A unified view of anisotropies in the astrophysical gravitational wave background

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In the literature different approaches have been proposed to compute the anisotropies of the astrophysical gravitational wave background. The different expressions derived, although starting from our work `1704.06184`, seem to differ. In this article, we compare the various theoretical expressions proposed so far and we provide a separate derivation based on a Boltzmann approach. We show that all the theoretical formula in the literature are equivalent and boil down to the one of `1704.06184` when a proper matching of terms and integration by parts are performed. The difference between the various predictions presented for anisotropies in a cosmological context can only lie in the astrophysical modeling of sources, and neither in the theory nor in the cosmological description of the large scale structures. Finally we comment on the gauge invariance of expressions.

Introduction

The anisotropic stochastic gravitational-wave (GW) background, generated by the superposition of various unresolved astrophysical and cosmological sources, has attracted a growing attention in the past years. It is probable that the astrophysical gravitational waves background (AGWB) from unresolved stellar-mass binaries may be detected within a few years of operation of the LIGO-Virgo network [1, 2].

While the homogeneous component of the AGWB, i.e. its monopole, in a perfectly homogeneous and isotropic spacetime has been studied for many years, the computation of its anisotropies and their angular power spectrum has only been derived recently. The computation of the anisotropies of the AGWB relies on *(i)* the underlying cosmology (assumed to be well described by a Friedmann-Lemaître cosmology) *(ii)* the large scales structure or galaxy clustering and its effect on GW propagation (described using linear perturbation and effectively including non-linearities in the matter evolution) and *(iii)* the local astrophysics on sub-galactic scales (given by the astrophysical modeling of the time-dependent GW luminosity $\mathcal{L}_{\rm G}$ of a given galaxy as a function of halo mass $M_{\rm G}$ and GW frequency at emission $\nu_{\rm G}$).

The first theoretical expression of anisotropies of the AGWB was derived in Refs. [3, 4], using a coarse-graining approach. In this work both a covariant expression valid in any spacetime and its application to the case of a perturbed Friedman-Lemaître universe are presented. Using this framework, the first predictions of the AGWB power spectrum have been presented in Ref. [5], for the contribution of binary black holes (BH) mergers and for frequencies in the LIGO-Virgo band. Making use of the astrophysical framework described in Refs. [6–8], the influence of various astrophysical parameters/functions (BH formation models, BH mass cut-off, stellar initial mass function, metallicity) on the angular power spectrum was investigated in Ref. [9] and extended to the LISA band in Ref. [10]. Similar predictions were proposed, on the basis of the same formalism, in Refs. [11, 12] (see Ref. [13] for a comment on the analytic approach used in these works). The first attempt to describe anisotropies of a GW with a Boltzmann approach was proposed by Ref. [14] while Ref. [15] refines it by introducing an emissivity function that realistically describes GW emission at the galactic scale.

Recently, Ref. [16] proposed a new derivation of anisotropies in a cosmological context, starting from the covariant expression presented in Ref. [3]. Using methods adapted to large scale structure observed in redshift space, the authors show the presence of new effects in their result, not previously taken into account in the literature, in particular a Kaiser-like projection term that would become relevant on large angular separation.

The goal of this article is to compare different expressions for anisotropies, obtained using the covariant approach of Ref. [3] as a starting point and making use of different perturbation methods. In particular we investigate the origin of the new projection terms recently found in Ref. [16]. To that purpose, we will first provide in § I a line-of-sight approach of our covariant expression [3]. Then, after a brief reminder of its implementation within cosmological perturbation theory in § II, we compare in § III the results of Ref. [3] in a cosmological context with the result recently presented in Ref. [16]. We show that the discrepancy arises from the different choices on how to model the GW luminosity of galaxies. In particular we show that projection effects, which play an important role in galaxy surveys, do not correspond to observable cosmological effects in the context of a background. To finish, we comment on the modeling of the luminosity perturbation in § IV.

Notation: We assume a standard Friedmann-Lemaître background spacetime with scale factor a, conformal time η and conformal Hubble function $\mathcal{H} \equiv {\rm d}\ln a/{\rm d}\eta$.

I. LINE OF SIGHT EXPRESSION

A simplified approach is to describe the AGWB observed today in terms of its energy density per units of observed frequency $\nu_{\rm O}$ and solid angle $\mathrm{d}\Omega_{\rm O}$, as

$$\frac{\mathrm{d}^3\rho_{\rm GW}(\boldsymbol{e},\nu_{\rm O})}{\mathrm{d}\nu_{\rm O}\mathrm{d}^2\Omega_{\rm O}} = \int \frac{n_{\rm G}[x^\mu(\lambda)]}{[1+z(\lambda)]^3}\,\frac{\mathcal{L}_{\rm G}[x^\mu(\lambda),\nu(\lambda)]}{4\pi}\frac{\mathrm{d}\tau}{\mathrm{d}\lambda}\mathrm{d}\lambda\,, \tag{1}$$

where $x^\mu(\lambda)$ is the geodesic followed by the GW observed in direction $\boldsymbol{e}$ and $z(\lambda)$ describes the redshift along that geodesic. It is defined as $1+z(\lambda) \equiv [u_\mu p^\mu_{\rm GW}]_{x^\mu(\lambda)}/[u_\mu p^\mu_{\rm GW}]_{\rm O}$, where $p^\mu_{\rm GW} = \mathrm{d}x^\mu/\mathrm{d}\lambda$ is the GW four-momentum and $u^\mu_{\rm O,G}$ the tangent vector to the observer (O) or source (G) geodesic. The physical number density of galaxies is denoted as $n_{\rm G}$ and the GW luminosity of galaxies as $\mathcal{L}_{\rm G}$. They are both defined in the matter comoving frame (i.e. comoving with $u^\mu_{\rm G}$ at $x^\mu(\lambda)$), and the latter depends on the redshifted frequency $\nu(\lambda) = \nu_{\rm O}(1+z(\lambda))$.

The expression (1) is fully covariant in the sense that it is valid for any spacetime metric. It was derived using the distance reciprocity relation $D_{\rm A}/D_{\rm L} = 1/(1+z)^2$ between the luminosity ($D_{\rm L}$) and angular ($D_{\rm A}$) distances, since the power received scales by definition as $1/D_{\rm L}^2$ while the area spanned by a solid angle scales like $D_{\rm A}^2$. The factor $\mathrm{d}\tau/\mathrm{d}\lambda = -u_\mu p^\mu_{\rm GW}$ accounts for the physical depth spanned (equal to the proper time of the source spanned $\mathrm{d}\tau$) in an infinitesimal change $\mathrm{d}\lambda$ of the affine parameter used in f integration. Instead of integrating with this affine parameter, one usually integrates over the conformal time η and makes use of the relation $(\mathrm{d}\tau/\mathrm{d}\lambda)\mathrm{d}\lambda = (\mathrm{d}\tau/\mathrm{d}\eta)\mathrm{d}\eta$.

Equation (1) is rather formal and requires some extra manipulations to obtain the general expression in a perturbed cosmological framework. However, in this general form it is apparent that it is the integral form of a Boltzmann equation with a source term but without any collision term (as was actually initially proposed in the context of GW by Ref. [14]). Indeed, defining a total derivative in phase space along a geodesic by $\frac{\mathrm{D}}{\mathrm{D}\eta} \equiv \partial_\eta + \frac{p^i}{p^0}\partial_i + \frac{\mathrm{d}p^i}{\mathrm{d}\eta}\frac{\partial}{\partial p^i}$, Eq. (1) is the integrated version of

$$\frac{\mathrm{D}}{\mathrm{D}\eta} f(x^\mu,\nu,\boldsymbol{e}) = \mathcal{E}(x^\mu,\nu,\boldsymbol{e})\,, \tag{2}$$

which is a Boltzmann-type equation for the distribution function

$$f(x^\mu,\nu,\boldsymbol{e}) \equiv \nu^{-3}\frac{\mathrm{d}^3\rho_{\rm GW}(x^\mu,\nu,\boldsymbol{e})}{\mathrm{d}\nu\mathrm{d}^2\Omega}\,, \tag{3}$$

with emissivity function

$$\mathcal{E}(x^\mu,\nu,\boldsymbol{e}) \equiv \frac{\mathrm{d}\tau}{\mathrm{d}\eta}(x^\mu,\boldsymbol{e}) n_{\rm G}(x^\mu)\frac{\mathcal{L}_{\rm G}(x^\mu,\nu)}{4\pi\nu^3}\,. \tag{4}$$

The galaxy number density $n_{\rm G}$ converts the galaxy luminosity $\mathcal{L}_{\rm G}$ into a luminosity per unit of volume. In this Boltzmann approach, the factor $\mathrm{d}\tau/\mathrm{d}\eta$ accounts for the fact that the rate of emission is defined, as it should, with respect to the time in the comoving frame of GW sources. The practical expression of this time conversion factor is

$$\frac{\mathrm{d}\tau}{\mathrm{d}\eta} = \frac{-u_\mu p^\mu}{p^0} = (1+z)\frac{\nu_{\rm O}}{p^0}\,. \tag{5}$$

Neither the line of sight expression (1) nor the Boltzmann equation (2) make any hypothesis on the spacetime symmetries. Note however that at the core of these descriptions is indeed hidden some kind of a coarse-graining since the source term involves the continuous galaxy number density field, which allows one to define a continuum of sources. All the works mentioned earlier make such an assumption, whether it is explicitly said or not. For comparison, we stress that this is not the case for the collision term of the CMB radiative transfer, which is based on the more fundamental microscopic form of Compton interactions.

II. LINEAR PERTURBATION THEORY

A. General expression

Let us now restrict to the framework of linear cosmological perturbation theory, with a metric perturbed in full generality (but with only scalar perturbations describing the large scale structure) as

$$\begin{aligned}\mathrm{d}s^2 = a^2\big[&-(1+2\psi)\mathrm{d}\eta^2 + 2\partial_i B\mathrm{d}x^i\mathrm{d}\eta \\ &+(1-2\phi)\delta_{ij}\mathrm{d}x^i\mathrm{d}x^j + 2\partial_i\partial_j E\mathrm{d}x^i\mathrm{d}x^j\big]\,.\end{aligned} \tag{6}$$

One needs to solve for the geodesic of gravitons, so as to determine their trajectory, and the frequency evolution and the redshift along a geodesic. In practice, when integrating with the parameter η, it is sufficient to consider the background geodesic which is a straight line, since time-delays and lensing appear only as second order contributions. For simplicity we omit here the contributions of the perturbations at the observer's position since observables do not depend on them. The redshift perturbation is

$$\begin{aligned}1+z &= (1+\bar{z})[1+\delta\ln(1+z)]\,, &(7)\\ \delta\ln(1+z) &\equiv -\psi + e^i\partial_i(v+B) &(8)\\ &\quad - \int_\eta^{\eta_{\rm O}}[\psi' + \phi' + e^i e^j\partial_i\partial_j(B-E')]\mathrm{d}\tilde{\eta}\,,\end{aligned}$$

with $1+\bar{z} \equiv 1/a$, and $u^i = a^{-1}v^i = a^{-1}\partial^i v$ is the spatial component of the matter velocity. The perturbed time conversion factor [Eq. (5)] is

$$\frac{\mathrm{d}\tau}{\mathrm{d}\eta} = a[1+\psi+e^i\partial_i(v+B)]\,. \tag{9}$$

Hence, with $\bar{\nu} = (1+\bar{z})\nu_{\rm O}$, we find from Eq. (1) that the linearly perturbed GW background energy density per units of observed frequency and angle is

$$\frac{\mathrm{d}^3\rho_{\rm GW}(\boldsymbol{e},\nu_{\rm O})}{\mathrm{d}\nu_{\rm O}\mathrm{d}^2\Omega_{\rm O}} = \int^{\eta_{\rm O}} \mathrm{d}\eta a^4 \bar{n}_{\rm G}\frac{\bar{\mathcal{L}}_{\rm G}(\eta,\bar\nu)}{4\pi} \times \Big[1+\delta_{\rm G}+\delta_{\mathcal{L}}+\left(\frac{\partial\ln\bar{\mathcal{L}}_{\rm G}}{\partial\ln\bar\nu}-2\right)e^i\partial_i(v+B)+\left(4-\frac{\partial\ln\bar{\mathcal{L}}_{\rm G}}{\partial\ln\bar\nu}\right)\psi + \left(3-\frac{\partial\ln\bar{\mathcal{L}}_{\rm G}}{\partial\ln\bar\nu}\right)\int_\eta^{\eta_{\rm O}}[\psi'+\phi'+e^ie^j\partial_i\partial_j(B-E')]\mathrm{d}\tilde\eta\Big], \qquad (10)$$

where the luminosity perturbation is defined by $\mathcal{L}_{\rm G}(\eta,\nu)=\bar{\mathcal{L}}_{\rm G}(\eta,\nu)[1+\delta_{\mathcal{L}}(\eta,\nu)]$, and the integration is performed on the background geodesic $x^i=e^i(\eta_{\rm O}-\eta)$. The terms with $\partial\ln\bar{\mathcal{L}}_{\rm G}/\partial\ln\bar\nu$ come by Taylor expanding $\mathcal{L}_{\rm G}(\nu,\eta)$ around $\bar\nu$. What remains to be specified, is the relation between the galaxy density contrast $\delta_{\rm G}$ and the underlying cosmological perturbations, and also a model for the luminosity perturbations $\delta_{\mathcal{L}}$.

B. Gauge invariance

Under a general gauge transformation generated by the vector field $\xi^\mu=(T,\partial^iL)$, the perturbations transform as $\psi\overset{\rm GT}{\to}\psi+\mathcal{H}T+T'$, $\phi\overset{\rm GT}{\to}\phi-\mathcal{H}T$, $E\overset{\rm GT}{\to}E+L$, $B\overset{\rm GT}{\to}B+L'-T$, $v\overset{\rm GT}{\to}v-L'$, $\delta_G\overset{\rm GT}{\to}\delta_G+(\ln\bar n_G)'T$ and $\delta_{\mathcal{L}}\overset{\rm GT}{\to}\delta_{\mathcal{L}}+T\partial\ln\bar{\mathcal{L}}_{\rm G}/\partial\eta$. It follows that two of the four scalar perturbations (ϕ,ψ,B,E) can always be set to zero by a proper choice of (T,L). Note that $\mathcal{L}_G$ is defined as the luminosity seen in the matter comoving frame, that is with respect to a tetrad field whose timelike vector is the matter velocity. This differs from the standard approach of the Boltzmann equation, where the distribution function, and thus its sources and associated collision term, are defined with respect to a tetrad whose timelike vector is proportional to $(\mathrm{d}\eta)_\mu$ [17]. The perturbations on these two different time slices are related by $\delta_{\mathcal{L}}^{\{\mathrm{d}\eta\}}=\delta_{\mathcal{L}}+e^i\partial_i(v+B)\partial\ln\bar{\mathcal{L}}_{\rm G}/\partial\bar\nu$, and $\delta_{\mathcal{L}}^{\{\mathrm{d}\eta\}}$ transforms under a general gauge transformation similarly to $\delta_{\mathcal{L}}$, but with an additional term $-e^i\partial_iT\partial\ln\bar{\mathcal{L}}_{\rm G}/\partial\bar\nu$ [18–20]. Using these transformation properties, it is easy to check that the general expression (10) is gauge invariant, as it should be the case since it describes an observable quantity.

C. Gauge fixing

Restricting to the Newtonian gauge (NG) by setting $B=E=0$ in Eq. (10), we get

$$\left.\frac{\mathrm{d}^3\rho_{\rm GW}(\boldsymbol{e},\nu_{\rm O})}{\mathrm{d}\nu_{\rm O}\mathrm{d}^2\Omega_{\rm O}}\right|_{\rm NG} = \int^{\eta_{\rm O}} \mathrm{d}\eta a^4 \bar{n}_{\rm G}\frac{\bar{\mathcal{L}}_{\rm G}(\eta,\bar\nu)}{4\pi}\times \Big[1+\delta_{\rm G}+\delta_{\mathcal{L}}+\left(4-\frac{\partial\ln\bar{\mathcal{L}}_{\rm G}}{\partial\ln\bar\nu}\right)\psi+\left(\frac{\partial\ln\bar{\mathcal{L}}_{\rm G}}{\partial\ln\bar\nu}-2\right)e^i\partial_iv + \left(3-\frac{\partial\ln\bar{\mathcal{L}}_{\rm G}}{\partial\ln\bar\nu}\right)\int_\eta^{\eta_{\rm O}}(\psi'+\phi')\mathrm{d}\tilde\eta\Big]. \qquad (11)$$

In Eq. (4) of Ref. [5], which was derived in Newtonian gauge, we introduced a phenomenological bias factor to relate $\delta_{\rm G}$ to matter overdensities (in practice relating the comoving density contrasts), and we set $\delta_{\mathcal{L}}=0$.

Even though the expression (11) is given in a specific (Newtonian) gauge, the perturbation variables can always be promoted to gauge invariant variables replacing $\psi\to\psi+\mathcal{H}(B-E')+B'-E''$, $\phi\to\phi-\mathcal{H}(B-E')$, $v\to v+E'$, $\delta_{\rm G}\to\delta_{\rm G}+(B-E')(\ln\bar n_G)'$ and $\delta_{\mathcal{L}}\to\delta_{\mathcal{L}}+(B-E')\partial\ln\bar{\mathcal{L}}_{\rm G}/\partial\eta$. This standard method allows one to re-express the result in an arbitrary gauge, and it is straightforward to check that after integration by parts of the integrated effects, one recovers the expression (10) in an arbitrary gauge.

Similarly, restricting to the synchronous gauge (SC) (with comoving condition on cold matter) by setting $\psi=v=B=0$ in Eq. (10), we get

$$\left.\frac{\mathrm{d}^3\rho_{\rm GW}(\boldsymbol{e},\nu_{\rm O})}{\mathrm{d}\nu_{\rm O}\mathrm{d}^2\Omega_{\rm O}}\right|_{\rm SC} = \int^{\eta_{\rm O}} \mathrm{d}\eta a^4 \bar{n}_{\rm G}\frac{\bar{\mathcal{L}}_{\rm G}(\eta,\bar\nu)}{4\pi}\times \qquad (12)$$
$$\left[1+\delta_{\rm G}+\delta_{\mathcal{L}}+\left(\frac{\partial\ln\bar{\mathcal{L}}_{\rm G}}{\partial\ln\bar\nu}-3\right)\int_\eta^{\eta_{\rm O}}(e^ie^j\partial_i\partial_jE'-\phi')\mathrm{d}\tilde\eta\right],$$

where the effect of metric perturbations appears as an integrated term.

III. COMPARISON WITH LITERATURE

Recently Bertacca *et. al.* [16] (see their Section III, and we follow here the notation of v1 of this preprint) also focus on a perturbed Friedmann-Lemaître spacetime and using the *cosmic rulers formalism* [21] find a general expression for the AGWB anisotropies on a perturbed cosmology. We start from Eq. (81) of [16] (in an arbitrary gauge) and for the sake of comparison we restrict it to the Newtonian gauge. This gives [setting to zero perturbations at the observer's position]

$$\left.\frac{\mathrm{d}^3\rho_{\rm GW}(\boldsymbol{e},\nu_{\rm O})}{\mathrm{d}\nu_{\rm O}\mathrm{d}^2\Omega_{\rm O}}\right|_{\rm Bertacca} = \int^{\eta_{\rm O}} \mathrm{d}\eta a^4 \bar{n}_{\rm G}\mathcal{K}(\bar z,\nu_{\rm O}(1+\bar z))$$
$$\Big\{1+\delta_{\rm G}+\left(-b_{\rm e}+3+\frac{\mathcal{H}'}{\mathcal{H}^2}-\frac{1}{\mathcal{H}}\frac{\mathrm{d}}{\mathrm{d}\eta}\right)\psi \qquad (13)$$
$$+\left(b_{\rm e}-1-\frac{\mathcal{H}'}{\mathcal{H}^2}+\frac{1}{\mathcal{H}}\frac{\mathrm{d}}{\mathrm{d}\eta}\right)e^i\partial_iv$$
$$+\frac{1}{\mathcal{H}}(\psi'+\phi')-\left(b_{\rm e}-2-\frac{\mathcal{H}'}{\mathcal{H}^2}\right)\int_\eta^{\eta_{\rm O}}(\phi'+\psi')\mathrm{d}\tilde\eta\Big\},$$

where the evolution bias is defined as $b_\mathrm{e} = \mathcal{H}^{-1}\mathrm{d}\ln(\bar{n}_\mathrm{G}a^3)/\mathrm{d}\eta$ and the function $\mathcal{K}$ corresponds to $\bar{\mathcal{L}}_\mathrm{G}/(4\pi)$ in our notation. This expression looks very different from our Eq. (11) but it can be easily compared with it. Using integration by parts, and noticing for instance that

$$a^4 n_\mathrm{G}\left(b_\mathrm{e} - \mathcal{H}'/\mathcal{H}^2 + \frac{1}{\mathcal{H}}\frac{\mathrm{d}}{\mathrm{d}\eta}\right)\psi = a\frac{\mathrm{d}}{\mathrm{d}\eta}\left(\frac{a^3 n_\mathrm{G}}{\mathcal{H}}\psi\right),$$

and replacing $\mathcal{K} \equiv \bar{\mathcal{L}}_\mathrm{G}/(4\pi)$, Eq. (13) reduces to[1]

$$\left.\frac{\mathrm{d}^3\rho_\mathrm{GW}(\boldsymbol{e},\nu_\mathrm{O})}{\mathrm{d}\nu_\mathrm{O}\mathrm{d}^2\Omega_\mathrm{O}}\right|_\mathrm{Bertacca} = \int^{\eta_\mathrm{O}} \mathrm{d}\eta a^4 \bar{n}_\mathrm{G}\frac{\bar{\mathcal{L}}_\mathrm{G}(\eta,\bar{\nu})}{4\pi}\times \quad (14)$$
$$\left[1 + \delta_\mathrm{G} + \left(4 - \frac{\partial \ln \bar{\mathcal{L}}_\mathrm{G}}{\partial \ln \bar{\nu}}\right)\psi + \left(\frac{\partial \ln \bar{\mathcal{L}}_\mathrm{G}}{\partial \ln \bar{\nu}} - 2\right)e^i\partial_i v \right.$$
$$\left. + \left(3 - \frac{\partial \ln \bar{\mathcal{L}}_\mathrm{G}}{\partial \ln \bar{\nu}}\right)\int_\eta^{\eta_\mathrm{O}}(\psi' + \phi')\mathrm{d}\tilde{\eta} - \frac{\partial \ln \bar{\mathcal{L}}_\mathrm{G}}{\partial \eta}\frac{\delta\ln(1+z)}{\mathcal{H}}\right].$$

We remark that the difference with Eq. (11) is the last term and the absence of $\delta_\mathcal{L}$. Hence Eq. (14) is equivalent to Eq. (11) with the choice $\delta_\mathcal{L} = (\eta_z - \eta)\partial \ln \bar{\mathcal{L}}_\mathrm{G}/\partial\eta$. In order to gain insight on the physical significance of this choice, let us define the luminosity perturbation on constant redshift hypersurfaces

$$\delta^z_\mathcal{L} \equiv \delta_\mathcal{L} - (\eta_z - \eta)\frac{\partial \ln \bar{\mathcal{L}}_\mathrm{G}}{\partial \eta}, \quad (15)$$

where

$$\eta_z \equiv \eta - \mathcal{H}^{-1}\delta\ln(1+z), \quad (16)$$

is the time at which a galaxy in a fiducial unperturbed cosmology would have the same redshift as a galaxy otherwise located at $x^\mu(\eta)$ in the (true) perturbed cosmology. It follows that $\delta_\mathcal{L} = (\eta_z - \eta)\partial \ln \bar{\mathcal{L}}_\mathrm{G}/\partial\eta$ corresponds to setting the perturbation of the galaxy luminosity to zero on constant redshift hyper surfaces $\delta^z_\mathcal{L} = 0$, which is the choice of [16] to describe the perturbation of the galaxy luminosity. The difference between Eqs. (11) and (14) is then reduced to

$$\left.\frac{\mathrm{d}^3\rho_\mathrm{GW}(\boldsymbol{e},\nu_\mathrm{O})}{\mathrm{d}\nu_\mathrm{O}\mathrm{d}^2\Omega_\mathrm{O}}\right|_\mathrm{Bertacca} \quad (17)$$
$$= \left.\frac{\mathrm{d}^3\rho_\mathrm{GW}(\boldsymbol{e},\nu_\mathrm{O})}{\mathrm{d}\nu_\mathrm{O}\mathrm{d}^2\Omega_\mathrm{O}}\right|_\mathrm{Cusin} - \int^{\eta_\mathrm{O}} \mathrm{d}\eta a^4 \bar{n}_\mathrm{G}\frac{\bar{\mathcal{L}}_\mathrm{G}(\eta,\bar{\nu})}{4\pi}\delta^z_\mathcal{L}.$$

We showed how to perform this comparison in the Newtonian gauge for simplicity. However, using the same types of integrations by parts, it is easy to check that the difference is the same in an arbitrary gauge, i.e. the difference between Eq. (10) and Eq. (81) of Ref. [16] is also given by Eq. (17). We stress that both these expressions are gauge invariant even though they differ, since the difference involves $\delta^z_\mathcal{L}$ which is itself a gauge invariant quantity, as $\delta\ln(1+z) \overset{\mathrm{GT}}{\to} \delta\ln(1+z) - \mathcal{H}T$. It also follows that the difference between the restriction to the synchronous gauge (12) and Eq. (85) of Ref. [16] is given by (17).

As a result, we find that the new terms found in Ref. [16] written in the form of typical projection terms in large scale structure observables [see for example the Kaiser-like term in the third line of Eq. (13)], can be removed with some integration by parts, hence do not correspond to measurable physical effects. This is a general result and applies to any background: projection effects, very relevant when dealing with large scale structure observables (i.e. sources that we can observe directly and localize in space) are not effective when dealing with a background. A background by definition is given by the superposition of signals from unresolvable sources and it is therefore *blind* to the location in redshift space of sources, which need to be described using a local model for galactic physics.

IV. LUMINOSITY PERTURBATION

In Ref. [16], the perturbation of the effective luminosity as a function of the observed redshift is set to zero on constant redshift hypersurfaces, i.e. $\delta^z_\mathcal{L} = 0$. Since we do not directly observe the luminosity of galaxies but only the integrated flux that we receive from them, in our work Ref. [3] we had assumed the luminosity of a galaxy to be a function of its local time η (even though it is only a coordinate and ideally one should prefer to use its proper time), rather than a function of redshift which depends on the way galaxies are observed later, and we had chosen an astrophysical model of GW sources such that the luminosity perturbation vanishes in the Newtonian gauge, i.e. $\delta_\mathcal{L} = 0$. A similar choice for source luminosity perturbations is made in the context of the cosmic infrared background [22]. Furthermore, constant redshift hypersurfaces are observer dependent, hence parametrizing the galaxy luminosity in terms of the observed redshift may be useful and well-motivated in the case we could directly access and measure the emitted luminosity, which is not the case in the context of a background.

Conclusion

We have compared different predictions for the anisotropies of the gravitational wave background proposed in the literature, starting from our covariant approach of Ref. [3]. We have shown that they are all equivalent and they all contain the same type of cosmologi-

[1] Note that instead of using integration by parts to obtain Eq. (14) from (13), we could consider that the time variable in the integration of Eq. (13) is in fact η_z, and then use the change of variable (16) to integrate on η. From first order expansion of all background quantities in $\eta_z - \eta$, one recovers Eq. (11) up to the difference (17).

cal effects. Differences in predictions can only arise in the way galactic physics is computed. In particular, the expression for the background anisotropies proposed in Ref. [16] reduces, after some integration by parts to the former expressions of Refs. [3, 4], except for a term [see Eq. (17)] which is the result of a different modeling of GW luminosity as a function of redshift and frequency. Moreover, we have shown that both the approaches of Ref. [16] and Refs. [3, 4] used to expand the covariant expression of Ref. [3] on a perturbed cosmological framework, give a gauge-invariant observable. This is a nice cross-check of the results derived so far: by construction, gauge invariance is built in the computation of an observable, and different choices of the time coordinate used to parametrize it should not change its transformation property under gauge transformations.

More generally, any difference between AGWB power spectra can arise from *(i)* the underlying cosmology ($\bar{n}_{\rm G}$) *(ii)* the large scales structure or galaxy clustering ($\delta_{\rm G}$ and the bias model) and/or *(iii)* the local astrophysics on sub-galactic scales encompassed by the luminosity function $\mathcal{L}_{\rm G}(\eta,\nu)$ (its background and perturbed values). As shown in this article, see also the previous note [13], since the physics on cosmological scale is quite well understood, the main source of uncertainties present in the predictions for the AGWB anisotropies is given by the description of galactic and sub-galactic physics. This is extremely interesting because we are left with an observable that is very sensitive to the details of the astrophysical modeling as shown in Refs. [9, 10], and that can be used to constrain astrophysics.

The expressions that we provided in Refs. [3, 4], correspond to an astrophysical model where $\delta_{\mathcal{L}} = 0$ in the Newtonian gauge (6). This is equivalent to assuming that the associated gauge invariant variable $\delta_{\mathcal{L}} + (B - E')\partial \ln \bar{\mathcal{L}}_{\rm G}/\partial\eta$ vanishes. This is a very natural choice in the framework of the coarse grained modeling used to describe anisotropies, since it allows one to compute both galaxy density and luminosity at the same time. Alternatively, in Ref. [16] the modeling corresponds to the choice that the gauge invariant luminosity perturbation on constant redshift hypersurfaces $\delta^{z}_{\mathcal{L}}$vanishes identically. The result of different choices for the $\delta_{\mathcal{L}}$ is expected to give little differences on observable scales since their difference (15) is proportional to the time perturbation $\delta\eta$ and to the derivative of the galaxy luminosity with respect to time, and in any realistic astrophysical model the galaxy luminosity is expected to have a smooth time evolution. There are of course other physically reasonable models to set $\delta_{\mathcal{L}}$, and one could for instance assume that it vanishes in the synchronous gauge. One would then use the very simple expression (12) in which the effect of metric perturbations would merely be interpreted as an integrated effect (with no physical projection effects present).

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