Gravitational waves generated by second order effects during inflation

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The generation of gravitational waves during inflation due to the non-linear coupling of scalar and tensor modes is discussed. Two methods describing gravitational wave perturbations are used and compared: a covariant and local approach, as well as a metric-based analysis based on the Bardeen formalism. An application to slow-roll inflation is also described.

I. INTRODUCTION

The generation of gravitational waves (GW) is a general prediction of an early inflationary phase [1]. Their amplitude is related to the energy scale of inflation and they are potentially detectable via observations of B-mode polarization in the cosmic microwave background (CMB) if the energy scale of inflation is larger than $\sim 3\times10^{15}$ GeV [2, 3, 4, 5, 6]. Such a detection would be of primary importance to test inflationary models.

Among the generic predictions of one-field inflation [7] are the existence of (adiabatic) scalar and tensor perturbations of quantum origin with an almost scale invariant power spectrum and Gaussian statistics. Even if non-linear effects in the evolution of perturbations are expected, a simple calculation [8], confirmed by more detailed analysis [9], shows that it is not possible to produce large non-Gaussianity within single field inflation as long as the slow-roll conditions are preserved throughout the inflationary stage. Deviations from Gaussianity can be larger in, e.g., multi-field inflation scenarios [8, 10] and are thus expected to give details on the inflationary era.

As far as scalar modes are concerned, the deviation from Gaussianity has been parameterized by a (scale-dependent) parameter, $f_{\rm NL}$. Various constraints have been set on this parameter, mainly from CMB analysis [12] (see Ref. [13] for a review on both theoretical and observational issues). Deviation from Gaussianity in the CMB can arise from primordial non-Gaussianity, i.e. generated during inflation, post-inflation dynamics or radiation transfer [14]. It is important to understand them all in order to track down the origin of non-Gaussianity, if detected.

Among the other signatures of non-linear dynamics is the fact that the Scalar-Vector and Tensor (SVT) modes of the perturbations are no longer decoupled. This implies in particular that scalar modes can generate gravity waves. Also, vector modes, that are usually washed out by the evolution, can be generated. In particular, second-order scalar perturbations in the post-inflation era will also contribute to B-mode polarization [15] or to multi-pole coupling in the CMB [16], and it is thus important to understand this coupling in detail.

In this article, we focus on the gravitational waves generated from scalar modes via second order dynamics. Second-order perturbation theory has been investigated in various works [17, 18, 19, 20, 21, 22, 23, 24, 25] and a fully gauge-invariant approach to the problem was recently given in Ref. [25]. Second-order perturbations during inflation have also been considered in Refs. [9, 26], providing the prediction of the bispectrum of perturbations from inflation.

Two main formalisms have been developed to study perturbations, and hence second order effects: the $1+3$ covariant formalism [27] in which exact gauge-invariant variables describing the physics of interest are first identified and exact equations describing their time and space evolution are then derived and approximated with respect to the symmetry of the background to obtain results at the desired order, and the coordinate based approach of Bardeen [28] in which gauge-invariants are identified by combining the metric and matter perturbations and then equations are found for them at the appropriate order of the calculation. In this article we carry out a detailed comparison of the two approaches up to second order, highlighting the advantages and disadvantages of each method, thus extending earlier work on the linear theory [29]. Our paper also extends the work of Ref. [22], in which the relation between the two formalisms on super-Hubble scales is investigated. In particular, we show that the degree of success of one formalism over the other depends on the problem being addressed. This is the first time a complete and transparent matching of tensor perturbations in the two formalisms at first and second order is presented. We also show, using an analytical argument, that the power-spectrum of gravitational waves from second-order effects is much smaller than the first order on super-Hubble scales. This is in contrast to the fact that during the radiation era the generation of GW from primordial density fluctuations can be large enough to be detected in principle, though this requires the inflationary background of GW to be sufficiently small [23].

This paper is organized as follows. We begin by reviewing scalar field dynamics in Section II within the 1 + 3 covariant approach. In Section III, we formulate the problem within the covariant approach followed by a reformulation in the coordinate approach in Section IV. A detailed comparison of the two formalisms is then presented in Section V. In Section VI, we study gravitational waves that are generated during the slow-roll period of inflation. In particular, we introduce a generalization of the f_{NL} parameter to take into account gravity waves and we compute the three point correlator involving one graviton and two scalars. Among all three point functions involving scalar and tensor modes, this correlator and the one involving three scalars are the dominant [9]. Finally, we conclude in Section VII.

II. SCALAR FIELD DYNAMICS

Let us consider a minimally coupled scalar field with Lagrangian density[1]

$$\mathcal{L}_\phi = -\sqrt{-g}\left[\tfrac{1}{2}\nabla_a\phi\nabla^a\phi + V(\phi)\right] \ , \qquad (2)$$

where $V(\phi)$ is a general (effective) potential expressing the self interaction of the scalar field. The equation of motion for the field ϕ following from $\mathcal{L}_\phi$ is the Klein-Gordon equation

$$\nabla_a\nabla^a\phi - V'(\phi) = 0 \ , \qquad (3)$$

where the prime indicates a derivative with respect to ϕ. The energy-momentum tensor of ϕ is of the form

$$T_{ab} = \nabla_a\phi\nabla_b\phi - g_{ab}\left[\tfrac{1}{2}\nabla_c\phi\nabla^c\phi + V(\phi)\right] \ ; \qquad (4)$$

provided $\phi_{,a} \neq 0$, equation (3) follows from the conservation equation

$$\nabla_b T^{ab} = 0 \ . \qquad (5)$$

We shall now assume that in the open region U of spacetime that we consider, the *momentum density* $\nabla^a\phi$ is *timelike*:

$$\nabla_a\phi\nabla^a\phi < 0 \ . \qquad (6)$$

This requirement implies two features: first, ϕ is not constant in U, and so $\{\phi = const.\}$ specifies well-defined surfaces in spacetime. When this is not true (i.e., ϕ is constant in U), then by (4),

$$\nabla_a\phi = 0 \quad \Leftrightarrow \quad T_{ab} = -g_{ab}V(\phi) \quad \Rightarrow \quad V = \text{const.} \ , \qquad (7)$$

in U, [the last being necessarily true due to the conservation law (5))] and we have an effective cosmological constant in U rather than a dynamical scalar field.

A. Kinematical quantities

Our aim is to give a formal description of the scalar field in terms of fluid quantities; therefore, we assign a 4-velocity vector u^a to the scalar field itself. This will allow us to define the dot derivative, i.e. the *proper time* derivative along the flow lines: $\dot{T}^{a\cdots b}{}_{c\cdots d} \equiv u^e\nabla_e T^{a\cdots b}{}_{c\cdots d}$. Now, given the assumption (6), we can choose the 4-velocity field u^a as the unique timelike vector with unit magnitude ($u^a u_a = -1$) parallel to the normals of the hypersurfaces $\{\phi = const.\}$ [31] [2] ,

$$u^a \ \equiv \ -\psi^{-1}\nabla^a\phi \ , \qquad (8)$$

where we have defined the field $\psi = \dot{\phi} = (-\nabla_a\phi\nabla^a\phi)^{1/2}$ to denote the magnitude of the momentum density (simply momentum from now on). The choice (8) defines u^a as the unique timelike eigenvector of the energy-momentum tensor (4).[3]

The kinematical quantities associated with the "flow vector" u^a can be obtained by a standard method [33, 34]. We can define a projection tensor into the tangent 3-spaces orthogonal to the flow vector:

$$h_{ab} \equiv g_{ab} + u_a u_b \ \Rightarrow h^a{}_b h^b{}_c = h^a{}_c \ , \ h_{ab}u^b = 0 \ ; \qquad (9)$$

with this we decompose the tensor $\nabla_b u_a$ as

$$\nabla_b u_a = \widetilde{\nabla}_b u_a - \dot{u}_a u_b \ , \quad \widetilde{\nabla}_b u_a = \tfrac{1}{3}\Theta h_{ab} + \sigma_{ab} \ , \qquad (10)$$

where $\widetilde{\nabla}_a$ is the spatially totally projected covariant derivative operator orthogonal to u^a (e.g., $\widetilde{\nabla}_a f =$

[1] We use conventions of Ref. [30]. Units in which $\hbar = c = k_B = 1$ are used throughout this article, Latin indices $a, b, c...$ run from 0 to 3, whereas Latin indices $i, j, k...$ run from 1 to 3. The symbol ∇ represents the usual covariant derivative and ∂ corresponds to partial differentiation. Finally the Hilbert-Einstein action in presence of matter is defined by

$$\mathcal{A} = \int dx^4\sqrt{-g}\left[\tfrac{1}{16\pi G}R + \mathcal{L}_\phi\right] \ . \qquad (1)$$

[2] In the case of more than one scalar field, this choice can still be made for each scalar field 4-velocity, but not for the 4-velocity of the total fluid. A number of frame choices exist for the 4-velocity of the total fluid, the most common being the energy frame, where the total energy flux vanishes (see [32] for a detailed description of this case.

[3] The quantity ψ will be positive or negative depending on the initial conditions and the potential V; in general ϕ could oscillate and change sign even in an expanding phase, and the determination of u^a by (8) will be ill-defined on those surfaces where $\nabla_a\phi = 0 \Rightarrow \psi = 0$ (including the surfaces of maximum expansion in an oscillating Universe). This will not cause us a problem however, as we assume the solution is differentiable and (6) holds almost everywhere, so determination of u^a almost everywhere by this equation will extend (by continuity) to determination of u^a everywhere in U.

${h_a}^b\nabla_b f$; see the Appendix of Ref. [35] for details), $\dot{u}_a$ is the acceleration ($\dot{u}_b u^b = 0$), and σ_{ab} the shear ($\sigma^a{}_a = \sigma_{ab}u^b = 0$). Then the expansion, shear and acceleration are given in terms of the scalar field by

$$\Theta = -\nabla_a(\psi^{-1}\nabla^a\phi) = -\psi^{-1}\left[V'(\phi) + \dot{\psi}\right], \quad (11)$$

$$\sigma_{ab} = -\psi^{-1}\, h_{\langle a}{}^c h_{b\rangle}{}^d \nabla_c\left[\nabla_d\phi\right], \quad (12)$$

$$a_a = -\psi^{-1}\,\widetilde{\nabla}_a\psi = -\psi^{-1}(\nabla_a\psi + u_a\dot{\psi}), \quad (13)$$

where the last equality in Eq. (11) follows on using the Klein - Gordon equation (3). We can see from Eq. (13) that ψ is an *acceleration potential* for the fluid flow [36]. Note also that the vorticity vanishes:

$$\omega_{ab} = -h_a{}^c h_b{}^d \nabla_{[d}\left(\psi^{-1}\nabla_{c]}\phi\right) = 0\,, \quad (14)$$

an obvious result with the choice (8), so that $\widetilde{\nabla}_a$ is the covariant derivative operator in the 3-spaces orthogonal to u^a, i.e. in the surfaces $\{\phi = \text{const.}\}$. As usual, it is useful to introduce a scale factor a (which has dimensions of length) along each flow - line by

$$\frac{\dot{a}}{a} \equiv \tfrac{1}{3}\Theta = H\,, \quad (15)$$

where H is the usual Hubble parameter if the Universe is homogeneous and isotropic. Finally, it is important to stress that

$$\widetilde{\nabla}_a\phi = 0 \quad (16)$$

which follows from our choice of u^a via equation (8), a result that will be important for the choice of gauge invariant (GI) variables and for the perturbations equations.

B. Fluid description of a scalar field

It follows from our choice of the four velocity (8) that we can represent a minimally coupled scalar field as a perfect fluid; the energy - momentum tensor (4) takes the usual form for perfect fluids

$$T_{ab} = \mu u_a u_b + p h_{ab}\,, \quad (17)$$

where the energy density μ and pressure p of the scalar field "fluid" are given by

$$\mu = \tfrac{1}{2}\psi^2 + V(\phi)\,, \quad (18)$$

$$p = \tfrac{1}{2}\psi^2 - V(\phi)\,. \quad (19)$$

If the scalar field is not minimally coupled this simple representation is no longer valid, but it is still possible to have an imperfect fluid form for the energy - momentum tensor [31].

Using the perfect fluid energy - momentum tensor (17) in (5) one obtains the energy and momentum conservation equations

$$\dot{\mu} + \psi^2\Theta = 0\,, \quad (20)$$

$$\psi^2\dot{u}_a + \widetilde{\nabla}_a p = 0\,. \quad (21)$$

If we now substitute μ and p from Eqs. (18) and (19) into Eq. (20) we obtain the 1+3 form of the Klein - Gordon equation (3):

$$\ddot{\phi} + \Theta\dot{\phi} + V'(\phi) = 0\,, \quad (22)$$

an exact ordinary differential equation for ϕ in any space - time with the choice (8) for the four - velocity. With the same substitution, Eq. (21) becomes an identity for the acceleration potential ψ. It is convenient to relate p and μ by the *index* γ defined by

$$p = (\gamma - 1)\mu \quad \Leftrightarrow \quad \gamma = \frac{p + \mu}{\mu} = \frac{\psi^2}{\mu}\,. \quad (23)$$

This index would be constant in the case of a simple one-component fluid, but in general will vary with time in the case of a scalar field:

$$\frac{\dot{\gamma}}{\gamma} = \Theta(\gamma - 2) - 2\frac{V'}{\psi}\,. \quad (24)$$

Finally, it is standard to *define* a speed of sound as

$$c_s^2 \equiv \frac{\dot{p}}{\dot{\mu}} = \gamma - 1 - \frac{\dot{\gamma}}{\Theta\gamma}\,. \quad (25)$$

C. Background equations

The previous equations assume nothing on the symmetry of the spacetime. We now specify it further and assume that it is close to a flat Friedmann-Lemaître spacetime (FL), which we consider as our background spacetime. The homogeneity and isotropy assumptions imply that

$$\sigma^2 \equiv \frac{1}{2}\sigma_{ab}\sigma^{ab} = 0\,,\ \omega_{ab} = 0\,,\ \ \widetilde{\nabla}_a f = 0\,, \quad (26)$$

where f is any scalar quantity; in particular

$$\widetilde{\nabla}_a\mu = \widetilde{\nabla}_a p = 0 \quad \Rightarrow \quad \widetilde{\nabla}_a\psi = 0\,,\ a_a = 0\,. \quad (27)$$

The background (zero - order) equations are given by [37]:

$$3\dot{H} + 3H^2 = 8\pi G\left[V(\phi) - \psi^2\right]\,, \quad (28)$$

$$3H^2 = 8\pi G\left[\tfrac{1}{2}\psi^2 + V(\phi)\right] \quad (29)$$

$$\dot{\psi} + 3H\psi + V'(\phi) = 0 \ \Leftrightarrow \dot{\mu} + 3H\psi^2 = 0\,, \quad (30)$$

where all variables are a function of cosmic time t only.

III. GRAVITATIONAL WAVES FROM DENSITY PERTURBATIONS: COVARIANT FORMALISM

A. First order equations

The study of linear perturbations of a FL background are relatively straightforward. Let us begin by defining the *first-order gauge-invariant* (FOGI) variables corresponding respectively to the spatial fluctuations in the energy density, expansion rate and spatial curvature:

$$X_a = \widetilde{\nabla}_a \mu \,, \quad Z_a = \widetilde{\nabla}_a \Theta \,, \quad C_a = a^3 \widetilde{\nabla}_a \tilde{R} \,. \tag{31}$$

The quantities are FOGI because they vanish exactly in the background FL spacetime [38, 40]. It turns out that a more suitable quantity for describing density fluctuations is the co-moving gradient of the energy density:

$$\mathcal{D}_a = \frac{a}{\mu} X_a \,, \tag{32}$$

where the ratio X_a/μ allows one to evaluate the magnitude of the energy density perturbation relative to its background value and the scale factor a guarantees that it is dimensionless and co-moving.

These quantities exactly characterize the inhomogeneity of any fluid; however we specifically want to characterize the inhomogeneity of the scalar field: this cannot be done using the spatial gradient $\widetilde{\nabla}_a \phi$ because it identically vanishes in any space-time by virtue of our choice of 4-velocity field u^a. It follows that in our approach the inhomogeneities in the matter field are completely incorporated in the spatial variation of the momentum density: $\widetilde{\nabla}_a \psi$, so it makes sense to define the dimensionless gradient

$$\Psi_a \equiv \frac{a}{\psi} \widetilde{\nabla}_a \psi \,, \tag{33}$$

which is related to $\mathcal{D}_a$ by

$$\mathcal{D}_a = \frac{\psi^2}{\mu} \Psi_a = \gamma \Psi_a \,, \tag{34}$$

where we have used Eq. (18) and γ is given by Eq. (23); comparing Eq. (33) and Eq. (13) we see that Ψ_a is proportional to the acceleration: it is a gauge-invariant measure of the spatial variation of proper time along the flow lines of u^a between two surfaces $\phi =$const. (see Ref. [33]). The set of linearized equations satisfied by the FOGI variables consists of the *evolution equations*

$$\dot{X}_a = -4HX_a - \psi^2 Z_a, \tag{35}$$
$$\dot{Z}_a = -3HZ_a - 4\pi G X_a + \widetilde{\nabla}_a \,\mathsf{div}\, \dot{u}, \tag{36}$$
$$\dot{\sigma}_{ab} - \widetilde{\nabla}_{\langle a} \dot{u}_{b\rangle} = -2H\sigma_{ab} - E_{ab}, \tag{37}$$
$$\dot{E}_{ab} - \mathsf{curl}\, H_{ab} = -4\pi G \psi^2 \sigma_{ab} - \Theta E_{ab}, \tag{38}$$
$$\dot{H}_{ab} + \mathsf{curl}\, E_{ab} = -3HH_{ab}; \tag{39}$$

and the *constraints*

$$0 = \tfrac{8\pi G}{3} X_a - \mathsf{div}\, E_a, \tag{40}$$
$$0 = \tfrac{2}{3} Z_a - \mathsf{div}\, \sigma_a, \tag{41}$$
$$0 = \mathsf{div}\, H_a, \tag{42}$$
$$0 = H_{ab} - \mathsf{curl}\, \sigma_{ab} \,, \tag{43}$$
$$0 = \mathsf{curl}\, X_a, \tag{44}$$
$$0 = \mathsf{curl}\, Z_a \,. \tag{45}$$

The curl operator is defined by $\mathsf{curl}\, \psi_{ab} = (\mathsf{curl}\, \psi)_{ab} = \varepsilon_{cd\langle a} \widetilde{\nabla}^c \psi^d_{b\rangle}$ where ϵ_{abc} is the completely antisymmetric tensor with respect to the spatial section defined by $\epsilon_{bcd} = \epsilon_{abcd} u^a$, ϵ_{abcd} being the volume antisymmetric tensor such that $\epsilon_{0123} = \sqrt{-g}$. The divergence div of a rank n tensor is a rank $n-1$ tensor defined by $(\mathsf{div}\, \psi)_{i_1 \dots i_{n-1}} \equiv \widetilde{\nabla}^{i_n} \psi_{i_1 \dots i_n}$.

Because the background is homogeneous and isotropic, each FOGI vector may be uniquely split into a *curl-free* and *divergence-free* part, usually referred to as scalar and vector parts respectively, which we write as

$$V_a = V_{\mathsf{S}a} + V_{\mathsf{V}a} \,, \tag{46}$$

where $\mathsf{curl}\, V_{\mathsf{S}a} = 0$ and $\mathsf{div}\, V_{\mathsf{V}} = 0$. Similarly, any tensor may be invariantly split into scalar, vector and tensor parts:

$$T_{ab} = T_{\mathsf{S}ab} + T_{\mathsf{V}ab} + T_{\mathsf{T}ab} \tag{47}$$

where $\mathsf{curl}\, T_{\mathsf{S}ab} = 0$, $\mathsf{div}\,\mathsf{div}\, T_{\mathsf{V}} = 0$ and $\mathsf{div}\, T_{\mathsf{T}a} = 0$. It follows therefore that in the above equations we can separately equate scalar, vector and tensor parts and obtain equations that independently characterize the evolution of each type of perturbation. In the case of a scalar field, the vorticity is exactly zero, so there is no vector contribution to the perturbations.

Let us now concentrate on scalar perturbations at linear order. It is clear from the above discussion that pure scalar modes are characterized by the vanishing of the magnetic part of the Weyl tensor: $H_{ab} = 0$, so the above set of equations reduce to a set of two coupled differential equations for X_a and Z_a:

$$\dot{X}_a + 4HX_a = -\psi^2 Z_a \tag{48}$$
$$\dot{Z}_a + 3HZ_a = -4\pi G X_a - \psi^{-2} \widetilde{\nabla}^2 X_a, \tag{49}$$

and a set of coupled evolution and constraint equations that determine the other variables

$$\dot{\sigma}_{ab} = -\psi^{-2} \widetilde{\nabla}_{\langle a} X_{b\rangle} - 2H\sigma_{ab} - E_{ab}. \tag{50}$$
$$\dot{E}_{ab} = -4\pi G\, \psi^2\, \sigma_{ab} - 3HE_{ab}, \tag{51}$$
$$\mathsf{div}\, \sigma_a = \tfrac{2}{3}\, Z_a, \tag{52}$$
$$\mathsf{curl}\, \sigma_{ab} = 0, \tag{53}$$
$$\mathsf{div}\, E_a = \tfrac{8\pi G}{3}\, X_a. \tag{54}$$

B. Gravitational waves from density perturbations

The preceding discussion deals with first-order variables and their behavior at linear order. It is important to keep in mind that we were able to set $H_{ab}=0$ only because pure scalar perturbations in the absence of vorticity implies that $\mathsf{curl}\,\sigma_{ab}=0$ at first order. The vanishing of the magnetic part then follows from equation (43). However, at second order $\mathsf{curl}\,\sigma_{ab}\neq 0$. We denote the non-vanishing contribution at second order by [21]

$$\Sigma_{ab}=\mathsf{curl}\,\sigma_{ab}.$$

The new variable is *second-order and gauge-invariant* (SOGI), as it vanishes at all lower orders [38]. It should be noted that the new variable is just the magnetic part of the Weyl tensor subject to the conditions mentioned above i.e.

$$\Sigma_{ab}=H_{ab}|_{\omega=0}. \tag{55}$$

We are interested in the properties inherited by the new variable from the magnetic part of the Weyl tensor. In particular, it can be shown that the new variable is transverse and traceless at this order and is thus a description of gravitational waves. It should be stressed that in full generality, there are tensorial modes even at first order. By assuming that there are none, we explore a particular subset in the space of solutions. From the "iterative resolution" point of view, this means that we constrain the equations in order to focus on second order GWs sourced by terms quadratic in scalar perturbations. In doing so, we artificially switch off GW perturbations at first order.

C. Propagation equation

The propagation of the new second-order variable now needs to be investigated using a covariant set of equations that are linearized to second-order about FL. We make use of Eqs. (20), (21) and the following evolution equations which are up to second order in magnitude;

$$\dot{E}_{ab} = -\Theta E_{ab}+\mathsf{curl}\,\Sigma_{ab}-4\pi G\psi^2\sigma_{ab}+3\sigma_{c\langle a}E_{b\rangle}{}^{c}, \tag{56}$$
$$\dot{\Sigma}_{ab} = -\Theta\Sigma_{ab}-\mathsf{curl}\,E_{ab}-2\epsilon_{cd\langle a}\dot{u}^{c}E_{b\rangle}{}^{d}, \tag{57}$$

together with the constraint

$$\dot{u}^a=-\psi^{-2}\tilde{\nabla}^a p=-\frac{3}{8\pi G\psi^2}\mathsf{div}\,E^a. \tag{58}$$

Unlike at first-order, where the splitting of tensors into their scalar, vector and tensor parts is possible, at second order this can only be achieved for SOGI variables.

We may isolate the tensorial part of the equations by decoupling Σ_{ab}: since it is divergence free it is already a pure tensor mode, whereas E_{ab} is not. The wave equation for the gravitational wave contribution can be found by first taking the time derivative of (57) and making appropriate substitutions using the evolution equations and keeping terms up to second order. The wave equation for Σ_{ab} then reads:

$$\ddot{\Sigma}_{ab}-\widetilde{\nabla}^2\Sigma_{ab}+7H\dot{\Sigma}_{ab}+(12H^2-16\pi G\psi^2)\Sigma_{ab}=S_{ab} \tag{59}$$

where the source is given by the cross-product of the electric-Weyl curvature and its divergence (or acceleration):

$$S_{ab} = -\left[2u^e\nabla_e+16H-15Hc_s^2\right]\left(\frac{1}{4\pi G\psi^2}\epsilon_{cd\langle a}E_{b\rangle}{}^{d}\,\mathsf{div}\,E^c\right). \tag{60}$$

To obtain this, we have used the fact that with a flat background space-time

$$\mathsf{curl}\,\mathsf{curl}\,T_{ab} = -\widetilde{\nabla}^2T_{ab}+\frac{3}{2}\widetilde{\nabla}_{\langle a}\mathsf{div}\,T_{b\rangle}$$

and used the commutation relation

$$(\mathsf{curl}\,T_{ab})^{\cdot} = \mathsf{curl}\,\dot{T}_{ab}-\frac{1}{3}\Theta\,\mathsf{curl}\,T_{ab}-\epsilon_{cd\langle a}\sigma^{ec}D_{|e|}T_{b\rangle}{}^{d} +\epsilon_{cd\langle a}[\dot{u}^c\dot{T}_b^d+\frac{1}{3}\Theta\dot{u}^cT_{b\rangle}{}^{d}].$$

We have also used Eqs. (24) and (25) to eliminate $\dot{\psi}/\psi$ from the source term. It can also be shown that S_{ab} is transverse, illustrating that Eq. (59) represents the gravitational wave contribution at second order. Note that this is a local description of gravitational waves, in contrast to the non-local extraction of tensor modes by projection in Fourier space. Since Σ_{ab} contains exactly the correct number of degrees of freedom possible in GW, any other variable we may choose to describe GW must be related by quadrature, making this a suitable master variable. The situation is analogous to the description of electromagnetic waves: Should we use the vector potential, the electric field, or the magnetic field for their description? Mathematically it doesn't matter of course – each variable obeys a wave equation and the others are related by quadrature. Physically, however, it's the electric and magnetic fields which drive charged particles through the Lorentz force equation – the electromagnetic analogue of the geodesic deviation equation.

In order to express the gravitational wave equation in Fourier space, we define our normalised tensor harmonics as

$$Q^{ab}=\frac{\xi^{ab}}{(2\pi)^{3/2}}e^{i\boldsymbol{k}\cdot\boldsymbol{x}}, \tag{61}$$

where ξ^{ab} is the polarization tensor, which satisfies the (background) tensor Helmholtz equation: $\tilde{\nabla}^2Q_{ab}=-(q^2/a^2)Q_{ab}$. As q_a is required to satisfy $q_au^a=0$ in the background, it can thus be identified with a 3-vector and will subsequently written in bold when necessary. We denote harmonics of the opposite polarization with an overbar. Amplitudes of Σ_{ab} may be extracted via

$$\Sigma(\boldsymbol{k},t)=\int \mathrm{d}^3\boldsymbol{k}\left[\Sigma_{ab}(\boldsymbol{x},t)Q^{*ab}(\boldsymbol{k},\boldsymbol{x})\right], \tag{62}$$

with an analogous formula for the opposite parity. This implies that our original variable may be reconstructed from

$$\Sigma_{ab} = \int \mathrm{d}^3\boldsymbol{k} \left[\Sigma(\boldsymbol{k},t) Q_{ab}(\boldsymbol{k},\boldsymbol{x}) + \bar{\Sigma}(\boldsymbol{k},t)\bar{Q}_{ab}(\boldsymbol{k},\boldsymbol{x})\right] . \tag{63}$$

The same relations hold for any transverse tensor. Hence, our wave equation in Fourier space is

$$\Sigma''(\boldsymbol{k},\eta) + 6\mathcal{H}\Sigma'(\boldsymbol{k},\eta) + [k^2 + 12\mathcal{H}^2 - 16\pi G\psi^2]\Sigma(\boldsymbol{k},\eta) = S(\boldsymbol{k},\eta), \tag{64}$$

with an identical equation for the opposite polarization. We have converted to conformal time η, where a prime denotes derivatives with respect to η, and we have defined the conformal Hubble parameter as $\mathcal{H} = a'/a$. The source term is composed of a cross-product of the electric part of the Weyl tensor and its divergence. At first-order, the electric Weyl tensor is a pure scalar mode, and can therefore be expanded in terms of scalar harmonics. To define these, let $Q^{(s)} = e^{i\boldsymbol{q}\cdot\boldsymbol{x}}/(2\pi)^{3/2}$, be a solution to the Helmholtz equation: $\tilde{\nabla}^2 Q^{(s)} = -(q^2/a^2)Q^{(s)}$. Beginning with this basis, it is possible to derive vectorial and (PSTF) tensorial harmonics by taking successive spatial derivatives as follows:

$$Q_a^{(s)} = \tilde{\nabla}_a Q^{(s)} = i\frac{q_a}{a}Q^{(s)}, \tag{65}$$

$$Q_{ab}^{(s)} = \tilde{\nabla}_{\langle a}\tilde{\nabla}_{b\rangle}Q^{(s)} = -a^{-2}\left(q_a q_b - \frac{1}{3}h_{ab}q^2\right)Q^{(s)}. \tag{66}$$

This symmetric tensor has the additional property $q^a q^b Q_{ab}^{(s)} = -(2q^4/3a^2)Q^{(s)}$. Using this representation we can express our source in Eq. (64) in terms of a convolution in Fourier space, by expanding the electric Weyl tensor as

$$E(\boldsymbol{q},\eta) = \int \mathrm{d}^3\boldsymbol{x} E_{ab} Q^{*ab}_{(S)}(\boldsymbol{q},\boldsymbol{x}). \tag{67}$$

Then, the right hand side of Eq. (60) expressed in conformal time, accompanied by appropriate Fourier decomposition of each term and making use of the normalization condition for orthonormal basis, yields:

$$S(\boldsymbol{k},\eta) = \int \mathrm{d}^3\boldsymbol{q} A(\boldsymbol{q},\boldsymbol{k}) \left\{2\left[E(\boldsymbol{q},\eta)E(\boldsymbol{k}-\boldsymbol{q},\eta)\right]' + (16 - 15c_s^2)\mathcal{H}\, E(\boldsymbol{q},\eta)E(\boldsymbol{k}-\boldsymbol{q},\eta)\right\} \tag{68}$$

where

$$A(\boldsymbol{q},\boldsymbol{k}) = \frac{i}{6\pi G a^3\psi^2}\epsilon_{cd\langle a}q_{b\rangle}q^d(k^c - q^c)|\boldsymbol{k}-\boldsymbol{q}|^2\xi^{ab}(\boldsymbol{k}), \tag{69}$$

with a similar expression for the other polarization.

In principle we can now solve for the gravitational wave contribution Σ_{ab}, and calculate the power spectrum of gravitational waves today. For this however, we need initial conditions for the electric Weyl tensor (or, alternatively Ψ_a).

IV. GRAVITATIONAL WAVES FROM DENSITY PERTURBATIONS: COORDINATE BASED APPROACH

In this formalism, we consider perturbations around a FL universe with Euclidean spatial sections and expand the metric as

$$\mathrm{d}s^2 = a^2(\eta)\left\{-(1+2A)\mathrm{d}\eta^2 + 2\omega_i \mathrm{d}x^i\mathrm{d}\eta + \left[(1+2C)\delta_{ij} + h_{ij}\right]\mathrm{d}x^i\mathrm{d}x^j\right\} , \tag{70}$$

where η is the conformal time and a the scale factor. We perform a scalar-vector-tensor decomposition as

$$\omega_i = D_i B + \bar{B}_i\, , \tag{71}$$

and

$$h_{ij} = 2\bar{\mathcal{E}}_{ij} + D_i\bar{\mathcal{E}}_j + D_j\bar{\mathcal{E}}_i + 2D_iD_j\mathcal{E}\ , \tag{72}$$

where $\bar{B}_i$, $\bar{\mathcal{E}}_i$ are transverse ($D_i\bar{\mathcal{E}}^i = D_i\bar{B}^i$), and $\bar{\mathcal{E}}_{ij}$ is traceless and transverse ($\bar{\mathcal{E}}^i_i = D_i\bar{\mathcal{E}}^i_j = 0$). Latin indices i, j, k... are lowered by use of the spatial metric, e.g. $B^i = \gamma^{ij}B_j$. We fix the gauge and work in the Newtonian gauge defined by $B_i = \mathcal{E} = B = 0$ so that $\Phi = A$ and $\Psi = -C$ are the two Bardeen potentials. As in the previous sections, we assume that the matter content is a scalar field ϕ that can be split into background and perturbation contributions: $\phi = \phi(\eta) + \delta\phi(\eta,\boldsymbol{x})$. The gauge invariant scalar field perturbation can be defined by

$$Q \equiv \delta\phi - \phi' \frac{C}{\mathcal{H}} , \tag{73}$$

where $\mathcal{H} \equiv a'/a \equiv aH$. We denote the field perturbation in Newtonian gauge by χ so that $Q = \chi + (\phi'/\mathcal{H})\Psi$. Introducing

$$\varepsilon = \frac{3}{2}\frac{\phi'^2}{\mu} , \tag{74}$$

the equation of state (23) takes the form $\gamma = w + 1 = 2\varepsilon/3$. We thus have two expansions: one concerning the perturbation of the metric and the other in the slow-roll parameter ϵ.

A. Scalar modes

Focusing on scalar modes at first order in the perturbation, it is convenient to introduce

$$v = aQ \tag{75}$$

and

$$z \equiv a\frac{\phi'}{\mathcal{H}}, \tag{76}$$

in terms of which the action (1) takes the form

$$S_{\rm scal} = \frac{1}{2}\int \mathrm{d}^3\boldsymbol{x}\mathrm{d}\eta \left[(v')^2 - (\partial_i v)^2 + \frac{z''}{z}v^2\right] , \tag{77}$$

when expanded to second order in the perturbations. It is the action of a canonical scalar field with effective square mass $m_v^2 = -z''/z$. v is the canonical variable that must be quantized [41]. It is decomposed as follows

$$\hat{v}(\boldsymbol{x},\eta) = \int \frac{\mathrm{d}^3\boldsymbol{k}}{(2\pi)^{3/2}} \left[v_k(\eta)\mathrm{e}^{i\boldsymbol{k}.\boldsymbol{x}}\hat{a}_{\boldsymbol{k}} + v_k^*(\eta)\mathrm{e}^{-i\boldsymbol{k}.\boldsymbol{x}}\hat{a}_{\boldsymbol{k}}^\dagger\right] . \tag{78}$$

Here v_k is solution of the Klein-Gordon equation

$$v_k'' + \left(k^2 - \frac{z''}{z}\right)v_k = 0 \tag{79}$$

and the annihilation and creation operators satisfy the commutation relation, $[\hat{a}_{\boldsymbol{k}}, \hat{a}_{\boldsymbol{k}'}^\dagger] = \delta(\boldsymbol{k} - \boldsymbol{k}')$. We define the free vacuum state by the requirement $\hat{a}_{\boldsymbol{k}}|0\rangle = 0$ for all $\boldsymbol{k}$.

From the Einstein equation, one can get the expression for the Bardeen potential (recalling that $\Psi = \Phi$)

$$\Delta\Phi = 4\pi G\frac{\phi'^2}{\mathcal{H}}\left(\frac{v}{z}\right)', \quad \left(\frac{a^2\Phi}{\mathcal{H}}\right)' = 4\pi G z v \tag{80}$$

and for the curvature perturbation in comoving gauge

$$\mathcal{R} = -v/z. \tag{81}$$

Once the initial conditions are set, solving Eq. (79) will give the evolution of $v_k(\eta)$ during inflation, from which $\Phi_k(\eta)$ and $\mathcal{R}_k(\eta)$ can be deduced, using the previous expressions.

Defining the power spectrum as

$$\langle \mathcal{R}_{\boldsymbol{k}}\mathcal{R}_{\boldsymbol{k}'}^*\rangle = \frac{2\pi^2}{k^3}\mathcal{P}_{\mathcal{R}}(k)\delta^{(3)}(\boldsymbol{k} - \boldsymbol{k}'), \tag{82}$$

one easily finds that

$$\mathcal{P}_{\mathcal{R}}(k) = \frac{k^3}{2\pi^2}\left|\frac{v_k}{z}\right|^2 . \tag{83}$$

Note also that z and ε are related by the simple relation

$$\sqrt{4\pi G}\, z = a\sqrt{\varepsilon} , \tag{84}$$

so that

$$\chi = Q - \frac{z}{a}\Phi = Q - \sqrt{\frac{\varepsilon}{4\pi G}}\Phi. \tag{85}$$

B. Gravitational waves at linear order

At first order, the tensor modes are gauge invariant and their propagation equation is given by

$$\bar{\mathcal{E}}_{ij}'' + 2\mathcal{H}\bar{\mathcal{E}}_{ij}' - \Delta\bar{\mathcal{E}}_{ij} = 0 \tag{86}$$

since a minimally coupled scalar field has no anisotropic stress. Defining the reduced variable

$$\mu_{ij} = \frac{a}{\sqrt{8\pi G}}\bar{\mathcal{E}}_{ij} , \tag{87}$$

the action (1) takes the form

$$S_{\rm tens} = \frac{1}{2}\int \mathrm{d}^3\boldsymbol{x}\mathrm{d}\eta \left[(\mu_{ij}')^2 - (\partial_k\mu_{ij})^2 + \frac{a''}{a}(\mu_{ij})^2\right] \tag{88}$$

when expanded tosecond order. Developing $\bar{\mathcal{E}}_{ij}$, and similarly μ_{ij}, in Fourier space:

$$\bar{\mathcal{E}}_{ij} = \sum_{\lambda=+,\times}\int \frac{\mathrm{d}^3\boldsymbol{k}}{(2\pi)^{3/2}}\mathcal{E}_\lambda\varepsilon_{ij}^\lambda(\boldsymbol{k})\mathrm{e}^{i\boldsymbol{k}.\boldsymbol{x}}, \tag{89}$$

where ε_{ij}^λ is the polarization tensor, the action (88) takes the form of the action for two canonical scalar fields with effective square mass $m_\mu^2 = -a''/a$

$$S_{\rm tens} = \frac{1}{2}\sum_\lambda \int \mathrm{d}^3\boldsymbol{x}\mathrm{d}\eta \left[(\mu_\lambda')^2 - (\partial_k\mu_\lambda)^2 + \frac{a''}{a}\mu_\lambda^2\right] . \tag{90}$$

If one considers the basis $(\mathbf{e}_1, \mathbf{e}_2)$ of the 2 dimensional space orthogonal to $\boldsymbol{k}$ then $\varepsilon_{ij}^\lambda = (e_i^1 e_j^1 - e_i^2 e_j^2)\delta_+^\lambda + (e_i^1 e_j^2 + e_i^2 e_j^1)\delta_\times^\lambda$.

μ_λ are the two degrees of freedom that must be quantized [41] and we expand them as

$$\hat{\mu}_{ij}(\boldsymbol{x},\eta) = \sum_\lambda \int \frac{\mathrm{d}^3\boldsymbol{k}}{(2\pi)^{3/2}} \Big[\mu_{k,\lambda}(\eta)\mathrm{e}^{i\boldsymbol{k}.\boldsymbol{x}}\hat{b}_{\boldsymbol{k},\lambda} + \mu^*_{k,\lambda}(\eta)\mathrm{e}^{-i\boldsymbol{k}.\boldsymbol{x}}\hat{b}^\dagger_{\boldsymbol{k},\lambda}\Big]\varepsilon^\lambda_{ij}(\boldsymbol{k}). \quad (91)$$

μ_k is solution of the Klein-Gordon equation

$$\mu_k'' + \left(k^2 - \frac{a''}{a}\right)\mu_k = 0, \quad (92)$$

where we have dropped the polarization subscript. The annihilation and creation operators satisfy the commutation relations, $[\hat{b}_{\boldsymbol{k},\lambda}, \hat{b}^\dagger_{\boldsymbol{k}',\lambda'}] = \delta(\boldsymbol{k}-\boldsymbol{k}')\delta_{\lambda\lambda'}$ and $[\hat{a}_{\boldsymbol{k}}, \hat{b}^\dagger_{\boldsymbol{k}',\lambda}] = 0$. We define the free vacuum state by the requirement $\hat{b}_{\boldsymbol{k},\lambda}|0\rangle = 0$ for all $\boldsymbol{k}$ and λ.

Defining the power spectrum as

$$\langle \mathcal{E}_{\boldsymbol{k},\lambda}\mathcal{E}^*_{\boldsymbol{k}',\lambda'}\rangle = \frac{2\pi^2}{k^3}\mathcal{P}_T(k)\delta^{(3)}(\boldsymbol{k}-\boldsymbol{k}')\delta_{\lambda\lambda'}, \quad (93)$$

one easily finds that

$$\mathcal{P}_T(k) = 16\pi G\frac{k^3}{2\pi^2}\left|\frac{\mu_k}{a}\right|^2, \quad (94)$$

where the two polarizations have the same contribution.

C. Gravitational waves from density perturbations

At second order, we split the tensor perturbation as $\bar{\mathcal{E}}_{ij} = \bar{\mathcal{E}}^{(1)}_{ij} + \bar{\mathcal{E}}^{(2)}_{ij}/2$. The evolution equations of $\bar{\mathcal{E}}^{(2)}_{ij}$ is similar to Eq. (86), but inherits a source term quadratic in the first order perturbation variables and from the transverse tracefree (TT) part of the stress-energy tensor

$$a^2\left[T^i_j\right]^{\mathrm{TT}} = \gamma^{ip}\left[\partial_j\chi\partial_p\chi\right]^{\mathrm{TT}}. \quad (95)$$

It follows that the propagation equation is

$$\bar{\mathcal{E}}^{(2)\prime\prime}_{ij} + 2\mathcal{H}\bar{\mathcal{E}}^{(2)\prime}_{ij} - \Delta\bar{\mathcal{E}}^{(2)}_{ij} = S^{\mathrm{TT}}_{ij}, \quad (96)$$

where S^{TT}_{ij} is a TT tensor that is quadratic in the first order perturbation variables.

Working in Fourier space, the TT part of any tensor can easily be extracted by means of the projection operator

$$\perp_{ij}(\hat{\boldsymbol{k}}) = \delta_{ij} - \hat{k}_i\hat{k}_j, \quad (97)$$

where $\hat{k}^i = k^i/k$ (note that $\perp_{ij}(\hat{\boldsymbol{k}})$ is not analytic in k and is a non-local operator) from which we get

$$S^{\mathrm{TT}}_{ij}(\boldsymbol{k},\eta) = \left[\perp^a_i\perp^b_j - \frac{1}{2}\perp_{ij}\perp^{ab}\right]S_{ab}(\boldsymbol{k},\eta) \equiv P^{ab}_{ij}(\boldsymbol{k})S_{ab}(\boldsymbol{k},\eta). \quad (98)$$

The source term is now obtained as the TT-projection of the second order Einstein tensor quadratic in the first order variables and of the stress-energy tensor

$$S_{ab} = S^{(2)}_{ab,\mathrm{SS}} + S^{(2)}_{ab,\mathrm{ST}} + S^{(2)}_{ab,\mathrm{TT}}. \quad (99)$$

The three terms respectively indicate terms involving products of first order scalar quantities, first order scalar and tensor quantities and first order tensor quantities. The explicit form of the first term is

$$S^{\mathrm{TT}}_{ij} = 4\left[\partial_i\Phi\partial_j\Phi + 4\pi G\,\partial_i\chi\partial_j\chi\right]^{\mathrm{TT}}. \quad (100)$$

The first term was considered in Ref. [43] and the second term was shown to be the dominant contribution for the production of gravitational waves during preheating [42]. In Fourier space, it is given by

$$S^{(2)}_{ab,\mathrm{SS}} = -4\left[\int \mathrm{d}^3\boldsymbol{q}\,q_bq_a\Phi(\boldsymbol{q},\eta)\Phi(\boldsymbol{k}-\boldsymbol{q},\eta) + 4\pi G\int \mathrm{d}^3\boldsymbol{q}\,q_bq_a\chi(\boldsymbol{q},\eta)\chi(\boldsymbol{k}-\boldsymbol{q},\eta)\right]. \quad (101)$$

$\mu^{(2)}_{ij}(\boldsymbol{x},\eta)$ can be decomposed as in Eq. (89), using the same definition (87) at any order. The two polarizations evolve according to

$$\mu^{(2)\prime\prime}_\lambda + \left(k^2 - \frac{a''}{a}\right)\mu^{(2)}_\lambda = -\frac{2a}{\sqrt{8\pi G}}P^{ab}_{ij}S^{(2)}_{ab,\mathrm{SS}}\varepsilon^{ij}_\lambda. \quad (102)$$

Since the polarization tensor is a TT tensor, it is obvious that $P^{ab}_{ij}\varepsilon^{ij}_\lambda = \varepsilon^{ab}_\lambda$, so that

$$\mu^{(2)\prime\prime}_\lambda + \left(k^2 - \frac{a''}{a}\right)\mu^{(2)}_\lambda = -\frac{4a}{\sqrt{8\pi G}}\varepsilon^{ij}_\lambda\int \mathrm{d}^3\boldsymbol{q}\,q_iq_j\left[\Phi(\boldsymbol{q},\eta)\Phi(\boldsymbol{q}-\boldsymbol{k},\eta) + 4\pi G\,\chi(\boldsymbol{q},\eta)\chi(\boldsymbol{q}-\boldsymbol{k},\eta)\right]. \quad (103)$$

From the equation (102), we deduce that the source term derives from an interaction Lagrangian of the form

$$S_{\mathrm{int}} = \int \mathrm{d}\eta\mathrm{d}^3\boldsymbol{x}\frac{4a}{\sqrt{8\pi G}}\left[\partial_i\Phi\partial_j\Phi + 4\pi G\,\partial_i\chi\partial_j\chi\right]\mu^{ij}. \quad (104)$$

It describes a two-scalars graviton interaction. In full generality the interaction term would also include, at lowest order, cubic terms of three scalars, two gravitons-scalar and three gravitons. They respectively correspond to second order scalar-scalar modes generated from gravitational waves and second order tensor modes. As emphasized previously, we do not consider these interactions here.

V. COMPARISON OF THE TWO FORMALISMS

Before going further it is instructive to compare the two formalisms and understand how they relate to each other. Note that we go beyond Ref. [35], where a comparison of the variables was made at linear order. Here we investigate how the equations map to each other and extend the discussion to second order for the tensor sector. At the background level the scale factors a and expansion rates H introduced in each formalism agree, which explains why we made use of the same notation.

The perturbations of the metric around FL space-time has been split into a first-order and a second-order part according to

$$X = X^{(1)} + \frac{1}{2} X^{(2)} . \tag{105}$$

We make a similar decomposition for the quantities used in the $1+3$ covariant formalism. As long as we are interested in the gravitational wave sector, we only need to consider the four-velocity of the perfect fluid describing the matter content of the universe which we decompose as

$$u^\mu = \frac{1}{a}(\delta^\mu_0 + V^\mu). \tag{106}$$

Its spatial components are decomposed as

$$V^i = \partial^i V + \bar{V}^i , \tag{107}$$

$\bar{V}^i$ being the vector degree of freedom and V the scalar degree of freedom. As V^μ has only three independent degrees of freedom since u^μ satisfies $u_\mu u^\mu = -1$, its temporal component is linked to other perturbation variables. We assume that the fluid has no vorticity ($\bar{V}^i = 0$), as it is the case for the scalar fluid we have in mind and consequently we will also drop the vectorial perturbations ($\bar{\mathcal{E}}_i = 0$).

A. Matching at linear order

At first order, the spatial components of the shear, acceleration and expansion are respectively given by

$$\sigma^{(1)}_{ij} = a\left(\partial_{\langle i}\partial_{j\rangle} V^{(1)} + \bar{\mathcal{E}}^{(1)\prime}_{ij}\right) , \tag{108}$$

$$\dot{u}^{(1)}_i = \partial_i\left(\Phi^{(1)} + \mathcal{H}V^{(1)} + V^{(1)\prime}\right) , \tag{109}$$

$$\delta\Theta^{(1)} = \frac{1}{a}\left(-3\Psi^{(1)\prime} - 3\mathcal{H}\Phi^{(1)} + \Delta V^{(1)}\right) . \tag{110}$$

The electric and magnetic part of the Weyl tensor take the form

$$E^{(1)}_{ij} = \partial_{\langle i}\partial_{j\rangle}\Phi^{(1)} - \frac{1}{2}\left(\bar{\mathcal{E}}''_{ij} + \Delta\bar{\mathcal{E}}_{ij}\right) , \tag{111}$$

$$H^{(1)}_{ij} = \eta_{kl\langle i}\partial^k \bar{\mathcal{E}}^{(1)\prime\ l}_{j\rangle} \equiv (\hat{\mathsf{curl}}\,\bar{\mathcal{E}}^{(1)\prime})_{ij} . \tag{112}$$

Note that η_{kli} is the completely antisymmetric tensor normalized such that $\eta_{123} = 1$, which differs from ε_{abc}. We deduce from the last expression that

$$\left(\mathsf{curl}\, E^{(1)}\right)_{ij} = -\frac{1}{2a}\left[\left(\hat{\mathsf{curl}}\,\bar{\mathcal{E}}^{(1)\prime\prime}\right)_{ij} + \left(\hat{\mathsf{curl}}\,\Delta\bar{\mathcal{E}}^{(1)}\right)_{ij}\right] , \tag{113}$$

where we have used simpler notation by writing $(\hat{\mathsf{curl}}\,\bar{\mathcal{E}})_{ij}$ as $\hat{\mathsf{curl}}\,\bar{\mathcal{E}}_{ij}$. We also note that the derivative along u_μ of a tensor T of rank (n,m), vanishing in the background, takes the form

$$\dot{T}^{i_1\dots i_n}_{\quad j_1\dots j_m} = \partial_t T^{i_1\dots i_n}_{\quad j_1\dots j_m} + (n-m)HT^{i_1\dots i_n}_{\quad j_1\dots j_m} \tag{114}$$

at first order, or alternatively

$$\frac{\left(a^{m-n}T^{i_1\dots i_n}_{\quad j_1\dots j_m}\right)^{\cdot}}{a^{m-n}} = \partial_t T^{i_1\dots i_n}_{\quad j_1\dots j_m} . \tag{115}$$

Again, recall that a dot refers to a derivative along u^μ. Indeed at first order, it reduces to a derivative with respect to the cosmic time but this does not generalize to second order.

Now, Eq. (39) can be recast a

$$a^{-2}\left(a^2 H_{ij}\right)^{\cdot} + \mathsf{curl}\, E_{ij} + H\, H_{ij} = 0 . \tag{116}$$

Using the expressions (111-112) for the geometric quantities, this equation takes the form

$$\hat{\mathsf{curl}}\left[\frac{1}{2a}\left(\bar{\mathcal{E}}''_{ij} + 2\mathcal{H}\bar{\mathcal{E}}'_{ij} - \Delta\bar{\mathcal{E}}_{ij}\right)\right] = 0 . \tag{117}$$

Similarly Eq. (59) can be recast as

$$\frac{\left(a^2 H_{ab}\right)^{\cdot\cdot}}{a^2} + 3H\frac{\left(a^2 H_{ab}\right)^{\cdot}}{a^2} + 2(H^2 + \dot{H})H_{ab} - \widetilde{\nabla}^2 H_{ab} = 0 , \tag{118}$$

so that it reduces at first order to

$$\hat{\mathsf{curl}}\left[\frac{1}{2a^2}\left(\bar{\mathcal{E}}^{(1)\prime\prime}_{ij} + 2\mathcal{H}\bar{\mathcal{E}}^{(1)\prime}_{ij} - \Delta\bar{\mathcal{E}}^{(1)}_{ij}\right)'\right] = 0 . \tag{119}$$

Thus, Eq. (119) maps to Eq. (86) with the identification (112), if there is no vector modes. This can be understood from the fact that in the Bardeen formalism, Eq. (86) is obtained from the Einstein equation as $\hat{\mathsf{curl}}^{-1}[\hat{\mathsf{curl}}\, G_{ij}] = 0$.

In the case where there are vector modes, Eq. (112) has to be replaced by

$$H^{(1)}_{ij} = (\hat{\mathsf{curl}}\,\bar{\mathcal{E}}^{(1)\prime})_{ij} + \frac{1}{2}\eta_{kl\langle i}\partial^k\partial_{j\rangle}\bar{\mathcal{E}}^{(1)\prime l}$$

and H_{ab} is no longer a description of the GW, i.e. directly related to the TT part of the spacetime metric and the matching is not valid anymore.

B. Matching at second order

At second order, the matching is much more intricate mainly because the derivative along u^μ does not match with the derivative respect to cosmic time any more.

Let us introduce the short hand notation

$$(X \times Y)_{ij} \equiv \eta_{kl\langle i} X^k Y_{j\rangle}{}^l \tag{120}$$

for any tensors X^k and Y^{lm}. If $Y^{lm} = \partial^l \partial^m Z$, or $X^k = \partial^k W$, we also use the short-hand notation $Y = \partial\partial Z$ $X = \partial W$.

Among the terms quadratic in first-order perturbations, those involving a first-order tensorial perturbation can be omitted, as we are only interested in second-order effects sourced by scalar contributions. At second order, the geometric quantities of interest read

$$H^{(2)}_{ij} = \left(\hat{\mathrm{curl}}\, \bar{\mathcal{E}}^{(2)\prime}\right)_{ij} - 4\left(\partial V^{(1)} \times \partial\partial\Phi^{(1)}\right)_{ij} \tag{121}$$

$$\left(\mathrm{curl}\, E^{(2)}\right)_{ij} = -\frac{1}{2a}\left[\left(\hat{\mathrm{curl}}\, \bar{\mathcal{E}}^{(2)\prime\prime}\right)_{ij} + \left(\hat{\mathrm{curl}}\, \Delta\bar{\mathcal{E}}^{(2)}\right)_{ij}\right] - \frac{2}{a}\left[\left(\partial\Phi^{(1)} \times \partial\partial\Phi^{(1)}\right)_{ij} + \mathcal{H}\left(\partial V^{(1)} \times \partial\partial\Phi^{(1)}\right)_{ij} - \left(\partial V^{(1)} \times \partial\partial\Phi^{(1)\prime}\right)_{ij}\right]. \tag{122}$$

From the latter expression, we remark that $H^{(2)}_{ij}$ has a term quadratic in first-order perturbations involving $V^{(1)}$ and $\Phi^{(1)}$. This terms arise from a difference between the two formalisms related to the fact that geometric quantities, such as H_{ij} E_{ij} etc., live on the physical space-time, whereas in perturbation theory, any perturbation variable at any order, such as $V^{(1)}$, $\mathcal{E}^{(2)}_{ij}$ etc., are fields propagating on the background space-time.

It follows that the splitting into tensor, vector and scalar modes is different. In the covariant formalism, the splitting refers to the fluid on the physical space-time, whereas in perturbation theory it refers to the comoving fluid of the background solution. Indeed, this difference only shows up at second order as the magnetic Weyl tensor vanishes in the background. The one to one correspondence at first order between equations of both formalisms disappears, as the second order equations of the covariant formalism contain the dynamics of the first order quantities.

When keeping terms contributing to the second order, Eq. (39) has an additionnal source term and reads

$$\dot{H}_{ab} + \mathrm{curl}\, E_{ab} + 3HH_{ab} = -2\epsilon_{cd\langle a}\dot{u}^c E_{b\rangle}{}^d \tag{123}$$

If first order tensorial perturbations are neglected then H_{ab} vanishes at first order and Eq. (114) still holds when applied to H_{ab}. Thus Eq. (123) can be recast as

$$\frac{\left(a^2 H_{ab}\right)^{\cdot}}{a^2} + \mathrm{curl}\, E_{ab} + \frac{\mathcal{H}}{a} H_{ab} = -2\epsilon_{cd\langle a}\dot{u}^c E_{b\rangle}{}^d\,. \tag{124}$$

Substituting the geometric quantities for their expressions at second order, and making use of Eq. (115) to handle the derivatives, Eq. (116) reads at second order

$$\hat{\mathrm{curl}}\left[\frac{1}{2a}\left(\bar{\mathcal{E}}^{(2)\prime\prime}_{ij} + 2\mathcal{H}\bar{\mathcal{E}}^{(2)\prime}_{ij} - \Delta\bar{\mathcal{E}}^{(2)}_{ij}\right)\right] = -\frac{2}{a}\left[\left(\partial\Phi^{(1)} \times \partial\partial\Phi^{(1)}\right)_{ij} - \left(\partial V^{(1)} \times \partial\partial\Phi^{(1)\prime}\right)_{ij} - \mathcal{H}\left(\partial V^{(1)} \times \partial\partial\Phi^{(1)}\right)_{ij}\right]. \tag{125}$$

Using the momentum and constraint equation (41) at first order

$$\Phi^{(1)\prime} + \mathcal{H}\Phi^{(1)} = \left(\mathcal{H}' - \mathcal{H}^2\right) V^{(1)} \tag{126}$$

and the background equation $\mathcal{H}' - \mathcal{H}^2 = -4\pi G\mu(1+w)a^2$, that we deduce from the Raychaudhuri equation and the Gauss-Codacci equation at first order, we can link it to Eq. (96) as it then reads

$$\frac{1}{a}\hat{\mathrm{curl}}\left[\frac{1}{2}\left(\bar{\mathcal{E}}^{(2)\prime\prime}_{ij} + 2\mathcal{H}\bar{\mathcal{E}}^{(2)\prime}_{ij} - \Delta\bar{\mathcal{E}}^{(2)}_{ij}\right)\right] = \frac{1}{a}\hat{\mathrm{curl}}\left[2\partial_i\Phi^{(1)}\partial_j\Phi^{(1)} + 8\pi G a^2(\mu+P)\partial_i V^{(1)}\partial_j V^{(1)}\right]. \tag{127}$$

When applied to a scalar field, this is exactly the gravitational wave propagation equation (96) with the source term (100).

C. Discussion

In conclusion, we have matched both the perturbation variables and equations at first and second order in the perturbations. This extends the work of Ref. [35] which considered the linear case, and has not been previously investigated.

Even though we restrict to the tensor sector, this comparison is instructive and illustrates the difference of approach between the two formalisms, in a clearer way than at first order. In the Bardeen approach, all perturbation variables live on the unperturbed spacetime. At each order, we write exact equations for an approximate spacetime. In particular, this implies that the time derivatives are derivative with respect to the cosmic time of the background spacetime. In the covariant approach, one derives an exact set of equations (assuming no perturbation to start with). These exact equations are then solved iteratively starting from a background solution which assumes some symmetries. The time derivative is defined in terms of the flow vector as $u^a\nabla_a$. Indeed, at first order for scalars, this derivative matches exactly with the derivative with respect to the background cosmic time. At second order, this is no longer the case. First the flow vector at first order does not coincide with its background value. This implies a (first-order) difference between the two time derivatives which must be taken into account. Then, the geometric quantities, such as H_{ij} E_{ij} etc., "live" on the physical space-time, whereas in perturbation theory, any perturbation variable at any order, such as $V^{(1)}$, $\mathcal{E}^{(2)}_{ij}$ etc., live on the background spacetime. This explains why e.g. $H^{(2)}_{ij}$ has a term quadratic in first-order perturbations involving $V^{(1)}$ and $\Phi^{(1)}$.

The master variables and corresponding wave equations in both formalisms are also different in nature. In the metric approach the wave equation with source is defined non-locally in Fourier space; in the covariant approach, we are able to derive a local tensorial wave equation which, because it is divergence-free, represents the gravitational wave contribution. Of course, we can make a non-local decomposition in Fourier space as required. Furthermore, on one hand the TT part of the metric in a particular gauge is a perturbative approach used to describe GW, and this tells us the shear of spatial lengths with respect to a homogenous and isotropic background, referring implicitly to a hypothetical set of averaged observers. On the other hand, the covariant description using H_{ab} which is built out of the Weyl tensor and the comoving observer's velocity, directly describes the dynamically free part of the gravitational field [39] (up to second-order when rotation is zero) as seen by the true comoving observers. This is part of the dynamic spacetime curvature which directly induces the motion of test particles through the geodesic deviation equation, and it accounts for effects due to the non-homogenous comoving fluid velocity.

There is one more difference between the two formalisms, concerning the initial conditions. In the Bardeen approach, as we recalled in section IV, there is a natural way to set up the initial conditions on sub-Hubble scales by identifying canonical variables, both for the scalar and tensor modes, and promoting them to the status of quantum operators. In the covariant formalism such variables have not been constructed in full generality (see however Ref. [44] for a proposal). Consequently this sets limitations to this formalism since it cannot account for both the evolution and the initial conditions at the same time.

VI. ILLUSTRATION: SLOW-ROLL INFLATION

A. Slow-roll inflation

In this section, we focus on the case of a single slow-rolling scalar field and we introduce the slow-roll parameters

$$\varepsilon = \frac{3}{2}\frac{\psi^2}{\mu}, \qquad \delta = -\frac{\dot\psi}{H\psi}. \tag{128}$$

Using the Friedmann equations (28-29), these parameters can be expressed in terms of the Hubble parameter as

$$\varepsilon = -\frac{1}{4\pi G}\left[\frac{H'(\phi)}{H(\phi)}\right]^2, \qquad \delta = \frac{1}{4\pi G}\frac{H''(\phi)}{H(\phi)}. \tag{129}$$

Interestingly Eq. (28) takes the form

$$H^2\left(1-\frac{1}{3}\varepsilon\right) = \frac{\kappa}{2}V(\phi), \tag{130}$$

which implies

$$\frac{\ddot a}{a} = (1-\varepsilon)H^2. \tag{131}$$

The equation of state and the sound speed of the equivalent scalar field are thus given by

$$w = -1+\frac{2}{3}\varepsilon, \qquad c_s^2 = -1+\frac{2}{3}\delta. \tag{132}$$

The evolution equations for ε and δ show that $\dot\varepsilon$ and $\dot\delta$ are of order 2 in the slow-roll parameters so that at first order in the slow-roll parameters, they can be considered constant. Using the definition of the conformal time and integrating it by parts, one gets

$$a(\eta) = -\frac{1}{H\eta}\frac{1}{1-\varepsilon}, \tag{133}$$

assuming ε is constant, from which it follows that

$$\mathcal{H} \equiv aH = -\frac{1}{\eta}(1+\varepsilon) + \mathcal{O}(2), \tag{134}$$

where η varies between $-\infty$ and 0. This implies that

$$\frac{a''}{a} = \frac{2+3\varepsilon}{\eta^2}, \qquad \frac{z''}{z} = \frac{2+6\varepsilon-3\delta}{\eta^2}. \tag{135}$$

The general solution of Eq. (79) is

$$v_k = \sqrt{-\pi\eta/4}\left[c_1 H_\nu^{(1)}(-k\eta) + c_2 H_\nu^{(2)}(-k\eta)\right], \tag{136}$$

with $|c_1|^2-|c_2|^2=1$, where $H_\nu^{(1)}$ and $H_\nu^{(2)}$ are Hankel functions of first and second kind and $\nu=3/2+2\varepsilon-\delta$. Among this family of solutions, it is natural to choose the one with $c_2=0$ which contains only positive frequencies [41]. It follows that the solution with these initial conditions is

$$v_k(\eta)=\frac{\sqrt{\pi}}{2}\sqrt{-\eta}H_\nu^{(1)}(-k\eta)\,. \tag{137}$$

On super-Hubble scales, $|k\eta|\ll 1$, we have

$$v_k\to 2^{\nu-3/2}\Gamma(\nu)/\Gamma(3/2)(2k)^{-1/2}(-k\eta)^{-\nu+1/2}.$$

Now, using Eq. (133) to express η and Eq. (84) to replace z in expression (83), we find that

$$\mathcal{P}_\mathcal{R}(k) \;=\; \frac{1}{\pi}\frac{H^2}{M_p^2\varepsilon}\left[2^{\nu-3/2}\frac{\Gamma(\nu)}{\Gamma(3/2)}\right]^2\left(\nu-\frac{1}{2}\right)^{-2\nu+1}\times\left(\frac{k}{aH}\right)^{-2\nu+3}, \tag{138}$$

where we have set $M_p^2=G^{-1}$. At lowest order in the slow-roll parameter, it reduces to

$$\mathcal{P}_\mathcal{R}(k) \;=\; \frac{1}{\pi}\frac{H^2}{M_p^2\varepsilon}\left(\frac{k}{aH}\right)^{2\delta-4\varepsilon}. \tag{139}$$

The evolution of the gravitational waves at linear order are dictated by the same equation but with $\nu_T=3/2+\varepsilon$, so that

$$\mu_k^{(1)}(\eta)=\frac{\sqrt{\pi}}{2}\sqrt{-\eta}H_{\nu_T}^{(1)}(-k\eta). \tag{140}$$

Similarly as for the scalar mode, we obtain

$$\mathcal{P}_T(k) \;=\; \frac{16}{\pi}\frac{H^2}{M_p^2}\left(\frac{k}{aH}\right)^{-2\varepsilon}. \tag{141}$$

B. Gravitational waves at second order

The couplings between scalar and tensor modes at second order imply that the second order variables can be expanded as

$$\mathcal{R}=\mathcal{R}^{(1)}+\frac{1}{2}\left(\mathcal{R}^{(2)}_{\mathcal{R}\mathcal{R}}+\mathcal{R}^{(2)}_{\mathcal{E}\mathcal{E}}+\mathcal{R}^{(2)}_{\mathcal{R}\mathcal{E}}\right)$$

and a similar expansion for $\mathcal{E}$, where, e.g., $\mathcal{R}^{(2)}_{\mathcal{R}\mathcal{E}}$ stands for the second order scalar modes induced by the coupling of first order scalar and tensor modes etc. The deviation from Gaussianity at the time η of the end of inflation can be characterized by a series of coefficients $f_{\rm NL}^{a,bc}$ defined for example as

$$\frac{1}{2}\mathcal{R}^{(2)}_{\mathcal{E}\mathcal{E}}(\boldsymbol{k},\eta) \;=\; \frac{1}{(2\pi)^{3/2}}\int\delta^3(\boldsymbol{k}_1+\boldsymbol{k}_2-\boldsymbol{k})\mathcal{E}(\boldsymbol{k}_1,\eta)\mathcal{E}(\boldsymbol{k}_2,\eta)\, f_{\rm NL}^{\mathcal{R},\mathcal{E}\mathcal{E}}(\boldsymbol{k},\boldsymbol{k}_1,\boldsymbol{k}_2,\eta)\mathrm{d}^3\boldsymbol{k}_1\mathrm{d}^3\boldsymbol{k}_2. \tag{142}$$

These six coefficients appear in different combinations in the connected part of the 3-point correlation function of $\mathcal{R}$ and $\mathcal{E}$. For instance

$$\langle\mathcal{E}_{\boldsymbol{k}_1}\mathcal{R}_{\boldsymbol{k}_2}\mathcal{R}_{\boldsymbol{k}_3}\rangle_c=\left[2f_{\rm NL}^{\mathcal{E},\mathcal{R}\mathcal{R}}(\boldsymbol{k}_1,\boldsymbol{k}_2,\boldsymbol{k}_3)P_\mathcal{R}(k_3)P_\mathcal{R}(k_2)+f_{\rm NL}^{\mathcal{R},\mathcal{E}\mathcal{R}}(\boldsymbol{k}_1,\boldsymbol{k}_2,\boldsymbol{k}_3)P_\mathcal{R}(k_3)P_\mathcal{E}(k_1)\right]\delta^3(\boldsymbol{k}_1+\boldsymbol{k}_2+\boldsymbol{k}_3) \tag{143}$$

and $f_{\rm NL}^{\mathcal{R},\mathcal{R}\mathcal{R}}$ is the standard $f_{\rm NL}$ parameter. One can easily check that $\langle\mathcal{R}_{\boldsymbol{k}_1}\mathcal{R}_{\boldsymbol{k}_2}\mathcal{R}_{\boldsymbol{k}_3}\rangle_c$ involves $f_{\rm NL}^{\mathcal{R},\mathcal{R}\mathcal{R}}$, $\langle\mathcal{E}_{\boldsymbol{k}_1}\mathcal{E}_{\boldsymbol{k}_2}\mathcal{R}_{\boldsymbol{k}_3}\rangle_c$ involves $f_{\rm NL}^{\mathcal{E},\mathcal{E}\mathcal{R}}$ and $f_{\rm NL}^{\mathcal{R},\mathcal{E}\mathcal{E}}$, and $\langle\mathcal{E}_{\boldsymbol{k}_1}\mathcal{E}_{\boldsymbol{k}_2}\mathcal{E}_{\boldsymbol{k}_3}\rangle_c$ involves $f_{\rm NL}^{\mathcal{E},\mathcal{E}\mathcal{E}}$.

C. Expression for $f_{\rm NL}^{\mathcal{E},\mathcal{R}\mathcal{R}}$

From our analysis, we can give the expression of $f_{\rm NL}^{\mathcal{E},\mathcal{R}\mathcal{R}}$. Starting from the fact that $-\varepsilon\mathcal{R}=\Phi(1+\varepsilon)+\Phi'/\mathcal{H}$ and from the expression (85), we get that $\Phi\sim-\varepsilon\mathcal{R}-\Phi'/\mathcal{H}$ or $\Phi=-\epsilon\eta\int\frac{\mathcal{R}}{\eta^2}\mathrm{d}\eta$, and $\sqrt{4\pi G}\chi\sim-\sqrt{\varepsilon}\left[\mathcal{R}-\Phi'/\mathcal{H}\right]$. It follows that the source term (100) reduces at lowest order in the slow-roll parameter to

$$S_{ij}^{\rm TT}=4\left[\varepsilon\partial_i\mathcal{R}\partial_j\mathcal{R}\right]^{\rm TT}.$$

The interaction Lagrangian is thus given by

$$S_{\rm int}=\int\mathrm{d}\eta\mathrm{d}^3\boldsymbol{x}\frac{4a}{\sqrt{8\pi G}}\,\varepsilon\,\partial_i\mathcal{R}\partial_j\mathcal{R}\,\mu^{ij}, \tag{144}$$

which reduces to

$$S_{\rm int}=\int\mathrm{d}\eta\mathrm{d}^3\boldsymbol{x}\,2\partial_i v\,\partial_j v\,\bar{\mathcal{E}}^{ij}. \tag{145}$$

This is the same expression as obtained in Ref. [9].

In full generality, during inflation, we should use the "in-in" formalism to compute any correlation function of the interacting fields. As was shown explicitly in Ref. [45]

for a self-interacting field and more generally in Ref. [47], the quantum computation agrees with the classical one on super-Hubble scales at lowest order. Note however that both computations may differ (see Ref. [26] versus Ref. [9]) due to the fact that in the classical approach the change in vacuum is ignored. The difference does not affect the order of magnitude but the geometric k-dependence. In order to get an order of magnitude, we thus restrict our analysis here to the classical description. This description is also valid when considering the post-inflationary era.

In the classical approach, we can solve Eq. (103) by mean of a Green function. Since the two independent solutions of the homogeneous equation are $\sqrt{-k\eta}H^{(1/2)}_{\nu_T}(-k\eta)$, the Wronskian of which is $4i/(\pi k)$, the Green function is given by

$$G(k,\eta,\eta') = -i\frac{\pi}{4}\sqrt{\eta\eta'}\left[H^{(1)}_{\nu_T}(-k\eta)H^{(2)}_{\nu_T}(-k\eta') - H^{(1)}_{\nu_T}(-k\eta')H^{(2)}_{\nu_T}(-k\eta)\right]. \quad (146)$$

It follows that the expression of the second order tensor perturbation is given by

$$\mu^{(2)}_{\boldsymbol{k},\lambda}(\eta) = \frac{2}{(2\pi)^{3/2}}\int_{-\infty}^{\eta}\mathrm{d}\eta'\frac{a(\eta')}{\sqrt{8\pi G}}\,\varepsilon\, G(k,\eta,\eta') \int \mathrm{d}^3\boldsymbol{q}\left(q_i q_j \varepsilon^{ij}_{\lambda}\right)\mathcal{R}_{\boldsymbol{q}}(\eta')\mathcal{R}_{\boldsymbol{k}-\boldsymbol{q}}(\eta'). \quad (147)$$

We thus obtain

$$f^{\mathcal{E}_\lambda\mathcal{R}\mathcal{R}}_{NL}(\boldsymbol{k},\boldsymbol{q_1},\boldsymbol{q_2},\eta) = \left[\mathcal{R}_{\boldsymbol{q_1}}(\eta)\mathcal{R}_{\boldsymbol{q_2}}(\eta)\right]^{-1}\frac{\varepsilon}{a(\eta)}\int_{-\infty}^{\eta}\mathrm{d}\eta' a(\eta')\, G(k,\eta,\eta')\left(q_{1i}q_{1j}\varepsilon^{ij}_{\lambda}(\boldsymbol{k})\right)\mathcal{R}_{\boldsymbol{q_1}}(\eta')\mathcal{R}_{\boldsymbol{q_2}}(\eta'). \quad (148)$$

If we want to estimate Eq. (143) in the squeezed limit $k_1 \ll k_2, k_3$ the contribution coming from the term involving $f^{\mathcal{E}_\lambda\mathcal{R}\mathcal{R}}_{NL}(\boldsymbol{k},\boldsymbol{q_1},\boldsymbol{q_2},\eta)$ can be computed by use of the super-Hubble limit of the Green function $|G(k,\eta,\eta')| \simeq \frac{\sqrt{\eta\eta'}}{2\nu_T}\left[\left(\frac{\eta'}{\eta}\right)^{\nu_T} - \left(\frac{\eta'}{\eta}\right)^{-\nu_T}\right]$. This contribution will be proportional to $\frac{H^4}{M_p^4\varepsilon}k_2^{-8}\left(k_{2i}k_{2j}\varepsilon^{ij}_{\lambda}\right)\delta^3(\boldsymbol{k}_1+\boldsymbol{k}_2+\boldsymbol{k}_3)$, which is the same order of magnitude as in Ref. [9], but do not have the same geometric dependence as it goes like $k_2^{-5}k_1^{-3}$ instead.

D. Orders of magnitude

When we want to estimate $\langle\mathcal{E}^{(2)}_{\boldsymbol{k},\lambda}(\eta)\mathcal{E}^{(2)*}_{\boldsymbol{k}',\lambda'}(\eta)\rangle$, we have to evaluate the connected part of $\langle\mathcal{R}_{\boldsymbol{q}}(\eta')\mathcal{R}_{\boldsymbol{k}-\boldsymbol{q}}(\eta')\mathcal{R}^*_{\boldsymbol{p}}(\eta'')\mathcal{R}^*_{\boldsymbol{k}'-\boldsymbol{p}}(\eta'')\rangle$, where $\boldsymbol{q}$ and $\boldsymbol{p}$ are the two internal momentum and η' and η'' the two integration times. From the Wick theorem, this correlator reduces to $\mathcal{R}(q,\eta')\mathcal{R}^*(q,\eta'')\mathcal{R}(|\boldsymbol{k}-\boldsymbol{q}|,\eta')\mathcal{R}^*(|\boldsymbol{k}-\boldsymbol{q}|,\eta'')\delta(\boldsymbol{k}-\boldsymbol{k}')[\delta(\boldsymbol{q}-\boldsymbol{p})+\delta(\boldsymbol{k}-\boldsymbol{q}-\boldsymbol{p})]$ and because $k^i\varepsilon_{ij}=0$ the two terms give the same geometric factor. Thus, the integration on $\boldsymbol{p}$ is easily done and we can factorize $\delta(\boldsymbol{k}-\boldsymbol{k}')$. Now, note that the terms in the integral involve only the modulus of $\boldsymbol{q}$ and $\boldsymbol{k}-\boldsymbol{q}$ so that it does not depend on the angle φ of $\boldsymbol{q}$ in the plane orthogonal to $\boldsymbol{k}$. This implies that the integration of φ will act on a term of $\cos^2 2\varphi$, $\sin^2 2\varphi$ and $\cos 2\varphi \sin 2\varphi$ respectively for $++$, $\times\times$ and $+\times$ so that it gives a term $\pi\delta_{\lambda\lambda'}$. In conclusion, defining the second order power spectrum $\mathcal{P}^{(2)}_T$ by

$$\frac{1}{4}\langle\mathcal{E}^{(2)}_{\boldsymbol{k},\lambda}\mathcal{E}^{*2}_{\boldsymbol{k}',\lambda'}\rangle = \frac{2\pi^2}{k^3}\mathcal{P}^{(2)}_T(k)\delta^{(3)}(\boldsymbol{k}-\boldsymbol{k}')\delta_{\lambda\lambda'}, \quad (149)$$

it can be expressed as

$$\mathcal{P}^{(2)}_T(k) = \frac{k^3}{(2\pi)^3\pi^2 a^2}\int \mathrm{d}\eta'\mathrm{d}\eta'' a(\eta')a(\eta'')\,\varepsilon^2\, G(k,\eta,\eta')G^*(k,\eta,\eta'') \times\int \mathrm{d}^3 q\left(q_i q_j\varepsilon^{ij}_{\lambda}\right)^2\mathcal{R}(q,\eta')\mathcal{R}^*(q,\eta'')\mathcal{R}(|\boldsymbol{k}-\boldsymbol{q}|,\eta')\mathcal{R}^*(|\boldsymbol{k}-\boldsymbol{q}|,\eta''). \quad (150)$$

Setting $\boldsymbol{k}\cdot\boldsymbol{q}=kq\mu$, this reduces to

$$\mathcal{P}_T^{(2)}(k) \;=\; \frac{k^3}{(2\pi)^3\pi a^2}\int q^6\mathrm{d}q\left(1-\mu^2\right)^2\mathrm{d}\mu\left|\int_{-\infty}^{\eta}\mathrm{d}\eta'\,a(\eta')\,\varepsilon\,G(k,\eta,\eta')\mathcal{R}(q,\eta')\mathcal{R}(|\boldsymbol{k}-\boldsymbol{q}|,\eta')\right|^2, \tag{151}$$

after integration over φ which gives a factor $\pi\left(1-\mu^2\right)^2 q^4$.

We can now take the super-Hubble limit of this expression at lowest order in the slow-roll parameters. In order to do so, we make use of the super-Hubble limit of the Green function given above, and we perform the time integral from $1/k$ to η and keep only the leading order contribution:

$$\mathcal{P}_T^{(2)}(k) \;=\; \frac{1}{3^4 2^3\pi^2}G^2H^4F(\epsilon,\delta)\left(\frac{k}{aH}\right)^{-2\epsilon}, \tag{152}$$

where, with the defnitions $\boldsymbol{y}\equiv\boldsymbol{q}/k$ and $\boldsymbol{n}\equiv\boldsymbol{k}/k$,

$$F(\epsilon,\delta)\equiv\int\left(y\,|\boldsymbol{n}-\boldsymbol{y}|\right)^{-3-4\epsilon+2\delta}y^6\mathrm{d}y\left(1-\mu^2\right)^2\mathrm{d}\mu \tag{153}$$

is a numerical factor. In this approximation, the ratio between the second order power spectrum and the first order power spectrum at leading order in the slow-roll parameters, is given by:

$$\frac{\mathcal{P}_T^{(2)}(k)}{\mathcal{P}_T^{(1)}(k)}=\frac{1}{2^7 3^4\pi}\left(\frac{H}{M_p}\right)^2F(\epsilon,\delta). \tag{154}$$

Indeed there are ultraviolet and infrared divergences hidden in $F(\epsilon,\delta)$. We expect the infrared divergence not to be relevant for observable quantities due to finite volume effects (see for instance Ref. [46]). The ultraviolet divergence, on the other hand, has to be carefully dimensionaly regularized in the context of quantum field theory (see e.g. Ref. [47]).

VII. CONCLUSIONS

In this article we have investigated the generation of gravitational waves due to second order effects during inflation. We have considered these effects both in the covariant perturbation formalism and in the more standard metric based approach. The relation between the two formalisms at second-order has been considered and we have discussed their relative advantages. This comparison leads to a better understanding of the differences in dynamics between the two formalisms.

As an illustration, we have focused on GW generated by the coupling of first order scalar modes. To characterize this coupling we have introduced and computed the parameter $f_{\rm NL}^{\mathcal{E},\mathcal{RR}}$. It enters in the expression of $\langle\mathcal{E}_{\boldsymbol{k}_1}\mathcal{R}_{\boldsymbol{k}_2}\mathcal{R}_{\boldsymbol{k}_3}\rangle_c$ that was shown to be of order $(H/M_p)^4/\varepsilon$, as $\langle\mathcal{R}_{\boldsymbol{k}_1}\mathcal{R}_{\boldsymbol{k}_2}\mathcal{R}_{\boldsymbol{k}_3}\rangle_c$. On the other hand the power spectrum of GW remains negligible.

This shows that the contribution of $\langle\mathcal{E}_{\boldsymbol{k}_1}\mathcal{R}_{\boldsymbol{k}_2}\mathcal{R}_{\boldsymbol{k}_3}\rangle_c$ to the CMB bispectrum is important to include in order to constrain the deviation from Gaussianity, e.g. in order to test the consistency relation [48].

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[1] A. Starobinsky, JETP Lett. **30**, 682 (1979).
[2] M. Kamionkowski, A. Kosowsky, and A. Stebbins Phys. Rev. Lett. **78**, 2058 (1997).
[3] U. Seljak and M. Zaldarriaga Astrophys. J. **469**, 437 (1996).
[4] M. Kesden, A. Cooray and M. Kamionkowski, Phys. Rev. Lett. **89**, 011304 (2002).
[5] L. Knox and Y. Song, Phys. Rev. Lett. **89**, 011303 (2002).
[6] U. Seljak and C. M. Hirata Astrophys. J. **463**, 1 (1996).
[7] V.F. Mukhanov and G.V. Chibisov, JRTP Lett. **33**, 532 (1981); A.A. Starobinsky, Phys. Lett. B **117**, 175 (1982); A. Guth and S.Y. Pi, Phys. Rev. Lett. **49**, 110 (1981); J.M. Bardeen, P.J. Steinhardt, and M.S. Turner, Phys. Rev. D **28**, 679 (1983).
[8] F. Bernardeau and J.-P. Uzan, Phys. Rev. D **66**, 103506 (2002).
[9] J. Maldacena, JHEP **0305**, 013 (2003).
[10] D.S. Salopek and R. Bond, Phys. Rev. D **42**, 3936 (1990); A. Linde and V.F. Mukhanov, Phys. Rev. D **56**, 535 (1997); F. Bernardeau and J.-P. Uzan, Phys. Rev. D **67**, 121301(R) (2003); F. Bernardeau, T. Brunier, and J.-P. Uzan, [arXiv:astro-ph/0604200]; N. Afshordi N. and R. Brandenberger Phys. Rev. D **63**, 123505 (2001); G.I. Rigoupoulos, E.P.S Shellard, and B.J. van Tent, Phys. Rev. D **73**, 083522 (2006).
[11] T.J. Allen, B. Grinstein, and M.B. Wise, Phys. Lett. B **197**, 66 (1987); L. Kofman *et al.*, in "Large scale structures and peculiar motions in the universe" ed D W Latham and L da Costa (San Francisco: Astronomical Society of the Pacific) p 339 (1989) , D.W. Latham and L. da Costa Eds., 339 (1989); P.J.E. Peebles, Astrophys. J. **483**, L1 (1997); M. Bucher and Y. Zhu, Phys. Rev. D **55**, 7415 (1997); A. Gangui and S. Mollerach, Phys. Rev. D **54**, 4750 (1996); A.A. Starobinsky, Grav. Cosmol. **4**, 88 (1998).
[12] E. Komatsu *et al.*, Astrophys. J. Suppl. **148**, 119 (2003);

N. Kogo and E. Komatsu, Phys. Rev. D **73**, 083007 (2006); M. Liguori *et al.*, Phys. Rev. D **73**, 043505 (2006); P. Creminelli *et al.*, [arXiv:astro-ph/0610600].

[13] N. Bartolo *et al.*, Phys. Rept. **402**, 103 (2004).

[14] N. Bartolo *et al.*, JCAP **0606**, 024 (2006); N. Bartolo *et al.*, [arXiv:astro-ph/0610110].

[15] S. Mollerach, D. Harari and S. Matarrese Phys. Rev. D **69**, 063002 (2004).

[16] S. Prunet *et al.*, Phys. Rev. D **71**, 083508 (2005).

[17] K. Tomita, Prog. Theor. Phys. **37**, 831 (1967).

[18] F. Mena, R. Tavakol and M. Bruni, Int. J. Mod. Phys. A **17**, 4239 (2002).

[19] S. Matarrese, S. Mollerach S. and M. Bruni, Phys. Rev. D **58**, 043504 (1998).

[20] F. Finelli *et al.*, Phys. Rev. D **74**, 083522 (2006); H. Noh and J. Hwang, Phys. Rev. D **69**, 104011 (2004); K.A. Malik and D. Wands, Class. Quant. Grav. **21**, L65 (2004); K. Tomita, Phys.Rev. D **71**, 083504 (2005).

[21] C. Clarkson, Phys. Rev. D **70**, 103524 (2004).

[22] D. Langlois and F. Vernizzi, Phys Rev D. **72**, 103501 (2006); D. Langlois and F. Vernizzi, Phys. Rev. Lett. **95**, 091303 (2005); F. Vernizzi, Phys. Rev D **71**, 061301 (2005).

[23] K.A. Ananda, C. Clarkson and D. Wands, [arXiv:gr-qc/0612013].

[24] V. F. Mukhanov, L. R. W. Abramo and R.H. Brandenberger, Phys. Rev. Lett. **78**, 1624 (1997).

[25] K. Nakamura, Phys. Rev. D **74**, 101301 (2006).

[26] V. Acquaviva, *et al.*, Nuc. Phys. B **667**, 119 (2003).

[27] G.F.R. Ellis and M. Bruni, Phys. Rev. D **40**, 1804 (1989); M. Bruni, P.K.S. Dunsby and G.F.R. Ellis, Astrophys. J. **395** 34 (1992).

[28] J. Bardeen, Phys. Rev. D **22**, 1882 (1980).

[29] M. Bruni, P.K.S. Dunsby and G.F.R. Ellis, Astrophys. J. **395**, 34 (1992).

[30] R.M. Wald, *General Relativity* (Chicago: The University of Chicago Press, 1984).

[31] M.S. Madsen, Class. Quant. Grav. **5**, 627 (1988).

[32] P. K. S. Dunsby, M. Bruni and G. F. R. Ellis, Astrophys. J. **395**, 54 (1992).

[33] G. F. R. Ellis, in *General Relativity and Cosmology*, ed. R. K. Sachs (New York: Academic Press, 1971), p. 104.

[34] D. Langlois and F. Vernizzi, [arXiv:astro-ph/0610064].

[35] M. Bruni, G. F. R. Ellis, and P. K. S. Dunsby, Class. Quant. Grav. **9**, 921 (1992).

[36] G.F.R. Ellis and H. van Elst, in "Theoretical and Observational Cosmology", Marc Lachièze-Rey Ed., (Kluwer, Dordrecht, 1999).

[37] M. Bruni, G.F.R. Ellis, and P.K.S Dunsby, Class. Quant. Grav. **9**, 921 (1992).

[38] J. M. Stewart, and M. Walker, Proc. R. Soc. London A **341**, 49 (1974).

[39] P. K. S. Dunsby, B. A. Bassett and G. F. R. Ellis, Class. Quant. Grav. **14**, 1215 (1997); R. Maartens, G. F. R. Ellis and S. T. C. Siklos, Class. Quant. Grav. **14**, 1927 (1997); R. Maartens and B. A. Bassett, Class. Quant. Grav. 15, 705 (1998); A. Challinor, Class. Quant. Grav. **17**, 871 (2000).

[40] M. Bruni, *et al.*, Class. Quant. Grav. **14**, 2585 (1997); M. Bruni and S. Sonego, Class. Quant. Grav. **16**, L29 (1999); C. F. Sopuerta, M. Bruni and L. Gaultieri, [arXiv:gr-qc/0306027].

[41] V.F. Mukhanov, H. Feldman, and R. Brandenberger, Phys. Rept. **215**, 204 (1992).

[42] L. Kofman and J.-P. Uzan, to be submitted.

[43] C. Carbone, C. Baccigalupi, and S. Matarrese, Phys.Rev. D **73**, 063503 (2006).

[44] C. Pitrou and J.-P. Uzan, [arXiv:astro-ph/0701121].

[45] T. Brunier, F. Bernardeau, and J.-P. Uzan, Phys. Rev. D **69**, 063520 (2004).

[46] F. Bernardeau, and J.-P. Uzan, Phys. Rev. D **70**, 043533 (2004).

[47] S. Weinberg, Phys. Rev. D **72**, 043514 (2005); D.H. Lyth and D. Seery, [arXiv:astro-ph/0607647].

[48] P. Creminelli and M. Zaldarriaga, Phys. Rev. D **70**, 083532 (2004).