$.

1.1. General form of the metric

Bianchi I spacetimes are the simplest anisotropic universe models. They allow for different expansion factors in three orthogonal directions. In comoving coordinates, the metric takes the general form

$$ds^2 = g_{\mu\nu} dx^\mu dx^\nu = -dt^2 + \sum_{i=1}^3 X_i^2(t)(dx^i)^2. \quad (1.1)$$

It includes Friedmann–Lemaître spacetimes as a subcase when the three scale factors are equal. The average scale factor, defined by

$$a(t) \equiv [X_1(t)X_2(t)X_3(t)]^{1/3}, \quad (1.2)$$

characterizes the volume expansion. It follows that we can recast the metric (1.1) as

$$ds^2 = -dt^2 + a^2(t)\gamma_{ij}(t) dx^i dx^j. \quad (1.3)$$

The ‘spatial metric’ γ_{ij} is the metric on constant time hypersurfaces. It can be decomposed as

$$\gamma_{ij} = \exp[2\beta_i(t)]\delta_{ij}, \quad (1.4)$$

with the constraints

$$\sum_{i=1}^3 \beta_i = 0. \quad (1.5)$$

Let us emphasize that β_i are not the components of a vector so that they are not subjected to the Einstein summation rule. Note also that all italic indices i, j, \dots are lowered with the metric γ_{ij} . The decomposition (1.4) implies that $\dot{\gamma}_{ij} = 2\dot{\beta}_i\gamma_{ij}$, where a dot refers to a derivative with respect to the cosmic time, and it can be checked that the spatial hypersurfaces are flat. This relation, together with the constraint (1.5), implies that the determinant of the spatial metric is constant

$$\dot{\gamma} = \gamma^{ij}\dot{\gamma}_{ij} = 0.$$

This simply means that any comoving volume remains constant during the expansion of the universe, even if this expansion is anisotropic. We define the shear as

$$\hat{\sigma}_{ij} \equiv \frac{1}{2}\dot{\gamma}_{ij} \quad (1.6)$$

and introduce the scalar $\hat{\sigma}^2 \equiv \hat{\sigma}_{ij}\hat{\sigma}^{ij}$. This definition is justified from the relation to the 1 + 3 covariant formalism (see appendix A.3). Let us emphasize at this point that $(\gamma^{ij})\dot{\gamma} = -2\hat{\sigma}^{ij}$ differs from $\dot{\gamma}^{ij} \equiv \gamma^{ip}\gamma^{jq}\dot{\gamma}_{pk} = +2\hat{\sigma}^{ij}$.

Introducing the conformal time as $dt \equiv a d\eta$, the metric (1.3) can be recast as

$$ds^2 = a^2(\eta)[-d\eta^2 + \gamma_{ij}(\eta) dx^i dx^j]. \quad (1.7)$$

We define the comoving Hubble parameter by $\mathcal{H} \equiv a'/a$, where a prime refers to a derivative with respect to the conformal time. The shear tensor, now defined as

$$\sigma_{ij} \equiv \frac{1}{2}\gamma'_{ij}, \quad (1.8)$$

is related to $\hat{\sigma}_{ij}$ by $\sigma_{ij} = a\hat{\sigma}_{ij}$. From the relation $(\gamma_{ij})' = \gamma'_{ij} = 2\beta'_i\gamma_{ij}$, the definition

$$\sigma^2 \equiv \sigma_{ij}\sigma^{ij} \quad (1.9)$$

is explicitly given by

$$\sigma^2 = \sum_{i=1}^3 (\beta'_i)^2, \quad (1.10)$$

and is related to its cosmic time analogous by $\sigma = \hat{\sigma}a$. Again, we stress that $(\gamma^{ij})' = -2\sigma^{ij}$ differs from $(\gamma')^{ij} \equiv \gamma^{ip}\gamma^{jk}\gamma'_{pk} = +2\sigma^{ij}$.

1.2. Background equations

We concentrate on an inflationary phase during which the matter content of the universe is assumed to be described by a minimally coupled scalar field, φ , with stress-energy tensor

$$T_{\mu\nu} = \partial_\mu\varphi\partial_\nu\varphi - \left(\frac{1}{2}\partial_\alpha\varphi\partial^\alpha\varphi + V\right)g_{\mu\nu}. \quad (1.11)$$

Making use of the expressions (A.6) and (A.7) (see appendix A.2), we easily obtain the Friedmann equations

$$\mathcal{H}^2 = \frac{\kappa}{3} \left[\frac{1}{2}\varphi'^2 + V(\varphi)a^2 \right] + \frac{1}{6}\sigma^2, \quad (1.12)$$

$$\mathcal{H}' = -\frac{\kappa}{3}[\varphi'^2 - V(\varphi)a^2] - \frac{1}{3}\sigma^2, \quad (1.13)$$

$$(\sigma_j^i)' = -2\mathcal{H}\sigma_j^i, \quad (1.14)$$

where $\kappa \equiv 8\pi G$. The first two are similar to the ones usually used in a Friedmann–Lemaître universe, up to the contribution of the shear (which acts as an extra massless field). The latter equation arises from the trace-free part of the ‘ ij ’-Einstein equations and gives an extra equation compared to the Friedmann–Lemaître case. We can easily integrate it and conclude that the shear evolves as

$$\sigma_j^i = \frac{\mathcal{S}_j^i}{a^2} \quad (1.15)$$

where \mathcal{S}_j^i is a constant tensor, $(\mathcal{S}_j^i)' = 0$. This implies that

$$\sigma^2 = \frac{\mathcal{S}^2}{a^4} \Rightarrow \hat{\sigma}^2 = \frac{\mathcal{S}^2}{a^6}, \quad (1.16)$$

(with $\mathcal{S}^2 \equiv \mathcal{S}_i^j\mathcal{S}_j^i$) from which we deduce that

$$\sigma' = -2\mathcal{H}\sigma. \quad (1.17)$$

Let us note that these equations can be combined to give

$$2\mathcal{H}^2 + \mathcal{H}' = \kappa a^2 V, \quad \kappa(\varphi')^2 = 2\mathcal{H}^2 - 2\mathcal{H}' - \sigma^2. \quad (1.18)$$

These equations are completed by a Klein–Gordon equation, which keeps its Friedmann–Lemaître form,

$$\varphi'' + 2\mathcal{H}\varphi' + a^2V_\varphi = 0. \quad (1.19)$$

The general solution for the evolution of the scale factor from these equations is detailed in appendix A.4.

2. Gauge invariant variables

This section is devoted to the definition of the gauge invariant variables that describe the perturbed spacetime. We follow a method *à la Bardeen*. In order to define scalar, vector and tensor modes, we will need to use a Fourier transform. We start, in section 2.1, by recalling its definition and stressing its differences with the standard Friedmann–Lemaître case. In section 2.2, we perform a general gauge transformation to identify the gauge invariant variables.

2.1. Mode decomposition

2.1.1. Definition of the Fourier transform. We decompose any quantity in Fourier modes as follows. First, we pick up a comoving coordinates system, $\{x^i\}$, on the constant time hypersurfaces. Then, we decompose any scalar function as

$$f(x^j, \eta) = \int \frac{d^3k_i}{(2\pi)^{3/2}} \hat{f}(k_i, \eta) e^{ik_ix^i}, \quad (2.1)$$

with the inverse Fourier transform

$$\hat{f}(k_j, \eta) = \int \frac{d^3x^i}{(2\pi)^{3/2}} f(x^i, \eta) e^{-ik_ix^i}. \quad (2.2)$$

In the Fourier space, the comoving wave co-vectors k_i are constant, $k'_i = 0$. We now define $k^i \equiv \gamma^{ij}k_j$ that is obviously a time-dependent quantity. Contrary to the standard Friedmann–Lemaître case, we must be careful not to trivially identify k_i and k^i , since this does not commute with the time evolution. Note however that $x_i k^i = x^i k_i$ remains constant so that there is no extra time dependence entering our definitions (2.1) and (2.2). In the following of this paper, we will forget the ‘hat’ and use the notation $f(x^j, \eta)$ and $f(k_j, \eta)$ both for a function and its Fourier transform.

It is easily checked, using the definition (1.8), that

$$(k^i)' = -2\sigma^{ip}k_p. \quad (2.3)$$

This implies that the modulus of the comoving wavevector, $k^2 = k^i k_i = \gamma^{ij}k_i k_j$, is now time dependent and that its rate of change is explicitly given by

$$\frac{k'}{k} = -\sigma^{ij}\hat{k}_i\hat{k}_j, \quad (2.4)$$

where we have introduced the unit vector

$$\hat{k}_i \equiv \frac{k_i}{k}. \quad (2.5)$$

This vector will turn to be particularly useful for our analysis and we note that it evolves as

$$(\hat{k}^i)' = (\sigma^{pq} \hat{k}_p \hat{k}_q) \hat{k}^i - 2\sigma^{ip} \hat{k}_p. \quad (2.6)$$

Indeed, we find that in the standard Friedmann–Lemaître limit ($\sigma_{ij} = 0$), k^i and k are constant.

2.1.2. Decomposition of the vector and tensor modes. We shall now decompose the perturbations in their scalar, vector and tensor modes.

Any (three-dimensional) vector field, V^i , can be decomposed as

$$V_i = \partial_i V + \bar{V}_i, \quad \text{with } \partial^i \bar{V}_i = 0. \quad (2.7)$$

Note that we have chosen orthogonal (but not Cartesian) coordinates on the (Euclidean) spatial sections (in particular spatial flatness and homogeneity imply that in these coordinates the Christoffel symbols vanish and that $\partial_k \gamma_{ij} = 0$). It follows that its Fourier components can be split as

$$V_i = k_i V + \bar{V}_i, \quad \text{with } k^i \bar{V}_i = 0, \quad (2.8)$$

so that \bar{V}^i lives in the subspace perpendicular to k^i . This is a two-dimensional subspace so that V_i has been split into 1 scalar (V) and two vector modes (\bar{V}_i) that correspond to transverse modes. Let us now consider the base $\{e^1, e^2\}$ of the subspace perpendicular to k^i . By construction, it satisfies the orthonormalization conditions

$$e_i^a k_j \gamma^{ij} = 0, \quad e_i^a e_j^b \gamma^{ij} = \delta^{ab}.$$

Such a basis is defined up to a rotation about the axis k^i . Now, the vector modes can be decomposed on this basis as

$$\bar{V}_i(k_i, \eta) = \sum_{a=1,2} V_a(\hat{k}_i, \eta) e_i^a(\hat{k}_i), \quad (2.9)$$

which defines the two degrees of freedom, V_a , which depend on \hat{k}^i since the decomposition differs for each wavenumber. The two basis vectors allow us to define a projection operator onto the subspace perpendicular to k^i as

$$P_{ij} \equiv e_i^1 e_j^1 + e_i^2 e_j^2 = \gamma_{ij} - \hat{k}_i \hat{k}_j. \quad (2.10)$$

It trivially satisfies $P_j^i P_k^j = P_k^i$, $P_j^i k^j = 0$ and $P^{ij} \gamma_{ij} = 2$. It is also the projector on vector modes so that we can always make the scalar–vector decomposition

$$V_i = [k^j V_j] \hat{k}_i + P_i^j V_j. \quad (2.11)$$

Analogously, any (three-dimensional) symmetric tensor field, V_{ij} , can be decomposed as

$$V_{ij} = T \gamma_{ij} + \Delta_{ij} S + 2\partial_{(i} \bar{V}_{j)} + 2\bar{V}_{ij}, \quad (2.12)$$

where $\Delta_{ij} \equiv \partial_i \partial_j - \Delta \gamma_{ij}/3$ and

$$\partial_i \bar{V}^i = 0, \quad \bar{V}_i^i = 0 = \partial_i \bar{V}^{ij}. \quad (2.13)$$

The symmetric tensor \bar{V}_{ij} is transverse and trace-free. Hence it has only two independent components and can be decomposed as

$$\bar{V}_{ij}(k_i, \eta) = \sum_{\lambda=+, \times} V_\lambda(k^i, \eta) \varepsilon_{ij}^\lambda(\hat{k}_i) \quad (2.14)$$

where the polarization tensors have been defined as

$$\varepsilon_{ij}^\lambda = \frac{e_i^1 e_j^1 - e_i^2 e_j^2}{\sqrt{2}} \delta_+^\lambda + \frac{e_i^1 e_j^2 + e_i^2 e_j^1}{\sqrt{2}} \delta_\times^\lambda. \quad (2.15)$$

It can be checked that they are traceless ($\varepsilon_{ij}^\lambda \gamma^{ij} = 0$), transverse ($\varepsilon_{ij}^\lambda k^i = 0$), and that the two polarizations are perpendicular ($\varepsilon_{ij}^\lambda \varepsilon_{\mu}^{ij} = \delta_\mu^\lambda$). This defines the two tensor degrees of freedom.

In order to deal with the properties of the polarization tensors, it is useful to define two new quantities

$$Q_{ij} \equiv e_i^1 e_j^2 - e_i^2 e_j^1, \quad \text{and} \quad \eta_{\lambda\mu} \equiv \delta_\lambda^+ \delta_\mu^\times - \delta_\mu^+ \delta_\lambda^\times. \quad (2.16)$$

The tensor Q_{ij} trivially satisfies

$$P_{ij} Q^{ij} = 0, \quad Q_{ij} Q^{ij} = 2. \quad (2.17)$$

They allow us to simplify the product of two and three polarization tensors as

$$\varepsilon_{ik}^\lambda \varepsilon_j^{k\mu} = \frac{1}{2} (P_{ij} \delta^{\lambda\mu} + Q_{ij} \eta^{\lambda\mu}), \quad \varepsilon_{ik}^\lambda \varepsilon_\mu^{kj} \varepsilon_{j\nu}^i = 0. \quad (2.18)$$

Introducing the projector operator on tensor modes by

$$\Lambda_{ij}^{ab} = P_i^a P_j^b - \frac{1}{2} P_{ij} P^{ab},$$

and the ‘trace extracting’ operator

$$T_i^j = \hat{k}_i \hat{k}^j - \frac{1}{3} \delta_i^j,$$

the scalar–vector–tensor terms in the decomposition of equation (2.12) are extracted as follows

$$V_{ij} = [\frac{1}{3} V_{ab} \gamma^{ab}] \gamma_{ij} + [\frac{3}{2} V_{ab} T^{ab}] T_{ij} + 2 \hat{k}_{(i} [P_{j)}^a \hat{k}^b V_{ab}] + \Lambda_{ij}^{ab} V_{ab}. \quad (2.19)$$

In this expression, V_{ij} has been split into two scalars (T and S), two vector modes (\bar{V}_i) and two tensor modes (\bar{V}_{ij}). Thus, we can always split any equation $V_i = 0$ by projecting along \hat{k}^i (scalar) and P_j^i (vector) and any equation $V_{ij} = 0$ by projecting along γ^{ij} (scalar), T^{ij} (scalar), $P_i^j \hat{k}^j$ (vector) and Λ_{ab}^{ij} (tensor).

2.1.3. Properties of the projectors, polarization vectors and tensors. The previous SVT decomposition matches the one used in the perturbation theory about a Friedmann–Lemaître spacetime. There is however an important difference that we will have to deal

with. As we pointed out, in a Bianchi I spacetime, the spatial metric is time dependent. It implies in particular that, in order to remain an orthonormal basis perpendicular to k^i during the time evolution, the polarization vectors, and thus the polarization tensors, must have a non-vanishing time derivative. Indeed, since $(k_i)' = 0$, the vector $(e_a^i)'$ is orthogonal to k_i and is thus a linear combination of e^1 and e^2 , that is

$$(e_a^i)' = \sum_b \mathcal{R}_{ab} e_b^i.$$

In each time hypersurface, there is a remaining freedom in the choice of this basis because of the rotational invariance around k^i . We can continuously fix the choice of the basis by imposing

$$\mathcal{R}_{[ab]} = 0.$$

The orthonormalization condition implies that $(e_a^i e_i^b)' = 0$ and thus that

$$\mathcal{R}_{ab} = -\sigma_{ij} e_a^i e_b^j. \quad (2.20)$$

Consequently, the time derivative of any polarization vector is given by

$$(e_a^i)' = \sum_b \mathcal{R}_{ab} e_b^i + 2\sigma_{ij} e_a^j, \quad (2.21)$$

from which we deduce

$$k^i (e_a^i)' = 2\sigma^{pi} k_p e_a^i. \quad (2.22)$$

This allows us to derive the expression of the time derivative of the polarization tensor. Starting from their definitions (2.15), we easily obtain that

$$(\varepsilon_{ij}^\lambda)' = -(\sigma^{kl} \varepsilon_{kl}^\lambda) P_{ij} - (\sigma^{kl} P_{kl}) \varepsilon_{ij}^\lambda + 4\sigma_{(i}^\lambda \varepsilon_{j)k}^\lambda, \quad (2.23)$$

from which we can deduce some useful algebra

$$k^i k^j (\varepsilon_{ij}^\lambda)' = 0, \quad \gamma^{ij} (\varepsilon_{ij}^\lambda)' = 2\sigma^{ij} \varepsilon_{ij}^\lambda, \quad k^i (\varepsilon_{ij}^\lambda)' = 2\sigma^{ip} k_p \varepsilon_{ij}^\lambda. \quad (2.24)$$

We also have that

$$(\varepsilon_j^{i\lambda})' \varepsilon_i^{j\mu} = 0. \quad (2.25)$$

We gather in appendix B some other useful relations concerning the polarization vectors and tensors.

For the sake of completeness, we shall define here two important matrices for the following of our computation,

$$\mathcal{M}_{ab}^\lambda \equiv \varepsilon_{ij}^\lambda e_a^i e_b^j, \quad (2.26)$$

which is manifestly symmetric in ab and

$$\mathcal{N}_{ab} \equiv Q_{ij} e_a^i e_b^j, \quad (2.27)$$

which is antisymmetric in ab . We stress that a and λ are not indices but only labels. It can easily be checked that

$$\mathcal{M}_{ab}^\lambda = \frac{1}{\sqrt{2}} \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix} \delta_+^\lambda + \frac{1}{\sqrt{2}} \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix} \delta_\times^\lambda, \quad (2.28)$$

and that

$$\sum_a \mathcal{M}_{aa}^\lambda = 0. \quad (2.29)$$

2.2. Defining gauge invariant variables

2.2.1. Gauge invariant variables for the geometry. Let us consider the most general metric of an almost Bianchi I spacetime. It can always be decomposed as

$$ds^2 = a^2[-(1 + 2A) d\eta^2 + 2B_i dx^i d\eta + (\gamma_{ij} + h_{ij}) dx^i dx^j]. \quad (2.30)$$

B_i and h_{ij} can be further conveniently decomposed as

$$B_i = \partial_i B + \bar{B}_i, \quad (2.31)$$

$$h_{ij} \equiv 2C \left(\gamma_{ij} + \frac{\sigma_{ij}}{\mathcal{H}} \right) + 2\partial_i \partial_j E + 2\partial_{(i} E_{j)} + 2E_{ij}, \quad (2.32)$$

with

$$\partial_i \bar{B}^i = 0 = \partial_i E^i, \quad E_i^i = 0 = \partial_i E^{ij}. \quad (2.33)$$

Note that this decomposition of h_{ij} involves the shear. This judicious choice is justified, *a posteriori*, by the simplicity of the transformation properties of the perturbation variables, as we shall now see.

Let us consider an active transformation of the coordinate system defined by a vector field ξ . The coordinates of any point change according to

$$x^\mu \rightarrow \tilde{x}^\mu = x^\mu - \xi^\mu(x^\nu) \quad (2.34)$$

so that the spacetime metric transforms as

$$g_{\mu\nu} \rightarrow g_{\mu\nu} + \mathcal{L}_\xi g_{\mu\nu}, \quad (2.35)$$

where $\mathcal{L}_\xi g_{\mu\nu}$ is the Lie derivative of $g_{\mu\nu}$ along ξ . At first order in the perturbations, we decompose the metric as $g_{\mu\nu} = \bar{g}_{\mu\nu} + \delta g_{\mu\nu}$ and it follows that

$$\delta g_{\mu\nu} \rightarrow \delta g_{\mu\nu} + \mathcal{L}_\xi \bar{g}_{\mu\nu}. \quad (2.36)$$

The vector field ξ is now decomposed into a scalar and vector part as

$$\xi^0 = T(x^i, \eta), \quad \xi^i = \partial^i L(x^j, \eta) + L^i(x^j, \eta), \quad (2.37)$$

with $\partial_i L^i = 0$. With the use of the expressions (C.1), we deduce that the perturbations of the metric transform as (in Fourier space)

$$A \rightarrow A + T' + \mathcal{H}T \quad (2.38)$$

$$B \rightarrow B - T + \frac{(k^2 L)'}{k^2} \quad (2.39)$$

$$C \rightarrow C + \mathcal{H}T \quad (2.40)$$

$$E \rightarrow E + L, \quad (2.41)$$

for the scalar variables, and as

$$\bar{B}_i \rightarrow \bar{B}_i + \gamma_{ij} (L^j)' - 2ik^j \sigma_{lj} P_i^l L \quad (2.42)$$

$$E_i \rightarrow E_i + L_i, \quad (2.43)$$

for the vector variables. We also obtain that the tensor modes are readily gauge invariant,

$$E_{ij} \rightarrow E_{ij}. \quad (2.44)$$

Had we not included the shear in the decomposition (2.31), this would not be the case.

Let us also note that the transformation rule of the vector modes is different from the one derived in [36] where the non-commutativity between the projection and the time evolution has been neglected.

From the gauge transformations (2.38)–(2.41), we can construct a set of gauge invariant variables for the scalar sector. Only two degrees of freedom remain, the two other being absorbed by the scalar part of the gauge transformation. We define the two gravitational potentials

$$\Phi \equiv A + \frac{1}{a} \left\{ a \left[B - \frac{(k^2 E)'}{k^2} \right] \right\}', \quad (2.45)$$

$$\Psi \equiv -C - \mathcal{H} \left[B - \frac{(k^2 E)'}{k^2} \right]. \quad (2.46)$$

From the gauge transformations (2.42) and (2.43), we deduce that a gauge invariant vector perturbation is given by

$$\Phi_i \equiv \bar{B}_i - \gamma_{ij}(E^j)' + 2ik^j \sigma_{lj} P_i^l E. \quad (2.47)$$

It is obvious from these expressions that when γ_{ij} is time independent, that is when $\sigma_{ij} = 0$, these variables reduce to the standard Bardeen variables defined in the Friedmann–Lemaître case. By analogy, we define the Newtonian gauge by the conditions

$$B = \bar{B}^i = E = 0, \quad (2.48)$$

so that

$$A = \Phi, \quad C = -\Psi, \quad \Phi^i = -(E^i)', \quad (2.49)$$

the latter condition being equivalent to $\Phi_i = -E_i' + 2\sigma_{ij}E^j$.

2.2.2. Gauge invariant variables for the matter. We focus our analysis on the scalar field case, which is the most relevant for the study of inflation. Under a gauge transformation of the form (2.34), it transforms as $\varphi \rightarrow \varphi + \mathcal{L}_\xi \varphi$. At first order in the perturbations, we get

$$\delta\varphi \rightarrow \delta\varphi + \mathcal{L}_\xi \bar{\varphi}, \quad (2.50)$$

that is

$$\delta\varphi \rightarrow \delta\varphi + \mathcal{L}_\xi \bar{\varphi} = \delta\varphi + \varphi' T, \quad (2.51)$$

with use of equation (2.37). Thus, we can define the two gauge invariant variables

$$Q \equiv \delta\varphi - \frac{C}{\mathcal{H}} \varphi' \quad (2.52)$$

and

$$\chi \equiv \delta\varphi + \left[B - \frac{(k^2 E)'}{k^2} \right] \varphi'. \quad (2.53)$$

They are related by

$$Q = \chi + \frac{\Psi}{\mathcal{H}} \varphi'. \quad (2.54)$$

3. Perturbations equations

Once the gauge invariant variables have been defined, we can derive their equations of evolution. The mode decomposition will require a decomposition of the shear tensor in a basis adapted to each wavenumber. We start by defining this decomposition and then we derive the perturbed Klein–Gordon and Einstein equations.

3.1. Decomposition of the shear tensor

The shear σ_{ij} is a symmetric trace-free tensor and, as such, has five independent components. In the coordinates system (1.3)–(1.4), it was expressed in terms of only two independent functions of time $\beta_i(\eta)$. The three remaining degrees of freedom are related to the three Euler angles needed to shift to a general coordinate system.

3.1.1. Components of the shear. As mentioned before, when working out the perturbations in Fourier space, it would be fruitful to decompose the shear in a local basis adapted to the mode we are considering. The shear, being a symmetric trace-free tensor, can be decomposed on the basis $\{\hat{k}_i, e_i^1, e_i^2\}$ as

$$\sigma_{ij} = \frac{3}{2}(\hat{k}_i \hat{k}_j - \frac{1}{3}\gamma_{ij})\sigma_{\parallel} + 2 \sum_{a=1,2} \sigma_{\nu a} \hat{k}_{(i} e_{j)}^a + \sum_{\lambda=+, \times} \sigma_{\text{T}\lambda} \hat{\epsilon}_{ij}^{\lambda}. \quad (3.1)$$

This decomposition involves five independent components of the shear in a basis adapted to the wavenumber k_i . We must stress however that $(\sigma_{\parallel}, \sigma_{\nu a}, \sigma_{\text{T}\lambda})$ must not be interpreted as the Fourier components of the shear, even if they explicitly depend on k_i . This dependence arises from the local anisotropy of space.

Using equation (3.1), it can be easily worked out that

$$\begin{aligned} \sigma_{ij} \gamma^{ij} &= 0, \\ \sigma_{ij} \hat{k}^i &= \sigma_{\parallel} \hat{k}_j + \sum_a \sigma_{\nu a} e_j^a, \quad \sigma_{ij} \hat{k}^i \hat{k}^j = \sigma_{\parallel}, \end{aligned}$$

and

$$\sigma_{ij} \hat{\epsilon}_{\lambda}^{ij} = \sigma_{\text{T}\lambda}, \quad \sigma_{ij} \hat{k}^i e_a^j = \sigma_{\nu a}.$$

The scalar shear is explicitly given by

$$\sigma^2 = \sigma_{ij} \sigma^{ij} = \frac{3}{2}\sigma_{\parallel}^2 + 2 \sum_a \sigma_{\nu a}^2 + \sum_{\lambda} \sigma_{\text{T}\lambda}^2, \quad (3.2)$$

which is independent of k_i . We emphasize that the local positivity of the energy density of matter implies that $\sigma^2/6 < \mathcal{H}^2$ and thus

$$\frac{1}{2}\sigma_{\parallel} \leq \frac{1}{\sqrt{6}}\sigma < \mathcal{H}. \quad (3.3)$$

This, in turn, implies that

$$\sigma_{\parallel} < 2\mathcal{H}, \quad (3.4)$$

a property that shall turn to be very useful in the following of our discussion. Analogously, we have that

$$\sigma_{\text{T}\lambda} < \sqrt{6}\mathcal{H}. \quad (3.5)$$

The following derivations will involve the contraction of the shear with the polarization vectors,

$$\sigma_{ij} e_a^i e_b^j = -\frac{1}{2} \sigma_{\parallel} \delta_{ab} + \sum_{\lambda} \sigma_{T\lambda} \mathcal{M}_{ab}^{\lambda}, \quad (3.6)$$

from which we deduce that

$$\sigma_{ij} P^{ij} = -\sigma_{\parallel}. \quad (3.7)$$

It will also involve the contraction of the shear with the polarization tensors,

$$\sigma_{il} \varepsilon_{\lambda}^{lj} = -\frac{1}{2} \sigma_{\parallel} \varepsilon_i^{j\lambda} + \sum_a \sigma_{va} \hat{k}_i e_l^a \varepsilon_{\lambda}^{lj} + \sum_{\mu} \sigma_{T\mu} \varepsilon_{il}^{\mu} \varepsilon_{\lambda}^{lj}, \quad (3.8)$$

which implies that

$$\sigma_{il} \varepsilon_{\lambda}^{lj} P_j^i = \sigma_{T\lambda}, \quad \sigma_{il} \varepsilon_{\lambda}^{lj} \varepsilon_j^{i\mu} = -\frac{1}{2} \sigma_{\parallel} \delta_{\mu}^{\lambda}. \quad (3.9)$$

To finish, we will make use of the following expression

$$e_l^b e_a^j \sigma_{jm} \varepsilon_{\lambda}^{lm} = -\frac{1}{2} \sigma_{\parallel} \mathcal{M}_{ab}^{\lambda} + \frac{1}{2} \delta_b^a \sigma_{T\lambda} + \frac{1}{2} \mathcal{N}_{ab} (\sigma_{T+} \delta_{\lambda}^{\times} - \sigma_{T\times} \delta_{\lambda}^+). \quad (3.10)$$

3.1.2. Time evolution of the components of the shear. In the previous paragraph we detailed the definition of the components of the shear in a basis adapted to the wave mode k_i . The time evolution of these modes is easily obtained from equation (1.14)

$$\sigma'_{\parallel} + 2\mathcal{H}\sigma_{\parallel} = -2 \sum_a \sigma_{va}^2, \quad (3.11)$$

$$\sigma'_{va} + 2\mathcal{H}\sigma_{va} = \frac{3}{2} \sigma_{va} \sigma_{\parallel} - \sum_{b,\lambda} \sigma_{vb} \sigma_{T\lambda} \mathcal{M}_{ab}^{\lambda}, \quad (3.12)$$

$$\sigma'_{T\lambda} + 2\mathcal{H}\sigma_{T\lambda} = 2 \sum_{a,b} \mathcal{M}_{ab}^{\lambda} \sigma_{va} \sigma_{vb}, \quad (3.13)$$

where the matrix $\mathcal{M}_{ab}^{\lambda}$ is defined in equation (2.26).

These equations allow us to derive some important constraints on the rate of change of σ_{\parallel} and $\sigma_{T\lambda}$. Since equation (3.11) implies that

$$\left| \frac{1}{a^2} (a^2 \sigma_{\parallel})' \right| = 2 \sum_a \sigma_{va}^2 < \sigma^2 < 6\mathcal{H}^2, \quad (3.14)$$

we can conclude that

$$\left| \frac{1}{a^2} (a^2 \sigma_{\parallel})' \right| < 6\mathcal{H}^2. \quad (3.15)$$

Identically, equation (3.13) implies that

$$\left| \frac{1}{a^2} (a^2 \sigma_{T\lambda})' \right| = 2 \sum_{a,b} \mathcal{M}_{ab}^{\lambda} \sigma_{va} \sigma_{vb} < \frac{\sigma^2}{\sqrt{2}},$$

so that

$$\left| \frac{1}{a^2} (a^2 \sigma_{T\lambda})' \right| < 3\sqrt{2}\mathcal{H}^2. \quad (3.16)$$

The two relations (3.15) and (3.16) will be important at the end of our analysis.

3.2. Klein–Gordon equation

The Klein–Gordon equation, $\square\varphi = V_\varphi$, can be rewritten under the form

$$g^{\mu\nu}\nabla_\mu\partial_\nu\varphi = V_\varphi(\varphi). \quad (3.17)$$

When expanded at first order in the perturbations, the rhs is trivially given by $V_\varphi(\bar{\varphi}) + V_{\varphi\varphi}(\bar{\varphi})\chi$. It follows that the Klein–Gordon equation at first order in the perturbations is then obtained to be

$$\chi'' + 2\mathcal{H}\chi' - \gamma^{ij}\partial_i\partial_j\chi + a^2V_{\varphi\varphi}\chi = 2(\varphi'' + 2\mathcal{H}\varphi')\Phi + \varphi'(\Phi' + 3\Psi'), \quad (3.18)$$

where $V_{\varphi\varphi}$ is the second derivative of the potential with respect to the scalar field. Surprisingly, it has the same form as in the Friedmann–Lemaître case. This can be understood if we remind that the d'Alembertian can be expressed as $\square\varphi = \partial_\nu[\sqrt{-g}g^{\mu\nu}\partial_\mu\varphi]/\sqrt{-g}$, and if we realize that $\sqrt{-g}$ does not involve the shear. Thus, at first order in the perturbations, the only place where the shear σ_{ij} could appear would be associated with δg^{ij} . But then it would multiply $\partial_i\bar{\varphi}$ which vanishes. Consequently the Klein–Gordon equation is not modified. This result is not specific to the scalar field case as the conservation equation in the fluid case is also the same as for a Friedmann–Lemaître spacetime, indeed only as long as the anisotropic stress vanishes (see equation (A.22)).

3.3. Einstein equations

The procedure to obtain the mode decomposition of the Einstein equations is somehow simple. We start from the general perturbed equation $\delta G^\mu_\nu = \kappa\delta T^\mu_\nu$ with the expressions (C.2)–(C.4) and (C.10)–(C.12) respectively for the stress-energy tensor and the Einstein tensor and we then project them, as described in section 2.1.2.

Special care must however be taken. In the Friedmann–Lemaître case, the projections on the scalar, vector and tensor modes commute with the time evolution. This no more the case in a Bianchi I universe, as explained in section 2.1.3. Let us take an example and consider the extraction of the vector part of an equation involving a term of the form $(\Phi^i)' + \mathcal{H}\Phi^i$. We project this equation on the polarization tensor e_i^a to get

$$e_i^a[(\Phi^i)' + \mathcal{H}\Phi^i] = (e_i^a\Phi^i)' - \Phi^i(e_i^a)' + \mathcal{H}\Phi_a.$$

We then use equation (2.21) to rewrite $\Phi^i(e_i^a)'$, and we develop the shear in the basis adapted to the mode k^i . This implies, in particular, that contrary to the Friedmann–Lemaître case, the scalar, vector and tensor modes will be coupled.

This being said, the extraction of the mode decomposition of the Einstein equation is a lengthy but straightforward computation that we carry in the Newtonian gauge. It reduces to (1) Fourier transforming the Einstein equations, (2) projecting them on the modes, (3) commuting the projection operators and the time evolution in order to extract the evolution of the polarizations and (4) finally expressing the decomposition of the shear.

3.3.1. *Scalar modes.* There are four scalar Einstein equations. The first is obtained from $\delta G_0^0 = \kappa \delta T_0^0$ and gives

$$k^2 \Psi + 3\mathcal{H}(\Psi' + \mathcal{H}\Phi) - \frac{\kappa}{2} (\varphi'^2 \Phi - \varphi' \chi' - V_\varphi a^2 \chi) = \frac{1}{2} \sigma^2 [X - 3\Psi] + \frac{1}{2} \frac{k^2}{\mathcal{H}} \sigma_{\parallel} \Psi - \frac{1}{2} k^2 \sum_a \tilde{\sigma}_{va} \Phi_a - \frac{1}{2} \sum_{\lambda} [\sigma_{T\lambda} E'_\lambda + (\sigma'_{T\lambda} + 2\mathcal{H}\sigma_{T\lambda}) E_\lambda], \quad (3.19)$$

where we have defined the extremely useful variable [2]

$$X \equiv \Phi + \Psi + \left(\frac{\Psi}{\mathcal{H}} \right)', \quad (3.20)$$

and the quantity

$$\sigma_{va} \equiv ik \tilde{\sigma}_{va}. \quad (3.21)$$

As an example, the only tricky term which appears when deriving this equation is $\sigma_j^i (E_i^j)'$, which is obtained from

$$\sigma_j^i (E_i^j)' = (\sigma_j^i E_i^j)' - E_i^j (\sigma_j^i)' = \sum (\sigma_{T\lambda} E_\lambda)' - E_i^j (-2\mathcal{H}\sigma_j^i) = \sum (\sigma_{T\lambda} E_\lambda)' + 2\mathcal{H}\sigma_{T\lambda} E_\lambda,$$

where we have used equation (1.14) to compute $(\sigma_j^i)'$. We will not detail these steps in the following.

The second equation is obtained from $k^i \delta G_i^0 = \kappa k^i \delta T_i^0$. We find

$$\Psi' + \mathcal{H}\Phi - \frac{\kappa}{2} \varphi' \chi = -\frac{1}{2\mathcal{H}} \sigma^2 \Psi + \frac{1}{2} \sigma_{\parallel} X + \frac{1}{2} \sum_{\lambda} \sigma_{T\lambda} E_\lambda. \quad (3.22)$$

The two remaining equations are obtained from

$$\delta_j^i \delta G_i^j = \kappa \delta_j^i \delta T_i^j, \quad (\hat{k}^i \hat{k}_j - \frac{1}{3} \delta_j^i) \delta G_i^j = \kappa (\hat{k}^i \hat{k}_j - \frac{1}{3} \delta_j^i) \delta T_i^j$$

and take the form

$$\begin{aligned} \Psi'' + 2\mathcal{H}\Psi' + \mathcal{H}\Phi' + (2\mathcal{H}' + \mathcal{H}^2)\Phi - \frac{1}{3} k^2 (\Phi - \Psi) + \frac{\kappa}{2} [\varphi'^2 \Phi - \varphi' \chi' + V_\varphi a^2 \chi] \\ = -\frac{1}{2} \sigma^2 [X - 3\Psi] + \frac{1}{6} \frac{k^2}{\mathcal{H}} \sigma_{\parallel} \Psi + \frac{1}{2} k^2 \sum_a \tilde{\sigma}_{va} \Phi_a \\ + \frac{1}{2} \sum_{\lambda} [\sigma_{T\lambda} E'_\lambda + (\sigma'_{T\lambda} + 2\mathcal{H}\sigma_{T\lambda}) E_\lambda], \end{aligned} \quad (3.23)$$

$$\frac{2}{3} k^2 (\Phi - \Psi) = \sigma_{\parallel} \left[X' - \frac{k^2 \Psi}{3\mathcal{H}} \right] + 4k^2 \sum_{\lambda, a, b} \mathcal{M}_{ab}^{\lambda} \tilde{\sigma}_v^a \tilde{\sigma}_v^b E_\lambda - 2k^2 \sum_a \tilde{\sigma}_{va} \Phi_a. \quad (3.24)$$

It can be checked that indeed equations (3.19) and (3.22)–(3.24) reduces to their well-known Friedmannian form when the shear vanishes.

3.3.2. *Vector modes.* The two vector equations are obtained from

$$e_a^i \delta G_i^0 = 0, \quad k_i e_a^j \delta G_j^i = 0.$$

They respectively give

$$\Phi_a = -2\tilde{\sigma}_{va} X + 4 \sum_{b,\lambda} \mathcal{M}_{ab}^\lambda \tilde{\sigma}_{vb} E_\lambda, \quad (3.25)$$

and

$$\begin{aligned} \Phi'_a + 2\mathcal{H}\Phi_a - \frac{5}{2}\sigma_{\parallel}\Phi_a + \sum_{b\lambda} \mathcal{M}_{ab}^\lambda \sigma_{T\lambda} \Phi_b &= -2\tilde{\sigma}_{va} X' + 4 \sum_{b,\lambda} \mathcal{M}_{ab}^\lambda \tilde{\sigma}_{vb} E'_\lambda \\ + 4 \sum_{b\lambda} \mathcal{N}_{ab} \tilde{\sigma}_{vb} (\sigma_{T+} \delta_\lambda^\times - \sigma_{T\times} \delta_\lambda^+) E_\lambda, \end{aligned} \quad (3.26)$$

where the matrix \mathcal{N}_{ab} is defined in equation (2.27). It can be shown that equation (3.26) results from the time derivative of equation (3.25) once equations (3.11)–(3.13) are used to express the time derivatives of the shear. This a consequence of the Bianchi identities.

3.3.3. *Tensor modes.* The equation of evolution of the tensor modes is obtained from $\varepsilon_i^{j\lambda} \delta G_j^i = 0$. To simplify, we shall use the shorthand notation $(1 - \lambda)$ for the opposite polarization of λ , i.e. it means that if $\lambda = +$, then $(1 - \lambda) = \times$, and vice versa. With the use of equation (B.6), we obtain

$$\begin{aligned} E''_\lambda + 2\mathcal{H}E'_\lambda + k^2 E_\lambda &= \sigma_{T\lambda} \left[k^2 \left(\frac{\Psi}{\mathcal{H}} \right) + X' \right] + 2k^2 \sum_{a,b} \mathcal{M}_{ab}^\lambda \tilde{\sigma}_{va} \Phi_b \\ - 2k^2 \sum_a \tilde{\sigma}_{va}^2 E_\lambda - 2\sigma_{T\times} \sigma_{T+} E_{(1-\lambda)} + 2\sigma_{T(1-\lambda)}^2 E_\lambda. \end{aligned} \quad (3.27)$$

It can be shown that in the long wavelength limit, the former equations (3.18), (3.19)–(3.24), (3.25)–(3.26) and (3.27) are equivalent to the ones obtained in a more general gradient expansion of Einstein equations on large scales [45].

4. Reduced equations and Mukhanov–Sasaki variables

The previous equations (3.18), (3.19)–(3.24), (3.25)–(3.26) and (3.27) form a coupled set of equations for the scalar, vector and tensor modes. In a Friedmann–Lemaître spacetime, the three kind of perturbations decouple and one can arbitrarily set one of the contributions to zero to focus on a given type of mode. This is no more possible here, and in particular, it is not possible to neglect the vector modes. Their contribution, as we shall see, is in fact central to get the correct set of reduced equations.

First, we introduce the Mukhanov–Sasaki variables [10] for scalar and tensor modes as

$$v \equiv aQ, \quad \sqrt{\kappa}\mu_\lambda \equiv aE_\lambda, \quad (4.1)$$

exactly in the same way as in a Friedmann–Lemaître spacetime. These three variables were shown to be the canonical degrees of freedom that shall be quantized during inflation when a Friedmann–Lemaître universe is assumed [5].

4.1. Scalar modes

Let us introduce these variables in our analysis and start by focusing on the scalar modes. First, we note that equation (3.22) can be recast under the more compact form

$$(2\mathcal{H} - \sigma_{\parallel})X = \frac{\kappa}{a}\varphi'v + \sum_{\lambda} \sigma_{\tau\lambda}E_{\lambda}. \quad (4.2)$$

If we now combine equation (3.19) with equation (3.23), replace the vector mode by its expression (3.25) and use the background equations (1.18), we obtain

$$\mathcal{H}X' + 2(\mathcal{H}' + 2\mathcal{H}^2)X + \kappa aV_{\varphi}v + k^2\Psi = \frac{k^2}{3}(\Phi - \Psi) + \frac{2k^2}{3\mathcal{H}}\sigma_{\parallel}\Psi. \quad (4.3)$$

Now, using equation (3.24) to simplify the rhs, and again replacing the vector mode by its expression (3.25), we get

$$(2\mathcal{H} - \sigma_{\parallel})\left(X' + \frac{k^2}{\mathcal{H}}\Psi\right) + 4\kappa a^2VX + 2\kappa aV_{\varphi}v = 4k^2\left(\sum_a \tilde{\sigma}_{Va}^2X - \sum_{a,b,\lambda} \mathcal{M}_{ab}^{\lambda}\tilde{\sigma}_{Va}\tilde{\sigma}_{Vb}E_{\lambda}\right). \quad (4.4)$$

Then, forcing Q in the Klein–Gordon equation (3.18), using also its background version, we obtain

$$Q'' + 2\mathcal{H}Q' + k^2Q + a^2V_{\varphi\varphi}Q + 2a^2V_{\varphi}X - \varphi'\left(X' + \frac{k^2}{\mathcal{H}}\Psi\right) = 0. \quad (4.5)$$

Now, we can replace the last term by using equation (4.4) and the next to last by using equation (4.2) to get

$$\begin{aligned} Q'' + 2\mathcal{H}Q' + k^2Q + a^2V_{\varphi\varphi}Q + 2a^2V_{\varphi}X \\ = \frac{\varphi'}{(2\mathcal{H} - \sigma_{\parallel})}\left[4k^2\left(\sum_a \tilde{\sigma}_{Va}^2X - \sum_{a,b,\lambda} \mathcal{M}_{ab}^{\lambda}\tilde{\sigma}_{Va}\tilde{\sigma}_{Vb}E_{\lambda}\right) - 4\kappa a^2VX - 2\kappa aV'v\right]. \end{aligned} \quad (4.6)$$

Introducing the definitions (4.1), we obtain, after some algebra which requires in particular equations (3.11)–(3.13) to express terms such as $\sum_{a,b} \mathcal{M}_{ab}^{\lambda}\tilde{\sigma}_{Va}\tilde{\sigma}_{Vb}$,

$$v'' + \left(k^2 - \frac{a''}{a} + a^2V_{,\varphi\varphi}\right)v = \frac{1}{a^2}\left(\frac{2a^2\varphi'^2}{2\mathcal{H} - \sigma_{\parallel}}\right)' \kappa v + \sum_{\nu} \frac{1}{a^2}\left(\frac{2a^2\varphi'\sigma_{\tau\nu}}{2\mathcal{H} - \sigma_{\parallel}}\right)' \sqrt{\kappa}\mu_{\nu}. \quad (4.7)$$

This equation is the first central result of this section.

4.2. Tensor modes

The scalar contribution of the tensor equation (3.27) is exactly given by the relation (4.4), so that it reduces, after replacing the vector mode by its expression (3.25), to

$$\begin{aligned} \mu''_\lambda + \left(k^2 - \frac{a''}{a}\right) \mu_\lambda &= -2\mu_{(1-\lambda)}\sigma_{T+}\sigma_{T\times} + 2\mu_\lambda\sigma_{T(1-\lambda)}^2 + \frac{1}{a^2} \left(\frac{2a^2\varphi'\sigma_{T\lambda}}{2\mathcal{H} - \sigma_\parallel}\right)' \sqrt{\kappa}v \\ &+ \sum_\nu \frac{1}{a^2} \left(\frac{2a^2\sigma_{T\nu}\sigma_{T\lambda}}{2\mathcal{H} - \sigma_\parallel}\right)' \mu_\nu + \frac{(a^2\sigma_\parallel)'}{a^2} \mu_\lambda. \end{aligned} \quad (4.8)$$

This equation is the second central result of this section.

4.3. Summary

We have reduced the perturbation equations to a set of three coupled equations for the variables v and μ_λ defined in equation (4.1). If we define two new functions z_s and z_λ by

$$\begin{aligned} \frac{z''_s}{z_s}(\eta, k_i) &\equiv \frac{a''}{a} - a^2V_{,\varphi\varphi} + \frac{1}{a^2} \left(\frac{2a^2\kappa\varphi'^2}{2\mathcal{H} - \sigma_\parallel}\right)' \\ \frac{z''_\lambda}{z_\lambda}(\eta, k_i) &\equiv \frac{a''}{a} + 2\sigma_{T(1-\lambda)}^2 + \frac{1}{a^2}(a^2\sigma_\parallel)' + \frac{1}{a^2} \left(\frac{2a^2\sigma_{T\lambda}^2}{2\mathcal{H} - \sigma_\parallel}\right)', \end{aligned} \quad (4.9)$$

the system reduces to

$$v'' + \left(k^2 - \frac{z''_s}{z_s}\right) v = \sum_\nu \frac{1}{a^2} \left(\frac{2a^2\varphi'\sigma_{T\nu}}{2\mathcal{H} - \sigma_\parallel}\right)' \sqrt{\kappa}\mu_\nu, \quad (4.10)$$

$$\begin{aligned} \mu''_\lambda + \left(k^2 - \frac{z''_\lambda}{z_\lambda}\right) \mu_\lambda &= \frac{1}{a^2} \left(\frac{2a^2\varphi'\sigma_{T\lambda}}{2\mathcal{H} - \sigma_\parallel}\right)' \sqrt{\kappa}v \\ &+ \left[\frac{1}{a^2} \left(\frac{2a^2\sigma_{T\times}\sigma_{T+}}{2\mathcal{H} - \sigma_\parallel}\right)' - 2\sigma_{T\times}\sigma_{T+} \right] \mu_{(1-\lambda)}. \end{aligned} \quad (4.11)$$

Formally, it can be rewritten as

$$V'' + k^2V + \Omega V = \Upsilon V, \quad (4.12)$$

where $V \equiv (v, \mu_+, \mu_\times)$. The matrices Ω and Υ are defined by

$$V'' + \begin{pmatrix} k^2 - \frac{z''_s}{z_s} & 0 & 0 \\ 0 & k^2 - \frac{z''_+}{z_+} & 0 \\ 0 & 0 & k^2 - \frac{z''_\times}{z_\times} \end{pmatrix} V = \begin{pmatrix} 0 & \aleph_+ & \aleph_\times \\ \aleph_+ & 0 & \beth \\ \aleph_\times & \beth & 0 \end{pmatrix} V, \quad (4.13)$$

and the functions $\aleph_\lambda(\eta, k_i)$ and $\beth(\eta, k_i)$ can be read on equations (4.10) and (4.11). This is one of the central results of our study.

When the shear vanishes, these equations decouple and we recover the usual equations [5] for the variables v and μ_λ so that we only have three physical degrees of

freedom. Now, the anisotropy of space is at the origin of some interesting effects. First the functions z_s and z_λ are not functions of time only. They depend on k_i explicitly through the components of the decomposition of the shear. Second, the two types of modes are coupled through a non-diagonal mass term. The mass term and the evolution operator cannot be diagonalized at the same time so that we expect the equivalent of a seesaw mechanism. The importance of the vector modes, that cannot be neglected, has to be emphasized again. Had we neglected them, the mass term would not be correct.

4.4. Sub-Hubble limit

Let us consider the behaviour of the mass term appearing in equations (4.10) and (4.11) in the sub-Hubble limit in which $k/\mathcal{H} \gg 1$. We introduce the two slow-roll parameters as

$$\epsilon \equiv 3 \frac{\varphi'^2}{\varphi'^2 + 2a^2 V}, \quad \delta \equiv 1 - \frac{\varphi''}{\mathcal{H}\varphi'}, \quad (4.14)$$

in terms of which the Friedmann equations take the form

$$\mathcal{H}^2 = \frac{\kappa}{3-\epsilon} V a^2 + \frac{1}{6} \sigma^2, \quad (3-\delta)\mathcal{H}\varphi' + V_\varphi a^2 = 0, \quad (4.15)$$

and

$$\mathcal{H}' = (1-\epsilon)\mathcal{H}^2 + \left(\frac{\epsilon-3}{6}\right)\sigma^2.$$

We now focus on the behaviour of the functions \aleph_λ , \beth , z_s''/z_s and z_λ''/z_λ in the sub-Hubble regime. We define $x \equiv \sigma/\sqrt{6}\mathcal{H}$ and use the fact that, since $\sigma_\parallel/2 \leq \sigma/\sqrt{6}$ (see equation (3.4)), there exists $\alpha < 1$ such that $0 \leq x < \alpha$ due to the positive energy condition (see equation (3.3)). Starting from the definition (4.10) for \aleph_λ we have

$$|\aleph_\lambda| < \left| \frac{1}{a^2} (a^2 \sigma_{\text{T}\lambda})' \right| \times \left| \frac{2\sqrt{\kappa}\varphi'}{2\mathcal{H} - \sigma_\parallel} \right| + 2|\sigma_{\text{T}\lambda}| \times \left| \left(\frac{\sqrt{\kappa}\varphi'}{2\mathcal{H} - \sigma_\parallel} \right)' \right|. \quad (4.16)$$

Now, the property (3.16) implies that the first term of the right-hand side of the inequality is smaller than

$$3\sqrt{2}\mathcal{H}^2 \left| \frac{2\sqrt{\kappa}\varphi'}{2\mathcal{H} - \sigma_\parallel} \right|.$$

Then,

$$\left| \frac{2\sqrt{\kappa}\varphi'}{2\mathcal{H} - \sigma_\parallel} \right| = \sqrt{2\epsilon} \frac{\sqrt{\mathcal{H}^2 - \sigma^2/6}}{\mathcal{H} - \sigma_\parallel/2} \leq \sqrt{2\epsilon} \sqrt{\frac{1+x}{1-x}}. \quad (4.17)$$

Now, since x varies in the range $0 \leq x < \alpha$, we deduce that $\sqrt{(1+x)/(1-x)} \leq \sqrt{(1+\alpha)/(1-\alpha)}$. Equation (3.5) then implies that the second term of the inequality (4.16) is smaller than

$$2\sqrt{6}\mathcal{H} \times \left| \left(\frac{\sqrt{\kappa}\varphi'}{2\mathcal{H} - \sigma_\parallel} \right)' \right|.$$

Then, the absolute value is bounded by

$$\left| \left(\frac{\sqrt{\kappa}\varphi'}{2\mathcal{H} - \sigma_{\parallel}} \right)' \right| < \left| \frac{\sqrt{\kappa}\varphi''}{2\mathcal{H} - \sigma_{\parallel}} \right| + \left| \frac{\sqrt{\kappa}\varphi'}{2\mathcal{H} - \sigma_{\parallel}} \right| \left| \frac{2\mathcal{H}' - \sigma'_{\parallel}}{2\mathcal{H} - \sigma_{\parallel}} \right|.$$

Using the fact that equation (3.4) implies $|\sigma'_{\parallel}| < 10\mathcal{H}^2$ ($|\sigma'_{\parallel}| < 6\mathcal{H}^2 + |2\mathcal{H}\sigma_{\parallel}|$), we obtain that

$$\left| \left(\frac{\sqrt{\kappa}\varphi'}{2\mathcal{H} - \sigma_{\parallel}} \right)' \right| \leq \frac{\sqrt{\epsilon}\mathcal{H}}{\sqrt{2}} \frac{\sqrt{1+\alpha}}{\sqrt{1-\alpha}} \left[(1-\delta) + \frac{2(1-\epsilon) + (1-\epsilon/3)\sqrt{6} + 10}{2(1-\alpha)} \right].$$

Gathering all these terms, we thus conclude that

$$|\aleph_{\lambda}| < \sqrt{\epsilon}\mathcal{H}^2 \sqrt{\frac{1+\alpha}{1-\alpha}} \left[6 + 2\sqrt{3} \left((1-\delta) + \frac{2(1-\epsilon) + (1-\epsilon/3)\sqrt{6} + 10}{2(1-\alpha)} \right) \right]. \quad (4.18)$$

To summarize, we have shown that

$$|\aleph_{\lambda}| < Z\mathcal{H}^2, \quad (4.19)$$

where Z is a finite constant. This constant can in principle be quite large since α can be arbitrarily close to unity in the worst case of an empty universe. A large Z also corresponds to a very ellipsoidal Hubble radius, and this explains why the short wavelength limit has to be taken much smaller than the average Hubble radius.

The same reasoning can be applied for $|\boxminus|$, $|z''_s/z_s|$ and $|z''_{\lambda}/z_{\lambda}|$. Thus, it follows that on sub-Hubble scales the three physical degrees of freedom decouple and behave as harmonic oscillators,

$$V'' + k^2V = 0. \quad (4.20)$$

5. Perturbation of the action

In order to construct canonical quantization variables and to properly normalize the amplitude of their quantum fluctuations, one needs to derive the action for the cosmological perturbations. We will now demonstrate that the previous equations (4.10) and (4.11) can be obtained from the expansion of the action, written in the ADM formalism [46], at second order. Another simpler route would have been to infer the action from the equations of motion, which is always possible up to an overall factor, that could then be fixed by considering a simple limiting case. Still, we prefer to work out the action at second order since it provides a check of the previous computations.

5.1. ADM formalism

In the ADM formalism, we expand the metric as

$$ds^2 = -(N^2 - N_i N^i) dt^2 + 2N_i dx^i dt + g_{ij} dx^i dx^j \quad (5.1)$$

and the Einstein–Hilbert action for a minimally coupled scalar field, takes the form

$$S = \frac{1}{2\kappa} \int dt d^3x \sqrt{-g} [NR^{(3)} + N(K_{ij}K^{ij} - K^2) - \kappa N(g^{ij}\partial_i\varphi\partial_j\varphi + 2V(\varphi)) + \kappa N^{-1}(\dot{\varphi} - N^i\partial_i\varphi)^2], \quad (5.2)$$

where $R^{(3)}$ is the Ricci scalar constructed with the metric g_{ij} and K_{ij} is the extrinsic curvature, defined as

$$K_{ij} \equiv \frac{N^{-1}}{2}(\dot{g}_{ij} - 2\nabla_{(i}N_{j)}), \quad K = K_i^i. \quad (5.3)$$

Every spatial index is now manipulated with the metric g_{ij} . The ADM metric is designed in such a way that the constraints arising from the Einstein equations can be immediately derived from the action. Varying equation (5.2) with respect to the lapse N and the shift N_i , we get the Hamiltonian and momentum constraints, respectively

$$R^{(3)} - (K_{ij}K^{ij} - K^2) - 2V - \kappa g^{ij}\partial_i\varphi\partial_j\varphi + N^{-2}\kappa(\dot{\varphi} - N^i\partial_i\varphi)^2 = 0, \quad (5.4)$$

$$\nabla_j (K_i^j - K\delta_i^j) - N^{-1}\kappa(\dot{\varphi} - N^j\partial_j\varphi)\partial_i\varphi = 0. \quad (5.5)$$

Comparing the form (2.30) of the metric in Newtonian gauge with equation (5.1), we conclude that the lapse N and the shift N_i are given by

$$N^2 = (1 + 2\Phi), \quad N_i = 0 \quad (5.6)$$

and that the metric g_{ij} is

$$g_{ij} = a^2 \left[\gamma_{ij} - 2\Psi \left(\gamma_{ij} + \frac{\hat{\sigma}_{ij}}{H} \right) + 2\partial_{(i}E_{j)} + 2E_{ij} \right]. \quad (5.7)$$

It follows that the Hamiltonian and momentum constraints reduce, at first order, to

$$\begin{aligned} \frac{2}{a^2}\Delta\Psi - \frac{1}{a^2}\hat{\sigma}^{ij}\partial_i\partial_j \left(\frac{\Psi}{H} \right) - 6H\dot{\Psi} + \left(\frac{\Psi}{H} \right) \hat{\sigma}^2 - 3\Psi\hat{\sigma}^2 - \Phi(6H^2 - \hat{\sigma}^2 - \dot{\varphi}^2) \\ + \frac{1}{a}\hat{\sigma}_j^i\partial_i\Phi^j - \hat{\sigma}_j^i(E_j^i) - \kappa V_\varphi\delta\varphi - \kappa\dot{\varphi}\delta\dot{\varphi} = 0, \end{aligned} \quad (5.8)$$

and

$$\begin{aligned} \hat{\sigma}^2\partial_i \left(\frac{\Psi}{H} \right) - \hat{\sigma}_i^j\partial_j \left[\Phi + \left(\frac{\Psi}{H} \right) \right] + 2\partial_i(\dot{\Psi} + H\Phi) \\ - \frac{1}{2a}\Delta\Phi_i + 2\hat{\sigma}^{jl}\partial_j E_{il} - \hat{\sigma}^{jl}\partial_i E_{jl} - \kappa\dot{\varphi}\partial_i\delta\varphi = 0, \end{aligned} \quad (5.9)$$

respectively. Once Fourier transformed, written in conformal time and projected along its scalar and vector components, we recover precisely equations (3.19) and (3.22).

In order to expand the action up to second order in all first order perturbed quantities, we expand the spatial metric as

$$g_{ij} = a^2(\gamma_{ij} + h_{ij}).$$

The inverse metric and its determinant are then given by

$$g^{ij} = a^{-2}(\gamma^{ij} - h^{ij} + h^{il}h_l^j), \quad \sqrt{g} = a^3[1 + \frac{1}{2}h + \frac{1}{8}h^2 - \frac{1}{4}h_j^i h_i^j],$$

where

$$h_{ij} = -2\Psi(\gamma_{ij} + \hat{\sigma}_{ij}/H) + 2\partial_{(i}E_{j)} + 2E_{ij}. \quad (5.10)$$

5.2. Action at zeroth and first orders

The expansions of the action at zeroth and first orders are

$$\begin{aligned} S_0 &= \frac{1}{2\kappa} \int dt d^3x [a^3(-6H^2 + \hat{\sigma}^2 - 2\kappa V + \kappa\dot{\varphi}^2)] \\ &= \frac{1}{2\kappa} \int dt d^3x \left[-4 \frac{d}{dt} (a^3 H) \right], \end{aligned} \quad (5.11)$$

$$\begin{aligned} S_1 &= \frac{1}{2\kappa} \int dt d^3x a^3 [R_1^{(3)} + \hat{\sigma}^{ij} \dot{h}_{ij} - 2\hat{\sigma}_{ij} \hat{\sigma}_i^j h^{jl} + 12H\dot{\Psi} + 3\Psi(6H^2 - \hat{\sigma}^2 + 2\kappa V - \kappa\dot{\varphi}^2) \\ &\quad + \Phi(6H^2 - \hat{\sigma}^2 - 2\kappa V - \kappa\dot{\varphi}^2) - 2\kappa V_\varphi \delta\varphi + 2\kappa\dot{\varphi}\delta\dot{\varphi}] \\ &= \frac{1}{2\kappa} \int dt d^3x \left\{ \partial_i \left[\partial^i (4a\Psi) - \partial^i \left(\frac{a\Psi}{H} \right) \right] \right. \\ &\quad \left. + \frac{d}{dt} \left[\Delta \left(\frac{a\Psi}{H} \right) + a^3 \hat{\sigma}^{ij} h_{ij} + 12a^3 H\Psi + 2a^3 \kappa\dot{\varphi}\delta\varphi \right] \right\}, \end{aligned} \quad (5.12)$$

where we use the notation X_n for the n th order term of the quantity X when expanded in perturbations. Note that we have used the background field equations to go from the first line to the second line in equations (5.11) and (5.12). As can be seen, these two terms can be rewritten in terms of total derivatives. It implies that the only non-trivial term will arise from the expansion of the action at second order.

5.3. Action at second order

A lengthy but straightforward computation shows that the expansion of the action at second order is

$$\begin{aligned} S_2 &= \frac{1}{2\kappa} \int dt d^3x a^3 \left[R_2^{(3)} + N_1 R_1^{(3)} + \frac{1}{2} h R_1^{(3)} + \mathcal{K}_2 + \frac{1}{2} h \mathcal{K}_1 + \frac{1}{8} h^2 \mathcal{K}_0 \right. \\ &\quad - \frac{1}{4} h_j^i h_i^j \mathcal{K}_0 - N_1 \mathcal{K}_1 - \frac{1}{2} N_1 h \mathcal{K}_0 + N_1^2 \mathcal{K}_0 + \kappa (-a^{-2} \partial_i \delta\varphi \partial^i \delta\varphi - V_{\varphi\varphi} \delta\varphi^2 \\ &\quad - 2N_1 V_\varphi \delta\varphi - h V_\varphi \delta\varphi - h N_1 V - \frac{1}{4} h^2 V + \frac{1}{2} h_j^i h_i^j V + \delta\dot{\varphi}^2 - 2N_1 \dot{\varphi} \delta\dot{\varphi} \\ &\quad \left. + N_1^2 \dot{\varphi}^2 + h \dot{\varphi} \delta\dot{\varphi} - \frac{1}{2} h N_1 \dot{\varphi}^2 + \frac{1}{8} h^2 \dot{\varphi}^2 - \frac{1}{4} h_j^i h_i^j \dot{\varphi}^2 \right], \end{aligned} \quad (5.13)$$

where

$$a^2 R_1^{(3)} = 4 \left(\Delta - \frac{\hat{\sigma}^{ij} \partial_i \partial_j}{2H} \right) \Psi, \quad (5.14)$$

$$\begin{aligned} a^2 R_2^{(3)} &= -\partial_l h^{lj} \partial_j h_i^i - 2h^{jl} \partial_j \partial_i h_i^i - 9\partial_i \Psi \partial^i \Psi - \frac{1}{4} \partial_l h^{ij} \partial^l h_{ij} - \frac{1}{2} \partial_l h_{ij} \partial^i h^{lj} \\ &\quad - 6\partial_i (h^{ji} \partial_j \Psi) + \frac{1}{2} \partial^i \partial_i (h^{jl} h_{jl}), \end{aligned} \quad (5.15)$$

$$\mathcal{K}_0 = -6H^2 + \hat{\sigma}^2, \quad (5.16)$$

$$\mathcal{K}_1 = -2H\dot{h} + \hat{\sigma}^{ij} \dot{h}_{ij} - 2\hat{\sigma}_{ij} \hat{\sigma}_i^j h^{li}, \quad (5.17)$$

$$\begin{aligned} \mathcal{K}_2 = & 2H\dot{h}_{ij}h^{ij} - 4H\hat{\sigma}_{ij}h^{il}h_l^j - 2\hat{\sigma}_i^l h^{im}\dot{h}_{ml} + 2\hat{\sigma}_{ij}\hat{\sigma}_i^j h^{im}h_m^l + \frac{1}{4}\dot{h}^{ij}\dot{h}_{ij} \\ & + \hat{\sigma}_{ij}\hat{\sigma}_{lm}h^{im}h^{jl} - \frac{1}{4}\dot{h}^2. \end{aligned} \quad (5.18)$$

The construction of the action at second order shall be pursued in Fourier space, since many non-local operators appear, such as inverse Laplacian Δ^{-1} or $(\sigma^{ij}\partial_i\partial_j)^{-1}$, when using the constraints. Also, it simplifies the use of the background equations (3.11)–(3.13) for the components of the shear σ_{\parallel} , $\sigma_{\nu a}$ and $\sigma_{T\lambda}$ which were defined in Fourier space. We recall that these components are *not* the Fourier transforms of the shear but its decomposition in a basis adapted to a given mode k_i .

The integral of any 3-divergence is clearly zero in Fourier space. For instance, let us consider a typical term like $\partial_l(\Psi\partial^l\Psi)$, then

$$\int d\eta d^3x \partial_l(\Psi\partial^l\Psi) = \int d\eta d^3\mathbf{k} d^3\mathbf{q} [-\mathbf{k} \cdot (\mathbf{k} + \mathbf{q})\Psi_{\mathbf{k}}\Psi_{\mathbf{q}}]\delta^{(3)}(\mathbf{k} + \mathbf{q}) = 0. \quad (5.19)$$

Thus, we first express S_2 in terms of the Fourier modes and then use conformal time. Next, h_{ij} is replaced by its expression (5.10) in function of the variables Ψ , E_j and E_{ij} . All terms involving E_j either vanish or have the form $(E^j)'$, and thus reduce to $-\Phi^j$ in Newtonian gauge. Then, we decompose Φ^j and E_{ij} according to equations (2.9) and (2.14). The constraint (5.9), once expressed in conformal time and in Fourier space, can be projected onto its scalar and vector parts in order to obtain the scalar constraint (4.2) and the vector constraint (3.25). Then, we replace Φ_a in function of X and E_λ using the vector constraint and substitute Φ by $X - \Psi - (\Psi/\mathcal{H})'$. We then eliminate X using the scalar constraint. Finally, we replace E_λ and $\delta\varphi$ by their expressions in terms of the variables μ_λ and v (see equation (4.1)).

After a tedious calculation, that strictly follows the recipe described just above, the action S_2 can be recast under a form that contains only the physical degrees of freedom,

$$\begin{aligned} S_2 = & \frac{1}{2} \int d\eta d^3k \left\{ v'v'^* + \left(\frac{z_s''}{z_s} - k^2 \right) vv^* + \sum_\nu \frac{1}{a^2} \left(\frac{2a^2\sqrt{\kappa}\varphi'\sigma_{T\nu}}{2\mathcal{H} - \sigma_{\parallel}} \right)' (v^*\mu_\nu + v\mu_\nu^*) \right. \\ & \times \sum_\lambda \left[\mu'_\lambda\mu_\lambda'^* + \left(\frac{z_\lambda''}{z_\lambda} - k^2 \right) \mu_\lambda\mu_\lambda^* \right. \\ & \left. \left. + \left[-2\sigma_{T\times}\sigma_{T+} + \frac{1}{a^2} \left(\frac{2a^2\sigma_{T\times}\sigma_{T+}}{2\mathcal{H} - \sigma_{\parallel}} \right)' \right] \mu_{(1-\lambda)}\mu_\lambda^* \right] + \mathcal{T} \right\}, \end{aligned} \quad (5.20)$$

that is, in a more compact way, as

$$S_2 = \frac{1}{2} \int d\eta d^3k (|V'|^2 - k^2|V|^2 + {}^tV(\Omega - \Upsilon)V^* + \mathcal{T}), \quad (5.21)$$

where \mathcal{T} is a total derivative which, for the sake of completeness, is explicitly given in appendix D.

It is clear under this form that the variation of this action with respect to the physical degrees of freedom leads directly to the equations of motion (4.7) and (4.8). More important, it shows that the overall factor is unity. It also follows from this action that the canonical momentum associated with v and μ_λ are $\pi_v = v'^*$ and $\pi_\lambda = \mu_\lambda'^*$.

6. Discussion and conclusions

In this paper, we have presented a full and complete analysis of the theory of cosmological perturbations around a homogeneous but anisotropic background spacetime of the Bianchi I type. We have described the scalar–vector–tensor decomposition and the construction of gauge invariant variables. We have reduced our analysis to a scalar field but it can be easily extended to include hydrodynamical matter.

After presenting the full set of evolution equations for the gauge invariant variables, we have shown that the vector modes can be algebraically expressed in terms of scalar and tensor modes, so that only three physical degrees of freedom remain, one for the scalar sector and two for the tensor sector. Contrary to the Friedmann–Lemaître case, the scalar, vector and tensor perturbation equations do not decorrelate and it was important for the consistency of the computation not to neglect the vector modes. We have shown that these physical degrees of freedom are the trivial generalization of the Mukhanov–Sasaki variables that were derived in a flat Friedmann–Lemaître universe.

We have also constructed the action for the cosmological perturbations up to second order and demonstrated that, after use of the constraints was made, it only contains the physical degrees of freedom and takes a canonical form. We have also shown that in the sub-Hubble limit the scalar and tensor degrees of freedom decouple and behave as standard harmonic oscillators. It follows that one can apply the standard quantization procedure [5] and properly define the normalization of the amplitude of their quantum fluctuations.

The anisotropy of the underlying space induces two physical effects: (1) the equations of motion explicitly involve the wavenumber k_i and (2) a non-diagonal mass term that describes the coupling between scalar perturbation and gravitational waves is at the origin of a scalar–tensor seesaw mechanism.

Since the shear decays as the inverse of the second power of the scale factor, the universe isotropizes and tends toward a Friedmann–Lemaître spacetime. The modes that exit the Hubble radius during inflation while the shear is non-negligible will experience the seesaw mechanism and will have the primordial anisotropy imprinted on their statistical properties. Modes of smaller wavelength will not reflect the anisotropy. It follows that an early Bianchi I phase may be at the origin of a primordial anisotropy of the cosmological perturbations, mainly on large angular scales. The companion article [49] describes such a scenario of early anisotropic slow-roll inflation. Since the post-inflationary evolution is well described by a Friedmann–Lemaître spacetime, observable effects, and in particular those related to the CMB anomalies we alluded to in the introduction, can be taken into account easily once the initial conditions are known. This investigation, that we plan to do later, is beyond the scope of the present work.

Our analysis extends and sheds some light on the robustness of the quantization procedure that was developed under the assumption of a Friedmann–Lemaître background, and thus on the predictions of the standard inflationary scenario. We emphasize that this work is very conservative and that no new speculative hypothesis was invoked. Indeed, we are not claiming that such a primordial anisotropy is needed. On the one hand, it can be used to set stronger constraints on the primordial shear. On the other hand it can also be a useful example for the study of second order perturbations, in which a shear appears only at first order and induces a correlation between scalar and

tensor at second order [47, 48], and more generally for the understanding of quantum field theory in curved (cosmological) spacetimes [19]. One may for instance wonder whether this analysis can be further extended to other Bianchi type or to non-spatially flat spacetimes.

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Appendix A. Details on Bianchi I universes

A.1. Geometrical quantities in conformal time

Starting from the metric (1.7) in conformal time, the expressions of the Christoffel symbols are

$$\Gamma_{00}^0 = \mathcal{H}, \quad \Gamma_{ij}^0 = \mathcal{H}\gamma_{ij} + \sigma_{ij}, \quad \Gamma_{0j}^i = \mathcal{H}\delta_j^i + \sigma_j^i, \quad (\text{A.1})$$

where we have used the definition of the shear to express $\gamma_{ij}' = 2\sigma_{ij}$ so that

$$(\gamma^{ij})' = -2\sigma^{ij}, \quad (\text{A.2})$$

and indeed trivially $(\gamma_j^i)' = (\delta_j^i)' = 0$.

We deduce that the non-vanishing components of the Ricci tensor are given by

$$a^2 R_0^0 = 3\mathcal{H}' + \sigma^2 \quad (\text{A.3})$$

$$a^2 R_j^i = (\mathcal{H}' + 2\mathcal{H}^2)\delta_j^i + 2\mathcal{H}\sigma_j^i + (\sigma_j^i)', \quad (\text{A.4})$$

where we recall that $\sigma^2 = \sigma_{ij}\sigma^{ij}$. The Ricci scalar is

$$a^2 R = 6(\mathcal{H}' + \mathcal{H}^2) + \sigma^2. \quad (\text{A.5})$$

The non-vanishing components of the Einstein tensor are thus given by

$$a^2 G_0^0 = -3\mathcal{H}^2 + \frac{1}{2}\sigma^2 \quad (\text{A.6})$$

$$a^2 G_j^i = -(2\mathcal{H}' + \mathcal{H}^2 + \frac{1}{2}\sigma^2)\delta_j^i + 2\mathcal{H}\sigma_j^i + (\sigma_j^i)'. \quad (\text{A.7})$$

For a general fluid with stress-energy tensor of the form

$$T_{\mu\nu} = \rho u_\mu u_\nu + P(g_{\mu\nu} + u_\mu u_\nu) + \pi_{\mu\nu}, \quad (\text{A.8})$$

where ρ is the energy density, P the isotropic pressure and $\pi_{\mu\nu}$ the anisotropic stress ($\pi_{\mu\nu}u^\mu = 0$ and $\pi_\mu^\mu = 0$), it implies that the Einstein equation takes the form

$$3\mathcal{H}^2 = \kappa a^2 \rho + \frac{1}{2}\sigma^2, \quad (\text{A.9})$$

$$\mathcal{H}' = -\frac{\kappa a^2}{6}(\rho + 3P) - \frac{1}{3}\sigma^2, \quad (\text{A.10})$$

$$(\sigma_j^i)' = -2\mathcal{H}\sigma_j^i + \kappa a^2 \tilde{\pi}_j^i, \quad (\text{A.11})$$

which correspond respectively to the ‘00’-component and trace and trace-free part of the ‘ ij ’-equation. The conservation equation for matter reads

$$\rho' + 3\mathcal{H}(\rho + P) + \sigma_{ij}\tilde{\pi}^{ij} = 0, \quad (\text{A.12})$$

where the ij -component of $\pi_{\mu\nu}$ has been defined as $a^2\tilde{\pi}_{ij}$ (so that $\tilde{\pi}_j^i = \gamma^{ik}\tilde{\pi}_{kj}$).

To close this system, one needs to specify, as usual, an equation of state for the fluid, that is an equation $P(\rho)$, but also to provide a description for $\pi_{\mu\nu}$. The latter vanishes for a perfect fluid and for a scalar field.

A.2. Bianchi I universes in cosmic time

Starting from the metric (1.3) in cosmic time, the expressions of the Christoffel symbols are

$$\hat{\Gamma}_{ij}^0 = a^2[H\gamma_{ij} + \frac{1}{2}\dot{\gamma}_{ij}], \quad \hat{\Gamma}_{0j}^i = a^2[H\delta_j^i + \frac{1}{2}\gamma^{ik}\dot{\gamma}_{kj}]. \quad (\text{A.13})$$

The Einstein equations take the form

$$3H^2 = \kappa\rho + \frac{1}{2}\hat{\sigma}^2, \quad (\text{A.14})$$

$$\frac{\ddot{a}}{a} = -\frac{\kappa}{6}(\rho + 3P) - \frac{1}{3}\hat{\sigma}^2, \quad (\text{A.15})$$

$$(\hat{\sigma}_j^i)^\cdot = -3H\hat{\sigma}_j^i + \kappa\tilde{\pi}_j^i, \quad (\text{A.16})$$

and the conservation equation for the matter reads

$$\dot{\rho} + 3H(\rho + P) + \hat{\sigma}_{ij}\tilde{\pi}^{ij} = 0. \quad (\text{A.17})$$

A.3. Bianchi I universes in the 1 + 3 formalism

The dynamics of Bianchi universes can be discussed in terms of the 1 + 3 covariant formalism (see e.g. [39, 42]). This description assumes the existence of a preferred congruence of worldlines representing the average motion of matter. The central object is the 4-velocity u^μ of these worldlines. The symmetries imply that it is orthogonal to the hypersurfaces of homogeneity,

$$u^\mu = -\delta_0^\mu, \quad u_\mu = \delta_{\mu 0}. \quad (\text{A.18})$$

The projection operator on the constant time hypersurfaces is defined as

$$\perp_{\mu\nu} = g_{\mu\nu} + u_\mu u_\nu.$$

Its only non-vanishing components being $\perp_{ij} = a^2(t)\gamma_{ij}(t)$.

The central kinematical quantities arise from the decomposition

$$\nabla_\mu u_\nu = -u_\mu \dot{u}_\nu + \frac{1}{3}\Theta \perp_{\mu\nu} + \Sigma_{\mu\nu} + \omega_{\mu\nu}. \quad (\text{A.19})$$

For a Bianchi I universe, homogeneity implies that $D_\mu f = 0$ for all scalar functions (where the spatial derivative operator is defined as $D_\mu T^\alpha = \perp_\mu^{\mu'} \perp_{\alpha'}^\alpha \nabla_{\mu'} T^{\alpha'}$ etc). Since $D_\mu P = 0$, the flow is geodesic and irrotational ($\omega_{\mu\nu} = 0$) so that the acceleration also vanishes, $a_\mu = 0$, and we are just left with the expansion, Θ , and the shear $\Sigma_{\mu\nu}$.

It is clear from the form (1.3) that

$$\Theta = 3H. \quad (\text{A.20})$$

The only non-vanishing components of the shear is expressed simply in terms of the trace-free part of the Christoffel symbol $\hat{\Gamma}_{ij}^0$ as

$$\Sigma_{ij} = a^2 \hat{\sigma}_{ij}$$

so that

$$\Sigma^2 = \hat{\sigma}^2 = \sum_{i=1}^3 \hat{\beta}_i^2. \quad (\text{A.21})$$

With the general form (A.8) for the stress-energy tensor, we get the conservation equation

$$\dot{\rho} + (\rho + P)\Theta + \Sigma_{\mu\nu} \pi^{\mu\nu} = 0, \quad (\text{A.22})$$

which reduces to equation (A.12).

The Raychaudhuri [42] equation simplifies to

$$\dot{\Theta} + \frac{1}{3}\Theta^2 = -\Sigma^2 - 4\pi G(\rho + 3P). \quad (\text{A.23})$$

Since $\Theta = 3H$, the rhs is simply $3\ddot{a}/a = 3\mathcal{H}'/a^2$ so that it reduces to equation (A.10).

The Gauss equation takes the form

$${}^{(3)}R_{\mu\nu} = -u^\alpha \nabla_\alpha \Sigma_{\mu\nu} - \Theta \Sigma_{\mu\nu} + \kappa \pi_{\mu\nu} + \frac{2}{3} \perp_{\mu\nu} (\kappa \rho - \frac{1}{3}\Theta^2 + \frac{1}{2}\Sigma^2). \quad (\text{A.24})$$

We have to be careful here since $u^\alpha \nabla_\alpha \Sigma_{\mu\nu}$ is not equal to $\partial_t \Sigma_{\mu\nu}$. It is given, for the ij -component by $u^\alpha \nabla_\alpha \Sigma_{ij} = (a^2 \hat{\sigma}_{ij})^\cdot - 2\hat{\Gamma}_{0j}^k a^2 \hat{\sigma}_{ik} = a^2 [(\hat{\sigma}_{ij})^\cdot - 2\hat{\sigma}_{ik} \hat{\sigma}_j^k]$.

In the particular case of a Bianchi I spacetime, ${}^{(3)}R_{\mu\nu} = 0$ so that the trace of the generalized Friedmann equation [42] reduces to

$$\kappa \rho - \frac{1}{3}\Theta^2 + \frac{1}{2}\Sigma^2 = {}^{(3)}R.$$

Shifting to conformal time, this gives equation (A.9) when ${}^{(3)}R = 0$. The trace-free part leads to equation (A.11). Note that this implies that when the anisotropic stress vanishes, $a^3 \Sigma_{\mu\nu}$ is constant for the $u^\alpha \nabla_\alpha$ time derivative but that it implies that $a^2 \sigma_j^i$ is constant in terms of the ordinary conformal time derivative. The identification of $u^\mu \nabla_\mu$ and ∂_t holds only for scalars (see e.g. [47]).

A.4. General solution of the background equations

It is useful to determine general solutions of the evolution of the background spacetime [2]. We concentrate on the particular case in which $\pi_{\mu\nu} = 0$ (relevant for scalar fields) and first set

$$\beta_i = B_i W(t). \quad (\text{A.25})$$

Equations (1.10) and (1.16) then imply that

$$\left(\sum B_i^2 \right) \dot{W}^2(t) = \frac{\mathcal{S}^2}{a^6} \quad \text{or} \quad \left(\sum B_i^2 \right) [W'(\eta)]^2 = \frac{\mathcal{S}^2}{a^4},$$

from which we deduce that

$$W(t) = \int \frac{dt}{a^3} \quad \text{or} \quad W(\eta) = \int \frac{d\eta}{a^2}. \quad (\text{A.26})$$

The constraints (1.5) and (1.10) imply that the B_i must satisfy

$$\sum_{i=1}^3 B_i = 0, \quad \sum_{i=1}^3 B_i^2 = \mathcal{S}^2, \quad (\text{A.27})$$

which are trivially solved by setting

$$B_i = \sqrt{\frac{2}{3}} \mathcal{S} \sin \alpha_i, \quad \alpha_p = \alpha + \frac{2\pi}{3} p, \quad p \in \{1, 2, 3\}. \quad (\text{A.28})$$

Thus, the general solution is of the form

$$\beta_i(t) = \sqrt{\frac{2}{3}} \mathcal{S} \sin \left(\alpha + \frac{2\pi}{3} i \right) \times W, \quad (\text{A.29})$$

where a is solution of

$$3H^2 = \kappa\rho + \frac{1}{2} \frac{\mathcal{S}^2}{a^6}. \quad (\text{A.30})$$

Once an equation of state is specified, the conservation equation gives $\rho[a]$ and we can solve for $a(t)$.

As an example, consider the case of a pure cosmological constant, $V = \text{const.}$ and $\dot{\varphi} = 0$. The Friedmann equation takes the form

$$H^2 = V_0 \left[1 + \left(\frac{a_*}{a} \right)^6 \right],$$

with $V_0 \equiv \kappa V/3$ and $a_* \equiv (\mathcal{S}/6V_0)^{1/6}$. It can be integrated easily to get

$$a(t) = a_* \left[\sinh \left(3\sqrt{V_0} t \right) \right]^{1/3}. \quad (\text{A.31})$$

Asymptotically, it behaves as the scale factor of a de Sitter universe, $a \propto \exp(\sqrt{V_0} t)$ but at early time the shear dominates and $a \propto t^{1/3}$.

Appendix B. Properties of the polarizations

We summarize here the main properties of the polarization tensors, defined in section 2.1. The time derivative of the polarization tensor is given by

$$(\varepsilon_{ij}^\lambda)' = -(\sigma^{kl} \varepsilon_{kl}^\lambda) P_{ij} - (\sigma^{kl} P_{kl}) \varepsilon_{ij}^\lambda + 4\sigma_{(i}^k \varepsilon_{j)k}^\lambda, \quad (\text{B.1})$$

or equivalently,

$$(\varepsilon_j^{i\lambda})' = -(\sigma^{kl} \varepsilon_{kl}^\lambda) P_j^i - (\sigma^{kl} P_{kl}) \varepsilon_j^{i\lambda} + 2\sigma_j^k \varepsilon_k^{i\lambda}. \quad (\text{B.2})$$

In terms of the decomposition (3.1) of the shear tensor, it takes the forms

$$(\varepsilon_{ij}^\lambda)' = -\sigma_{\tau\lambda} P_{ij} + \sigma_{\parallel} \varepsilon_{ij}^\lambda + 4\sigma_{(i}^k \varepsilon_{j)k}^\lambda, \quad (\text{B.3})$$

and

$$(\varepsilon_j^{i\lambda})' = -\sigma_{T\lambda} P_j^i + \sigma_{\parallel} \varepsilon_j^{i\lambda} + 2\sigma_j^k \varepsilon_k^{i\lambda}, \quad (\text{B.4})$$

and the time derivative of the projection operator P_{ij} is given by

$$(P^{ij})' = -2\sigma_{T+} \varepsilon_+^{ij} + 2\sigma_{\parallel} P^{ij}. \quad (\text{B.5})$$

Let us also give the following relations that turn to be useful for the derivation of the evolution equation of the tensor modes

$$\begin{aligned} \sigma_{il} \varepsilon_{\lambda}^{lj} \sigma_{jp} \varepsilon_{\lambda}^{pi} &= \frac{1}{4} \sigma_{\parallel}^2 + \frac{1}{2} (\sigma_{\lambda}^2 - \sigma_{(1-\lambda)}^2) \\ \sigma_{il} \varepsilon_{\lambda}^{lj} \sigma_{jp} \varepsilon_{(1-\lambda)}^{pi} &= \sigma_{T \times} \sigma_{T+} \\ \sigma_{il} \varepsilon_{\lambda}^{lj} \sigma^{ip} \varepsilon_{jp}^{\lambda} &= \frac{1}{4} \sigma_{\parallel}^2 + \frac{1}{2} (\sigma_{T+}^2 + \sigma_{T \times}^2) + \frac{1}{2} \sum_a \sigma_{va}^2 \\ \sigma_{il} \varepsilon_{\lambda}^{lj} \sigma^{ip} \varepsilon_{jp}^{(1-\lambda)} &= 0. \end{aligned} \quad (\text{B.6})$$

Appendix C. Perturbed quantities

For the sake of completeness, let us give the expression of the Lie derivative (2.36) of the displacement ξ (2.37)

$$\begin{aligned} \mathcal{L}_{\xi} \bar{g}_{00} &= -2a^2 (T' + \mathcal{H}T) \\ \mathcal{L}_{\xi} \bar{g}_{0i} &= a^2 (\xi'_i - \partial_i T - 2\sigma_{ji} \xi^j) \\ \mathcal{L}_{\xi} \bar{g}_{ij} &= a^2 [2\partial_{(i} \xi_{j)} + 2\mathcal{H}T \gamma_{ij} + 2T \sigma_{ij}]. \end{aligned} \quad (\text{C.1})$$

The expression of the components of the stress-energy tensor of the scalar field at first order is

$$a^2 \delta T_0^0 = \varphi'^2 \Phi - \varphi' \chi' - V_{\varphi} a^2 \chi, \quad (\text{C.2})$$

$$a^2 \delta T_i^0 = -\partial_i [\varphi' \chi], \quad (\text{C.3})$$

$$a^2 \delta T_i^j = -\delta_j^i [\varphi'^2 \Phi - \varphi' \chi' + V_{\varphi} a^2 \chi]. \quad (\text{C.4})$$

Note that they are exactly the same expressions than in a Friedmann–Lemaître spacetime. It comes from the fact that δg_{ij} never appears. Indeed, the δT_{ij} etc components will be different compared to the Friedmann–Lemaître case.

In Newtonian gauge the Christoffel symbols at first order take the form

$$\delta \Gamma_{00}^0 = \Phi' \quad (\text{C.5})$$

$$\delta \Gamma_{0j}^0 = \partial_j \Phi \quad (\text{C.6})$$

$$\delta \Gamma_{ij}^0 = \mathcal{H} h_{ij} + \frac{1}{2} h'_{ij} - 2\mathcal{H} \Phi \gamma_{ij} - 2\Phi \hat{\sigma}_{ij} \quad (\text{C.7})$$

$$\begin{aligned} \delta \Gamma_{0j}^i &= \frac{1}{2} h_j^{i'} - \hat{\sigma}_{kj} h^{ki} \\ &= \frac{1}{2} (h_j^i)' - \hat{\sigma}_{kj} h^{ki} + h_{kj} \hat{\sigma}^{ki} \end{aligned} \quad (\text{C.8})$$

$$\delta \Gamma_{jk}^i = \frac{1}{2} \gamma^{li} (\partial_j h_{lk} + \partial_k h_{jl} - \partial_l h_{jk}). \quad (\text{C.9})$$

In Newtonian gauge, the expressions of the components of the Einstein tensor at first order are

$$a^2 \delta G_0^0 = -2\Delta\Psi + 6\mathcal{H}\Psi' + 2\sigma^2\Psi - \left(\frac{\Psi}{\mathcal{H}}\right)' \sigma^2 + \frac{\sigma^{ij}}{\mathcal{H}} \partial_i \partial_j \Psi - \sigma^{ij} \partial_i \Phi_j + (E_j^i)' \sigma_i^j + (6\mathcal{H}^2 - \sigma^2)\Phi, \quad (\text{C.10})$$

$$a^2 \delta G_i^0 = -\sigma^2 \frac{\partial_i \Psi}{\mathcal{H}} + \sigma_i^j \partial_j \left[\Phi + \Psi + \left(\frac{\Psi}{\mathcal{H}}\right)' \right] - 2\partial_i(\Psi' + \mathcal{H}\Phi) + \frac{1}{2}\Delta\Phi_i - 2\sigma^{jk} \partial_j E_{ik} + \sigma^{jk} \partial_i E_{jk}, \quad (\text{C.11})$$

$$a^2 \delta G_j^i = \delta_j^i \left[2\Psi'' + (2\mathcal{H}^2 + 4\mathcal{H}')\Phi + \Delta(\Phi - \Psi) + 2\mathcal{H}\Phi' + 4\mathcal{H}\Psi' \right] + \partial^i \partial_j (\Psi - \Phi) - \frac{2}{\mathcal{H}} \sigma_k^{(i} \partial_j) \partial^k \Psi + \sigma_j^i \left[-\mathcal{H} \left(\frac{\Psi'}{\mathcal{H}^2}\right)' + \left(\frac{\mathcal{H}'}{\mathcal{H}^2}\right)' \Psi + \frac{\Delta\Psi}{\mathcal{H}} - \Phi' - \Psi' \right] + \delta_j^i \left[\sigma^2 \left(\Phi + \left(\frac{\Psi}{\mathcal{H}}\right)' - 2\Psi \right) + \frac{\sigma^{kl}}{\mathcal{H}} \partial_k \partial_l \Psi \right] + (E_j^i)'' + 2\mathcal{H}(E_j^i)' - \Delta E_j^i + 2[\sigma_k^i (E_j^k)' - \sigma_j^k (E_k^i)'] - [(E_l^k)' \sigma_k^l] \delta_j^i + \delta_j^i \sigma^{kl} \partial_k \Phi_l - \gamma^{ik} [\partial_{(k} (\Phi_{j)})' + 2\mathcal{H} \partial_{(k} \Phi_{j)} - 2\sigma_{(k}^l \partial_{|l} \Phi_{j)}]. \quad (\text{C.12})$$

Appendix D. Details concerning the expansion of the action

The total time derivative \mathcal{T} that appears in equation (5.20) is explicitly given by

$$\mathcal{T} = \left[-a^2 \sigma_i^l h_{ml} h^{im} + a^2 h_{ij} h^{ij} \mathcal{H} + \mathcal{H}(a\delta\varphi)^2 - \frac{(\varphi'v)^2}{2\mathcal{H} - \sigma_{\parallel}} - \frac{2a\varphi'v\sigma^{ij} E_{ij}}{2\mathcal{H} - \sigma_{\parallel}} - \frac{a^2(\sigma^{ij} E_{ij})^2}{2\mathcal{H} - \sigma_{\parallel}} - \mathcal{H} \mu_{ij} \mu_{kl} \gamma^{ik} \gamma^{jl} - 4a^2 \sigma^{jk} E_{ik} \Psi \left(\gamma_j^i + \frac{\sigma_j^i}{\mathcal{H}} \right) + \frac{(2\mathcal{H} - \sigma_{\parallel})a^2 k^2 \Psi^2}{\mathcal{H}^2} - 18\Psi^2 \mathcal{H} a^2 + \frac{7\Psi^2 \sigma^2 a^2}{\mathcal{H}} + \frac{2\Psi^2 a^2 \sigma_i^j \sigma_j^k \sigma_k^i}{\mathcal{H}^2} - \frac{2va\Psi\varphi''}{\mathcal{H}} - 6\Psi\varphi' av \right]. \quad (\text{D.1})$$

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