BARS IN GALAXIES : A DYNAMICAL FORMATION

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ABSTRACT

This essay addresses the origins of bars in galaxies in terms of a linear instability via the adiabatic alignment of resonant orbital streams. Three models of increasing complexity are discussed:

- A simple alignment instability model which provides a Jeans-like criterion;

- A model involving only the inner Lindblad resonance which shows that the orbits are also distorted as they re-align; this model is illustrated for the isochrone potential.

- A global azimuthal instability model re-introducing all the resonances via the full Boltzmann and Poisson equations; it gives an accurate estimate of the stability of any azimuthal mode.

Each of these models give both an instability criterion and the pattern speed at which the instability propagates. It is concluded that the bi-symmetrical orbital instability provides a complementary approach to the WKB theory for the study of the eigenmodes of galactic discs.
Paris, January 11th

I hereby certify that the present work, which has been supervised by Professor Lynden-Bell is original, unless stated otherwise.

Part of the contents of sections III and IV will be published in the Monthly Notices and were done in collaboration with Donald Lynden-bell and Jim Collet.

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I  INTRODUCTION

Many spiral galaxies show developed bars. They appear as elongated isophotes beyond the central core. The photometric analysis of these isophotes seems to imply that they are made of late type stars, which in turn suggests that they are not produced by transient phenomena. This also suggests that pure stellar dynamics ought to be a sufficient theoretical framework to understand their evolution and ultimately their formation. Some considerable amount of work has been done to understand the stability of these structures, but the problem of their formation has to my knowledge always been a puzzle. While numerical simulations tend to show that bi-symmetrical instability is fairly common and quite robust, only forty percent of observed spirals show strong bars. Why should some galaxies form bars and others not? The present essay is an attempt to address this question.

I 1  Physical concepts

The mechanism of bar formation relies on three basic concepts:

▷ What orbits form a bar?

“If a particular star has an orbit with radial period \(2\pi/\kappa\) and an average period around the galaxy \(2\pi/\Omega\), then when viewed from axes that rotate at the rate \(\Omega - \kappa/2\), it will close in a figure not unlike a centred ellipse. If this whole orbit is populated by an orbital stream, it will look stationary in the rotating frame but will tumble over and over at the angular rate \(\Omega - \kappa/2\) in the fixed frame.” (Lynden-Bell 92) I shall consider our galaxy to be an assembly of many such tumbling orbital streams. Observation suggest the stationary pattern of such orbits builds the core of what a bar is.

▷ How do they interact?

“If two orbital streams tumble at the same rate, \(\Omega_\ell\), but one has its lobes a little ahead of the other, then the gravitational interaction of the streams will produce a forwards torque on the hindmost stream and a backwards torque on
the foremost stream. Because the two tumbling rates are equal (resonant), this
torque will not reverse but will continue until it modifies the orbits. If however
the tumbling rates are significantly different, then the torques will reverse as the
angle between the lobes changes so the effects will cancel. Near-resonant orbits
with a small difference of tumbling rates may nevertheless bring about significant
changes before the difference can lead to any reversal of the torque.” (Lynden-Bell
92) The key issue is therefore to calculate the relative dispersion in the number of
near resonant orbits. Let us assume for a while that this dispersion is narrow. It
follows that the period corresponding to the relative libration of these orbits around
their mean pattern speed $\Omega_p$ should be large as they almost resonate. Many orbits
are to be found for which the ratio of the orbital period to the oscillation period will
be very small. Let us therefore construct the corresponding adiabatic invariant $J$.
Strictly speaking, $J$ is only an invariant for the quasi-resonant orbits See Appendix
A. On the other hand non-resonant orbits are not expected to play any significant
role in restructuring the galaxy. I shall therefore consider $J$ as a strict invariant.
I shall show that this argument is self-consistent with the claim that the galaxy is
azimuthally unstable through the reorientation of orbital streams.

> How does a backward torque affect a tumbling orbit?

The orbits follow a remarkable generic behaviour as described by Lynden-Bell
and Kalnajs (72): A given orbit accelerates when adiabatically pulled backwards
and slows down when pulled forwards. The underlying physical property is that
their adiabatic moment of inertia is generally negative This apparent paradox is
quite similar to the problem of a satellite falling through the atmosphere and con-
serving its angular momentum: it rotates faster and faster as the frictional drag
increases.. Therefore no collective alignment is to be expected from such “donkey”
orbits. Collet nevertheless pointed out that a given orbit could possibly anti-align
with the potential trough created by the other orbits, and widely oscillate, there-
fore spending most of its time at ninety degrees to the minimum of the potential,
creating a pattern. This would be the azimuthal equivalent to the two-stream in-
stability in plasma physics. I shall not consider here such a process, but note that,
for the so called inner Lindblad resonance, it turns out that the adiabatic moment
of inertia is positive in the central parts of galaxies, especially in the rising parts of
rotation curves. The lobes of two orbital streams may then attract and cooperate, vibrating around exact alignment and so forming a deeper potential well to trap further orbits into that orientation, and make a bar.

I 2 Context

Consider a rotating dissipative protogalactic cloud collapsing into what will be our galaxy. That cloud will rapidly lose its energy of motion along the axis of rotation, but will conserve its total angular momentum. It will therefore settle into a flat system. If it is also assumed that the process of baryonic matter injection from the halo is quasi-static and isotropic, it can be argued that the disc will be thin and axisymmetric. While cooling, the gas will produce stars. The surface density of these stars increases and their velocity dispersion decreases as more and more gas is injected. This process goes on until eventually an azimuthal instability occurs in order to redistribute angular momentum. This scenario is compatible with the observations, and provides a symmetry necessary for the analytical treatment. It does not rely on unknown initial conditions.

I shall therefore consider a infinitely flat disc made of point-like stars evolving in their self-consistent gravitational field. The influence from the external fields from the halo and the other galaxies will undoubtedly amplify the growth rate of unstable eigenmodes. I shall nevertheless restrict myself in this essay to the normal modes of an isolated galaxy. The influence of the gas will be neglected because of the colour of bars, though the inherent instability of the scarce gaseous component might trigger the stellar instability.

The motion of each star will therefore be imposed by their self-consistent gravitational field which I may mentally split into two components:

- A mean field, satisfying Poisson equation for density, calculated by averaging locally on a Dirac distribution corresponding to point-like stars.

- The fluctuation of this field around its mean value due to the neighbouring stars, which can be evaluated statistically in order to quantify its relevance.
Indeed, it can be shown that due to the long range interaction of gravitation and the numbers of stars to be found in galaxies, the latter component can be neglected. I shall therefore the kinetic theory. This theory provides a way to follow statistically the behaviour of all stars as they describe their orbits. While the mean field description is attractive to describe the global effect of the other stars on a given star, it is obviously insufficient to describe two stars tracing orbits which have a constant relative phase delay (i.e. orbits which resonate). Indeed, in stellar dynamics, as opposed to fluid mechanics, the mean free path of a given particle is quite large compared to the size of the system; therefore this resonance will generally last, until the weak but cumulative effect of the couple modifies the nature of the two orbits. It is this reorganization of resonant orbits which, I believe, would restructure the galaxy. From an energy point of view, it can be argued that the disc was maintained far from its minimum energy state by the conservation of angular momentum. Hence the dominant factor controlling its dynamical evolution is the way in which specific angular momentum may be redistributed within the disc. The mean field theory claims on the other hand that no angular momentum is exchanged between stars except on resonance.

For that purpose, I shall take the two fundamental equations of galactic dynamics, namely

- Boltzmann equation, which describes the evolution of the distribution function in a given potential,

- Poisson equation, which describes the mean field for a given distribution of stars.

These equations may be re-expressed in order to identify the contribution from the resonances of orbits, leaving aside the contribution corresponding to the detailed phase of the stars on each orbit. This trick enables me to decouple the Poisson and Boltzmann equations. I shall therefore not consider the gravitational interactions of individual stars but shall replace them by the gravitational interactions of whole orbital streams, i.e. steady streams of stars that follow the same orbit. The net effect of this interaction will then be to distort and reorientate the
orbits. Such instabilities will restructure the galaxy only if the orbits cooperate by moving in the direction of the torques on them. Orbits corresponding to the symmetry of bars present such a cooperative behaviour. I therefore expect a collective realignment of orbits to produce the azimuthal analogue to Jeans gravitational instability. Jeans’s instability works by trapping particles in a growing potential well which may be moving relative to fixed axes. Azimuthal gravitational instability works not by trapping the stars but rather by trapping the lobes of orbital streams of stars azimuthaly in a growing potential well which may be rotating relative to fixed axes. I will identify the rotation velocity of this frame with the pattern speed of the bar.

The actual instability criterion shall be obtained by very classical means. The starting point will be a given galaxy specified by its distribution function. This distribution function will be slightly perturbed, in order to linearize the equations and therefore find a dispersion relation for the normal modes of the system. I shall concentrate on the condition for the existence of linear metastable modes, and explicit the constraints they would then impose on the underlying galaxy.

This will lead to the following results:

• an azimuthal instability criterion for a given galaxy.

• the angular frequency at which this instability is propagating.

In order to evaluate these results, it is then possible to work out observables from the distribution function, and compare them to observed barred galaxies (assuming the bar has not been drastically modified the underlying galaxy). The efficiency of this criterion may also be evaluated numerically with existing codes.
I 3 Overview

This essay is divided into six sections. Section II, III and IV, describe three models of increasing complexity. Section V is devoted to an example.

Section II introduces a toy model, the purpose of which is to give a general idea of what are the most relevant physical concepts; I shall describe in details the behaviour of an assembly of tumbling ellipses as they go through bi-symmetric azimuthal instability. I introduce for this purpose an effective interaction potential between the ellipses. This leads to a Jeans-like instability criterion.

Section III focuses on a fictitious galaxy in which the only relevant resonance is the inner Lindblad. Introducing a effective interaction potential similar to that of the first model, together with some arbitrary assumptions on its analytical expression leads to an analogous criterion under less stringent conditions. This implies the orbits should be distorted as they reorient themselves with respect to the potential trough.

I re-introduce all the resonances in section IV, and work out what the interaction potentials are from Poisson equation. This will give an exact criterion for an azimuthal instability (which can be made explicit for pattern speeds small compared to the relevant frequency of the galaxy).

One application to a two parameters family of galaxies (the Isochrone) is given in the fifth section.

*Note that a short introduction to Angle and Action formalism is given in Appendix A.*
II THE ALIGNMENT INSTABILITY MODEL

The toy model described in the following section provides a simplified insight into the mechanism involved in the production of bars in galaxies. Many aspects of the complexity of what is a real galaxy will be disregarded, making several arbitrary assumptions. Many of the over-simplifications made in this model will be dealt with in the other sections.

II.1 Equations

Let us consider a set of rotating ellipses, which represents our quasi-resonant inner Lindblad resonant orbits viewed from a rotating frame $\Omega_p$. They will all have the same characteristic (i.e. same geometry and same mass). Each ellipse will therefore be entirely specified by the orientation of its semi-major axis with respect to that rotating frame $\varphi$, and the angular frequency $\Omega_\ell$.

27.1pc by 12.9pc (Fig1 scaled 1000) Figure 1 represents the relative torque amongst the orbits as a function of their shape

The relative interaction of these ellipses will be assumed to derive from an effective alignment potential:

$$\psi_{12} = G A^2 \cos 2(\varphi_1 - \varphi_2)$$

where $G$ is the constant of gravitation, and $\varphi_1 - \varphi_2$ measures the relative azimuthal orientation of the two orbital lobes. $A^2$ is in this model taken to be a constant. The torque that ellipse “two” operates on ellipse “one” may then be written:

$$Dh_1/Dt = \partial \psi_{12}/\partial \varphi_1$$
The response in angular velocity to such a torque can be worked out through the moment of inertia \( \alpha \) of the ellipse which I will identify with the adiabatic moment of inertia of the orbit this ellipse is supposed to represent.

Let \( F^*(\varphi, \Omega_{\ell}, t) d\varphi d\Omega_{\ell} \) be the number of ellipses with an orientation \( \varphi \) between \( \varphi \) and \( \varphi + d\varphi \) and an angular frequency \( \Omega_{\ell} \) between \( \Omega_{\ell} \) and \( \Omega_{\ell} + d\Omega_{\ell} \). This distribution function satisfies a continuity equation in \((\varphi, \Omega_{\ell})\) space.

\[
\frac{\partial F^*}{\partial t} + \frac{\partial}{\partial \varphi} (\Omega_{\ell} F^*) + \frac{\partial}{\partial \Omega_{\ell}} \left( D\Omega_{\ell}/Dt F^* \right) = 0 \tag{2-4}
\]

Let us now assume that \( \alpha \) is a constant which I shall take to be \( \pm 1 \), depending on the cooperative or “donkey” behaviour of the orbits.

Then \( D\Omega_{\ell}/Dt \) becomes independent of \( \Omega_{\ell} \), therefore \( F^* \) satisfies a Boltzmann equation:

\[
\frac{\partial F^*}{\partial t} + \Omega_{\ell} \frac{\partial F^*}{\partial \varphi} + \frac{\partial F^*}{\partial \Omega_{\ell}} \left( \alpha \frac{\partial \psi}{\partial t} \right) = 0 \tag{2-5}
\]

the density of ellipses which have orientation \( \varphi \) may then be defined by:

\[
\rho(\varphi, t) = \int F^*(\varphi, \Omega_{\ell}, t) d\Omega_{\ell} \tag{2-6}
\]

The interaction potential generated by the “two” ellipses may be obtained by multiplying 2-1 with \( F^*_2 \) and integrating over them.

\[
\psi(\varphi, t) = G A^2 \int \rho(\varphi_2, t) \cos 2(\varphi - \varphi_2) d\varphi_2 \tag{2-7}
\]

The stationary axisymmetric unperturbed state obey \( F^* = F^*_0(\Omega_{\ell}) \) and \( \rho = \rho_0 = \text{constant} \). \( \Psi \) is therefore identically zero and 2-4, 2-5 and 2-6 are trivially verified.
Let us now study the stability of such a state with respect to bi-symmetrical instabilities. Let us write \( F^* = F_0^* + f^* \), and linearize the equation with respect to the perturbation \( f^* \) and \( \psi \). Equation 2-5 becomes
\[
\frac{\partial f^*}{\partial t} + \Omega_\ell \frac{\partial f^*}{\partial \varphi} + \alpha \frac{\partial F_0^*}{\partial \Omega_\ell} \frac{\partial \psi}{\partial \varphi} = 0.
\]

II 2 Criterion

Let us Fourier expand \( f^* \)
\[
f^* = \sum_{m=-\infty}^{\infty} e^{im\varphi} f_m^* (\Omega_\ell, t)
\]

Then
\[
\rho = \sum_{m=-\infty}^{\infty} e^{im\varphi} \rho_m
\]

where
\[
\rho_m = \int f_m^* d\Omega_\ell
\]

Putting 2-12 into 2-7 gives:
\[
\psi = \pi G A^2 \left( \rho_2 e^{2i\varphi} + \rho_{-2} e^{-2i\varphi} \right).
\]

When the perturbation has no \( |m| = 2 \) component, 2-7 becomes
\[
\frac{\partial f^*}{\partial t} + \Omega_\ell \frac{\partial f^*}{\partial \varphi} = 0.
\]

The general solution is \( f^* = g(\varphi - \Omega_\ell t) \) where \( g \) is an arbitrary function without \( |m| = 2 \) component. As 2-8 is linear, its complete solution corresponds to a sum of such a solution (which propagates in \((\Omega, \varphi)\) space) and a solution for the resonant modes \( |m| = 2 \). These modes are responsible for the gravitational instability. Let us look for growing modes, when \( f_m^* \propto e^{i\omega t} \) strictly speaking, I ought to Laplace transform 2-13, and get the instability criterion by avoiding the poles as it is usually done for Landau damping. As I am only interested in the dispersion relation for
marginally stable modes rather than the transient behaviour, this trick leads faster to the answer, where $\omega$ has a negative imaginary part. From 2-8, 2-9 et 2-13, it follows

$$
i(\omega + m\Omega_\ell)f^* = -i m \pi G \rho_m A^2 \alpha \partial F_0^*/\partial \Omega_\ell \quad \text{for } |m| = 2$$

Let us divide by $i(\omega + m\Omega_\ell)/A$ and integrate over all $\Omega_\ell$ in order to build $\rho_m$ on the r.h.s. Calling $\Omega_p = -\omega/m$ (which corresponds to the angular frequency at which the perturbation is propagating), it follows

$$1 = \pi G \int \frac{-\alpha A^2 \partial F_0^*/\partial \Omega_\ell}{\Omega_\ell - \Omega_p} d\Omega_\ell \quad 2-16$$

Recall that $\Omega_p$ has a positive imaginary part (for $m = 2$) corresponding to the growth rate of the instability.

Let us concentrate on marginal instability by letting $\Omega_p$’s imaginary part vanish. Some work has been done in the framework of Landau damping to calculate the growth rate of the instability when $\Omega_p$’s imaginary part is small but non zero (Landau (76) & Lynden-Bell (67)). The integral may then be written as the sum of a Cauchy principal part, and the residue corresponding to the pole at $\Omega_\ell = \Omega_p$.

$$1 = \pi G \int \frac{-\alpha A^2 \partial F_0^*/\partial \Omega_\ell}{\Omega_\ell - \Omega_p} d\Omega_\ell - i \pi^2 G \left[ \frac{\alpha A^2 \partial F_0^*/\partial \Omega_\ell}{\Omega_\ell - \Omega_p} \right]_{\Omega_p} \quad 2-18$$

where $\Omega_p$ is now real.

Identifying real and imaginary parts, it follows

$$1 = \pi G \int \frac{-\alpha A^2 \partial F_0^*/\partial \Omega_\ell}{\Omega_\ell - \Omega_p} d\Omega_\ell \quad 2-19$$

$$\partial F_0^*/\partial \Omega_\ell = 0 \quad \text{for } \Omega_\ell = \Omega_p \quad 2-20$$

This is the condition which fixes $\Omega_p$, the pattern speed of the bi-symmetrical instability
II 3 Discussion

To the extrema for $F_0^*$ correspond candidates for angular frequency of the perturbation. Equation 2-19 tells which (if any) is going to be most dominant.

The pattern speed of the perturbation as well as of its global fate is therefore a characteristic of the unperturbed galaxy.

When $\alpha$ is positive the equivalent to the two stream instability corresponds here to a negative $\alpha$ and a minimum of the distribution function $\partial^2 F_0^*/\partial \Omega^2_\ell \geq 0$ i.e. few repealing orbits widely librating to anti-align to the potential valley, I may rewrite 2-18 as

$$\sigma^2_\Omega \leq G M \alpha,$$

where $\sigma^2_\Omega$ measures the weighted dispersion of the distribution function in the neighbourhood of the resonance

$$\sigma^{-2}_\Omega = \int \left[ \frac{A^2 \alpha \partial F_0^*/\partial \Omega_\ell}{\Omega_p - \Omega_\ell} \right] d\Omega_\ell / \int A^2 \alpha F_0^* d\Omega_\ell,$$

Equation 2-21 is very much like Jeans instability. It implies that the dispersion in $\Omega_\ell$ around $\Omega_p$ should be less than an effective $G$ density, built out from the adiabatic moment of inertia $\alpha$, and $GA^2$, the amplitude of the effective interaction potential. $\alpha$ scales like $[\text{Length}^{-2}]$ and $A^2$ like $[\text{Length}^{-1}]$. If $\sigma^2_\Omega$ represents the pressure corresponding to the dispersion in angular velocity in the frame of the maximum of stationary ellipses, then the criterion therefore implies that the lower the pressure the more efficient the instability. Indeed, if I assume for instance that $F^*$ is a Gaussian with characteristics $(\omega, \sigma_\omega)$, then 2-18 and 2-19 imply $\Omega_p = \omega$ and $\sigma_\Omega = \sigma_\omega$

26.26pc by 20.58pc (Fig2 scaled 1000) Figure 2 is a schematic plot of the instability as viewed in $(\phi, \Omega)$ space. The perturbed iso-density contours of the distribution do not fade away via phase mixing (because of the relative difference
of frequency) as self-gravity triggers a collapse at $\phi = \Omega p t$: the iso-density contours wind up.

Let us now turn ourselves back to the problem of making bars in galaxies. I expect the above model to describe most of the relevant physics I would like to introduce in this essay, but I shall now show how it is related to the azimuthal stability of a galaxy.
III THE INNER LINDBLAD RESONANCE MODEL

The model described in the following section intends to recover the toy model criterion for a fictitious axisymmetric galaxy.

III 1 Equations

Let us consider a galaxy made of orbital streams of inner Lindblad resonances. Each orbital stream is characterized by its orientation $\varphi$, its specific angular momentum, $h$, and its circulation $4\pi J = 4\pi (J_R + h)$ See Appendix A & Lynden-Bell (79). As such streams interact gravitationally, $\varphi$ and $h$ change, but $J$ is taken to be adiabatically invariant. Let us assume that any two orbital streams interact through their mutual potential energy, the angularly dependent part of which is approximated by $-\Psi_{12}$ where

$$\Psi_{12} = G A_1 A_2 \cos 2(\varphi_1 - \varphi_2)$$

Here $\varphi_1 - \varphi_2$ is the relative azimuth of the orbital lobes. The amplitude of the alignment potential must vanish if either orbit is circular, and for small ellipticities it should be proportional to their product. The critical (and arbitrary) assumption in this section is that the amplitude depends on the orbits through a product $A_1 A_2$ with $A_1$ depending on orbit one only and $A_2$ on orbit two. The specific torque on stream “one” due to unit mass on stream “two”

$$D_h/ Dt = \partial \Psi / \partial \varphi_1$$

Let $F^*(\varphi, \phi, h, J, t)d\varphi d\phi dh dJ$ be the mass weighted distribution function of stars belonging to an orbital stream with lobe azimuths in the range $\varphi$ to $\varphi + d\varphi$, with orbital phase between $\phi$ and $\phi + d\phi$, with angular momenta in the range $h$ to $h + dh$ and invariant between $J$ and $J + dJ$.

This distribution function obeys a Boltzmann equation corresponding to the conservation of mass in phase space. Let us assume that the only relevant resonance is the inner Lindblad ($i.e.$ I disregard all other resonances, both between
orbits and in the relative orbital phases). I may then apply the averaging principle
to this equation in order to write a Boltzmann equation for the number of orbital
streams, which are the relevant objects according to our assumptions. Calling
\( F = \int F^* d\phi \) it follows:
\[
\frac{\partial F}{\partial t} + \Omega \frac{\partial F}{\partial \varphi} + \frac{\partial \Psi}{\partial \varphi} \frac{\partial F}{\partial h} = 0
\]
3 - 3
\( F \) is a distribution function in phase-space since \( \varphi, h \) are canonically conjugate
variables. The orientation potential \( \Psi \) due to all the orbital streams is found by
weighting 2-1 with \( F^2 \) and integrating over all streams ‘two’
\[
\Psi(\varphi, h, J, t) = G A(h, J) \int \rho(\varphi_2, t) \cos 2(\varphi - \varphi_2) d\varphi_2
\]
where I have defined an “effective” density of streams at orientation \( \varphi \) by
\[
\rho(\varphi, t) = \int \int A(h, J) F(\varphi, h, J, t) dh dJ
\]
in order to identify these equations formally with those of section II.

Notice that with this set of variables, \( \Psi \) depends on \( h \) and \( J \) as well as
orientation. This is because the torque depends on the shape and size of the orbit
on which the potential acts.

Unperturbed axially-symmetric steady states have \( F = F_0(h, J) \) and \( \rho = \rho_0 = \text{const.} \) I investigate here again the stability of such equilibria to bar formation.
Writing \( F = F_0 + f \) and linearizing in the perturbed quantities \( f \) and \( \psi \) equation
3-3 becomes
\[
\frac{\partial f}{\partial t} + \Omega \frac{\partial f}{\partial \varphi} + \frac{\partial \Psi}{\partial \varphi} \frac{\partial F_0}{\partial h} = 0.
\]
3 - 6
Let us note that
\[
\frac{\partial F_0}{\partial h} = \frac{\partial F_0}{\partial \Omega_{\ell}} \cdot \alpha
\]
3-6 may then be formally identified to 2-8. 3-5 is the stellar equivalent to 2-
6. Notice that the sum over \( d\Omega \) is replaced by a double sum over \( dh dJ \). The
assumption that \( A \) is separable with respect to the characteristics of orbit one and
two enables us to follow precisely the same steps as section 2 and write :

\[
1 = \pi G \int \int \frac{A^2[-\alpha \partial F_0/\partial \Omega_{\ell}]}{\Omega_{\ell} - \Omega_p} dh dJ
\]
3 - 7
III 2 Discussion

Let us now show how this criterion relates to the toy model criterion 2-20. Let

\[ F^\ell(\Omega^\ell, J) = F_0(J, h(\Omega^\ell, J)) \]
\[ A^\ell(\Omega^\ell, J) = A(J, h(\Omega^\ell, J)) \]

Then

\[ \int - (\partial F^\ell / \partial \Omega^\ell) A^2 dJ = - \frac{d}{d\Omega^\ell}\left( \int F^\ell A^2 dJ \right) - [F^\ell A^2] + \int F^\ell (\partial A^2 / \partial \Omega^\ell) dJ \]

The square brackets correspond to the limits of integration, for which the argument vanishes because these are non-interacting orbits (circular orbits for instance) \([F^\ell A^2] \equiv 0\)

Let us introduce

\[ F_0^*(\Omega^\ell) = \frac{1}{A^2} \int F^\ell(\Omega^\ell, J) A^2 dJ \]

and

\[ \frac{d}{d\Omega^\ell} \delta F_0^*(\Omega^\ell) = -\frac{1}{A^2} \sum_\ell \int F^\ell(\Omega^\ell, J) \partial A^2 / \partial \Omega^\ell dJ \]

where the arbitrary \(<A^2>\) constant (say the averaged \(A^2\)) is here in order to identify 3-11 with 2-20. 3-7 becomes

\[ \frac{1}{\pi G} = \left( \int - \frac{<A^2>}{\Omega^\ell - \Omega_p} d\Omega^\ell \right) \]

The effective number of orbital stream

Let us emphasize that \(F^\ell d\Omega^\ell dJ = \alpha F_0 dh dJ\). Therefore \(F^\ell d\Omega^\ell dJ\) is the distribution of orbital stream with angular frequency in the range \(\Omega^\ell\) to \(\Omega^\ell + d\Omega^\ell\) and invariants between \(J\) and \(J + dJ\) weighted by the corresponding adiabatic moment of inertia. \(F_0^*(\Omega^\ell)\) is therefore the “effective” number of resonant orbits at the frequency \(\Omega^\ell\). As \(\alpha(J)\) generally changes sign, the above summation is algebraic. Remember that negative \(\alpha(J)\) corresponds to inner Lindblad orbits which fail to cooperate this shall be illustrated in section V.
Let us now understand the new contribution from $\delta F_0^* (\Omega_\ell)$. The integrant in 3-10 may be rewritten as:

$$F_0^* \left( \frac{\partial A^2_m}{\partial \Omega_\ell} \right)_J = F_0^* \left( \frac{\partial A^2_m}{\partial h} \right)_J \left( \frac{\partial h}{\partial \Omega_\ell} \right)_J$$

Now $\left( \frac{\partial A^2_m}{\partial h} \right)_J$ measures the distortion of the resonant orbit when adiabatically pulled forward, and $\left( \frac{\partial h}{\partial \Omega_\ell} \right)_J$ cancels with the implicit weighting by $\alpha$ in 3-9. This is not surprising since a linear theory should include both the alignment and the distortion independently. It therefore turns out that $\delta F_0^*$ is the correction to the effective number of orbits because of the distortion of the resonant orbits.

I have produced here an $|m| = 2$ instability criterion for a galaxy made of orbital streams of inner Lindblad resonant orbits. I have shown that as the orbits align, they are distorted by the growing potential. I have given a precise physical meaning to the effective number of orbital stream, which is the stellar equivalent of the distribution function of ellipses in the toy model.
IV THE GLOBAL AZIMUTHAL INSTABILITIES MODEL

The previous models were introduced with simplifications to present a physical picture of the phenomenon of bar formation. In order to apply this theory to galaxies, I must now investigate the instability within the full Boltzmann and Poisson equations.

IV 1 Boltzmann Equation

Let us first concentrate on the Boltzmann equation. If $F$ is the mass-weighted distribution function in phase space, this reads

$$\frac{\partial F}{\partial t} + [H, F] = 0$$  \hspace{1cm} 4 - 1

where $H$ is the Hamiltonian for the motion of one star and the square bracket is the Poisson bracket

$$[H, F] = \frac{\partial H}{\partial p} \cdot \frac{\partial F}{\partial q} - \frac{\partial H}{\partial q} \cdot \frac{\partial F}{\partial p}$$  \hspace{1cm} 4 - 2

Let us write $F = F_0 + f$ and $\Psi = \psi_0 + \psi$ and linearize the Boltzmann equation in $f$ and $\psi$

$$\frac{\partial f}{\partial t} + [H_0, f] - [\psi, F_0] = 0$$  \hspace{1cm} 4 - 3

In polar coordinates $(r, \phi)$, the unperturbed equation $[H_0, F_0] = 0$ is solved if $F_0$ is any function of $H_0$ and $h (= Rv_\phi)$, according to Jeans theorem. The unperturbed Hamiltonian is

$$H_0 = (p_R^2 + p_\phi^2) - \psi_0(R)$$

where $p_R = v_R$ and $p_\phi = h$.

Following Lynden-Bell & Kalnajs (1972), I shall employ the angle and action variables of the unperturbed Hamiltonian $H_0$ as my coordinates in phase-space. Indeed, the unperturbed Hamilton’s equation are quite trivial in these variables, which makes them suitable for perturbation theory in order to study quasi-resonant orbits See Appendix A.
These are $\mathbf{J} = (J_R, J_\phi) = (J_R, h)$, where

$$J_R = \frac{1}{2\pi} \oint [\dot{R}] dR, \quad J_\phi = h = R^2 \dot{\phi}. \quad 4 - 6$$

Here $[\dot{R}]$ is the function of $R$, the specific energy $E$, and the specific angular momentum $h$ given by $[\dot{R}] = \sqrt{2E + 2\Psi_0(R) - h^2/R^2}$. The phase-angles conjugate to $J_R$ and $h$ are $\varphi = (\varphi_R, \varphi_\phi)$, where

$$\varphi_R = \kappa \int R [\dot{R}]^{-1} dR, \quad \varphi_\phi = \phi + \int R (h/R^2 - \Omega)/[\dot{R}] dR \quad 4 - 7$$

$$2\pi/\kappa = \oint [\dot{R}]^{-1} dR = P, \quad 4 - 8$$

and $P$ is the (radial) period between apocentres. The angle between apocentres is

$$\Phi = \oint h/(R^2[\dot{R}]) dR \quad 4 - 9$$

and

$$\Omega = \Phi/P. \quad 4 - 10$$

Hamilton’s equations for the unperturbed orbits give us, writing $\Omega = (\kappa, \Omega)$

$$\dot{\varphi} = (\partial H_0 / \partial \mathbf{J}) = \Omega(\mathbf{J}) \quad 4 - 11$$

while the $\mathbf{J}$ are constant because the actions were designed so that $H_0$ is independent of the phases $\varphi$ the actions are combination of integrals of the unperturbed motion by construction. See appendix A for details. The stationary unperturbed Boltzmann equation 4-1 is solved by $F = F_0(\mathbf{J})$, since $[H_0, \mathbf{J}] = 0$, so our unperturbed distribution function is of that form.

The perturbation of the distribution function expanded in Fourier series reads:

$$\partial f_m / \partial t + i \mathbf{m} \cdot \Omega f_m + i \mathbf{m} \cdot \partial F_0 / \partial \mathbf{J} \psi_m = 0. \quad 4 - 13$$
where $\mathbf{m}$ is a vector integer with components $(\ell, m)$. $\psi_m$ is similarly the Fourier transform of $\psi$ with respect to $(\varphi_r, \varphi_\phi)$

Let us look for growing instabilities with $f_m$ and $\psi_m$ both proportional to $e^{i\omega t}$, $\omega$ having a negative imaginary part. Thus 4-13 becomes

$$f_m = \frac{-m^{-1}\mathbf{m} \cdot \partial F_0 / \partial \mathbf{J}}{\Omega_\ell - \Omega_p} \psi_m,$$

where $\Omega_\ell = m^{-1}\mathbf{m} \cdot \mathbf{\Omega} = \Omega + \ell \kappa/m$ and $\Omega_p = -\omega/m$. When $\ell = -1$ this coincides with the lobe tumbling rate of the orbits discussed in section III. When $\ell = 1$ it gives the lobe tumbling rate of the type of orbits encountered at outer-Lindblad-resonance while $\ell = 0$ gives $\Omega_\ell = \Omega$, the tumbling rate of co-rotating epicycles.

There is evidently a resonant response from all those parts of phase-space for which $\Omega_\ell(J)$ is close to $R_s(\Omega_p)$. For different $\ell$ values these are distinct parts of phase-space because $\Omega_\ell - \Omega_{\ell'} = (\ell' - \ell)\kappa/m \neq 0$, for $\ell \neq \ell'$ as different $m$'s do not interact according to 4-25. In dealing with each resonant term I take new angle and action variables which are more appropriate and make the particular resonance labelled by $(\ell, m)$ especially simple. These ‘slow’ and ‘fast’ variables were applied in the context of galactic dynamics by Lynden-Bell (1973) and are obtained from the canonical transformation generated by the generating function:

$$S(J_s, J_f, \varphi_R, \varphi_\phi) = J_s (\varphi_\phi - \Omega_p t - \ell \varphi_R / m) + J_f \varphi_R$$

$$\begin{align*}
\frac{\partial S}{\partial \varphi_\phi} &= J_\phi = h \quad \text{gives} \quad J_s = h \\
\frac{\partial S}{\partial \varphi_R} &= J_R \quad \text{gives} \quad J_f = J_R - \ell h / m \\
\frac{\partial S}{\partial J_s} &= \varphi_s \quad \text{gives} \quad \varphi_s = \varphi_\phi + \frac{\ell}{m} \varphi_R - \Omega_p t \\
\frac{\partial S}{\partial J_f} &= \varphi_f \quad \text{gives} \quad \varphi_f = \varphi_R
\end{align*}$$

Notice that

$$\Omega_s = \dot{\varphi}_s = \Omega - \ell \kappa/m - \Omega_p = \Omega_\ell - \Omega_p$$

In axes that rotate at $\Omega + \ell \kappa/m$ the orbit closes with $m$ lobes after $-\ell$ turns about the centre.
From 4-17, $\Omega_s$ is zero on resonance and for all near-resonant orbits $\varphi_s$ varies slowly. In fact $\varphi_s$ is the azimuth of a lobe of the orbit in the frame that rotates with angular velocity $\Omega_p$. It is convenient to define

$$\varphi_\ell \equiv \varphi_\phi + \ell \varphi_R/m$$

which is the lobe’s direction in absolute space. Then $\varphi_s = \varphi_\ell - \Omega_p t$. It is interesting to see that $(\partial H_0/\partial h)_{J_f} = \Omega_\ell = \dot{\varphi}_\ell$ for every $\ell$.

Returning to 4-14 and evaluating it using 4-16

$$m^{-1} m \cdot \partial F_0 / \partial J = \left( \frac{\partial F_0}{\partial h} \right)_{J_R} + \ell m \frac{\partial F_0}{\partial J_R}$$

$$= \left( \frac{\partial F_0}{\partial h} \right)_{J_R} + \left( \frac{\partial F_0}{\partial J_R} \right)_h \left( \frac{\partial J_R}{\partial h} \right)_{J_f}$$

$$= \left( \frac{\partial F_0}{\partial h} \right)_{J_f}$$

Thus

$$f_\ell = \frac{[-\partial F_0/\partial h]_{J_f}}{\Omega_\ell - \Omega_p} \psi_\ell$$
IV 2  Poisson equation

The Poisson equation relates the potential, \( \psi \), to the density perturbation:

\[
\psi(R', \phi') = \int \int \frac{f(R, \phi)}{|R - R'|} dR d\phi d\nu_R d\nu_\phi
\]

Let us rewrite this equation in order to explicit the contribution from the interaction of orbits. There again angle-action turns out to be useful, as a given unperturbed orbit is entirely specified by its action. It is therefore straightforward to identify on Poisson equation the contribution corresponding to the interaction of orbits. Let us express this equation in terms of angle and actions \((\varphi, J)\) and Fourier transform it with respect to \(\varphi\). it follows that:

\[
\psi_m(J) = \pi G \sum_{m'} \int f_{m'}(J') A_{mm'}(J, J') d^2J'
\]

where

\[
A_{mm'} = \frac{1}{4\pi^3} \int \int \frac{\exp \left( i(m' \cdot \varphi' - m \cdot \varphi) \right)}{|R - R'|} d^2\varphi' d^2\varphi
\]

The (double) sum in 4-22 extends for both \(\ell\) and \(m\) going from minus infinity to infinity. \(R = R(\varphi, J)\) and \(R' = R(\varphi', J')\) are radii re-expressed as functions of our variables. Let us emphasize that a \(m\) fold symmetry family of orbits may contribute to the \(m\)th component of the potential according to 4-22. Now \(|R - R'|\) depends on \(\varphi_\phi\) and \(\varphi'_\phi\) in the combination \(\varphi'_\phi - \varphi_\phi = \Delta \varphi\) only. But

\[
(m' \cdot \varphi' - m \cdot \varphi) = m' \Delta \varphi - (m - m') \varphi_\phi - \ell' \varphi'_R + \ell \varphi_R
\]

As \(d\varphi'_\phi d\varphi_\phi = d\Delta \varphi d\varphi_\phi\), I may then reverse the order of integration doing the \(\varphi_\phi\) integration with \(\Delta \varphi\) fixed (Lynden-Bell, personal communication). This yields \(2\pi \delta_{mm'}\), so \(m'\) becomes \((\ell', m)\) in the surviving terms

\[
A_{mm'} = \frac{\delta_{mm'}}{2\pi^2} \int \int \int \frac{\exp \left( i(m \Delta \varphi - \ell' \varphi'_R + \ell \varphi_R) \right)}{|R - R'|} d\Delta \varphi d\varphi'_R d\varphi_R
\]
Now $\varphi_\ell = \varphi_\phi + \ell \varphi_R/m$ and $\varphi_f = \varphi_R$, so we may rewrite this again

$$A_{mm'} = \frac{\delta_{mm'}}{2\pi^2} \int \int \frac{\exp im(\varphi'_{\ell'} - \varphi_\ell)}{|R - R'|} d(\varphi'_{\ell'} - \varphi_\ell) d\varphi_f d\varphi_{f'}$$

$$= \frac{\delta_{mm'}}{2\pi^2} \int \int \frac{d\varphi_f d\varphi'_{f'}}{|R - R'|} \exp im(\Delta \varphi_\ell) d(\Delta \varphi_\ell)$$

where $|R - R'|$ is thought of as a function of $J, J'$, the angle between the lobes of the two orbits, $\Delta \varphi_\ell$, and the two phases around those orbits $\varphi_f$ and $\varphi'_{f'}$.

4-26 is an interesting result; It implies that two orbits with different m-fold azimuthal symmetry do not interact. $GA_{mm'}$ has a simple physical interpretation. Imagine a unit mass orbital stream on the orbit specified by $m, \ell', J', \varphi'_{\ell'}$. In the gravitational potential of that stream place another stream specified by $m, \ell, J, \varphi_\ell$; then their mutual gravitational potential energy is some function

$$V (m, \ell', J', \ell, J, \Delta \varphi_\ell).$$

If $V$ is expanded as a Fourier series in $\Delta \varphi_\ell$, the angle between the lobes of the two orbits, then the first non-zero angularly-dependent term is

$$-A_{mm'} \cos(m \Delta \varphi_\ell).$$

This is therefore the analogue of the orientation potential $\Psi$ introduced in section III.
Let us now find our criterion. Putting 4-20 into 4-22 leads to the integral equation:

$$\psi_{\ell_1}(J_1) = 2\pi^2 G \sum_{\ell_2} \frac{-\partial F_0}{\partial h} \int \int A_{mm'}(J_1, J_2) \frac{\partial F_0}{\partial h} \psi_{\ell_2}(J_2) d^2J_2$$

Note that I may write such an equation for each m mode.

This integral equation could be solved once again by assuming that there exists a way to split \( A_{mm'}(J_1, J_2) = A_{\ell_1}(J_1)A_{\ell_2}(J_2) \) I shall follow a more rigorous route which provides an elegant way to solve this integral equation, but assumes that

- the bar is rotating slowly compared to the orbital period of the resonant orbits. This can be achieved by choosing \( F_0 \).
- the orbits are well described in the epicyclic approximation. This should not put too tight a constraint for galaxies in which this linear instability regime apply.
- the potential created by a given orbit may be approximated by its first non zero multipole component. I therefore underestimate the interaction of quasi-identical orbitsSee Appendix C for details.

Let me sketch the method. The crucial fact is that the kernel of the integral equation is under these assumptions a Green’s Kernel as shown in appendix B. Therefore the integral equation may be re-expressed as an eigenvalue problem, where the so called Sturm Liouville operator does depend only on the nature of the interaction, not on the properties of the galaxy. When the solution to the eigenvalue problem is expanded over the eigenfunctions for this operator, a determinant-like dispersion equation follows.

The actual calculation is rather tedious and may be found in appendix C. Let us analyze the outcome:

$$D(\Omega_p) \equiv \det |A + H| = 0$$

4 – 28
IV 4 Description

This is our new dispersion relation. It is currently a function of a free parameter \( \Omega_p \) corresponding to the pattern velocity of the growing bar. Our purpose is to put constraints on \( \Omega_p \) when the bar is marginally stable, i.e. when \( \text{Im}(\Omega_p) \to 0 \).

\( \Lambda \) is a diagonal matrix of known positive constants (the eigenvalues \( \lambda_i \) of the Sturm Liouville operator). \( H \) is a matrix, which coefficient \( H_{(n,n')}^{(\ell,\ell')} \) are given by:

\[
H_{(n,n')}^{(\ell,\ell')} = \gamma_{(\ell,\ell')} \pi G \int \int A_{n,n'}^{(\ell,\ell')} (J) \left[ -\frac{\partial F_0}{\partial h} J_f \right] \Omega_\ell - \Omega_p d^2 J
\]

I have defined the \( \gamma_{(\ell,\ell')} \) constants and the generalized effective \( A_{n,n'}^{(\ell,\ell')} (J) \) in Appendix C. The latter are expressed in terms of the eigenfunctions \( \psi_n, \psi'_n \) of the Sturm Liouville operator.

Lynden-Bell suggested the following description to see how the criterion now operates. Consider the complex \( \Omega_p \) plane and a contour that traverses along the real axis and then closes around the circle at \( \infty \) with \( \text{Im}(\Omega_p) \) positive. The determinant \( D \) is a continuous function of the complex variable \( \Omega_p \); so, as \( \Omega_p \) traces out that closed contour, so \( D(\Omega_p) \) will trace out a closed contour in the complex \( D \) plane. If the \( D(\Omega_p) \) contour encircles the origin, then the system is unstable. An intuitive way of seeing this is to imagine turning up the strength of \( G \) for the perturbation (i.e. turn on self gravity, which is physically equivalent to wait for the gravitational interaction to play its role). Starting with small \( G \), all the \( H_{\ell\ell'}^{nn'} \) are small, so for all \( \Omega_p \) the determinant \( D(\Omega_p) \) remains on a small contour close to \( \prod_i \lambda_i \neq 0 \) as shown in Appendix D. As \( G \) is increased to its full value, either the \( D \) contour crosses through the origin to give a marginal instability or it does not. If it does not, then by continuous change of \( G \) we can not modify the stability, therefore the self-gravitating system has the same stability as in the zero \( G \) case.
i.e. is stable. If, however, it crosses and remains circling the origin then it has passed beyond the marginally stable case and is therefore unstable.

32.26pc by 12.73pc (Fig5 scaled 1000) figure 5 is a schematic representation of how the criterion operates via $D(\Omega^\ell)$

Although the $D$ contour statement underlined above is the general criterion for un-symmetrical instability, it is much less intuitive than the criterion 2-20 which makes good physical sense.

\textbf{Connection to the previous models}

Let us look at 4-28 when $|m| = 2$ and assume that only one (!) eigen mode $(\psi^0(R), \lambda)$ of the Sturm Liouville operator $L^\dagger$ is sufficient to describe accurately the behaviour of $H(R)$ in C8. $(H)$ then reduces to $(\mathcal{H})$.

I expect our galaxy to be likely to form bars through the alignment of inner Lindblad orbits. I shall therefore order the $\mathcal{H}_\ell \equiv \int H_\ell(R)\psi^2(R) dR \ (= H^{(0,0)}_{(\ell, \ell')}/\gamma_{(\ell, \ell')})$ so that $\mathcal{H}_1 = \epsilon \mathcal{H}_{-1}$ and $\mathcal{H}_0 = \epsilon \mathcal{H}_{-1}$, where $\epsilon \ll 1$.

To second order in $\epsilon$, 4-28 then reads

$$\lambda = \gamma_{(-1,-1)} \mathcal{H}_{-1} + \epsilon \left[ \gamma_{(1,1)} \mathcal{H}_1 + \gamma_{(0,0)} \mathcal{H}_0 \right]$$

$$+ \epsilon^2 \left[ (\gamma_{(-1,0)}^2 - \gamma_{(0,0)} \gamma_{(-1,-1)}) \mathcal{H}_{-1} \mathcal{H}_0 + (\gamma_{(-1,1)}^2 - \gamma_{(-1,-1)} \gamma_{(1,1)}) \mathcal{H}_{-1} \mathcal{H}_1 \right] \frac{1}{\lambda}$$

The second order contribution involves cross interaction terms via $\gamma_{(-1,0)}$ and $\gamma_{(1,0)}$. These correspond to the interaction of orbits caught in different resonances in the same frame.

To first order in $\epsilon$, 4-28 be re-expressed as:

$$\gamma_{(-1,-1)} \int \int \frac{-A_{-1}^2 \partial F_{-1}/\partial h}{\Omega_{-1} - \Omega_p} d^2 J + \epsilon \left( \sum_{\ell=0,1} \gamma_{(\ell,\ell)} \int \int \frac{-A_{\ell}^2 \partial F_{\ell}/\partial h}{\Omega_{\ell} - \Omega_p} d^2 J \right) = \frac{\lambda}{2\pi^2 G} \left[ \frac{4}{4 - 32} \right]$$
\(A^2_\ell\) is given in Appendix C and \(F_\ell\) is the underlying distribution function, as expressed in terms of the adiabatic invariant corresponding to the \((\ell, m)\) resonance. The identification with 3-7 is straightforward. The second term on the r.h.s. corresponds to the contribution from the other resonances. If the dominant process is the alignment of ILR, the contribution from the other resonances may be interpreted as a dynamical friction drag, as the inner Lindblad orbital stream gets caught into other type of resonance. This has been described by Weinberg & Tremaine (84).

### IV 5 Discussion

The above model has provided us with a rigorous generalization of the criterion of section III for all azimuthal instabilities.

When concentrating on the inner Lindblad, I have shown how the other resonances could perturb the alignment process, and given methods to calculate the interaction coefficients. The purpose of the determinant-like dispersion relation 4-28 is to show how to solve rigorously the integral equation 4-27, without assuming that \(A_{mn}\) is separable in \(J\) and \(J'\). For realistic applications, I believe that the product approximation should be preferable as it is much simpler and does not assume that the bar is rotating slowly.

\[\bullet \circ \bigcirc \circ \bullet.\]
V APPLICATION

In section II, I have given a criterion for the formation of bi-symmetric structures via collective alignment of quasi-resonant orbits. This criterion relied on the knowledge of the distribution function for the underlying galaxy.

I shall now apply this criterion for parameterized distribution functions. This yields to relating parameterized observables to a formation law for bars. In practice, I faced some difficulties in finding realistic distribution functions which can be expressed analytically both in angle and action, and in position and velocity. Indeed, the criterion involves derivatives of the distribution function expressed in terms of adiabatic invariants and angular momenta. As for the observables (i.e. velocity field, projected density, etc), they involve integrals of the distribution function, as expressed in terms of positions and velocities. Finally, Jeans theorem allows us to write simply the distribution in terms of specific energies and the specific angular momenta. A compromise for a simple expression for \( F_0 \) either in terms of \((v, R)\), \((h, E)\) or \((J, \Omega)\) must be found.

I will consider a parameterized galaxy described by the Isochrone potential, which is to my knowledge the only non trivial potential which leads to algebraic actions. For the sake of simplicity, I have omitted the contribution from the distorted orbits in 3-10.
V 1 Isochrone Potential Equations

Let us consider an isochrone disc of Mass $M$ with a scaling parameter $b$. Its potential is given by

$$\Psi(r) = \frac{GM}{b + s(r)} \quad \text{with} \quad s(r) = \sqrt{b^2 + r^2}$$

5 - 1

corresponding to a surface density

$$\Sigma(r) = \frac{Mb}{r^3} \left( \log(r + s) - \frac{r}{s} \right)$$

5 - 2

▷ Angle and Action for the Isochrone

For the Isochrone, 4-6 provides via 5-1 an algebraic expression for the radial action $J_R(E, h)$. Its inversion (Binney & Tremaine 84) in terms of the specific energy can be re-expressed as a function of the fast and slow actions ($h, J = J_R + 1/2h$)

$$E(J, h) = -\frac{(GM)^2}{2(J + 2k(h))^2} \quad \text{with} \quad k(h) = \sqrt{h^2 + 4GMb}$$

5 - 3

I may deduce the Lindblad frequency from 5-3:

$$\Omega_\ell(h, J) = \left( \frac{\partial \epsilon}{\partial h} \right)_J = (GM)^2 \frac{2h}{k(2J + k)^3}$$

5 - 4

22.25pc by 17.73pc (Fig3 scaled 1000)

picture 3 represents the iso-$\Omega_\ell$ in $(h, J)$ space for the inner Lindblad resonance. The upper part correspond to large $\Omega_\ell$. I have added the $J = 1/2h$ line corresponding to circular orbits.
Let us introduce the dimension-less quantities

\[ h^\dagger = \frac{h}{\sqrt{4GMb}} \quad k^\dagger = \frac{k}{\sqrt{4GMb}} \quad E^\dagger = \frac{E}{GM/2b} \]

\[ r^\dagger = \frac{r}{b} \quad s^\dagger = \frac{s}{b} \quad \Psi^\dagger = \frac{\Psi}{GM/2b} \quad \Sigma^\dagger = \frac{\Sigma}{M/b^2} \]

\[ \Omega^\dagger \ell = \frac{\Omega^\ell}{\sqrt{GM/16b^2}} \quad F_0^\dagger(h, E) = \frac{F_0(h, E)}{1/Gb} \]

5-1, 5-2, 5-3, and 5-4 then read

\[ \Psi^\dagger = \frac{2}{1 + s^\dagger} \]

\[ \Sigma^\dagger = \frac{1}{r^\dagger^3} \left( \log(r^\dagger + s^\dagger) - \frac{r^\dagger}{s^\dagger} \right) \]

\[ E^\dagger = -\frac{1}{(2J^\dagger + k^\dagger)^2} \]

\[ \Omega(h, J) = \frac{2h}{k(2J + k)^3} \]

I shall drop the \( ^\dagger \) superscript, but keep in mind that there exist two degrees of freedom in the scaling of the dimensions via \((b, M)\).

**Distribution Function for the Isochrone**

Lynden-Bell has suggested an inversion method to produce formally parameterized distribution functions compatible with a given surface density. The general idea is to reduce the functional degree of freedom in \( F_0(h, E) \) by choosing a power law component for the energy. This enables us to solve the integral equation for the unknown function \( f(h) \) in

\[
\Sigma(R) = \int \int F_0(h, E) d^2v = 2 \int^{+\sqrt{2R^2\Psi}}_{-\sqrt{2R^2\Psi}} \int^0_{\frac{1}{2R^2} - \Psi} \frac{f(h)(-E)^n dE dh}{\sqrt{2(E + \Psi)R^2 - h^2}}
\]

I have applied this method for \( n = 3/2 \), and obtained the following distribution function: Note that the integral equation for the surface density constrains only the
even part (with respect to $h$) of the distribution function. Another similar integral equation for $\Sigma v_\phi$ should be inverted in order to constrain the total distribution function.

$$F_0(E, h) = \frac{5 + 2h^2 + 9h^4 + 8h^6 + 2h^8}{(1 + h^2)^4 \sqrt{2 + h^2}} \cdot (-E)^{3/2} \quad 5 - 8$$

From 5-7 and 5-8, I find $E = E(\Omega_\ell, h)$

$$E = -\left(\frac{k\Omega_\ell}{2h}\right)^{2/3} \quad 5 - 9$$

Putting 5-9 into 5-8 provides us with $F_0(h, \Omega_\ell)$.

19.07pc by 18.32pc (Fig4 scaled 1000) figure 4 represents the density contour of $F_0$ in $(h, J)$ space

▷ Interaction Potential $A^2_\ell$

I shall in this context restrict myself to a product approximation for $A^2_\ell$, which presents the right asymptotic behaviour for small ellipticities, and extrapolates the quasi-circular approximation given in Appendix B when the two mean radii are of comparable size. I will therefore write: according to some recent work by Lynden-Bell, a more rigorous expression should be $A^2_\ell = \frac{(a/\Delta)^2}{R_h}$, where $\Delta$ is some smoothing radius.

$$A^2_\ell = \frac{(a/R_h)^2}{R_h} \quad 5 - 10$$

where $h = \Omega R_h$ and $J_R = \kappa a^2$. This, together with 5-8 allows us to write $A^2_\ell(h, \Omega)$. 
V 2 Effective Number of Orbital Streams

Let us now concentrate on the effective distribution function $F_0^*$ given by 3-9

$$F_0^*(\Omega_\ell) = 1/\langle A^2 \alpha \rangle \int F^\ell(\Omega_\ell, J) A^2_\ell \, dJ$$

According to the identity

$$\left( \frac{\partial J}{\partial h} \right)_\Omega \cdot \left( \frac{\partial h}{\partial \Omega} \right)_J \cdot \left( \frac{\partial \Omega}{\partial J} \right)_h = -1$$

applied to the isochrone, it follows that cooperative inner Lindblad orbits (i.e. $\alpha > 0$) correspond to the rising part of $J = J(h)$, $\Omega$ being kept constant. Conversely, “donkey” orbits correspond to the decreasing branch. In order to keep the summation over all types of orbits algebraic, it is easier to use $h$ rather than $J$ as an integration variable, as $J(h)$ is a function. In shall therefore rewrite 5-11 as:

$$F_0^*(\Omega_\ell) = 1/\langle A^2 \alpha \rangle \int F_0(\Omega_\ell, h) A^2_\ell(\Omega_\ell, h) \left( \frac{\partial J}{\partial h} \right) \, dh$$

The $(\partial J/\partial h)$ factor provides the right sign for the algebraic summation. It can be evaluated from 5-8.

Let $(h_1, h_2)$ be the limits of integration corresponding to the circular orbits. They are positive 8th order roots of 5-8 when $J = 1/2 h$ (i.e. $J_R \equiv 0$). The actual integration is therefore done numerically.

The integral 5-12 follows the path corresponding to $\Omega = const$ on figure 3, weighted by the slope of this function, the $A^2_\ell$ factor, and the number of orbits of this type given by the corresponding value on figure 4. Notice that the $J_r/h = const$ lines correspond to families of a given eccentricity. The larger the slope, the larger the eccentricity. The $A^2_\ell$ factor smooths out the contribution from small eccentricities. Hence a family of slowly rotating (i.e. $h \ll 1$), radially extended (i.e. $J_r \gg 1$) orbits will contribute quite significantly to the summation corresponding to small $\Omega_\ell$ in 5-12. Hence $F_0^*(\Omega_\ell)$ is expected to be significant for small $\Omega_\ell$. Conversely, for $\Omega_\ell = \Omega_{crit}$ (corresponding to the heighest rotation allowed by
bound motion), \(F^*_0(\Omega_\ell)\) vanishes because of \(A^2_\ell\) (as these orbits are fairly circular. Indeed, more eccentric orbits would reach the escape velocity.

\[\triangleq\] Preliminary Results

I have computed numerically 3-9, and recovered results compatible with the above qualitative remarks. This suggest the bar is not rotating, whatever the values for \((b, M)\), though I believe this is a consequence of some of our limiting assumptions (not to include the effect of the distortion for instance) \(\sigma_\Omega\)
VI CONCLUSION AND PROSPECTS

VI 1 Conclusion

In this essay, I have tried to show how to relate the instability of any azimuthal component of the galaxy to its physical characteristics (i.e. its distribution function). In particular, I have defined a criterion for the bi-symmetric ($m = 2$) global instability which, I believe could be responsible for observed bars in many spiral galaxies. This criterion involves a mechanism which aligns the lobes of cooperative inner Lindblad orbital streams. It provides a new approach complementary to the WKB theory for the study of the eigenmodes of galactic discs.

VI 2 Prospects

$\triangleright$ Examples

Most of my recent work concentrated on finding a good analytical illustration for the criterion. As stated in section V, I face some difficulties in providing realistic applications for galaxies which do form rotating bars. In order to address these, I am currently constructing a new distribution function constrained by both the surface density and the azimuthal velocity field, the aim being to obtain an effective distribution function $F_0^* + \delta F_0^*$ the profile of which should look like figure 6. I could then calculate the dimensionless $\sigma_\Omega$ given by 2-21, and conclude about the likelihood of an instability for various $(b,M)$.

25.26pc by 15.73pc (Fig6 scaled 1000) This is the expected effective distribution function. It should present a sharp maximum for what will be the pattern speed of the bar. The width of the maximum and its relative position would then become a function of the parameters of the model via $\Omega^\dagger_\ell = \frac{\Omega_\ell}{\sqrt{GM/16b^3}} F_0^\dagger(h,E) = \frac{F_0(h,E)}{1/Gb}$

$\triangleright$ Numerical Work

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I am currently testing an N-body program in order to assess some of the limiting assumptions of my model. In particular, I would like to evaluate the in-accuracy in the description of interacting orbits rather than interacting stars. The actual program launches two particles on ILR resonant trajectories with initially no self-gravity. The effect of self-gravity is to deviate the stars from their initial orbits. If the initial conditions lead to doubly resonant stars (i.e. the orbits are resonant and the stars complete their orbits on phase), the deviation should be quite violent. Conversely, I expect a smaller deviation for simply resonant stars. My purpose is to investigate the linear component of this deviation. The adiabaticity of the resonant orbits assumes that the initial orbital phase is irrelevant in the process of the orbital interaction. I therefore intend to integrate the trajectories long enough for phase mixing to operate on the fast (orbital) angle, before the effect of self-gravity becomes dominant. The stars’ own mass should therefore be small enough to achieve this hierarchy of phenomena.

In practice, the integration has to be carried for quite a long time, which makes it difficult to assess the contribution from numerical noise. I shall address this problem by implementing a leapfrog algorithm on a lattice, which involves integer arithmetic rather than floating point, following some work by Earn & al.

\(\triangleright\) Statistical Mechanics

Another way to approach the problem of interacting ellipses is to build an effective potential in order to be able to calculate the statistical properties of an assembly of such interacting ellipses against temperature. I expect a “phase transition” to occur for some critical temperature, when the ensemble becomes azimuthally unstable. One major apparent difficulty is that gravitating systems are generally not extensive, which makes the standard canonical statistical mechanics formalism unapplicable. Yet I believe this should not be of any concern here, as only a small fraction of quasi-resonant ellipses contribute to the self-interaction. This fact is the equivalent of shielding in momentum space, and provides the extensivity.
In practice, I shall consider the Hamiltonian:

\[ H = \sum_i \frac{h_i}{a} + \sum_i \sum_j A \frac{\cos(2\varphi_i - 2\varphi_j)}{\text{ch}\left(\frac{h_i - h_j}{H}\right)} \]

where \( a \) is the moment of inertia of the ellipses, and \( \varphi \) and \( h \) their orientation and angular momentum respectively. The constant \( A \) measures the strength of the interaction, and \( H \) is the “shielding” momentum (two ellipses with relative momentum larger than a few \( H \) do not feel each other, as only quasi-resonant orbits interact). The partition function \( Z_N \) should then be calculated, and all the statistical properties would follow. All this is very preliminary work.

▷ **Non Linear evolution.**

In an infinite uniform medium, gravitational waves corresponding to the Jeans length are metastable. Consider a set of such adiabatically adjacent metastable waves, ordered by their initial specific energy. Those corresponding to large amplitudes should reflect the fate of the linear modes as they become non linear. An azimuthal equivalent to this (very) simple problem is to be implemented.

▷ **The Adiabatic Pendulum Problem.**

I would also like to address the problem of a set of penduli in a varying gravity field as further penduli are trapped into libration. This follows some work by Lynden-Bell Tremaine and Toomre. The actual trapping is not adiabatic as the orbital period of the critical penduli goes to infinity. This model should allow us to understand the fate of the weakly bound orbits which eventually get un-adiabatically caught by the bar potential well of the bar.
APPENDIX A

A Brief Introduction to Angle Action Formalism

What are Angle Action variables?

I shall here summarize briefly some work by Earn in order to introduce Angle Action, and generating functions (See Born (27) Arnold (89) and Goldstein (80) for references).

Let \( q \) and \( p \) be some known coordinates and momentum for Hamilton’s equation. They obey the equation of motion:

\[
\dot{q} = \frac{\partial H}{\partial p}, \quad \dot{p} = -\frac{\partial H}{\partial q}
\]

where \( H \) is the Hamiltonian for this set of variables. This can be expressed as a variational principle, writing

\[
\delta \int_{t_1}^{t_2} (p \cdot \dot{q} - H) dt = \int_{t_1}^{t_2} \left\{ \delta p (\dot{q} - \frac{\partial H}{\partial p}) - \delta q (\dot{p} + \frac{\partial H}{\partial q}) \right\} dt + [p \cdot \delta q]_{t_1}^{t_2}
\]

the r.h.s vanishes because the variation is assumed to vanish at the end points, and \( q \) and \( p \) obey Hamilton’s equation.

Let us derive a new set of variables for time independent Hamilton equations. A new set of variables, say \( P \) and \( Q \) with \( \mathcal{H} \) the new Hamiltonian expressed in these variables, will satisfy formally the same variational principle.

\[
\delta \int_{t_1}^{t_2} (P \cdot \dot{Q} - \mathcal{H}) dt = 0
\]

This will have the same physical content as A1 only if there exists a function \( S \) so that

\[
P \cdot \dot{Q} - \mathcal{H} = p \cdot \dot{q} - H + \frac{dS}{dt}
\]

where \( d/dt \) is a total derivative with respect to time.
Let us restrict ourselves to new sets of variables for which \( S(q, P) = S_q(q) + S_P(P) - P \cdot Q \). A3 is satisfied only if

\[
p = \frac{\partial S}{\partial q} \quad Q = \frac{\partial S}{\partial P} \quad \mathcal{H} = H
\]

which can be written in a more compact form as:

\[
H(q, \frac{\partial S_q}{\partial q}) = \mathcal{H}(\frac{\partial S_P}{\partial P}, P)
\]

Notice that \( \mathcal{H} \) is a function of \( P \) only, by choice of \( S \). The new equations of motion therefore reads:

\[
\dot{P} = -\frac{\partial H}{\partial Q} \equiv 0 \quad \dot{Q} = \frac{\partial H}{\partial P}
\]

which leads to the immediate solution:

\[
P = \text{constant} \quad Q = \frac{\partial H}{\partial P} t + Q_0
\]

Therefore \( S \) is a generating function to a set of variables for which the momenta are integral of the motions.

Now restricting myself to multiply periodic bound motion, I may construct some new momenta, so that the associated coordinates are the phase of the oscillation of each degree of freedom. Let us write this new set of variables \((\varphi, J)\). It follows

\[
2\pi = \oint d\varphi_i = \oint \left( \frac{\partial \varphi_i}{\partial q} \cdot dq + \frac{\partial \varphi_i}{\partial p} \cdot dp \right) = \oint \left( \frac{\partial}{\partial q} \frac{\partial S}{\partial J_i} \cdot dq + \frac{\partial}{\partial p} \frac{\partial S}{\partial J_i} \cdot dp \right)
\]

\[
= \frac{\partial}{\partial J_i} \oint \frac{\partial S}{\partial q} \cdot dq = \frac{\partial}{\partial J_i} \oint p \cdot dq
\]

where the loop integral refers to integration over one complete period of oscillation for the corresponding degree of freedom. This relation will be satisfied if

\[
J_i = \frac{1}{2\pi} \oint p_i dq_i
\]

These momenta are called actions. They correspond to the circulation of each degree of freedom.
For instance, it is possible to build such new momenta in the context of galactic dynamics for the radial and azimuthal motion of a star in the disc of the axisymmetric galaxy whose potential is $\Psi_0$. Namely

$$\mathbf{J} = (J_R, J_\phi) = (J_R, h),$$

where

$$J_R = \frac{1}{2\pi} \oint [\dot{R}]dR, \quad J_\phi = h = R^2 \dot{\phi}.$$ 

Here $[\dot{R}]$ is the function of $R$, the specific energy $E$, and the specific angular momentum $h$ of the star given by $[\dot{R}] = \sqrt{2E + 2\Psi_0(R) - h^2/R^2}$.

Why use Angle Action?

- The equations of motion are trivial in these variables when self-gravity is zero as the unperturbed actions are by construction integrals of the motion. The perturbation theory is therefore easy to implement.
- The actions label unperturbed orbits.
- The actions can be adiabatically invariant.

What is the Averaging Principle?

Let $F$ be a distribution function in phase space parameterized by angles and actions. Let us write this function as the sum over its Fourier components with respect to the phase $\varphi$.

$$F(\mathbf{J}, \varphi, t) = \sum_m F_m(\mathbf{J}, t)e^{im\varphi}$$

Now

$$\frac{D}{Dt}F = \frac{\partial F}{\partial t} + \frac{\partial H}{\partial \mathbf{J}} \cdot \frac{\partial F}{\partial \varphi} - \frac{\partial H}{\partial \varphi} \cdot \frac{\partial F}{\partial \mathbf{J}} = 0$$

implies that for each $m$

$$F_m(\mathbf{J}, t) = F_m(\mathbf{J})e^{-im\frac{\partial H}{\partial \mathbf{J}}t}$$

- if $\mathbf{m} \cdot \frac{\partial H}{\partial \mathbf{J}} \neq 0$ the time average of $F_m(\mathbf{J}, t)$ will be zero.
If there exists \( \mathbf{J} \) so that \( \mathbf{m} \cdot \frac{\partial H}{\partial \mathbf{J}} \equiv \mathbf{m} \cdot \mathbf{\Omega} = 0 \) for a given \( \mathbf{m} \), i.e. if there is a \textit{resonance} between the two degrees of freedom, the time average of the corresponding \( F_m \) will not vanish.

The averaging principle as applied in Section III assumes that our distribution function does not have any resonant component but the inner Lindblad, for all \( \mathbf{J} \). Strictly speaking, this statement implies that \( F \equiv 0 \). Nevertheless, I may in practice assume that for a sufficiently long time (of the order of \( 1/|\mathbf{\Omega m}| \)), the contribution from the higher \( m \)'s is negligible. For finite time, I may therefore consider that \( F(\mathbf{J}, \varphi, t) \) is well described by \( F_0 = \langle F(\mathbf{J}, \varphi, t) \rangle \) when restricting myself to the other low order resonances.

\( \triangleright \) \textbf{What is the Adiabatic Invariance?}

The adiabatic invariance implies that when a system is slowly perturbed, its actions still behave as a first integral to second order in the ratio of the natural frequency of the corresponding phase to the characteristic time scale of the perturbation. This relies on the averaging principle, and therefore assumes no low order resonances are involved between these motions. In this essay, the libration of a given orbit in the potential well of the bar is taken to be an external slowly varying perturbation for the stream motion of a star along its orbit. The circulation corresponding to the latter motion is therefore assumed to be adiabatically invariant. This reads:

\[ \frac{\dot{j}}{j} = o \left[ \left( \frac{\Omega_p - \Omega_p^*}{\kappa} \right)^2 \right] \]

using the same notation as those of section II, III, and IV.
APPENDIX B

Explicit calculation for the Interaction Components of the Potential

I show here how to calculate the interaction coefficients $A_{m_1m_2}$ assuming the perturbation may rotate slowly, and the galaxy is cool i.e. The epicyclic approximation holds. These assumptions will allow me to write the $A_{m_1m_2}$ as sums of so-called Green functions.

Let us rewrite 4-23.

$$A_{m_1m_2} = \frac{\delta_{m_1m_2}}{4\pi^3} \int \int \int \frac{d\varphi_1 d\varphi_2}{|R_1 - R_2|} \exp(-im(\Delta \varphi_\ell) d(\Delta \varphi_\ell))$$

Let us try to re-express $A_{m_1m_2}$ when both the interacting orbits can be accurately described in the epicyclic approximation. Then

$$R = R_h + a \cos(\varphi_f)$$
$$\varphi = \varphi_\phi - 2 \frac{a}{R_h \kappa} \sin(\varphi_f)$$

$a$ is the epicyclic radius, $R_h$ the guiding center radius, $\varphi_\phi$ the orientation of the guiding center.

The resonance condition implies

$$\frac{\Omega}{\kappa} = \frac{\Omega_p}{\kappa} + \frac{\ell}{m}$$

Let us assume that for the relevant orbits of this galaxy $m\Omega_p/\kappa \ll 1$ When it is not the case, the following calculation holds, up to B-10 with $\ell \to \ell + m\Omega_p/\kappa$

$$|R_2 - R_1|$$ can be expended and reads

$$|R_2 - R_1| = |R_1^2 + R_2^2 - 2R_1 R_2 \cos(\varphi_2 - \varphi_1)|^{1/2}$$
but  
\[ \varphi_2 - \varphi_1 = \Delta \varphi = \Delta \varphi_{\ell} + \frac{\ell}{m} \left[ \varphi_f - 2 \frac{a}{R_h} \sin(\varphi_f) \right] \]  

where the \( \Delta \) sign represents the difference between 2 and 1 indices. At constant \( \varphi_{f1} \) and \( \varphi_{f2} \), we may perform the inner integral in B1 which corresponds to the Fourier transform with respect to \( \Delta \varphi_{\ell} \).

\[
\int \frac{\exp(-im(\Delta \varphi_{\ell}))}{|R_2 - R_1|} d(\Delta \varphi_{\ell}) = \int |R_1^2 + R_2^2 - 2R_1R_2 \cos(\Delta \varphi)|^{-1/2} \exp(-im\Delta \varphi) d \Delta \varphi
\]

\[
\exp \frac{im\Delta \ell}{m} \left[ \varphi_f - 2 \frac{a}{R_h} \sin(\varphi_f) \right]
\]

Let us call

\[ G_m(R_1, R_2) = \frac{1}{2\pi} \int |R_1^2 + R_2^2 - 2R_1R_2 \cos(\varphi)|^{-1/2} \exp(-im\varphi) d \varphi \]  

\[ B5 \]

\( A_{m_1m_2} \) now reads:

\[ A_{m_1m_2} = \frac{\delta_{m_1m_2}}{2\pi^2} \int \int d\varphi_{f1} d\varphi_{f2} \]

\[ G_m(R_1[\varphi_{f1}], R_2[\varphi_{f2}]) \exp i\Delta \ell \left[ \varphi_f - 2 \frac{a}{R_h} \sin(\varphi_f) \right] \]  

\[ B6 \]

Let us now drop the \( f \) subscript. In order to evaluate this double integral, I am now going to take into account the approximation on which the epicyclic approximation relies, namely \( e \equiv a/R_h \ll 1 \). Let us expend into Taylor series both \( G_m(R_1[\varphi_1], R_2[\varphi_2]) \) and the exponential with respect to \( a/R_h \). This enables us to write : \( A_{m_1m_2} = \sum_i A_{m_1m_2}^{(i)} \) In this sum, the contribution from all possible interaction of resonant orbits may be identified. My purpose is to find the first non-zero contribution for each pair of resonances. This will provide us with the right power law in \( e_1 \) and \( e_2 \) for each of the type of interaction. Let us proceed to second order to recover the most relevant contributions for a bar.
To zeroth order in $e$, $B_6$ reads:

$$A^{(0)}_{m_1 m_2} = \frac{\delta_{m_1 m_2}}{2\pi^2} \int \int G_m(R_{h1}, R_{h2}) \exp i(\ell_2 \varphi_2 - \ell_1 \varphi_1) d\varphi_2 d\varphi_1$$

$$= \frac{\delta_{m_1} \delta_{m_2}}{2\pi^2} G_m(R_{h1}, R_{h2})$$

this expression corresponds to the interaction of two stars at co-rotation $A_{CR}$. It does not involve any eccentricity, as the orbits need not be eccentric to interact at co-rotation.

To first order in $e$, $B_6$ reads:

$$A^{(1)}_{m_1 m_2} = \frac{\delta_{m_1 m_2}}{2\pi^2} \int \int \exp i(\ell_2 \varphi_2 - \ell_1 \varphi_1) d\varphi_2 d\varphi_1$$

$$G_m(R_{h1}, R_{h2}) (i \sin(\varphi_1) e_1 \ell_1 - i \sin(\varphi_2) e_2 \ell_2) + \cos(\varphi_2) R_{h2} e_2 G_m^{(0,1)}(R_{h1}, R_{h2}) + \cos(\varphi_1) R_1 e_1 G_m^{(1,0)}(R_{h1}, R_{h2})$$

where $G_m^{(i,j)}(R_{h1}, R_{h2}) = \frac{\partial^i}{\partial R_1^i} \left( \frac{\partial^j}{\partial R_2^j} \right) G_m(R_{h1}, R_{h2})$

performing the double integral over $d\varphi_1 d\varphi_2$ leads to

$$A^{(1)}_{m_1 m_2} = \frac{\delta_{m_1 m_2}}{2}$$

$$\delta_{\ell_1} \delta_{\ell_2} \begin{bmatrix} G_m(R_{h1}, R_{h2}) \ell_1 + R_{h1} G_m^{(1,0)}(R_{h1}, R_{h2}) \end{bmatrix} e_1$$

$$+ \delta_{\ell_1} \delta_{\ell_2} \begin{bmatrix} R_{h1} G_m^{(1,0)}(R_{h1}, R_{h2}) - G_m(R_{h1}, R_{h2}) \ell_1 \end{bmatrix} e_1$$

$$+ (1 \leftrightarrow 2)$$

These contributions correspond to the cross interaction of orbits trapped in different resonances $\ell = -1, 1$ with orbits at co-rotation. It needs only the inner or the outer Lindblad orbit to be eccentric.

To second order in $e$ appears what I believe are the most relevant terms to make bars. Namely:

$$A^{(2)}_{m_1 m_2} = \frac{\delta_{m_1 m_2}}{4} e_1 e_2$$

$$\delta_{\ell_1} \delta_{\ell_2} \begin{bmatrix} G_m \ell_1 \ell_2 + \ell_1 R_{h2} G_m^{(0,1)} + \ell_2 R_{h1} G_m^{(1,0)} + R_{h1} R_{h2} G_m^{(1,1)} \end{bmatrix} + \delta_{\ell_1} \delta_{\ell_2} \begin{bmatrix} G_m \ell_1 \ell_2 - \ell_1 R_{h2} G_m^{(0,1)} - \ell_2 R_{h1} G_m^{(1,0)} + R_{h1} R_{h2} G_m^{(1,1)} \end{bmatrix} + \ldots$$
the \ldots stand here for terms corresponding to interaction between orbits in
different resonances, already taken into account in the lower order terms.
The first group in B10 corresponds to the interaction of inner Lindblad orbits
$A_{\text{ILR}}$, the second to outer Lindblad orbits $A_{\text{OLR}}$. It requires both orbits to
be eccentric.

Let us have a closer look to the brackets in B9 and B10. They are functions
of the $R$’s through $G_m$ and its successive derivatives. Recognizing in the integrant
of B5 the generator to Legendre polynomials, it follows I may easily find the \{$\Gamma_n$\} so that

$$G_m(R_1, R_2) = \sum_n \Gamma_n \frac{R_{1}^{2n}}{R_{2}^{2n+1}} \quad \text{if} \quad R_2 \geq R_1$$

$$\sum_n \Gamma_n \frac{R_{2}^{2n}}{R_{2}^{2n+1}} \quad \text{if} \quad R_1 \geq R_2$$

Now $R_{1}^{2n}/R_{2}^{2n+1}$ and $R_{2}^{2n}/R_{2}^{2n+1}$ are eigenfunctions for the operators $R_i \partial/\partial R_i \quad i = 1, 2$. It is therefore straightforward to find the new $\gamma_n$’s corresponding to expressions like the brackets in B9 and B10.

Formally I may therefore write:

$$A_{\ell_1, \ell_2}(e_1, e_2, R_1, R_2) = a_{\ell_1}(e_1)a_{\ell_2}(e_2) \sum_n \gamma_{n, \ell_1, \ell_2} g_n(R_1, R_2) \quad B11$$

where

$$g_n(R_1, R_2) = g_1^n(R_1) g_2^n(R_2) = \frac{R_1^{2n}}{R_2^{2n+1}} \quad \text{if} \quad R_2 \geq R_1$$

$$g_1^n(R_2) g_2^n(R_1) = \frac{R_2^{2n}}{R_1^{2n+1}} \quad \text{if} \quad R_1 \geq R_2 \quad B12$$

For instance

$$A_{\text{ILR}}(e_1, e_2, R_1, R_2) = e_1 e_2 \sum_n \gamma_n^{\text{ILR}} g_n(R_1, R_2)$$

\ldots
APPENDIX C
A Dispersion Relation from Green’s Kernels

This Appendix and the following one present a method to actually solve the integral equation 4-27.

\[ \psi_{\ell_1}(J_1) = 2\pi^2 G \sum_{\ell_2} \int \int A_{mm'}(J_1, J_2) \frac{-\partial F_0/\partial h}{\Omega_{\ell_2} - \Omega_p} \psi_{\ell_2}(J_2) d^2 J_2 \quad 4 - 27 \]

Let us proceed. Taking into account B12, I may rewrite it as:

\[ \psi_{\ell_1}(J_1) = a_{\ell_1}(e_1) 2\pi^2 G \sum_{\ell_2,n} \gamma_{n,\ell_1,\ell_2} \int \int a_{\ell_2}(e_2) G_n(R_1, R_2) \frac{[-\partial F_0/\partial h]_{J_f}}{\Omega_{\ell_2} - \Omega_p} \psi_{\ell_2} d^2 J_2 \]

Let us now rewrite the double integral in terms of the new variables \((e, R_h)\) where \(e\) is the ratio of the epicyclic radius to \(R_h\).

\[ d^2 J = d h d J_f = \left| \frac{\partial J_f}{\partial e} \frac{\partial h}{\partial R_h} \right| d e d R_h \]

Let us call

\[ K_{\ell_2}(R_2, e_2) = \left| \frac{\partial J_f}{\partial e} \frac{\partial h}{\partial R_h} \right| \frac{[-\partial F_0/\partial h]_{J_f}}{\Omega_{\ell_2} - \Omega_p} C2 \]

equation C1 becomes

\[ \psi_{\ell_1}(R_1, e_1) = a_{\ell_1}(e_1) 2\pi^2 G \sum_{\ell_2,n} \gamma_{n,\ell_1,\ell_2} \int G_n(R_1, R_2) \left[ \int a_{\ell_2}(e_2) K_{\ell_2}(R_2, e_2) \psi_{\ell_2}(R_2, e_2) d e_2 \right] d R_2 \]

Let us multiply C3 by \(a_{\ell_1}(e_1)K_{\ell_1}(R_1, e_1)\) and integrate over \(e_1\).

\[ \varphi_{\ell_1}(R_1) = H_{\ell_1}(R_1) \sum_{\ell_2,n} \gamma_{n,\ell_1,\ell_2} \int G_n(R_1, R_2) \psi_{\ell_2}(R_2) d R_2 \quad C4 \]
where
\[
\varphi_{\ell_i}(R_i) = \int a_{\ell_i}(e_i) K_{\ell_i}(R_i, e_i) \psi_{\ell_i}(R_i, e_i) \, de_i \quad i = 1, 2 \quad C5
\]
and
\[
H_{\ell_1}(R_i) = 2\pi^2 G \int a_{\ell_1}^2(e_1) K_{\ell_1}(R_1, e_1) \, de_1
\]
\[
= 2\pi^2 G \int a_{\ell_1}^2(e_1) \left| \frac{\partial J_f}{\partial e} \frac{\partial h}{\partial R_h} \right| \frac{[-\partial F_0/\partial h] J_f}{\Omega_{\ell_1} - \Omega_p} \, de_1 \quad C6
\]

Let us now assume the sum over \( n \) may be truncated in \( C4 \). This leads to the under-estimation of the interaction of orbits of similar size which may play a significant role in this process. On the other hand, I expect each decade of \( R_1/R_2 \) to contribute to the same amount in the interaction potential, very much like what happens to the dynamical friction drag Gaunt factor.

\( C4 \) may then be re-written more elegantly as:
\[
\varphi(R_1) = \int G(R_1, R_2) \mathcal{H}(R_2) \cdot \varphi(R_2) \, dR_2 \quad C7
\]

where \( \varphi \) is an unknown \( 2m - 1 \) vector. \( \mathcal{H} \) is a specified \((2m-1)(2m-1)\) matrix, with \( m \) the order of the resonance.

\[
\mathcal{H} = \begin{pmatrix}
\gamma(-m+1,-m+1)H(-m+1) & \gamma(-m+1,-m+2)H(-m+2) & \cdots & \gamma(-m+1,m-1)H(m-1) \\
\gamma(-m+2,-m+1)H(-m+1) & \gamma(-m+2,-m+2)H(-m+2) & \cdots & \gamma(-m+2,m-1)H(m) \\
\vdots & \vdots & \ddots & \vdots \\
\gamma(m-1,-m+1)H(-m+1) & \gamma(m-1,-m+2)H(-m+2) & \cdots & \gamma(m-1,m-1)H(m-1)
\end{pmatrix} \quad C8
\]

the \( \gamma \)'s are constant which can be worked out for each resonance. where

\[
H_\ell(R) = 2\pi^2 G \int a_\ell^2(e) \left| \frac{\partial J_f}{\partial e} \frac{\partial h}{\partial R_h} \right| \frac{[-\partial F_0/\partial h] J_f}{\Omega_\ell - \Omega_p} \, de
\]

and
\[
G(R_1, R_2) = \begin{cases}
\frac{R_1^2}{R_2^3} & \text{if } R_2 \geq R_1 \\
\frac{R_2^2}{R_1^3} & \text{if } R_1 \geq R_2
\end{cases}
\]

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This function has the properties of a Green Function because of the discontinuity of its derivative. There hence exists a differential operator $L^\dagger$ so that:

$$L^\dagger \varphi = \int \delta(R_1 - R_2) \mathcal{H}(R_2) \cdot \varphi(R_2) \, dR_2 = \mathcal{H} \cdot \varphi$$  \hspace{1cm} C9

I calculate in appendix D the operator $L^\dagger$. I also discuss the existence of a complete set of orthogonal eigenfunctions and eigenvalues compatible with the integral equation. These eigenfunctions for $L^\dagger$ do not depend on the resonance $\ell$ one is looking at. Let us therefore expand all $\varphi_\ell$ over the same set of eigenfunctions for $L^\dagger$.

$$\varphi_\ell = \sum_n x_{\ell,n} \psi_n$$  \hspace{1cm} C10

Inserting C10 into C9 leads to $2m - 1$ relations such as:

$$\sum_{n'} x_{\ell,n'} \lambda_n' \psi_n' + \sum_{n'} \sum_{\ell'} H_{\ell,\ell'} x_{\ell',n'} \psi_n' = 0$$

Let us multiply by $\psi_n$ and integrate over $R$. This implies

$$x_{\ell,n} \lambda_n + \sum_{n'} \sum_{\ell'} x_{\ell',n'} \int \psi_n(R) H_{\ell,\ell'}(R) \psi_n'(R) \, dR = 0$$  \hspace{1cm} C11

Let us now call the matrix of $(2m - 1)(2m - 1)$-matrix

$$\mathbf{A} = \text{Diag}(\mathbf{A}^1, \ldots, \mathbf{A}^n, \ldots) = \text{Diag}(\text{Diag}(\lambda_1, \ldots, \lambda_1), \ldots, \text{Diag}(\lambda_n, \ldots, \lambda_n), \ldots)$$

$$\mathbf{H}(\Omega_p) = (\mathcal{H}^{n,n'}) = \left( \int \psi_n(R) \mathcal{H}(R) \psi_n'(R) \, dR \right)$$  \hspace{1cm} C12

where $\mathcal{H}$ is given by C9. $\mathbf{H}$’s coefficient $H^{(n,n')}_{(\ell,\ell')}$ are functions of $\Omega_p$:

$$H^{(n,n')}_{(\ell,\ell')} = \gamma_{(\ell,\ell')} \pi G \left\{ \int a_\ell^2(c) \frac{\partial J_f}{\partial e} \frac{\partial h}{\partial R_h} \left[ \frac{-\partial F_0/\partial h}{\Omega_\ell - \Omega_p} \right] \, d\Omega \right\}$$

I may rewrite C13 as

$$H^{(n,n')}_{(\ell,\ell')} = \gamma_{(\ell,\ell')} \pi G \int A^{n,n'}_{\ell}(\mathbf{J}) \left[ \frac{-\partial F_0/\partial h}{\Omega_\ell - \Omega_p} \right] \, d^2\mathbf{J}$$  \hspace{1cm} C14
in order to make the analogy with section III more striking. This defines the generalized effective $A^{n,n'}_{\ell}$:

$$A^{n,n'}_{\ell}(J) = a_\ell^2 [e(J)] \psi_n[R(J)] \psi_{n'}[R(J)]$$  \hspace{1cm} C15

The system of equations C11 has a non trivial solution for the $x_{n,\ell}$’s only if

$$D(\Omega_p) = \det |\Lambda + H| = 0$$  \hspace{1cm} C16

This is the constraint for the existence of metastable modes.

The actual convergence of this determinant is analyzed in Appendix D.
APPENDIX D

The Eigenvalues and eigenfunctions

Let us now concentrate on the eigenfunctions for the Sturm Liouville equation corresponding to a Green Kernel integral equation.

I have the integral equation

$$\psi(R_1) = \int G(R_1, R_2) \psi(R_2) dR_2$$ \hspace{1cm} D1

where

$$G(R_1, R_2) = G_1(R_1) G_2(R_2) = \frac{R_2^2}{R_1^3} \text{ if } R_2 \geq R_1$$

$$= G_1(R_2) G_2(R_1) = \frac{R_1^2}{R_2^3} \text{ if } R_1 \geq R_2$$

which I may re-write:

$$\psi(R_1) = G_2(R_1) \Psi_1(R_1) + G_1(R_1) \Psi_2(R_1)$$ \hspace{1cm} D2

where

$$\Psi_1(R) = \int_{R}^{\infty} G_1(R) \psi(R) dR$$ \hspace{1cm} D3$$

$$\Psi_2(R) = \int_{R}^{\infty} G_2(R) \psi(R) dR$$

Double derivation of this equation leads to

$$\psi = G_2 \Psi_1 + G_1 \Psi_2$$

$$\psi' = G_2' \Psi_1 + G_1' \Psi_2$$

$$\psi'' = G_2'' \Psi_1 + G_1'' \Psi_2 + (G_2' G_1 - G_1' G_2) \psi$$

Eliminating $\Psi_1$ and $\Psi_2$ between the first two equations gives the so-called Sturm Liouville equation:

$$L^\dagger \psi \equiv (p \psi')' + q \psi = 0$$ \hspace{1cm} D4

with

$$p(R) = G_1' G_2 - G_2' G_1 = 5/R^2$$

$$q(R) = G_1'' G_2' - G_2'' G_1' - p^2 = 55/R^4$$
This differential equation is equivalent to the integral equation D1, when associated to the initial conditions which follows from D1, namely that \( \Psi_1 = [\psi G_1' - \psi' G_1] \) and \( \Psi_2 = [\psi G_2' - \psi' G_2] \) vanish at the end points.

Let us now turn ourselves to the eigenvalue problem.

\[ L^\dagger \psi = \lambda \psi \quad D5 \]

which may be re-expressed as:

\[ R^2 \phi'' - 2R \phi' + (11 - \lambda R^4) \phi = 0 \quad D6 \]

This differential equation has mixed boundary conditions. As such, it allows only for a discrete set of possible eigenvalues which satisfy these boundary conditions. As \( L^\dagger \) is hermitian (by construction), the eigenvalues are real, and positive. Assuming the spectrum has no accumulation point, it is legitimate to expand \( \mathcal{H} \) over the eigenfunctions of \( L^\dagger \). It can be checked that D6 does not allow for a solution compatible with the initial conditions when \( \lambda = 0 \). Therefore \( \prod \lambda_i \neq 0 \). The convergence of the determinant C16 may be worked out from the asymptotic behaviour of the solution to the differential equation. Indeed, the solution scales like \( \cos(nR^2) \). Therefore \( H_{\ell\ell'}^{nn'} = o\left(\frac{1}{nn'}\right) \) according to Lebegue-Riemann Lemma.

It is very unfortunate that D6 has to my knowledge no tabulated solution (though the analogy with Bessel’s differential equation is important). A numerical analysis ought to be developed. In practice, I do not believe the detailed spectral analysis is relevant to our criterion as it stands today.
Bibliography


