Quantization of $\mathcal{W}$ Gravities

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A thesis submitted in partial fulfillment of the requirements for the degree
of Doctor of Philosophy at the University of London
September 1994

PREFACE

The work presented in this thesis was carried out in the Theoretical Physics Group at Imperial College between October 1990 and September 1994 under the supervision of Dr. Kelly Stelle. This thesis has not been submitted for a degree or diploma at any other university.

The material contained in Chapter 3 will be appearing as: R. Mohayaee, "Path-integral quantization of W_∞ gravity", *Phys. Lett.* **B**, in press.

The material contained in Appendix C will be appearing as: R. Mohayaee, C. N. Pope, K. S. Stelle and K. W. Xu, "Canonical BRST Quantisation of Worldsheet Gravities", *Nucl. Phys.* **B**, in press.

The material contained in Chapter 5 closely parallels that in Appendix C; Chapter 5 is concerned with the theory of W_∞ gravity while Appendix C is concerned with the theory of W_3 gravity.

The material contained in Section 4.2 is part of an on-going collaboration with Lucy Wenham at Imperial College.

I am indebted to Dr Kelly Stelle without whose patient supervision, constant help and permanent encouragement this would not have been possible. I thank him in particular for reading this thesis more than once and making numerous corrections and invaluable comments. I am also very grateful to Dr Luis Garay for kindly reading this thesis and giving me helpful advice, especially on its preparation and presentation.

This thesis is dedicated to my parents.

Abstract

In this thesis we study the anomalies of the theories of $\mathcal{W}$ gravity in the context of path integral and BRST quantization. These theories—which include w_∞, W_∞ and W_3 gravity— are of particular interest in this context because they contain anomalies at higher loop orders.

The Fujikawa method can be satisfactorily used to derive the one-loop anomalies that arise in these theories. Furthermore, this method can be complimented by the Wess-Zumino consistency condition to provide a method of anomaly derivation to all loop orders. The Wess-Zumino consistency condition plays a fundamental role in the derivation and evaluation of the anomalies. We shall find that this condition narrows down the freedom in choosing a regulating operator in the Fujikawa scheme by restricting the choice to those that lead to consistent anomalies. In the ghost sector, the anomalies of the theories of w_∞ and W_∞ gravity suffer from divergent coefficients. The Wess-Zumino consistency condition determines which scheme of zeta-function regularization is allowed: it restricts the ghost-sector anomalies to have diagonal form and further relates the coefficients of the diagonal anomalies in all spin sectors to the coefficient of the Virasoro-sector anomaly.

We also discuss the anomalies from the point of view of the BRST charge. The non-nilpotence of the BRST charge results in an expression for the anomaly that is different from the one previously derived from the path-integral quantization scheme. The two different notions of anomalies can be related via a construction involving the BRST "gauge fermion". In the conformal gauge, the BRST charge fails to generate the transformation rule of the gauge fields. We overcome this problem by using a time-derivative gauge. In this gauge, the BRST charge is the true generator of all transformation rules including those of the gauge fields. We show that the anomaly in the BRST Ward identity is obtained by application of the anomalous operator given by the square of the BRST charge—which is calculated using operator products—to the gauge fermion.

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Chapter 1

Introduction

The concept of symmetry forms the foundation of our understanding of the way our universe behaves. It plays the central role in the formulation of the physical laws of nature. Many achievements of physics, in particular modern theoretical physics, spring from the concept of symmetry.

In quantum field theory, the symmetries of a physical system form the core of a theory. A classical action, describing the evolution of the system, is formulated on the basis that it respects the symmetries. The action and its symmetries satisfactorily predict the behaviour of the physical system. However, as one goes to higher energies or lower scales the classical theory fails to give a complete and consistent description of the events. A fuller description can then be obtained by quantizing the theory. In a quantization procedure the action and its symmetries are usually modified by finite additional terms. These terms are identified by various powers of the Planck constant h. Thus a classical theory can be thought of as the zero-h limit of a more general quantum theory.

The process of renormalization, although appearing to be a natural way of extending a classical theory, can cause many problems. One might find it impossible to accomplish a quantization procedure consistently. That is to say, there might be no way of renormalizing the action and its symmetries

in an invariant way. If a theory behaves in this manner it is said to be anomalous. The non-invariance of the quantum action under the classical symmetries is represented by terms of various orders in h. These terms are known as anomalies.

Anomalies can arise in many different ways. Instead of starting from the symmetries of the action, one can start from the generators of these symmetries. One can associate a Noether charge with any local symmetry of the action. The structure of the anomalies that arise in the process of quantization of a theory can be directly analyzed from the point of view of the Noether charge. BRST charge is an example of a Noether charge which is associated with the symmetries of a gauge-fixed action. This charge is classically nilpotent. This means that a single Wick contraction of the operator Q with itself vanishes. However, at the level of multiple Wick contractions the BRST charge might fail to be conserved. A multiple Wick contraction is a quantum process which may be represented by loops in the Feynman diagrams, while a single Wick contraction is a purely classical process which is represented by tree diagrams. Therefore, the non-nilpotence of the BRST charge at the level of multiple contractions represents the breakdown of a classical law under a quantization scheme. This breakdown is indicated by the presence of anomalies.

Since anomalies arise from the breakdown of classical symmetries, they can play different roles in a theory depending on the nature of the broken symmetry. The breakdown of gauge symmetries, which are of importance to the consistency of a theory, can render a quantum theory unphysical. On the other hand, if a theory is renormalizable without depending upon the conservation of a symmetry, then the related anomaly can create physically measurable phenomena. Such is the case in the decay of the neutral π meson into two photons. We take up this example for illustrating the fundamental importance of the anomalies.

A pion decaying into two photons is a classical example of an anomalous

system. This system is invariant under two classes of symmetries: gauge and chiral symmetries. The generators of these symmetries are the classical vector and axial currents. Since the axial current is partially conserved, the decay of the pion is classically forbidden. However, the experimental value of the decay rate is approximately 7.4 eV [38, 52, 60]. This disagreement between the experimental and the theoretical results is resolved by taking into account the contributions from the anomalous processes. A pion can decay into two photons via a process represented by the following triangle diagram:

Figure 1.1: The lowest order term in the expansion of the $\pi \to 2\gamma$ decay amplitude.

The fermions travelling in the triangle loop are the 3-colour constituent quarks that make up the neutral π meson (fig. 1.1). In the quantized theory, the axial and the vector currents are modified by the above one-loop corrections. The conservations of both of these currents cannot be preserved simultaneously by any renormalization program. Since preserving the gauge symmetry is crucial to this theory, the choice between the two alternatives is easy to make. The vector current is conserved at the price of loosing the chiral symmetry. The axial current is an external current, i.e. there is no gauge field of the theory coupling to it. Hence, the anomalous variation of the axial current makes a finite contribution to the decay rate of the pion

[38, 52, 60]. This finite contribution, assuming that each flavour of a quark comes in three colours, is of the order of 7.6 eV which is in good agreement with the experimental data.

The above process can alternatively be analyzed by a path-integral treatment. Instead of the current we can take the action (S) as our basic quantity. In the path-integral quantization scheme, one proceeds from the classical to the quantum field theory by means of functional integration. The partition function corresponding to the decay of the neutral pion ψ is given by

$$\mathcal{Z} = \int \mathcal{D}\psi \mathcal{D}\bar{\psi} e^{-S/\hbar}. \tag{1.1}$$

The anomalies originate from the non-invariance of the path-integral measure under the symmetries of the action [16, 17]. Under these symmetries, the measure transforms through a Jacobian factor which contains a product of delta functions and their derivatives and is thus ill-defined. However, the Jacobian can be regularized using a proper regulator. The anomaly $\mathcal{A}$ is given by the logarithm of the regularized Jacobian. This anomaly cannot be removed by the addition of counterterms to the action or the modification of its symmetries and will eventually contribute to the process of the pion decay. It follows therefore that under the symmetries of the action, the partition function transforms as:

$$\mathcal{Z} \longrightarrow \mathcal{Z}' = \int \mathcal{D}\psi \mathcal{D}\bar{\psi} \exp(S + \hbar\mathcal{A})/\hbar. \tag{1.2}$$

The transformed partition function is then used to calculate the S-matrix, from which the rate of the pion decay is obtained [60]. This method of anomaly derivation is an alternative to the usual Feynman diagram technique and is known as the Fujikawa method. The Fujikawa derivation of the chiral anomaly proves that the anomaly is a non-perturbative effect, even though the original discovery of the triangle anomaly was noted in a perturbative context [52]. The two different ways in which we have described the decay of the neutral π meson predict the same decay rate.

Similar triangle anomalies resolve the problem of U(1) in QCD and predict the existence of the top quark in the Glashow-Salam-Weinberg model [38]. The triangle anomaly, discussed so far, is the example of an anomaly whose presence plays a positive role in a theory. If in a four-dimensional theory anomalies arise as a result of the breakdown of local gauge symmetries, then the theory becomes non-renormalizable and so unphysical. The present theories of all known long-range forces, from electromagnetism to gravitation are based on a gauge principle. Anomalies that result when gauge symmetries cannot be maintained in the quantum theory are of a major importance to the theory and are usually studied extensively. A complete understanding of these anomalies is essential for the application of all gauge theories to physical problems.

In recent years, it has become clear that two-dimensional theories provide good examples for the study of anomalies. Many undiscovered features of anomalies are revealed in these theories. This dissertation is mainly concerned with the anomalies of the theory of W_∞ gravity. This is a two-dimensional theory of gravity in which conserved currents exist at all spin sectors up to infinity. These currents are the one-scalar realization of the W_∞ algebra [62, 64, 65, 66, 68–71] and are the generators of the symmetries of the W_∞ gravity action [8, 61, 62, 66]. The multiple-scalar realization of the W_∞ algebra remains an interesting open problem. However, at this stage, it seems that the multiple-scalar realization of the W_∞ algebra contains anomalies that cannot be removed by any renormalization scheme.

The one-scalar realization of the W_∞ algebra is obtained by removing the matter-dependent anomalies from the theory of w_∞ gravity [8]. The w_∞ algebra is also a linear higher-spin extension of the Virasoro algebra [1, 2] but, unlike the W_∞, it only admits central terms in the spin-two sector. There is also no realization of the gauged w_∞ algebra at the full quantum level [8]. By a full quantum level we mean the level of multiple, rather than single, Wick contractions. Because the one-scalar realization of the

w_∞ currents does not form a closed algebra at the full quantum level, w_∞ gravity contains anomalies. These divide into two main categories: matter-dependent and universal anomalies. The universal anomalies do not have functional dependence on the scalar fields. The matter-dependent anomalies, on the other hand, are functionally dependent on the matter-fields. However, all the matter-dependent anomalies can be removed by the addition of finite local counterterms to the action and by modifications of the symmetries. As a result of such a renormalization procedure, the theory of w_∞ gravity is replaced by the theory of W_∞ gravity. The new theory is free from all the matter-dependent anomalies by construction [8].

The W_∞ gravity is taken as the starting point for our discussion of the universal anomalies. We derive the universal anomalies via two different procedures: first by taking the action as the starting point for the Fujikawa derivation of the universal anomalies and second by deriving these anomalies from the BRST charge. The expressions for the anomalies that are derived in these two different ways are different. After deriving them, we will discuss a possible way of relating the two forms of the anomalies. We will check the validity of this relationship using a different gauge-fixing condition. Having derived and examined the structure of W_∞ anomalies in different ways, we will then prove that these anomalies can be eliminated from the theory via a BRST construction and a suitable regularization scheme [70, 71, 89]. Thus, the theory of W_∞ gravity is free from anomalies.

This dissertation is organized as follows. The second chapter is concerned with the theory of w_∞ gravity. We recall that w_∞ gravity is the one-scalar realization of the w_∞ algebra at the classical level (i.e. at the level of single Wick contractions). We shall derive the anomalies of the theory from the measure of the path integral. Under the symmetries of the w_∞ action, the measure transforms through an ill-defined Jacobian factor [16, 17]. This factor, which potentially carries the possible anomalies of the theory, is initially divergent. A Fujikawa regularization procedure can be used to extract the

finite part of the ill-defined Jacobian. The fact that the Fujikawa regulators have to be Hermitian, but are not constrained by any additional conditions, leaves one with an unlimited number of regulators. This can cause problems because the expressions for the anomalies depend on the choice of the regulator. The ambiguity in the choice of a regulator is resolved by using the Wess-Zumino consistency condition [3, 86]. Although the regulators are not tightly constrained by any conditions, the anomalies are. Since the expression for the anomalies has to satisfy the Wess-Zumino consistency condition, the choice of a regulator is indirectly constrained. Only the regulators that result in consistent anomalies (i.e. those that satisfy the consistency condition) are chosen. These regulators arise in a different scheme of regularization known as Pauli-Villars scheme. One is certain to obtain consistent anomalies if one chooses the Pauli-Villars regulators in the Fujikawa method [14].

The anomalies thus obtained occur in two sectors: the matter-sector anomalies and the ghost-sector anomaly. The matter-sector anomalies occur at all spin levels and are all finite expressions. The ghost-sector anomaly occurs at the Virasoro level only and suffers from infinities even after the Fujikawa regularization. These further infinities appear in the form of divergent sums. The summations are taken over the infinite number of ghost-spins. The divergences in these sums require a further regularization scheme.

The zeta-function regularization scheme can be used for removing such divergences [71, 89]. This regularization scheme is not free from ambiguities either. Let us consider a simple example. The sum $\sum_{n=0}^{\infty}(n+1)^2$, for instance, can be written as

$$\sum_{n=0}^{\infty}((n+3)-2)^2 = \sum_{n=0}^{\infty}(n+3)^2 - 4(n+3) + 4. \tag{1.3}$$

The alternative forms of this sum give different values after regularization. This ambiguity can be resolved by requiring that the regularized value of the sum remains invariant under the interchange $(n+1) \leftrightarrow (-n-2)$. This is a spin symmetry of the ghost/antighost system [89] and is discussed in detail

in Chapter 2. After deriving a finite expression for the anomalies in both the matter and the ghost sectors, we will discuss possible ways of removing them. The matter-dependent anomalies can be removed by modifying the action and its symmetries by finite local counterterms. As a result of the removal of the matter-dependent anomalies, the spin-2 matter-sector universal anomaly is modified by an extra factor. This modified spin-2 matter-sector anomaly cancels against its counterpart in the ghost-sector. Therefore, after the renormalization, only the higher-spin matter-sector universal anomalies remain in the theory. The renormalization procedure replaces the w_∞ currents by the renormalized W_∞ currents. The new currents satisfy the W_∞ algebra rather than the w_∞ algebra [8].

W_∞ gravity is the subject of Chapter 3. We derive the universal anomalies of the W_∞ gravity by the Fujikawa method. The ghost-sector universal anomalies are derived to all spin orders. Although these occur at all orders in $\hbar$, they only arise from single-loop diagrams. This is why they can be derived by the Fujikawa method. (The Fujikawa method is restricted to the one-loop anomalies.) The anomalies in the ghost sector belong to two general classes: the diagonal and the off-diagonal anomalies. The total spin of the ghost and the gauge fields has the value 1 in the expression for the diagonal anomalies, and otherwise in the expression for the off-diagonal anomalies. The ghost-sector anomalies have divergent coefficients which appear in the form of infinite sums. These sums suffer from ambiguities similar to those discussed for the w_∞ gravity earlier on. These infinite sums can be written in infinitely different ways. Each of these methods of summations then gives a different result upon regularization. In addition, as one goes up the spin ladder, these ambiguities multiply and a consistent zeta-function regularization scheme becomes more difficult to devise. The symmetry argument, by means of which these ambiguities have been removed for the Virasoro anomaly, becomes more complicated. However, these ambiguities are resolved by means of the Wess-Zumino consistency condition [3, 86].

The unambiguous result at the Virasoro level, can be extrapolated to all higher-spin levels by the Wess-Zumino consistency condition. That is to say, all higher-loop anomalies can be derived as a result of the implementation of the Wess-Zumino consistency condition on the one-loop anomaly. Not only does this condition provide a method of consistently regularizing the divergent sums at all the spin sectors but it also puts an additional constraint on the anomalies. A further consequence of the Wess-Zumino consistency condition is that off-diagonal anomalies are forbidden. A proper zeta-function regularization scheme is therefore required to give a vanishing coefficient for all the off-diagonal ghost-sector anomalies.

In the matter sector, only the spin-2 universal anomaly can be derived by means of the Fujikawa method. This is the only one-loop anomaly of this sort in the theory. Universal anomalies at higher-spin levels occur at higher-loop orders and cannot be derived by using the Fujikawa method. However, they can be obtained by the Wess-Zumino consistency condition. The matter-sector anomalies do not require any form of the zeta-function regularization scheme and automatically cancel against the ghost-sector anomalies. One thus achieves the cancellation of all the anomalies in the theory of W_∞ gravity.

At the beginning of the introduction, we discussed several different ways of deriving the anomalies. We described two possible methods of deriving the triangle anomalies. Having devoted Chapters 2 and 3 to the derivation of the anomalies from a path-integral point of view, we shall take an alternative route to this derivation in Chapter 4. There we shall discuss possible ways of deriving the anomaly from the BRST charge Q. The non-nilpotence of the BRST charge (i.e. $Q^2 \neq 0$) is an indication that there is an anomaly in the theory [40]. In this method of anomaly derivation, we encounter two main difficulties. The anomaly that results from the non-nilpotence of the BRST charge is not of the same form as that which we derived from the path-integral method. These two forms of the anomalies initially seem to be unrelated to each other. However, we propose that the path-integral

anomaly is obtained by the application of the anomalous operator Q^2—which is calculated using operator-products—to the gauge fermion ψ [59]. The gauge fermion is obtained from the BRST-trivial term in the action (i.e. that for which $Q(Q\psi) = 0$). The validity of this proposal for the theory of the W_∞ gravity is then confirmed by an explicit calculation. The second difficulty is that, contrary to what we had expected, the standard BRST charge Q does not generate all the symmetries of the action. The BRST charge fails to generate the transformations of the gauge fields [59]. This difficulty is discussed and resolved in the next chapter.

In Chapter 5, we consider the w_∞ theory in a different gauge. In the previous chapters, we had chosen the background-field gauge [i.e. $A_i = A_i^{back}$, where A_i is the gauge field of spin $(i+2)$] as the gauge-fixing condition. Here, we choose a time-derivative gauge. We then derive the BRST charge in this gauge and prove that it is the true generator of all the BRST transformations of the classical action, i.e. that the BRST charge also generates the symmetry transformation of the gauge fields.

We also introduce sources into the theory and discuss a more general and rigorous treatment of the Wess-Zumino consistency condition. We derive the anomalies both from the path-integral and the BRST-charge (Q) points of view. This provides us with an extra proof that the relationship proposed earlier (relating the path-integral anomalies to the BRST-charge anomalies) holds also in different gauges. In the derivative gauge, too, the path-integral anomalies can be obtained by application of the operator Q^2, calculated using operator-products, to the gauge-fermion. Although the gauge fermion takes a different form in the derivative gauge, the above relationship still holds.

The final chapter is devoted to a summary of the results of this thesis and its conclusion.

Appendices are included at the end of this thesis. Appendix A is concerned with the Pauli-Villars regularization scheme. In this appendix, we discuss the choice of the regulating operator in the Fujikawa method and

formulate a prescription for choosing regulators in the Fujikawa scheme that would automatically lead to consistent anomalies [36, 85]. Appendix B, is devoted to the zeta-function regularization scheme. We discuss different ways of summing the divergent coefficients of ghost-sector anomalies in spin-2, spin-3 and spin-2/spin-4 off-diagonal anomalies. We then conclude that, of the several methods considered, only one method of summation complies with the Wess-Zumino consistency. In Appendix C, we discuss the theory of W_3 gravity. This nonlinear theory, has been studied extensively in recent years [43–51, 58, 63, 73–75, 85]. The main results that we have obtained for the w_∞ and W_∞ gravity are re-examined briefly for the W_3 gravity. We study this theory in both the derivative and conformal gauges, and highlight the differences between this theory and the theories of w_∞ and W_∞ gravity.

Chapter 2

Path-integral quantization of chiral w_∞ gravity

The purpose of this chapter is to find whether the theory of w_∞ gravity is anomalous or not. Since anomalies arise in the process of quantization, one cannot consider them without considering a quantization program. It was shown by Fujikawa that in the path-integral quantization scheme, anomalies arise as a result of the non-invariance of the measure of the path integral under the symmetries of the action [16, 17]. Under these symmetries, the measure transforms through an ill-defined Jacobian factor. This factor, when properly regularized, results in a finite expression, namely the anomaly [17–31, 39, 53, 55, 83, 87, 88].

We start this chapter by considering the classical action and symmetries of w_∞ gravity [1, 2, 9]. We then fix the gauge by choosing a background-field gauge [45]. The process of gauge fixing replaces the original action and symmetries by their BRST counterparts. The measure of the new partition function, i.e. that corresponding to the BRST-invariant action, involves ghosts and anti-ghosts as well as the matter and the gauge fields.

The path-integral measure does not respect the BRST symmetries. The Jacobian that arises when the measure is varied under the symmetries of the

action is a singular expression. It involves the product of a delta function with its own derivative. This ill-defined quantity has to be regularized to become meaningful. The regularization procedure is complicated by the freedom one has in choosing a regulator. The final answer for the regularized Jacobian (i.e. the anomaly) depends on the choice of regulator. Therefore it is important to choose a proper regulator. It is known that anomalies have to satisfy the so-called Wess-Zumino consistency condition [3, 86]. We shall consider a regulator to be proper if it results in an expression for the anomaly that satisfies this condition.

The regulators that result in consistent anomalies are constructed from the propagators of the theory [14, 30, 36, 85]. The consistent regulators, i.e. those that result in consistent anomalies, are certain to arise in the Pauli-Villars regularization scheme. The point of studying Pauli-Villars regularization here is not to regularize the infinities of the theory by means of this method. The Pauli-Villars scheme is studied so as to obtain an insight into how one can choose the Fujikawa regulator in a consistent manner. Let us briefly mention the key points about Pauli-Villars regularization. In this scheme, massive Pauli-Villars fields are introduced into the action. These fields contribute in such a way as to cancel the infinite contributions from the original fields in the loops. Hence they remove the divergences from the theory. The Pauli-Villars terms decouple from the theory as one lets the regulating mass tend to infinity. In the process of such regularization, certain regulating operators arise. One can show that these regulators automatically result in consistent anomalies [36, 85].

The anomalies that arise can be categorized into two groups. The first group of such anomalies are the universal anomalies. We recall that these anomalies correspond to Feynman diagrams with external gauge field lines only. The second type of anomaly is the matter-dependent anomalies. Unlike the universal anomalies, these anomalies correspond to Feynman diagrams with external matter as well as gauge-field lines.

The universal anomalies themselves dissociate into two kinds; ghost and matter-sector anomalies. The ghost-loop universal anomalies give an infinite contribution. This is because there are an infinite number of ghosts in the theory. A proper zeta-function regularization scheme exists which can be used to regularize the infinite sums arising in the ghost sector [71, 89]. The process of zeta-function regularization is complicated by the fact that an infinite sum can be written in many different ways. Each of these gives a different finite value upon regularization. One then has no way of telling which of these values is preferable. However, there is a symmetry in the ghost/anti-ghost system that must be satisfied by the regularized infinite sum. Such a symmetry argument leaves us with only a few allowed methods of summation. All of the allowed summation methods give the same result upon regularization.[1] The matter-sector anomalies do not cause such problems.

Anomalies are, in essence, a measure of the degree of deformation of a theory under a quantization program. However, the anomalies that we will be calculating have a deeper significance. These anomalies are the result of the breakdown of gauge invariance. Because of the importance of the gauge symmetries, the gauge anomalies should be studied in detail. These anomalies have different impacts on different theories and their implications can vary from theory to theory. However, unless one can show that there is an acceptable way of cancelling these anomalies or of handling the anomalous dynamics implied by them, one is not in general able to put forward a viable theory.

The matter-dependent anomalies can be removed by the addition of finite local counter terms to the action and by corresponding modifications of the BRST transformation rules [8]. The renormalization program that is used to cancel the matter-dependent anomalies also cancels the spin-2 universal

[1] In Chapter 3 we shall discuss how the zeta-function regularization adopted in the spin-2 sector can be extended to all higher-spin sectors by means of the Wess-Zumino consistency conditions.

anomaly. The process of renormalization has a very significant effect on the theory of the w_∞ gravity. As a result of the anomaly cancellation, the currents that generate the w_∞ algebra are renormalized. The renormalized currents, however, no longer satisfy the w_∞ algebra. They instead satisfy the W_∞ algebra [8]. The theory of chiral W_∞ gravity will be discussed in Chapter 3.

2.1 The BRST action of chiral w_∞ gravity

Consider a single matter scalar field ϕ propagating in a two-dimensional spacetime with the light-cone coordinates $(+,-) = (\bar{z}, z)$. The Lorentz-invariant free Lagrangian for this field is given by

$$\mathcal{L} = \frac{1}{2}\partial\phi\bar{\partial}\phi, \tag{2.1}$$

where $\partial = \frac{\partial}{\partial z} = \partial_-$ and $\bar{\partial} = \frac{\partial}{\partial \bar{z}} = \partial_+$. The action corresponding to this Lagrangian is invariant under the following holomorphic transformation:

$$\delta\phi(z,\bar{z}) = \sum_{l=0}^{\infty} k_l(z)\Big(\partial\phi(z,\bar{z})\Big)^{l+1}. \tag{2.2}$$

We now lift the restriction on the parameter k and make it a function of $\bar{z}$ as well as z. The action corresponding to (2.1) is not invariant under this new local transformation. To restore the invariance of the action, we are forced to introduce gauge fields. Thus, a spin $(i+2)$ gauge field A_i is coupled to the current $v^i = (\partial\phi)^{i+2}/(i+2)$. The new action,

$$S = \frac{1}{\pi}\int d^2z\Big(\frac{1}{2}\partial\phi\bar{\partial}\phi - \sum_{i=0}^{\infty} A_i\frac{(\partial\phi)^{i+2}}{i+2}\Big), \tag{2.3}$$

is then invariant under the following gauge transformations [9, 56, 76]:

$$\delta\phi(z,\bar{z}) \;=\; \sum_{l=0}^{\infty} K_l(z,\bar{z})\Big(\partial\phi(z,\bar{z})\Big)^{l+1},$$

$$\delta A_l(z,\bar{z}) \;=\; \bar{\partial}K_l(z,\bar{z}) - \sum_{j=0}^{l}\Big((j+1)\partial K_{l-j}A_j - (l-j+1)K_{l-j}\partial A_j\Big). \tag{2.4}$$

The currents v^i generate the w_∞ algebra at the classical level (i.e. at the level of taking Poisson brackets). The classical w_∞ algebra is given by [1, 2]

$$[v^i_m, v^j_n] = [(j+1)m - (i+1)n]v^{i+j}_{m+n}, \tag{2.5}$$

where v^i_m denotes the mth Fourier mode of the spin-$(i+2)$ current $v^i(z)$. One can verify that the Jacobi identities permit only the usual Virasoro central term in this algebra.

Next, we shall address the issue of whether the classical symmetries of the action are preserved under quantization. In the path integral formalism, the basic quantity is the partition function which, for the theory of w_∞ gravity, is given by

$$\mathcal{Z} = \int \mathcal{D}\phi\mathcal{D}A\exp(S/\hbar), \tag{2.6}$$

where $\mathcal{D}A$ stands for $\prod_i \mathcal{D}A_i$. In deriving this form of the partition function, one starts from the Heisenberg equation describing the evolution of a state:

$$< q_f, t_f|q_i, t_i >=< q_f|\exp -\frac{i\hat{H}(t_f - t_i)}{\hbar}|q_i > . \tag{2.7}$$

Subsequent manipulations result in an expression for the partition function in phase space. Once this is obtained, one can integrate out the momenta in order to replace the Hamiltonian form of the partition function by its Lagrangian form. This integration usually, though not always, results in an infinite constant which can then be absorbed into the normalization of the path integral. Our reason for mentioning this point is to emphasize that one cannot start from just any action and write an expression like (2.6) as the corresponding path integral. Extra non-trivial terms can arise from the integration over the momenta [40]. However, the phase space actions for both

w_∞ and W_∞ are linear in momenta. Thus, the momentum integration makes a contribution to the normalizing factor only and finally one is left with the expression (2.6).

The partition function (2.6) suffers from singularities. It is not well-defined, since the path integral contains a sum over all possible gauge-field configurations including those related by gauge transformations. In other words, when integrating over the gauge fields one also integrates over the volume of the gauge group and as a result the integral diverges. Therefore, one has to fix the gauge. That is, one has to choose a representative for each orbit of the gauge group. Let $\mathcal{F}(A)$ be the gauge-fixing condition. The imposition of such a condition in the path integral is accompanied by the introduction of the Faddeev-Popov ghost and antighost fields. The Faddeev-Popov ghost term that must be added to the action is given by [15]

$$S_{FPG} = \sum_{i,j} \frac{-1}{\pi} \int d^2z d^2w \Big\{ b_i(z) \frac{\delta \mathcal{F}[A_i(z) + \delta A_i(z)]}{\delta K_j(w)} c_j(w) \Big\}. \tag{2.8}$$

The gauge-fixing condition is incorporated into the action by a Lagrange multiplier π_i. The gauge-fixing term is thus given by

$$S_{gf} = \frac{1}{\pi} \int \pi_i \mathcal{F}(A_i). \tag{2.9}$$

The full action, which now includes Faddeev-Popov ghost and gauge-fixing terms, is then invariant under BRST symmetries rather than the original gauge transformations (2.4). We shall initially use the background gauge ($A_i - A_i^{back} = 0$) as our gauge fixing condition $\mathcal{F}(A_i)$. The BRST action for w_∞ gravity is given by

$$\begin{aligned} S_{BRST} \;=\; & \frac{1}{\pi} \int \frac{1}{2} \partial\phi \bar{\partial}\phi + \sum_{i=0}^{\infty} \Big(- A_i \frac{(\partial\phi)^{i+2}}{(i+2)} - b_i \bar{\partial} c_i + \sum_{j=0}^{i} [(j+1) A_j b_i \partial c_{i-j} \\ & -(i-j+1) b_i \partial A_j c_{i-j}] + \pi_i (A_i - A_i^{back}) \Big). \end{aligned} \tag{2.10}$$

The BRST transformations for the scalar field and the gauge fields are the same as the gauge transformations (2.4) with the gauge parameters K_i

replaced by the ghosts c_i. Let us now concentrate on the transformation laws of the ghosts c_i, of the antighosts b_i and of the Lagrange multipliers π_i. The transformation rules of the antighost fields are derived by requiring the invariance of the BRST action. The original part of the action without the gauge-fixing term is invariant by itself for systems such as (2.10) for which the tree-level BRST transformations are obtained by replacing the parameters of the gauge transformations by the corresponding ghost fields. This implies that the variation of the ghost action has to cancel against that of the gauge-fixing term. The variation of these two terms can be written as follows:

$$\delta(S_{FPG} + S_{gf}) = \sum_{i=0}^{\infty} \Big[\Big(- \delta b_i + \pi_i \Big) \delta A_i + A_i \delta \pi_i \Big]. \tag{2.11}$$

We easily see that in order for these variations to vanish, δb_i has to equal the Lagrange multiplier π_i, a quantity whose variation vanishes. This illustrates an important characteristic of the tree-level BRST transformations, i.e. that they are nilpotent. This means that two successive transformations result in a null transformation (i.e. $\delta^2 = 0$). The transformations of ϕ and A_i are indeed also nilpotent provided that the ghost fields c_i transform in an appropriate way as given below. We find that the BRST action is invariant under the following BRST symmetries

$$\begin{aligned}
\delta\phi &= \sum_{l=0}^{\infty} \lambda c_l (\partial\phi)^{l+1}, \\
\delta A_i &= \lambda \bar{\partial} c_i - \lambda \sum_{j=0}^{i} \Big((j+1) \partial c_{i-j} A_j - (i-j+1) c_{i-j} \partial A_j \Big), \\
\delta c_i &= \lambda \sum_{j=0}^{i} (j+1) c_j \partial c_{i-j}, \\
\delta b_i &= \lambda \pi_i, \\
\delta \pi_i &= 0,
\end{aligned} \tag{2.12}$$

where λ is the constant anticommuting BRST parameter. In this thesis, it

only becomes necessary to write the BRST parameter explicitly in deriving the Noether charge. Since this is not the concern of this chapter, we shall drop λ for the sake of brevity.

2.2 The anomalies of w_∞ gravity

The aim of this section is to derive the anomalies that arise in the path-integral quantization of the theory of w_∞ gravity. We recall that the anomalies can render our theory inconsistent. It is therefore essential to see whether w_∞ gravity entails any anomalies.

The partition function corresponding to the BRST-invariant action (2.10) is:

$$\mathcal{Z} = \int \mathcal{D}\phi \mathcal{D}A \mathcal{D}c \mathcal{D}b \exp(S_{BRST}). \tag{2.13}$$

If the partition function is not invariant under the symmetries of the action, the theory suffers from anomalies [16, 17]. Under the BRST symmetries (2.12), the partition function $\mathcal{Z}$ transforms into $\mathcal{Z}'$ as follows:

$$\mathcal{Z} \longrightarrow \mathcal{Z}' = \int \mathcal{J} \mathcal{D}\phi \mathcal{D}A \mathcal{D}c \mathcal{D}b \exp(S_{BRST}), \tag{2.14}$$

where $\mathcal{J}$ is the Jacobian of the BRST transformation.

In saying that the path-integral measure transforms through a Jacobian factor in the above manner, we assume that the path-integral behaves like an ordinary integral. Stated differently, we assume that the measure of the path-integral can be treated separately from the action. Whether these assumptions are justified or not can be settled by considering the discretized version of the path-integral [6, 35, 84]. However, we shall not discuss the discretization of the path integral in this thesis and shall carry on with our calculations assuming that the Jacobian $\mathcal{J}$ is the only source of any possible one-loop anomaly. The Jacobian $\mathcal{J}$ of the BRST transformations is given by

$$\mathcal{J} = \prod_{l,r} \det \frac{\partial(\phi + \delta\phi, A_l + \delta A_l, c_l + \delta c_l, b_l + \delta b_l)}{\partial(\phi, A_r, c_r, b_r)}. \tag{2.15}$$

The above determinant can be written in the standard exponential-trace-log form, so that the transformed partition function acquires the form:

$$\mathcal{Z}' = \int \mathcal{D}\phi\mathcal{D}A\mathcal{D}c\mathcal{D}b \exp\left(S_{BRST} + \sum_{l=0}^{\infty}\sum_{r=0}^{\infty} \mathrm{Tr}\frac{\partial(\delta\phi, \delta A_l, \delta c_l, \delta b_l)}{\partial(\phi, A_r, c_r, b_r)}\right), \quad (2.16)$$

where the trace includes integration over spacetime as well as summation over the internal indices. Thus, our theory is anomaly-free if the trace is zero and anomalous otherwise. Using the BRST symmetries (2.12), the anomaly, i.e. the trace appearing in the exponential of the above expression, can be written in the form:

$$\mathcal{A} = \sum_{l,r} \mathrm{Tr}\mathcal{N}^{lr}\delta^2(z-w), \quad (2.17)$$

where

$$\mathcal{N}^{lr} = \begin{pmatrix} \mathcal{N}^{l}_{\phi\phi} & 0 & \mathcal{N}^{lr}_{\phi c} \\ 0 & \mathcal{N}^{lr}_{gg} & \mathcal{N}^{lr}_{gc} \\ 0 & 0 & \mathcal{N}^{lr}_{cc} \end{pmatrix}, \quad (2.18)$$

and

$$\begin{aligned} \mathcal{N}^{l}_{\phi\phi} &= (l+1)c_l(\partial\phi)^l\partial, \\ \mathcal{N}^{lr}_{gg} &= -(r+1)\partial c_{l-r} + (l-r+1)c_{l-r}\partial, \\ \mathcal{N}^{lr}_{cc} &= (r+1)\partial c_{l-r} - (l-r+1)c_{l-r}\partial. \end{aligned} \quad (2.19)$$

The matrix elements $\mathcal{N}^{lr}_{\phi c}$ and $\mathcal{N}^{lr}_{gc}$ are given similarly. As we shall see later, these two off-diagonal elements do not contribute to the anomaly. However, at this stage, the above trace cannot be evaluated since it diverges. Indeed, it involves the product of a delta function (coming from the definition of the trace) and its derivative (coming from the action of the matrix $\mathcal{N}$ on the delta function in (2.17)). Therefore it has to be regularized. We shall use the Fujikawa method for extracting the finite part of the singular expression (2.17). In this method, the delta function in (2.17) is, in essence, replaced by

a regularized expression [10, 11]. The procedure leading to this replacement starts by expanding the delta function in a plane-wave basis:

$$\delta^2(z-w) = \int \frac{d^2k}{4\pi^2} e^{ik\cdot(z-w)}, \tag{2.20}$$

where $k \cdot (z-w) = \frac{1}{2}[\bar{k}(z-w) + k(\bar{z}-\bar{w})]$. The above integral can be made meaningful everywhere by smearing out the integrand with a Gaussian damping factor $e^{-\beta(k)/M^2}$ as follows: [2]

$$\delta^2(z-w) = \lim_{M\to\infty} \int \frac{d^2k}{4\pi^2} e^{-\beta(k)/M^2} e^{ik\cdot(z-w)}. \tag{2.21}$$

Substituting this expression in (2.17) gives

$$\mathcal{A} = \lim_{M\to\infty} \sum_{l,r} \mathrm{Tr}\, \mathcal{N}^{lr} \exp(-\beta^{lr}(k)/M^2), \tag{2.22}$$

where the trace is evaluated in the plane-wave basis.[3] The above expression for the anomaly proves the statement made earlier that the regularization scheme effectively replaces the delta function by a regularized expression.

Next, we discuss the choice of the regulating operator. The function β in (2.22) is the real eigenvalue of a Hermitian operator H^{lr}. This leaves us with a wide range of possible operators H^{lr}. Since the final form of the anomaly depends on the choice of H^{lr}, the freedom in choosing this operator implies that there are many different expressions for the anomaly. In order to choose the appropriate Hermitian operator H^{lr}, one resorts to the Wess-Zumino consistency condition which has to be satisfied by the anomalies [3, 86].

The Wess-Zumino consistency condition states that the gauge transformations can form a group and has the following general form:

$$\delta\mathcal{A} = 0. \tag{2.23}$$

[2] In general any function $f(\frac{\beta}{M^2})$ can be used [19]. However, it is most convenient to use Gaussian regularization.

[3] The trace can be evaluated in any basis defined by a complete set of functions [17]. However, the plane-wave basis is most practical from the calculational point of view.

The role that this condition plays in choosing an appropriate regulator is discussed in detail in Appendix A. The fact that the anomalies have to satisfy this condition indirectly limits the choices that one has for the regulating operator H^{lr}. In principle, one can initially choose any H^{lr} that is Hermitian. But at the end of the calculation, one has to check that the resulting anomalies are consistent. If the expression for the anomaly does not satisfy the consistency condition, one can adopt one of two alternative procedures: either one modifies the anomaly by finite terms such that the modified expression satisfies the consistency condition, or one chooses a different regulator H^{lr} and repeats the procedure. Thus, the proper choice of a consistent regulator can enormously simplify the task of anomaly calculation.

The Fujikawa regularization scheme does not automatically provide a route to consistent anomalies. In this scheme, one is free to choose the regulating operator and can easily end up with an operator which does not result in consistent anomalies. On the contrary, in the Pauli-Villars scheme, the regulators[4] emerge from the theory itself and one can show that these regulators necessarily result in consistent anomalies (see Appendix A). Thus, in order to ensure that the anomalies that are derived in the Fujikawa scheme are consistent, one can use the regulators that arise in the Pauli-Villars scheme [14].

The chiral part of the consistent operator H^{lr} for the w_∞ gravity (see Appendix A) is given by

$$H^{lr} = \begin{pmatrix} \mathcal{H}^j_{\phi\phi} & 0 & 0 \\ 0 & \mathcal{H}^{lr}_{gg} & 0 \\ 0 & 0 & \mathcal{H}^{lr}_{cc} \end{pmatrix}, \tag{2.24}$$

where

$$\mathcal{H}^j_{\phi\phi} \;=\; -\Big(\partial\bar{\partial} + \partial\Big[(j+1)A_j(\partial\phi)^j\Big]\partial + \Big[(j+1)A_j(\partial\phi)^j\Big]\partial^2\Big),$$

[4]In this thesis, the word regulator denotes the regulating operator e^{H/M^2}.

$$
\begin{aligned}
\mathcal{H}^{lr}_{cc} &= -\partial\bar{\partial} + (r-l)\partial A_{r-2l}\partial - (l+1)\partial^2 A_{r-l} + (r-l+1)A_{r-l}\partial^2, \\
\mathcal{H}^{lr}_{gg} &= -\partial\bar{\partial} + (r-l+1)A_{r-l}\partial^2 - (l+1)\partial A_{r-l}\partial. \qquad (2.25)
\end{aligned}
$$

Due to the diagonal nature of the regulator H^{lr}, the anomaly dissociates into the sum of its diagonal elements as follows:

$$
\mathcal{A} = \mathcal{A}_\phi + \mathcal{A}_c + \mathcal{A}_g, \qquad (2.26)
$$

where

$$
\begin{aligned}
\mathcal{A}_\phi &= \lim_{M\to\infty} \mathrm{Tr} \sum_{j=0}^{\infty}\sum_{l=0}^{\infty} \mathcal{N}^{l}_{\phi\phi} \exp -\mathcal{H}^{j}_{\phi\phi}/M^2, \\
\mathcal{A}_c &= \lim_{M\to\infty} \mathrm{Tr} \sum_{r=0}^{\infty}\sum_{l=0}^{r} \mathcal{N}^{lr}_{cc} \exp -\mathcal{H}^{lr}_{cc}/M^2, \\
\mathcal{A}_g &= \lim_{M\to\infty} \mathrm{Tr} \sum_{r=0}^{\infty}\sum_{l=0}^{r} \mathcal{N}^{lr}_{gg} \exp -\mathcal{H}^{lr}_{gg}/M^2, \qquad (2.27)
\end{aligned}
$$

$\mathcal{N}$ is given in (2.22), and the trace includes integration over spacetime and is evaluated in the plane-wave basis as before. The anomaly always dissociates, in the above manner, if either the regulator or the matrix $\mathcal{N}$ is diagonal. Otherwise there will be off-diagonal contributions to the anomaly.

The matter-sector anomaly $\mathcal{A}_\phi$ can be evaluated in the plane-wave basis. Upon taking the trace in this basis, one obtains:

$$
\begin{aligned}
\mathcal{A}_\phi &= \lim_{M\to\infty} \sum_{l,j=0}^{\infty} \int d^2z \int \frac{d^2k}{4\pi^2}(l+1)c_l(\partial\phi)^l \\
&\qquad \times e^{-ik\cdot z}\partial\Big[\exp\Big(\frac{\partial\bar{\partial} - \partial B_j\partial - B_j\partial^2}{M^2}\Big)e^{ik\cdot z}\Big], \qquad (2.28)
\end{aligned}
$$

where we have used B_j to represent $(j+1)A_j(\partial\phi)^j$. The action of the regulator on $e^{ik\cdot z}$ replaces ∂ by $\partial + i\bar{k}/2$ and $\bar{\partial}$ by $\bar{\partial} + ik/2$. Next, one rescales k to $2kM$ and $\bar{k}$ to $2\bar{k}M$. This gives

$$\mathcal{A}_\phi = \lim_{M\to\infty}\int d^2z\int\frac{M^2d^2k}{\pi^2}\sum_{l=0}^{\infty}\sum_{j=0}^{\infty}(l+1)c_l(\partial\phi)^l(\partial+ikM)\exp\left(\frac{-k\bar{k}}{M^2}\right)$$
$$\times\exp\left(\frac{\partial\bar{\partial}-\partial B_j\partial-B_j\partial^2}{M^2}+\frac{ik\partial+i\bar{k}\bar{\partial}-i\bar{k}\partial B_j-2i\bar{k}B_j\partial}{M}+B_j\bar{k}^2\right). \tag{2.29}$$

The exponential is now expanded. It is obvious that the nth term in the expansion must yield no worse a factor than M^{-3} in order to avoid being discarded in the $M\to\infty$ limit. The only terms that meet this criterion are given by

$$\mathcal{A}_\phi = \int d^2z\sum_{l=0}^{\infty}(l+1)c_l(\partial\phi)^l\partial^3B_j\int\frac{d^2k}{\pi^2}\left(\frac{k\bar{k}}{2}-\frac{k^2\bar{k}^2}{3}+\frac{k^3\bar{k}^3}{24}\right)e^{-k\bar{k}}. \tag{2.30}$$

The essence of all regularization schemes is to extract the divergences of a theory. The infinities so isolated are then discarded from the theory. It may seem that we have not encountered any divergent terms in our regularization scheme. However, this is not really the case. There is a divergent term of order M^3 in the expansion of the exponential which is multiplied by a vanishing k integral. Therefore, in our regularization scheme the divergences are automatically discarded without any need for them to be thrown away by hand.

The k integrals in (2.30) can be performed easily by using the following general form of the Gaussian integral:

$$\int d^2k e^{-k\bar{k}}k^m\bar{k}^n = \delta_{m,n}n!. \tag{2.31}$$

The final expression for the chiral one-loop matter-sector anomaly is given by

$$\mathcal{A}_\phi = \frac{1}{12\pi}\int d^2z\ \sum_{l=0}^{\infty}\sum_{j=0}^{\infty}(l+1)(j+1)c_l(\partial\phi)^l\partial^3[A_j(\partial\phi)^j]. \tag{2.32}$$

We recall that the expression for the anomaly (2.26) includes the ghost-sector anomaly $(\mathcal{A}_g + \mathcal{A}_c)$ as well as the matter-sector anomaly. Likewise, the expression for the ghost-sector anomaly is given by

$$\begin{aligned}\mathcal{A}_c &= \lim_{M\to\infty}\int d^2z \mathrm{Tr}\sum_{l=0}^{\infty}\Big(-(l+1)\partial c_0 + c_0\partial\Big) \\ &\quad \times \exp\Big(\frac{\partial\bar{\partial} + l\partial A_0\partial + (l+1)\partial^2 A_0 - A_0\partial^2}{M^2}\Big), \end{aligned} \tag{2.33}$$

and the anomaly coming from the Jacobian of the gauge field

$$\begin{aligned}\mathcal{A}_g &= \lim_{M\to\infty}\int d^2z \mathrm{Tr}\sum_{l=0}^{\infty}\Big((l+1)\partial c_0 - c_0\partial\Big) \\ &\quad \times \exp\Big(\frac{\partial\bar{\partial} + (l+1)\partial A_0\partial - A_0\partial^2}{M^2}\Big), \end{aligned} \tag{2.34}$$

Note that only the spin-2 gauge field and the corresponding ghosts contribute to the ghost-sector anomalies. This is because, although the expression (2.19) for $\mathcal{N}_{cc}$ implies $l \geq r$, the expression (2.25) for $\mathcal{H}_{cc}$ implies $r \geq l$. Obviously these two contradict one another unless $r = l$. A similar situation arises in the case of $\mathcal{A}_g$. The expressions for $\mathcal{A}_c$ and $\mathcal{A}_g$ (2.27) can be evaluated in a similar manner to that for $\mathcal{A}_\phi$. The total ghost-sector anomaly is then given by

$$\mathcal{A}_c + \mathcal{A}_g = -\sum_{l=0}^{\infty}\Big(\frac{6(l+1)^2 + 6(l+1) + 1}{6\pi}\Big)\int d^2z c_0\partial^3 A_0. \tag{2.35}$$

The summation in (2.35) is strongly divergent. It might at first seem that the ghost-sector anomaly is infinite. However, this infinite sum can be rendered finite by a zeta-function regularization scheme as we shall now show.

2.3 Zeta-function regularization of the ghost-sector anomalies

The generalized Riemann zeta function is defined by [37]:

$$\zeta(\alpha, q) = \sum_{l=0}^{\infty} \frac{1}{(l+q)^{\alpha}} \qquad q \neq 0, -1, -2 \cdots, \operatorname{real} \alpha > 1. \tag{2.36}$$

The divergent sum in (2.35) is of the same form as (2.36) except that it involves a negative value of α. By using an analytic continuation of (2.36), we can write

$$\zeta(-\alpha, q) = \sum_{l=0}^{\infty} (l+q)^{\alpha}. \tag{2.37}$$

Taking into account that $\zeta(-\alpha, q)$ can be expressed in terms of Bernoulli polynomials as:

$$\zeta(-\alpha, q) = \frac{B_{\alpha+1}(q)}{(\alpha+1)}, \tag{2.38}$$

the sum appearing in (2.35) can be regularized as follows:

$$\begin{aligned} \sum_{l=0}^{\infty} 6(l+1)^2 + 6(l+1) + 1 &= 6\zeta(-2,1) + 6\zeta(-1,1) + \zeta(0,1) \\ &= 6\frac{B_3(1)}{3} + 6\frac{B_2(1)}{2} + 6B_1(1) \\ &= -1 \end{aligned} \tag{2.39}$$

There is, however, an ambiguity in the above regularization. This arises from the freedom in writing the infinite summation in powers of $(j+a)$ (for arbitrary a) instead of $(j+1)$ [89]. That is to say, the summation in (2.35) can also be expressed as:

$$\sum_{l=0}^{\infty} 6(l+a)^2 + (12 - 12a + 6)(l+a) + 6(1-a)(2-a) + 1, \tag{2.40}$$

which on using the Bernoulli polynomials gives

$$2B_3(a) + (9-6a)B_2(a) + [6(1-a)(2-a)+1]B_1(a) = 2a^3 - 9a^2 + 13a - 5. \tag{2.41}$$

Therefore, the regularized value of the infinite sum in (2.35) depends on the choice of the parameter a. This ambiguity is removed by using a symmetry of the ghost/antighost system. The original Faddeev-Popov action is invariant under the interchange of the ghost and antighost spins, i.e. the exchange $(-j-1) \leftrightarrow (2+j)$.[5] To find the value of a that respects this symmetry, we replace l by $-l-3$ in (2.39) and rewrite the new expression in powers of $l+a$. This gives

$$-\sum_{-\infty}^{-3} 6(l+a)^2 + (18-12a)(l+a) + 6(2-a)(1-a) + 1. \tag{2.42}$$

A subsequent change in the limits of the sum and use of the Bernoulli polynomial gives

$$-2a^3 + 9a^2 - 13a + 7. \tag{2.43}$$

Three solutions for a are then obtained by equating (2.41) with (2.43) and solving the resulting cubic equation. These are $a = 1, a = 2$ and $a = \frac{3}{2}$, as can be verified by substitution. All of these give the same result upon regularization, by construction. Once the sum is regularized in a fashion that respects the ghost/antighost spin interchange symmetry, its value can be extrapolated to all the higher spins by using the Wess-Zumino consistency conditions. This will be discussed in the next chapter.

Putting everything together, we can write the final expression for the one-loop anomaly of w_∞ gravity, which is

$$\mathcal{A} = \frac{1}{12\pi}\int d^2z \sum_{l=0}^{\infty}\sum_{j=0}^{\infty}(l+1)(j+1)c_l(\partial\phi)^l\partial^3(A_j(\partial\phi)^j) + \frac{1}{6\pi}\int c_0\partial^3 A_0. \tag{2.44}$$

2.4 Higher-loop anomalies

The Fujikawa anomaly given by the above expression corresponds to the one-loop diagrams only. Unlike what was originally believed [52], this does

[5] The ghost c_i and the antighost b_i have spins $(-1-i)$ and $(2+i)$, respectively.

not mean that an anomaly is a non-perturbative effect and can only occur at one-loop level. The theories of w_∞ and W_∞ gravity provide very good examples for showing that anomalies also have perturbative aspects. The derivation of higher-loop anomalies from the Fujikawa method remains an open problem. Whether the higher-loop anomalies also originate in some sense from the non-invariance of the measure of the path integral is still unknown. However, in the next chapter we shall discuss a possible way of resolving this problem. We shall use the Wess-Zumino consistency condition as a complement to the Fujikawa scheme and then derive all the multiple-loop anomalies for the theory of W_∞ gravity. This method can only be used for the derivation of the universal anomalies. Since the theory of w_∞ gravity contains multiple-loop matter-dependent anomalies, we shall use the method of operator-product expansions for the derivation of the anomalies to all loop orders.

The multiple-loop w_∞ anomalies can occur in the matter sector only. The ghost action is bilinear in ghosts and, as a result, can only generate 1-loop diagrams. The partition function corresponding to the matter-sector action S_0 (2.3) can be written as

$$\mathcal{Z} = \int \mathcal{D}\phi \Big(1 + S_0 + \frac{1}{2} S_0^2 + \frac{1}{3!} S_0^3 + \cdots \Big). \tag{2.45}$$

The two-point functions are given by the second term in the above expansion. This is the only term that is relevant for the calculation of the anomalies. This is because all the higher-point functions can be written as the product of two-point functions. The two-point function Γ can be written as [32]

$$\Gamma = \frac{1}{2} \int d^2 z \int d^2 w \sum_{i,j=0}^{\infty} \Big\langle A_i(z) \frac{[\partial\phi(z)]^{i+2}}{i+2}, A_j(w) \frac{[\partial\phi(w)]^{j+2}}{j+2} \Big\rangle. \tag{2.46}$$

Using the short-distance expansion for the scalar field [i.e. $\phi(z)\phi(w) = \log(z-w)$] and inserting the symmetry factors that arise from the combinatorics of

the contractions, we obtain

$$\begin{aligned}\Gamma &= \frac{1}{2}\sum_{i,j=0}^{\infty}\sum_{n}\frac{(i+1)!(j+1)!}{(i+2-n)!(j+2-n)!n!} \\ &\int d^2z\int d^2w A_i(z)(\partial\phi)^{i+2-n}(z)(\partial\phi)^{j+2-n}(w)A_j(w)\frac{1}{(z-w)^{2n}}.\end{aligned} \tag{2.47}$$

[In the above expression, the limits of summation over n are: $2 \leq n \leq (i+2)$ if $i \leq j$ and $2 \leq n \leq (j+2)$ if $j \leq i$.] The use of the standard relation,

$$\frac{1}{(z-w)^m} = \frac{(-1)^{m-1}}{\pi(m-1)!}\frac{\partial^{m-1}}{\bar{\partial}}\delta(z-w), \tag{2.48}$$

in the expression (2.47) gives

$$\begin{aligned}\Gamma &= \frac{-1}{2\pi}\sum_{i,j=0}^{\infty}\sum_{n}\frac{(i+1)!(j+1)!}{(i+2-n)!(j+2-n)!(2n-2)!n!} \\ &\times\int d^2z\int d^2w A_j(w)(\partial\phi)^{j+2-n}\frac{\partial^{2n-1}}{\bar{\partial}}\Big[(\partial\phi(z))^{i+2-n}A_i(z)\Big].\end{aligned} \tag{2.49}$$

The anomaly $\mathcal{A}$ is obtained by taking the variation of the quantum action Γ with respect to the gauge fields. The final expression for this anomaly is

$$\begin{aligned}\mathcal{A} &= \frac{-1}{2\pi}\sum_{i,j=0}^{\infty}\sum_{n}\int d^2z\frac{(i+1)!(j+1)!}{(i+2-n)!(j+2-n)!(2n-1)!n!} \\ &\times\Big(A_j(\partial\phi)^{j+2-n}\partial^{2n-1}\Big[c_i(\partial\phi)^{i+2-n}\Big] - c_j(\partial\phi)^{j+2-n}\partial^{2n-1}\Big[A_i(\partial\phi)^{i+2-n}\Big]\Big).\end{aligned} \tag{2.50}$$

The above expression gives anomalies to all orders in $\hbar$. Although the operator-product expansion is a valid, and probably more convenient, method of anomaly derivation, it does not give as much insight into the nature of anomalies as the Fujikawa method does. The Fujikawa method is limited to one-loop anomalies but shows that the measure of the path integral is responsible for the existence of the anomalies. It also shows how regulators enter the derivation of the anomalies and where the Wess-Zumino consistency condition contributes in the process of anomaly derivation.

2.5 Cancellation of the anomalies

In the preceding sections, we have been concerned with the calculation of the anomalies. In this section, we shall study the possible ways of removing the anomalies.

The renormalization process for anomalies involves the addition of finite local counter terms to the action and modifications of the transformation rules. This is equivalent to adjusting the currents $v^i[(\partial\phi)^{i+2}/(i+2)]$ by the addition of $\hbar$-dependent terms involving fewer ϕ fields but the same number of derivatives [8].

For the purpose of renormalization, it is only necessary to consider the matter part of the action (2.10). The renormalized Faddeev-Popov action and the BRST transformations of the ghosts and antighosts can be obtained directly from the renormalized transformations of the gauge fields. This is because the Faddeev-Popov action is constructed by using the transformation rule of the gauge fields. Similarly, the renormalized transformation rules of the ghosts can be obtained by requiring the nilpotency of the renormalized transformation rules of the gauge fields (see the next chapter).

From the matter part of the action (2.10), we can see that the matter field ϕ has the dimension $\hbar^{1/2}$ and the gauge fields A_i have dimensions $\hbar^{-i/2}$. The renormalization program involves the replacement of the scalar fields by powers of $\hbar$. This process continues until all the scalar fields are exhausted. This is apparent from the expression below:

$$\begin{aligned} S_\phi^{\rm ren} &= \frac{1}{\pi}\int \frac{1}{2}\partial\phi\bar{\partial}\phi - A_0\Big(\frac{(\partial\phi)^2}{2} + \sqrt{\hbar}\alpha\partial^2\phi\Big) \\ &\quad -A_1\Big(\frac{(\partial\phi)^3}{3} + \sqrt{\hbar}\beta\partial^2\phi\partial\phi + \hbar\gamma\partial^3\phi\Big) + \cdots, \end{aligned} \tag{2.51}$$

where $\alpha, \beta, \gamma \cdots$ are the renormalization constants that we shall find from the requirement of anomaly cancellation.

The modifications of the action resulting from the renormalization of the currents v^i lead to a modification of the transformation rule of the matter field. The transformation rules of the scalar field can be obtained by taking its operator-product expansion with the currents [8, 67]. Using the propagator for the matter field $\phi(z)\phi(w) = \log(z-w)$ and the renormalized currents (i.e. the terms coupling to the gauge fields in (2.51)), we can obtain the renormalized transformation rules of the scalar field as follows:

$$\begin{aligned}
\delta\phi(w) &= \oint \frac{dz}{2\pi i}\Big\langle c_0(z)\Big(\frac{(\partial\phi)^2}{2} + \sqrt{\hbar}\alpha\partial^2\phi\Big), \phi(w)\Big\rangle \\
&\quad + \Big\langle c_1(z_)\Big(\frac{(\partial\phi)^3}{3} + \sqrt{\hbar}\beta\partial^2\phi\partial\phi + \hbar\partial^3\phi\Big), \phi(w)\Big\rangle, \\
&= \Big(c_0\partial\phi - \sqrt{\hbar}\alpha\partial c_0\Big) + \Big(c_1(\partial\phi)^2 - \sqrt{\hbar}\beta\partial c_1\partial\phi + \hbar\gamma\partial^2 c_1\Big),
\end{aligned} \tag{2.52}$$

where the operators inside $\langle\rangle$ undergo a short distance expansion.

Next, we shall derive the renormalization constants α, γ and β from the requirement of anomaly cancellation. Consider the cancellation of the term $j = l = 0$ in the expression for the anomaly (2.44). This term corresponds to the spin-2 universal anomaly and is given by

$$\mathcal{A}_{c_0 A_0} = \frac{1}{4\pi}\int c_0\partial^3 A_0. \tag{2.53}$$

This is of the order of $\hbar$ and can be represented by the following Feynman diagram:[6]

Figure 2.1: The w_∞ gravity anomaly at the Virasoro level.

[6] Although we are not showing the factors of $\hbar$ explicitly, these factors are present and can be used as loop-counting parameters. The ith-loop anomaly is of order $\hbar^i$.

This anomaly can be cancelled by terms arising from the following variation:

$$\delta_{1/2}S_{1/2} + \delta_0 S_1 + \delta_1 S_0, \tag{2.54}$$

where S_i is the $\hbar^i$ part of the action S_0^{ren} (2.51) and δ_i is the term of order $\hbar^i$ in the variation of the scalar field (2.52). The only term in the above variation that can cancel against (2.44) is

$$\delta_{1/2}S_{1/2} = -\alpha^2 \int A_0 \partial^3 c_0. \tag{2.55}$$

The requirement of anomaly cancellation results in the following relationship for the parameter α:

$$\alpha^2 = \frac{1}{4} \Rightarrow \alpha = \pm\frac{1}{2}. \tag{2.56}$$

To find the renormalization constant γ in (2.51), we consider the cancellation of the terms $l = 1$ and $j = 0$ in the expression for the anomaly (2.44). This term is given by

$$\mathcal{A}_{A_0\phi c_1} = \frac{\hbar}{6\pi} \int c_1 \partial\phi \partial^3 A_0. \tag{2.57}$$

The terms containing c_1 , A_0 and ϕ that arise in the variation (2.54) are given by

$$-A_0\gamma\partial\phi\partial^3 k_1 - 2\gamma k_1 \partial A_0 \partial^3\phi + \gamma\partial^3\phi\partial k_1 A_0. \tag{2.58}$$

The cancellation of the above expression against (2.44) is only achieved if $\gamma = 1/12$.

Similarly, by considering the cancellation of the terms $l = 0$ and $j = 1$ in the expression (2.44), we find that the renormalization constant β has to assume the values $\pm 1/2$.

Also the modification

$$\delta_0 A_0 = \frac{\hbar}{20}(2\partial^3 A_1 - 3\partial^2 A_1 \partial c_1 + 3\partial A_1 \partial^2 c_1 - 2A_1 \partial^3 c_1), \tag{2.59}$$

together with the addition of the term

$$\frac{\hbar}{\pi}A_2\Big(\frac{1}{5}\partial\phi\partial^3\phi-\frac{1}{20}(\partial^2\phi)^2\Big), \tag{2.60}$$

to the action (2.51) are needed in order to cancel the term $l=j=1$ in the expression for the anomaly (2.44).

The fact that the transformation rules of the gauge fields must also be modified in order to cancel the anomalies has an important consequence for the anomalies themselves. The modification of the gauge transformation means that the Wess-Zumino consistency condition is modified. As a result, the expression for the anomaly, prior to renormalization, might no longer be consistent. The anomaly itself must therefore be modified to meet the requirements of the modified consistency condition. That is to say, the renormalization of the gauge-field transformation rules, for the purpose of anomaly cancellation, can lead to the modification of the anomaly itself. This point will be demonstrated in the next chapter.

We now continue with our renormalization program. In order to discuss the effect of the current renormalizations on the structure of the algebra, we first write them down explicitly:

$$\begin{aligned}
V^0 &= \frac{1}{2}(\partial\phi)^2+\frac{\sqrt{\hbar}}{2}\partial^2\phi,\\
V^1 &= \frac{1}{3}(\partial\phi)^3+\frac{\hbar}{2}\partial\phi\partial^2\phi+\frac{\hbar}{12}\partial^3\phi,\\
V^2 &= \frac{1}{4}(\partial\phi)^4+\alpha(\partial\phi)^2\partial^2\phi+-\frac{1}{20}(\partial^2\phi)^2+\frac{1}{5}\partial\phi\partial^3\phi+\frac{1}{60}\partial^4\phi.
\end{aligned} \tag{2.61}$$

By taking the operator-product expansion of these currents, one can easily establish that they do not satisfy the operator-product form of the w_∞ algebra, which is

$$v^i(z)v^j(w)\sim\Big((j+1)\partial_z-(i+1)\partial_w\Big)\frac{v^{i+j}(w)}{z-w}. \tag{2.62}$$

Instead, the currents satisfy the following relations:

$$
\begin{aligned}
V^0(z)V^0(w) &= \frac{\partial V^0}{z-w} + \frac{2V^0}{(z-w)^2} - \frac{1}{(z-w)^4}, \\
V^0(z)V^1(w) &= \frac{\partial V^1}{z-w} + \frac{3V^1}{(z-w)^2}, \\
V^1(z)V^2(w) &= \frac{\partial V^2}{z-w} + \frac{4V^2}{(z-w)^2} + \frac{12}{5}\frac{V^0}{(z-w)^4}, \\
V^1(z)V^1(w) &= \frac{2\partial V^2}{z-w} + \frac{4V^2}{(z-w)^2} + \frac{1}{10}\Big(\frac{\partial^3 V^0}{(z-w)^2} + 15\frac{\partial V^0}{(z-w)^3} \\
&\quad + \frac{9}{2}\frac{\partial^2 V^0}{(z-w)^2} + 30\frac{V^0}{(z-w)^4}\Big) - \frac{1}{(z-w)^6}.
\end{aligned}
\tag{2.63}
$$

The above operator-product expansions strongly suggest that the renormalized currents may form a linear algebra. However, it seems that for obtaining a closed algebra one needs to introduce higher- and higher-spin currents. In fact, the above operator-product expansions resemble the operator-product expansions generated by the $W_{1+\infty}$ currents [8, 57].

The currents of $W_{1+\infty}$ in the basis corresponding to an arbitrary value of the parameter α are given by

$$
V^i = \sum_{j=0}^{i+1} a_j(i,\alpha)\partial^j\bar{\psi}\partial^{i+1-j}\psi, \tag{2.64}
$$

where the coefficients $a_j(i,\alpha)$ are as follows:

$$
a_j(i,\alpha) = \begin{pmatrix} i+1 \\ j \end{pmatrix} \frac{(i+2\alpha+2-j)_j(2\alpha-i-1)_{i+1-j}}{(i+2)_{i+1}}. \tag{2.65}
$$

Here, $(a)_n = (a+n-1)!/[(a-1)!]$. The real scalar field ϕ can be obtained by the following bosonization of the complex fermion field ψ:

$$
\psi =: e^{\phi} : \qquad \bar{\psi} =: e^{-\phi} :, \tag{2.66}
$$

where :: denotes normal ordering with respect to the modes of ϕ. In the bosonized form, the $W_{1+\infty}$ currents for the first few spins are given as follows:

$$
V^{-1} = \partial\phi,
$$

$$
\begin{aligned}
V^0 &= \frac{1}{2}(\partial\phi)^2 + \sqrt{\hbar}\alpha\partial^2\phi, \\
V^1 &= \frac{1}{3}(\partial\phi)^3 + \sqrt{\hbar}\alpha\partial\phi\partial^2\phi + \frac{\hbar}{3}\alpha^2\partial^3\phi, \\
V^2 &= \frac{1}{4}(\partial\phi)^4 + \sqrt{\hbar}\alpha(\partial\phi)^2\partial^2\phi + \frac{\hbar}{20}(8\alpha^2-3)(\partial^2\phi)^2 \\
&\quad + \frac{\hbar}{10}(4\alpha^2+1)\partial\phi\partial^3\phi - \frac{\hbar^2\alpha}{60}(4\alpha^2+1)\partial^4\phi. \qquad (2.67)
\end{aligned}
$$

Therefore, the renormalized w_∞ currents (2.61) correspond to the above bosonized form of the $W_{1+\infty}$ currents with the spin-1 current truncated and with the parameter α set at $\pm\frac{1}{2}$. However, the positive value of α is the only value for which one can truncate the spin-1 current V^{-1} (without truncating others) from the $W_{1+\infty}$ algebra. The truncated currents satisfy the W_∞ algebra. Since all the higher spin currents of w_∞ are renormalized in the same way as the spin-2, spin-3 and spin-4 currents, we can extrapolate our result and conclude that the renormalized w_∞ currents are not just a series of unrelated currents but they indeed form a realization of the W_∞ algebra [8]. The W_∞ algebra and its gauge theory, W_∞ gravity, are the subjects of the next chapter.

Chapter 3

Path-integral quantization of chiral W_∞ gravity

In the last chapter, we derived the anomalies of w_∞ gravity using the Fujikawa method. We also showed that the transformation rules of the scalar and gauge fields can be renormalized in order to eliminate the matter-dependent anomalies by the introduction of finite local counter terms. This was equivalent to adjusting the currents $v^i[(\partial\phi)^{i+2}/(i+2)]$ by the addition of $\hbar$-dependent terms involving fewer scalar fields but the same number of derivatives. The modifications of the currents implied that they did not generate the original w_∞ algebra, but instead generated the W_∞ algebra. In this chapter, we consider the anomalies of W_∞ gravity. We start with the gauge-invariant W_∞ action. We then fix the gauge by choosing a background-field gauge. As discussed in the last chapter, this leads to the replacement of the gauge-invariant action by a BRST-invariant action. Starting from this action, we write down the partition function and derive the anomalies of the W_∞ gravity.

In the derivation of the anomalies, once again we employ the Fujikawa method. Since all the matter-dependent anomalies are cancelled prior to the formulation of W_∞ gravity, the only anomalies left in this theory are

universal. We recall that these are the anomalies with external gauge-field lines only.

Universal anomalies occur in both the matter and the ghost sectors. In the matter sector, the only universal anomaly that arises from the non-invariance of the measure of the path integral is the one-loop Virasoro (spin-2) anomaly. In order to test that the theory is genuinely free from all higher-loop anomalies, we use the Wess-Zumino consistency condition [3, 86]. The Virasoro anomaly, like all other consistent anomalies, is constrained by this condition. We find, however, that the Virasoro anomaly is not consistent in this theory unless it is extended to include all the higher-spin anomalies. In this way, we derive the higher-loop anomalies by the implementation of the Wess-Zumino consistency condition on the one-loop anomaly.

In the ghost sector, although anomalies occur at all orders in $\hbar$, they only arise from single-loop diagrams. This explains why they can be derived using the Fujikawa method. In the ghost sector, one also encounters, in principle, off-diagonal as well as diagonal anomalies.[1] Since all the matter-sector anomalies are diagonal, there is no way of cancelling the off-diagonal ghost-sector anomalies. These cannot be renormalized either, since the ghost action is only bilinear in ghosts. However, the absence of off-diagonal anomalies in the matter-sector is a requirement of the Wess-Zumino consistency condition. That is to say, off-diagonal anomalies are not allowed by this condition.

As in the case of w_∞ gravity, the ghost-sector anomalies of W_∞ gravity are multiplied by divergent coefficients. The off-diagonal ghost-sector anomalies would be absent if one were to choose a proper zeta-function regularization scheme that puts their divergent coefficients to zero. By using the Wess-Zumino consistency condition, the zeta-function regularization scheme used already in the spin-2 sector can be extended to all higher spins. In the spin-2 sector, the regularization scheme was chosen such that the spin symmetry in

[1] By off-diagonal anomalies we mean those of the form $c_i \partial^{2j+3} A_{2j-i}$ for $j \neq i$.

the ghost and antighost system was preserved (see Section 2.3). As one goes to higher spins, it becomes more difficult to identify a scheme of summation that respects this symmetry. However, once we have established the regularization method in the spin-2 sector, we can extend it to all the higher spins by the Wess-Zumino consistency condition. This is because all the higher-loop anomalies can be derived from the spin-2 anomaly. This method of zeta-function regularization eliminates all the off-diagonal anomalies as well as ensuring the cancellation of all the diagonal parts.

3.1 BRST-invariant W_∞ gravity action

The action for the W_∞ gravity is given by [8]

$$S = \frac{1}{\pi}\int\Big(\frac{1}{2}\partial\phi\bar{\partial}\phi - \sum_{i=0}^{\infty} A_i V^i\Big)d^2z, \tag{3.1}$$

where $\partial = \frac{\partial}{\partial z} = \partial_-$ and $\bar{\partial} = \frac{\partial}{\partial \bar{z}} = \partial_+$ (adopting Euclidean signature on the world sheet) and V^i are the renormalized spin $(i+2)$ currents (2.61) of the form[2]

$$V^i = \frac{(\partial\phi)^{i+2}}{i+2} + \frac{1}{2}\partial^2\phi(\partial\phi)^i + \cdots \tag{3.2}$$

These currents satisfy the operator-product expansion form of the W_∞ algebra and generate the following renormalized symmetry transformations of the ϕ field:

$$\begin{aligned} V^i(z)V^j(w) &= -\sum_{l=0}^{\infty} f_{2l}^{ij}(\partial_z,\partial_w)\frac{V^{i+j-2l}(w)}{z-w} - 4^{-2i}C_i\delta^{ij}\partial_z^{2i+3}\frac{1}{z-w}\,, \\ \delta\phi &= \sum_{l=0}^{\infty} K_l(\partial\phi)^{l+1} - \frac{1}{2}\partial K_0 + \cdots, \end{aligned} \tag{3.3}$$

where K_i are the parameters of the gauge transformations, C_i is given by

$$C_i = \frac{2^{2i-3}i!(i+2)!c}{(2i+1)!!(2i+3)!!}, \tag{3.4}$$

[2]From now on, we shall set $\hbar$ equal to 1.

and c is the central charge of the algebra. The f_{2l}^{ij} are the structure constants:[3]

$$f_{2l}^{ij}(m,n) = \frac{4^{-2l}\,\phi_{2l}^{ij}}{2(2l+1)!} M_{2l}^{ij}(m,n)\,. \tag{3.5}$$

In this expression, ϕ_{2l}^{ij} is given in terms of the Saalschutzian hypergeometric function ${}_4F_3$ [65]:

$$\begin{aligned} \phi_{2l}^{ij} &= {}_4F_3\left[\begin{array}{cccc} -\frac{1}{2} & \frac{3}{2} & -l-\frac{1}{2} & -l \\ & & & ;1 \\ -i-\frac{1}{2} & -j-\frac{1}{2} & i+j-2l+\frac{5}{2} & \end{array}\right] \\ &= \sum_{k=0}^{l} \frac{(\frac{-1}{2})_k(\frac{3}{2})_k(-l-\frac{1}{2})_k(-l)_k}{k!(-i-\frac{1}{2})_k(-j-\frac{1}{2})_k(i+j-2l+\frac{5}{2})_k}, \end{aligned} \tag{3.6}$$

where $(a)_n = \Gamma(a+n)/\Gamma(a)$. Here, M is a polynomial of degree $2l+1$ in the variables m and n :

$$M_{2l}^{ij}(m,n) = \sum_{k=0}^{2l+1} \mathcal{M}_{2l,k}^{ij}\, m^{2l+1-k}\, n^k\,, \tag{3.7}$$

where

$$\mathcal{M}_{2l,k}^{ij} = (-)^k \begin{pmatrix} 2l+1 \\ k \end{pmatrix} (2i-2l+2)_k [2j+2-k]_{2l+1-k}\,, \tag{3.8}$$

and $[a]_n = \Gamma(a+1)/\Gamma(a-n+1)$. The notation $f_{2l}^{ij}(\partial_z, \partial_w)$ in (3.3) indicates that the m and n arguments in $f_{2l}^{ij}(m,n)$ are to be replaced by partial derivatives acting either on functions of z only, or on functions of w only. The transformation rules of the gauge fields are derived by the renormalization of the w_∞ gauge transformations. However, since the currents transform under the adjoint representation of the algebra, it follows that the gauge fields must transform under the coadjoint action [7]. Therefore, we can derive a complete form of the gauge transformation by requiring that the variation

[3]The factors 4^{-2i} and 4^{-2l} in equations (3.3) and (3.4) correct a scaling error in [8].

of the action cancels the matter-dependent anomalies. The variation of the action is given by

$$\delta S = \frac{1}{\pi}\int \delta(\frac{1}{2}\partial\phi\bar{\partial}\phi) - \delta A_i V^i - A_i \delta V^i. \tag{3.9}$$

The transformation rules of the currents V^i are obtained by using the operator-product expansion (3.3):

$$\begin{aligned} \delta V^j(w) &= \oint \frac{dz}{2\pi i} K_n(z) V^n(z) V^j(w) \\ &= -\oint \frac{dz}{2\pi i} \sum_{l=0}^{\infty}\sum_{n=0}^{2l-j} f^{nj}_{n-j+2l}(\partial_z, \partial_w) \frac{V^{n+j-2l}(w)}{z-w}. \end{aligned} \tag{3.10}$$

In writing the variations of the currents, we have used terms that involve multiple contractions. Only the single-contraction part of the operator-product expansion of the currents is used in the variation of the action. The higher-contraction terms give the matter-dependent anomalies. Therefore, the variation of the action plus the matter-dependent anomalies has the following form:

$$\sum_{l=0}^{\infty}\sum_{j=0}^{i+2l}\int d^2w\Big(\delta A_i V^i - f^{ji-j+2l}_{2l}(\partial_A, \partial_K) A_j K_{i-j+2l} V^i + \delta\phi\partial\bar{\partial}\phi - c_i A_i \partial^{2i+3} K_i\Big). \tag{3.11}$$

Thus the cancellation of the matter-dependent anomalies requires that the gauge fields transform as follows:

$$\delta A_i = \bar{\partial} K_i + \sum_{j=0}^{i+2l}\sum_{l=0}^{\infty} f^{ji-j+2l}_{2l}(\partial_A, \partial_K) A_j K_{i-j+2l}. \tag{3.12}$$

We now perform the BRST quantization of this theory. The gauge-fixing procedure is conveniently carried out by using a background gauge $A_i = A_i^{back}$ which is imposed by a Lagrange multiplier π_i in the action. The Faddeev-Popov ghost term is obtained, as in the last chapter, by using the above

gauge transformations. After fixing the gauge the action, the symmetries and the partition function are replaced by their BRST counterparts:

$$
\begin{aligned}
S_{BRST} &= S - \frac{1}{\pi}\int \sum_{i=0}^{\infty}\Big(b_i\bar{\partial}c_i + \sum_{l=0}^{\infty}\sum_{j=0}^{i+2l} b_i f_{2l}^{ji-j+2l}(\partial_A,\partial_c)A_j c_{i-j+2l} \\
&\quad -\pi_i(A_i - A_i^{back})\Big), \\
\delta\phi &= \sum_{l=0}^{\infty} c_l(\partial\phi)^{l+1} - \frac{1}{2}\partial c_0 + \cdots\,, \\
\delta A_i &= \sum_{l=0}^{\infty}\sum_{j=0}^{i+2l}\Big(\bar{\partial}c_i + f_{2l}^{ji-j+2l}(\partial_A,\partial_c)A_j c_{i-j+2l}\Big), \\
\delta c_i &= -\sum_{l=0}^{\infty}\sum_{j=0}^{i+2l}\frac{1}{2}f_{2l}^{ji-j+2l}(\partial_{c_j},\partial_{c_{i-j+2l}})c_j c_{i-j+2l}\,, \\
\delta b_i &= \pi_i\,, \\
\delta\pi_i &= 0\,, \\
Z &= \int \mathcal{D}\phi\mathcal{D}A\mathcal{D}b\mathcal{D}c\,\exp\left(S_{BRST}\right).
\end{aligned}
\tag{3.13}
$$

The BRST symmetries of c_i and b_i follow from nilpotence of δA_i and invariance of the action, respectively.

3.2 The anomalies of W_∞ gravity

Under the BRST symmetries, the partition function transforms as follows:

$$
\mathcal{Z} \longrightarrow \mathcal{Z}' = \int \mathcal{J}\mathcal{D}\phi\mathcal{D}A\mathcal{D}c\mathcal{D}b\exp\left(S_{BRST} + \delta S_{BRST}\right), \tag{3.14}
$$

where

$$
\mathcal{J} = \det\sum_{n,k=0}^{\infty}\frac{\partial^l(\phi+\delta\phi, A_n+\delta A_n, c_n+\delta c_n, b_n+\delta b_n)}{\partial(\phi, A_k, c_k, b_k)} \tag{3.15}
$$

and ∂^l denotes left differentiation. The anomaly $\mathcal{A}$ is given by the logarithm of the above Jacobian. Expressing the Jacobian as the standard exponential-

trace-log, we obtain a formal expression for the transformed partition function:

$$\mathcal{Z}' = \int \mathcal{D}\phi \mathcal{D}A\mathcal{D}c\mathcal{D}b \, \exp\left(\mathcal{A} + S_{BRST} + \delta S_{BRST}\right). \tag{3.16}$$

Since the Faddeev-Popov and the gauge-fixed action are invariant under the BRST symmetries, the only non-invariant part of the BRST action comes from the variation of the original part of the action δS. However, this variation, by construction, cancels the matter-dependent part of the anomaly $\mathcal{A}$. Let us now proceed to find the remaining anomalies of the W_∞ gravity.

Using the BRST transformations (3.13), the explicit form of the anomaly is

$$\mathcal{A} = \mathrm{Tr} \sum_{n,l=0}^{\infty} \sum_{j=0}^{i+2l} \begin{pmatrix} \mathcal{N}^{l}_{\phi\phi} & 0 & \mathcal{N}_{\phi c} \\ 0 & \mathcal{N}^{jln}_{gg} & \mathcal{N}^{jln}_{gc} \\ 0 & 0 & \mathcal{N}^{jln}_{cc} \end{pmatrix} \delta^2(z-w), \tag{3.17}$$

where

$$\begin{aligned} \mathcal{N}^{l}_{\phi\phi} &= (l+1)c_l(\partial\phi)^l\partial + \cdots, \\ \mathcal{N}^{jln}_{cc} &= -\mathcal{N}^{jln}_{gg} = f^{jn-j+2l}_{2l}(\partial_c, \partial_\delta)c_j, \end{aligned} \tag{3.18}$$

and the expressions for $\mathcal{N}^{jln}_{\phi c}$ and $\mathcal{N}^{jln}_{gc}$ follow in exactly the same way.

Since $\mathcal{N}^{l}_{\phi\phi}$ cannot be written down explicitly as an infinite sum, we shall initially consider only the classical terms (i.e. terms with no factor of $\hbar$) in the matter sector. This means that the matter-sector anomaly, that we will be deriving from the measure, has exactly the same form as that we derived in the last chapter. We do not need at this stage to consider one-loop diagrams carrying higher powers of $\hbar$ because these will need to be considered later with the higher-loop anomalies.[4]

The trace operation in (3.17) includes integrations over z and w. As a result, the expression (3.17) involves the product of a delta function and its derivative. Therefore, this is an ill-defined object that requires regularization.

[4]In the later sections of this chapter, we shall derive the expression for the anomalies to all orders from the Wess-Zumino consistency condition.

As we have already shown in the last chapter, the Fujikawa regularization scheme replaces the δ function in (3.17) by a regularized operator [10, 11]:

$$\mathcal{A} = \lim_{M\to\infty} \mathrm{Tr} \sum_{n,l=0}^{\infty} \sum_{j=0}^{n+2l} \left(\mathcal{N}^{jln}\, e^{-H^{jln}/M^2} \right), \tag{3.19}$$

where $\mathcal{N}$ stands for the matrix in (3.17).

In Appendix A, we discuss a way of choosing the regulating operator H such that the resulting expression for the anomaly satisfies the Wess-Zumino consistency condition. We explain there how such consistent regulators arise in the Pauli-Villars scheme. Using the result of Appendix A, we have the operator H_ϕ acting on the field ϕ from $\partial^2 S_0/\partial\phi^2$:

$$H_\phi^j = -\partial\bar{\partial} + \partial[(j+1)A_j(\partial\phi)^j\partial]\,, \tag{3.20}$$

where S_0 is the part of S which is of zeroth order in $\hbar$. Since Fujikawa regularization is restricted to the one-loop anomalies, the Jacobian and the regulators in the matter sector can initially be limited to contain terms of zeroth order in $\hbar$ only. The operator H_ϕ, containing a d'Alembertian (i.e. the d'Alembert operator $\bar{\partial}\partial$), is Hermitian and positive semi-definite and is acceptable as it stands. On the other hand, the ghost sector operator derived from $(\partial^2 S_{BRST})/(\partial b_k \partial c_n)$, which is given by

$$D = \begin{pmatrix} 0 & (D_1)^{jln} \\ (D_1^\dagger)^{jln} & 0 \end{pmatrix} \tag{3.21}$$

with $(D_1)^{jln} = -\bar{\partial} - \sum_{r=0}^{\infty} f_{2r}^{2(l+r)-jn}(\partial_A, \partial)A_{2(r+l)-j}$, is not positive semi-definite. Consequently, one needs to use the square of D as the regulator in the ghost/anti-ghost sector.[5] Remembering that our present concern is with a chiral gauging of W_∞, we need from the regulators only the terms involving

[5]Hermitian conjugation and the inner product are defined by $(H\psi, \varphi) = (\psi, H^\dagger\varphi)$ and $(\psi, \varphi) = \int \bar{\psi}\varphi$.

A and not $\bar{A}$. Making the corresponding truncation in D^2, one obtains the following expression for the Hermitian operator H:

$$H = \begin{pmatrix} H_\phi^j & 0 & 0 \\ 0 & (D_1 D_1^\dagger)^{jln} & 0 \\ 0 & 0 & (D_1^\dagger D_1)^{jln} \end{pmatrix}_{|chiral} , \tag{3.22}$$

where the subscript chiral means that only terms containing A, and not $\bar{A}$, contribute to H.

Because the operator H is diagonal, the calculation is simplified and the anomaly $\mathcal{A}$, given in (3.19), dissociates into its individual components in the matter, ghost and gauge sectors :

$$\mathcal{A} = \mathcal{A}_\phi + \mathcal{A}_c + \mathcal{A}_g \, , \tag{3.23}$$

in which

$$\begin{aligned} \mathcal{A}_\phi &= \lim_{M\to\infty} \sum_{l,j=0}^{\infty} \mathrm{Tr}\Big(\mathcal{N}_{\phi\phi}^j e^{-H_\phi^l/M^2}\Big), \\ \mathcal{A}_c &= \lim_{M\to\infty} \sum_{l,n=0}^{\infty} \sum_{j=0}^{n+2l} \mathrm{Tr}\Big(\mathcal{N}_{cc}^{jln} e^{-(D_1^\dagger D_1)^{jln}/M^2}\Big), \\ \mathcal{A}_g &= \lim_{M\to\infty} \sum_{l,n=0}^{\infty} \sum_{j=0}^{n+2l} \mathrm{Tr}\Big(\mathcal{N}_{gg}^{jln} e^{-(D_1 D_1^\dagger)^{jln}/M^2}\Big), \end{aligned} \tag{3.24}$$

and the traces are evaluated in the plane-wave basis. The expression for the matter-sector anomaly $\mathcal{A}_\phi$ is exactly the same as that we derived for w_∞ gravity (2.32). This is because we have restricted ourselves to terms of zeroth order in $\hbar$. However, in the present case all the matter-dependent anomalies cancel against the variations of the W_∞ action (see Section 2.5), leaving us with the spin-2 universal anomaly:

$$\mathcal{A}_\phi^u = \frac{-1}{6\pi} \int dz \, c_0 \partial^3 A_0 \, . \tag{3.25}$$

This anomaly results in part from the non-invariance of the measure and in part from the following variation of the counter term:

$$\delta_{\frac{1}{2}}(\mathcal{L}_{\frac{1}{2}})_0 = -\frac{1}{4}c_0\partial^3 A_0, \tag{3.26}$$

where $\delta_{\frac{1}{2}}$ is given in (2.52) and $(\mathcal{L}_{\frac{1}{2}})_0$ is the counter term at spin-2 level given in expression (2.51).

Let us now concentrate on the derivation of $\mathcal{A}_c$ and $\mathcal{A}_g$. The expression for the $\mathcal{A}_c$ part of the ghost-sector anomaly is given by

$$\begin{aligned}
\mathcal{A}_c &= \lim_{M\to\infty}\sum_{l,n=0}^{\infty}\sum_{s=0}^{2l+1}\sum_{j=0}^{n+2l}\mathrm{Tr}\Big(-f_{2l}^{j,n-j+2l}(\partial_{c_j},\partial_\delta)c_j\delta(z-w)\Big) && (3.27)\\
&= \lim_{M\to\infty}\sum_{l,n=0}^{\infty}\sum_{s=0}^{2l+1}\sum_{j=0}^{n+2l}\mathrm{Tr}\Big(\Phi_{2l}^{j,n-j+2l}\partial^{2l+1-s}c_j\partial^s e^{\frac{-(D_1^\dagger D_1)^{jln}}{M^2}}\Big), && (3.28)
\end{aligned}$$

where the Φs are the deformed structure constants of the W_∞ algebra :

$$\Phi_{2\alpha,\lambda}^{\beta\gamma} = \frac{4^{-2l}}{2(2\alpha+1)!}\phi_{2\alpha}^{\beta\gamma}\,\mathcal{M}_{2\alpha,\lambda}^{\alpha\beta}\,. \tag{3.29}$$

In the above expression, $\phi_{2\alpha}^{\beta\gamma}$ is the hypergeometric function ${}_4F_3$ and $\mathcal{M}$ is the coefficient of the polynomial part of the structure constant. These are given in (3.6) and (3.8), respectively. Substituting for the regulator $(D_1^\dagger D_1)^{jln}$ in (3.28) and writing the trace in plane-wave basis, we obtain

$$\begin{aligned}
\mathcal{A}_c = &\sum_{l,n=0}^{\infty}\sum_{s'=0}^{2r+1}\sum_{s=0}^{2l+1}\sum_{j=0}^{2(r+l)}\mathrm{Tr}\\
&\Big(\Phi_{2l,s}^{j,n-j+2l}\partial^{2l+1-s}c_j\int\frac{d^2k}{4\pi^2}e^{-k\bar{k}/4M^2}(\partial+\frac{i\bar{k}}{2})^s\\
&\times\exp\frac{-1}{M^2}\Big\{-\partial\bar{\partial}-\frac{(ik\partial+i\bar{k}\bar{\partial})}{2}\\
&-\Big[\partial^{2r+2-s'}A_{2(r+l)-j}+\partial^{2r+1-s'}A_{2(r+l)-j}(\partial+\frac{i\bar{k}}{2})\Big]\\
&\times\Big[\Phi_{2r,s'}^{2(r+l)-j,n}(\partial+\frac{i\bar{k}}{2})^{s'}\Big]\Big\}\Big).
\end{aligned} \tag{3.30}$$

After rescaling $k(\bar{k}) \to 2kM(2\bar{k}M)$ and expanding the exponential, the only terms that are of interest to us are those linear in the gauge field. Upon truncating the terms nonlinear in the gauge fields, the above expression can be written as

$$\begin{aligned}\mathcal{A}_c &= \sum_{l,m,n=0}^{\infty}\sum_{s'=0}^{2r+1}\sum_{s=0}^{2l+1}\sum_{j=0}^{2(r+l)}\mathrm{Tr}\Big\{\Phi_{2l,s}^{j,n-j+2l}\frac{\Phi_{2r,s'}^{2(r+l)-j,n}}{M^2(m+1)!}\partial^{2l+1-s}c_j \\ &\times\int\frac{M^2d^2k}{\pi^2}e^{-k\bar{k}}\Big[(\partial+i\bar{k}M)^s(\frac{ik\partial}{M})^m\Big] \\ &\times\Big[\partial^{2r+2-s'}A_{2(r+l)-j}(i\bar{k}M)^{s'}+\partial^{2r+1-s'}A_{2(r+l)-j}(i\bar{k}M)^{s'+1}\Big]\Big\}.\end{aligned} \tag{3.31}$$

The binomial expansion

$$(\partial+i\bar{k}M)^s=\sum_{q=0}^{s}\frac{s!}{q!(s-q)!}\partial^q(i\bar{k}M)^{s-q} \tag{3.32}$$

is then substituted back in (3.31). The finite terms in the expression (3.31) are those for which the power of M is zero. The finite part of the expression (3.31) is given by

$$\begin{aligned}\mathcal{A}_c &= \sum_{l,m,n=0}^{\infty}\sum_{s'=0}^{2r+1}\sum_{s=0}^{2l+1}\sum_{j=0}^{2(r+l)}\mathrm{Tr} \\ &\Big(\frac{(-1)^{m+s+1}(s+1)!}{\pi(m+1)(s+s'+1-n)!(m-s')!}\Big) \\ &\times\Big(\Phi_{2l,s}^{j,n-j+2l}\Phi_{2r,s'}^{2(r+l)-j,n}\int c_j\partial^{2(r+l)+3}A_{2(r+l)-j}\Big).\end{aligned} \tag{3.33}$$

The other part of the ghost-sector anomaly $\mathcal{A}_g$ is obtained in exactly the same way and is given by

$$\mathcal{A}_g = \sum_{s'=0}^{2r+1}\sum_{s=0}^{2l+1}\sum_{l,n=0}^{\infty}\sum_{j=0}^{2(r+l)}\sum_{m=0}^{\infty}\mathrm{Tr} \tag{3.34}$$

$$\Big(\frac{(-1)^{m+s}(s)!}{\pi(m+1)(s+s'+1-m)!(m-s'-1)!} \times\Phi_{2l,s}^{j,n-j+2l}\Phi_{2r,s'}^{2(r+l)-j,n}\int c_j\partial^{2(r+l)+3}A_{2(r+l)-j}\Big). \tag{3.35}$$

3.3 Zeta-function regularization of the ghost-sector anomaly

In the last section we calculated the ghost-sector anomalies. The relevant quantity $\mathcal{A}_c+\mathcal{A}_g$ is given by[6]

$$\mathcal{A}_c+\mathcal{A}_g = \frac{-1}{\pi}\sum_{l,r,n=0}^{\infty}\sum_{j=0}^{2l+n}\sum_{s=0}^{2l+1}\sum_{s'=0}^{2r+1}\sum_{m=s'}^{s+s'}\mathrm{Tr} \Big(\frac{(-1)^{m+s}s!}{(m+1)(s+s'-m)!(m-s')!}\Phi_{2l,s}^{jn-j+2l}\Phi_{2r,s'}^{2(r+l)-jn}\Big) \times\Big(c_j\partial^{2(r+l)+3}A_{2(r+l)-j}\Big). \tag{3.36}$$

Since the ghost action is entirely determined by the structure constants of the algebra, it is bilinear in the ghosts. So, although anomalies will occur at all orders in $\hbar$, they do not actually arise from multi-loop diagrams. This explains why these anomalies can be derived directly from the measure of the path integral.

For the spin-2 ghost-sector anomaly ($j=r=l=0$), we obtain

$$(\mathcal{A}_c+\mathcal{A}_g)_{(2)} = \frac{1}{12\pi}\int\sum_{n=0}^{\infty}[6(n+1)^2+6(n+1)+1]c_0\partial^3A_0\,. \tag{3.37}$$

[6]This expression contains terms of all orders in $\hbar$. There is a factor $\hbar^{l+r+1}$ in (3.36) which we have set equal to 1.

This is exactly the same as the expression in (2.35), which means that the spin-2 anomaly is not affected by the renormalization of w_∞ to W_∞ gravity. This is because the Virasoro part of the ghost action given in (3.13) does not have to be modified and retains its form through renormalization.

The summation in (3.37) is strongly divergent. In the last chapter we regularized the sum by using a zeta-function regularization scheme. We recall that, the generalized Riemann zeta function is defined by [37]

$$\zeta(s-l,\alpha) \equiv \sum_{n=0}^{\infty}(n+\alpha)^{l-s} \qquad \alpha \neq 0,-1,-2\cdots, \qquad s > l+1, \quad (3.38)$$

and is calculated using Bernoulli's polynomials (2.38). The formally divergent sum $\sum_{n=0}^{\infty}(n+\alpha)^l$ can be interpreted as the analytic continuation of the zeta function to $s=0$ [i.e. $\zeta(-l,\alpha)$]. The regularized value of the divergent sum depends on the parameter α. Because the sum in (3.37) is not convergent, there is an ambiguity in the grouping of terms before regularization. This leads to different regularization schemes employing different α's and consequently resulting in different values for the divergent sum. This ambiguity was resolved in the last chapter for the spin-2 anomaly by requiring that the result obtained by regularizing the sum in (3.37) respect the symmetry under the interchange $n+1 \leftrightarrow -(n+2)$ existing in the (b,c) system. Upon choosing a scheme of regularization that respects this symmetry, one obtains the value 2 for the infinite sum in (3.37). This is the value of the ghost central charge.

As one goes on to higher spins, the freedom in choosing a scheme of regularization increases and the ambiguities multiply. However, the regularization schemes for different spins must be consistent with one another since, as we shall show later, all the higher-spin anomalies can be obtained from the spin-2 anomaly using the Wess-Zumino consistency condition. An extension of the spin-2 regularization scheme has been proposed in [71] where it was shown to give consistent results at least up to the spin-18 level for the diago-

nal anomalies (anomalies of the form $c_i\partial^{2i+3}A_i$). This scheme of summation is discussed in detail in Appendix B.

One also needs, however, to consider the off-diagonal anomalies (anomalies of the form $c_j\partial^{2(r+l)+3}A_{2(r+l)-j}$ for $r+l \neq j$). These anomalies, as can be seen from (3.36), are generically present in the theory. Since the anomalies in the matter sector are purely diagonal, the presence of off-diagonal ghost-sector anomalies would be problematic. We shall show, however, that these off-diagonal anomalies are not allowed by the Wess-Zumino consistency condition. This provides a strong reason for choosing a certain scheme of zeta-function regularization in preference to others.

3.4 Higher-loop anomalies from the Wess-Zumino consistency conditions

In the preceding calculation, we derived the one-loop matter-sector anomaly ($\mathcal{A}^u_\phi$) from the measure. We shall now show that all the higher matter-loop anomalies can be derived by the implementation of the Wess-Zumino consistency condition on $\mathcal{A}^u_\phi$.

We start with the commutation relation

$$[\delta_{c_m}, \delta_{c_n}]A_i = \delta_{c_{[m,n]}}A_i + g_{m,n,j}A_j \qquad j \neq i\,, \tag{3.39}$$

where $c_{[m,n]} = (n+1)c_n\partial c_m - (m+1)c_m\partial c_n$. This relation is then applied to the generator of one-particle-irreducible diagrams with external gauge fields only (Γ) :

$$[\delta_{c_m}, \delta_{c_n}]\Gamma = \delta_{c_{[m,n]}}\Gamma + \int g_{m,n,j}A_j\frac{\delta\Gamma}{\delta A_i}\,. \tag{3.40}$$

This form of the Wess-Zumino consistency condition can be used for diagrams with external gauge-field lines only [3, 86]. In the BRST formalism, the ghosts play the role of the parameters of the gauge transformations. As a result, anomalies which were originally functionals of the gauge fields only

now become functionals of the ghost fields as well. In the case of W_∞ gravity, where the gauge-field transformations are linear, the BRST transformations are obtained directly from the gauge transformations but with a substitution of ghosts for the original parameters. Consequently, the original Wess-Zumino commutator form of the consistency condition can be used, modulo the treatment of the ghosts as gauge parameters. It should be noted that this simplified treatment of the consistency conditions is not always possible. In more complicated theories like W_3 gravity the BRST transformations cannot always simply be rewritten as gauge transformations with ghosts replacing the original parameters (see Appendix C). In these circumstances, an anti-bracket form of the Wess-Zumino consistency condition has to be developed from the Ward identities and used for analyzing the anomalies.[7] In W_∞ gravity, on the other hand, the BRST transformations of the gauge fields involve terms at most linear in the gauge and ghost fields. As a result, the anti-bracket form of the Wess-Zumino consistency condition reduces to the commutator form (3.40) of the condition by virtue of the linearity of the W_∞ algebra.

Let us now proceed with our calculations, keeping in mind the limitations imposed by the above form of the Wess-Zumino consistency conditions. The second term in (3.40) is a non-local expression. All the non-local anomalies should cancel independently of the local ones. For this reason, we shall not consider them in our search for the local universal anomalies. Insertion of the spin-2 universal anomaly $\mathcal{A}^u_\phi$ (3.25) into (3.40) gives

$$\frac{1}{6\pi}\int \partial^3 c_0 f_{2l}^{2l-m,m}(\partial_A,\partial_c)A_{2l-m}c_m - \delta_0\int c_m\Lambda_{c_m} = \int (c_0\partial c_m - (m+1)c_m\partial c_0)\Lambda_{c_m}, \tag{3.41}$$

[7]The anti-bracket form of the consistency condition for W_∞ gravity is presented in Chapter 5.

where we have used the notation

$$\delta_\alpha \Gamma = \mathcal{A}_\alpha = \int \alpha \Lambda_\alpha \,. \tag{3.42}$$

Rearranging the expression (3.41) and writing out the structure constant explicitly, we obtain

$$\frac{1}{6\pi} \int \partial^3 c_0 f_{2l}^{2l-m\,m}(\partial_A, \partial_c) A_{2l-m} c_m = \int c_m (\delta_{c_0} - (m+2)\partial c_0 - c_0 \partial) \Lambda_{c_m} \,. \tag{3.43}$$

On the basis of dimensional analysis, we can see that Λ_{c_m} must be of the form

$$\Lambda_{c_m} = \alpha_{l,m} \partial^{2l+3} A_{2l-m} \,, \tag{3.44}$$

where $\alpha_{l,m}$ is a coefficient to be found. To derive the form of $\alpha_{l,m}$, this expression is inserted into the consistency condition (3.41) giving

$$\begin{aligned}
&\frac{-1}{12} \int \frac{\Phi_{2l}^{2l-m,m}}{(2l+1)!} c_0 \partial^3 (\partial^{2l+1-s} A_{2l-m} \partial^s c_m) = \\
&\Big(- \alpha_{l,m} \partial A_{2l-m} \partial^{2l+3} c_m - (2l-m+1) \alpha_{l,m} \partial^{2l+4} c_m \partial A_{2l-m} \\
&+ (m+2) \alpha_{l,m} \partial c_m \partial^{2l+3} A_{2l-m} + (m+2) \alpha_{l,m} c_m \partial^{2l+4} A_{2l-m} \\
&- \alpha_{l,m} c_m \partial^{2l+4} A_{2l-m} \Big) c_o.
\end{aligned} \tag{3.45}$$

By equating different powers of A on both sides of (3.45), one can see that there is no consistent value for α, apart from zero, if $m \neq l$. That is to say, off-diagonal anomalies are not allowed by the consistency condition. Since the consistency condition has exactly the same form in the ghost sector, off-diagonal anomalies are not allowed in the ghost sector either. As a result, a proper zeta-function regularization is the one that not only gives the right value for the central charge but also puts the coefficients of all off-diagonal anomalies in the ghost sector equal to zero. It is not easy to find a scheme that works for all spins. As explained earlier, the regularization ambiguities multiply as we go to higher orders and a generic zeta-function regularization will not lead to cancellation of all the off-diagonal anomalies. However,

appropriately chosen schemes that give cancellation of the off-diagonal anomalies do exist; an example of such a scheme is given in [61, 70, 71]. Thus, our Wess-Zumino consistency condition considerations provide a stringent requirement on the summation scheme used to combine the contributions from different spin sectors to the anomalies. This consistency condition is important because the sum of the contributions from the different spin levels is otherwise ambiguous.

The diagonal part of anomalies has the following coefficients

$$\alpha_{m,m} = \frac{-\Phi^{mm}_{2m,0}}{12\pi(2m+1)!(m+1)}\,, \tag{3.46}$$

where

$$\Phi^{mm}_{2m,0} = -(2m+2)!\,{}_3F_2\left[\begin{array}{ccc} -\frac{1}{2} & \frac{3}{2} & -m \\ & -m-\frac{1}{2} & \frac{5}{2} \end{array};1\right]. \tag{3.47}$$

[In writing the above, we have used the definition of $\Phi^{\beta,\gamma}_{2\alpha}$ from (3.29).] Because of the simple relationship between the indices on Φ, the hypergeometric function ${}_4F_3$ appearing in the definition of $\Phi^{mm}_{2m,0}$ is reducible to the hypergeometric function ${}_2F_1$ [65, 78]:

$${}_4F_3\left[\begin{array}{cccc} -\frac{1}{2} & \frac{3}{2} & -m & -m-\frac{1}{2} \\ -m-\frac{1}{2} & -m-\frac{1}{2} & \frac{5}{2} & \end{array};1\right] = \frac{(3)_m}{(\frac{5}{2})_m}\,{}_2F_1\left[\begin{array}{cc} -\frac{1}{2} & -m \\ & -m-\frac{1}{2} \end{array};1\right], \tag{3.48}$$

where we have used the following transformation rule for the hypergeometric functions:

$${}_3F_2\left[\begin{array}{ccc} a & b & -n \\ c & d & \end{array};1\right] = \frac{(d-a)_n}{(d)_n}\,{}_3F_2\left[\begin{array}{ccc} a & c-b & -n \\ c & 1+a-d-n & \end{array};1\right]. \tag{3.49}$$

The function ${}_2F_1$, on the other hand, can be calculated using Gauss' law for

hypergeometric functions:

$$ {}_2F_1\left[\begin{array}{cc} a & b \\ & c \end{array};1\right] = \frac{\Gamma(c)\Gamma(c-a-b)}{\Gamma(c-a)\Gamma(c-b)}. \tag{3.50} $$

After some manipulations, we obtain the final answer for $\alpha_{m,m}$:

$$ \alpha_{m,m} = -\frac{2^{2m-2}}{\pi}\frac{m!(m+2)!}{(2m+1)!!(2m+3)!!} \ . \tag{3.51} $$

Note that this is exactly the same as the central charge C_m of the algebra given in (3.4). Consequently the final local matter anomalies to all loop orders are given by:

$$ \mathcal{A}^{\mathrm{u}}_{\phi} = \int \sum_{m=0}^{\infty} \alpha_{m,m} c_m \partial^{2m+3} A_m \ . \tag{3.52} $$

We can check order-by-order that $\mathcal{A}^{\mathrm{u}}_{\phi}$ cancels against the regularized anomalies in the ghost sector $(\mathcal{A}_c + \mathcal{A}_g)$. Since the matter and ghost sector anomalies cancel at the Virasoro level, we can conclude that anomaly cancellation is achieved to all loop orders. This is because all the higher-loop anomalies, in both sectors, can be derived by using the Wess-Zumino consistency condition.

Chapter 4

Anomalies from the BRST charge

In the preceding chapters, we considered the anomalies in the context of path-integral quantization. We showed that non-invariance of the measure of the path integral under the symmetries of the action can lead to anomalies. In this chapter, we discuss the anomalies from a different point of view.

Instead of the partition function, we take the BRST charge as our basic quantity. This charge is derived by using the Noether theorem. This theorem associates a conservation law with any symmetry of the action. The BRST charge is the generator of the BRST transformations [1] and is classically nilpotent. The nilpotence of the BRST charge at the classical level means that a single Wick contraction of the charge with itself vanishes. However, at the full quantum level, one takes multiple as well as single Wick contractions of the BRST charge and consequently the charge may fail to be nilpotent. The failure of nilpotence of the BRST charge at the full quantum level is represented by terms which involve various powers of $\hbar$. These terms represent the anomalies.

[1] Later in this chapter, we shall find that the W_∞ BRST charge, in the conformal gauge, does not actually generate the symmetries of the gauge fields. The reason behind this problem and its solution will be discussed in the next chapter.

The expression for the anomaly obtained from the BRST charge is different from that obtained via path-integral quantization. Although both expressions represent the same quantity (i.e. the anomaly), they appear to be unrelated. To bridge this gap, we propose a relationship that connects the two forms of the anomalies [59]. We shall see that the path-integral anomaly $\mathcal{A}$ can be obtained by application of the anomalous operator Q^2, calculated using operator-products, to the gauge fermion. The gauge fermion is defined by the BRST-trivial terms in the action (i.e. those terms which can be written as $Q\psi$). Although, there is still no general proof for the validity of this relationship, it has been explicitly derived in various theories. Our results also confirm the validity of this relationship for the theory of W_∞ gravity.

4.1 The anomalies of W_∞ gravity from the BRST charge

The BRST charge can be derived in the same manner as for any Noether charge [12]. The Noether charge is defined as:

$$Q = \frac{\delta}{\delta(\bar{\partial}\lambda)}(\mathcal{L} + \delta\mathcal{L}), \tag{4.1}$$

where, for the BRST symmetries, λ is the parameter of the BRST transformations (3.13), $\mathcal{L}$ is the BRST Lagrangian given in (3.13) and $\delta\mathcal{L}$ is its variation with respect to the BRST transformations. Using the BRST Lagrangian and symmetries from (3.13), we obtain

$$\begin{aligned} Q \;=\; & \int \frac{\delta}{\delta(\bar{\partial}\lambda)} \sum_{l=0}^{\infty} \sum_{j=0}^{i+2l} \\ & \Big[-\lambda c_l (\partial\phi)^{l+1} \partial\bar{\partial}\phi - \bar{\partial}(\lambda c_i) V^i - b_i \bar{\partial}\delta c_i - \delta\lambda \pi_i \bar{\partial} c_i \\ & + \pi_i \bar{\partial}(\lambda c_i) - b_i f_{2l}^{j,i-j+2l} (\partial_{c_j}, \partial_{c_{i-j+2l}}) \bar{\partial}(\lambda c_j) c_{i-j+2l} \Big], \end{aligned} \tag{4.2}$$

where we have neglected terms that do not contain $\bar{\partial}$. Note that the BRST charge is obtained by first ensuring the invariance of the action under global symmetries (i.e. under a constant λ) and then by making the parameter a local function of time (here $\bar{z}$). Proceeding in this way, we obtain

$$Q = \sum_{i,l=0}^{\infty} \sum_{j=0}^{2l-i} \int \Big(c_i V_i - \frac{1}{2} f_{2l}^{ij} (\partial_{c_j}, \partial_{c_i}) c_i c_j b_{i+j-2l} \Big). \tag{4.3}$$

Before discussing the structure of the anomalies, let us mention that, contrary to expectation, the above BRST charge is not in fact the generator of the full BRST transformations (3.13). It fails to generate the transformations of the gauge fields. This is because the BRST charge contains nothing with non-vanishing operator product with the gauge fields.

We shall continue with our derivation of the anomalies, bearing the above difficulty in mind. We recall that anomalies appear when the nilpotency of the BRST charge cannot be maintained at the full quantum level (i.e. at the level of multiple-contractions in the operator-product expansion). Interpreting the BRST charge Q (4.3) as a normal-ordered quantum operator, one may evaluate Q^2 by standard operator-product expansion techniques. We can Write (4.3) as the integral of a normal-ordered operator current,

$$Q = \oint \frac{dz}{2\pi i} J_B(z), \tag{4.4}$$

where this integral is now evaluated over a closed loop around the origin and, in order to recover equivalence with the standard mode-expansion result, one should collect the simple poles in the above expression using Cauchy's theorem; hence the factor of $(2\pi i)^{-1}$ in the measure [37]. Calculating Q^2 using standard operator-product rules, one obtains

$$
\begin{aligned}
Q^2 &= \oint \frac{dz}{2\pi i} \langle J_B J_B \rangle_1(z); \\
\langle J_B J_B \rangle_1(z) &= \sum_{i=0}^{\infty} C_i c_i(z) \partial^{2i+3} c_i(z) \\
&\quad + \sum_{i,r,l=0}^{\infty} \sum_{j=0}^{2(r+l)} \sum_{s=0}^{2l+1} \sum_{s'=0}^{2r+1} \frac{\Phi_{2l}^{ij} \Phi_{2r}^{2(r+l)-j,i+j-2l}}{(2l+1)!(2r+1)!} \frac{(-1)^{s+s'+1} s'!}{(s+s'+1)!} \\
&\quad \times c_{2(r+l)-j}(z) \partial^{2(r+l)+3} c_i(z). \qquad (4.5)
\end{aligned}
$$

In evaluating (4.5), we have taken the operator product $J_B(z)J_B(w)$, as usual, and have extracted the residue of the first pole $(z-w)^{-1}$ in the resulting Laurent series; the outcome of this procedure is here denoted by $\langle J_B J_B \rangle_1$. In addition, we have used the operator-product expansion $c_i(z)b_j(w) = \delta_{ij}(z-w)^{-1}$. The C_i in (4.5) are the coefficients of the central terms (3.4) and the Φ_{2l}^{ij} are the deformed structure constants of the W_∞ algebra (3.29).

The first term in the expression (4.5) corresponds to the matter-sector anomalies and the second term to the ghost-sector anomalies. The matter-sector anomalies, as can be seen from the above expression, are diagonal. In other words, the constituent ghosts in the first term have the same spin. On the other hand, the ghost-sector anomalies can be off-diagonal as well as diagonal. The diagonal part of the ghost-sector anomalies cancel against the matter-sector anomalies via a zeta-function regularization scheme similar to that discussed in the preceding chapters.[2] The same zeta-function regularization scheme sets the coefficient of the off-diagonal anomalies to zero. This is justified by the fact that the off-diagonal anomalies are not allowed by the Wess-Zumino consistency condition (see Chapter 3, Section 3.4).

[2]Note that the coefficient of the ghost-sector anomalies derived from the BRST charge and from the partition function are the same. The difference between the two expressions for the anomalies comes solely from their functional dependence. As a result, the divergent coefficients of the ghost-sector anomalies derived from the BRST charge can be regularized by a method identical to that employed for regularizing the partition-function anomalies in the preceding chapters.

4.2 Relating the BRST-charge anomalies to the path-integral anomalies

In Chapter 3, we derived the expression for the matter-sector anomalies to all loop orders. This was given by

$$\mathcal{A}_\phi = \sum_{m=0}^{\infty} \frac{\alpha_{m,m}}{\pi} \int d^2w c_m(w) \partial^{2m+3} A_m(w), \tag{4.6}$$

where the constants $\alpha_{m,m}$ are the same as the central charges of the algebra C_i (3.4). In this chapter, however, we have derived a different expression for the matter-sector universal anomalies. This can be inferred from (4.5) and has the form

$$\begin{aligned} Q_\phi^2 &= \oint \frac{dz}{2\pi i} \langle J_B J_B \rangle_1^\phi; \\ \langle J_B J_B \rangle_1^\phi &= \sum_{i=0}^{\infty} C_i c_i(z) \partial^{2i+3} c_i(z). \end{aligned} \tag{4.7}$$

Our task in this section is to relate the above two expressions. In order to find the connection between these two forms of the anomalies, let us recall how each of them was derived.

The path-integral expression for the anomaly, although not obvious from the Fujikawa method, is obtainable by taking the full contraction between $A_i(z)V^i(z)$ and $A_j(w)V^j(w)$. We shall calculate this operator product explicitly. We recall that the anomalies are basically the variation of the effective action with respect to the gauge fields (see Section 2.4). The effective action for the gauge fields in the theory of W_∞ gravity may be written as

$$\Gamma = \frac{1}{2} \sum_{i,j=0}^{\infty} \int \int d^2z d^2w \Big\langle V^i(z), V^j(w) \Big\rangle A_i(z) A_j(w). \tag{4.8}$$

Since in W_∞ gravity the matter-dependent anomalies are absent by construction, the only terms that we need consider in taking the operator-product expansion of the currents V^i are the universal anomalies. These are given by

the central term of the W_∞ algebra (3.4). Substitution of this term in the above expression yields

$$\Gamma = \frac{1}{2}\sum_{i=0}^{\infty}\int\int d^2z d^2w A_j(w)A_i(z)\delta_{ij}C_i\frac{\partial^{2i+3}}{\bar{\partial}}\delta(z-w). \tag{4.9}$$

The anomaly $\mathcal{A}_\phi$ can be obtained by varying the effective action with respect to the gauge fields, as follows:

$$\mathcal{A}_\phi = \sum_k \int \frac{\delta\Gamma}{\delta A_k(w)}\delta A_k(w) = -\sum_{i=0}^{\infty}\frac{C_i}{\pi}\int \partial^{2i+3}A_i c_i \; d^2z. \tag{4.10}$$

On the other hand, the BRST charge anomaly is derived by taking the full contraction between $c_i(z)V^i(z)$ and $c_j(w)V^j(w)$. This is why the gauge fields in the path-integral anomaly (4.10) are replaced by the ghosts in the BRST-charge anomaly (4.7). In the light of this fact, the following relationship between the two forms of the anomalies is proposed:

$$\mathcal{A}_\phi = \frac{1}{2\pi}\int \Big\langle \langle J_B J_B\rangle_1^\phi, b_i A_i \Big\rangle_1, \tag{4.11}$$

This relation can be verified for the anomalies given in (4.7) and (4.10):

$$\begin{aligned}\frac{1}{2\pi}\int d^2z \Big\langle \langle J_B J_B\rangle_1^\phi, b_i A_i \Big\rangle_1 &= \sum_{i,j=0}^{\infty}\frac{C_i}{2\pi}\int \langle c_i\partial^{2i+3}c_i, b_j A_j\rangle_1 \\ &= \sum_{i=0}^{\infty}\frac{C_i}{\pi}\int d^2z c_i(z)\partial^{2i+3}A_i(z) \\ &= \mathcal{A}_\phi. \end{aligned} \tag{4.12}$$

To pinpoint the origin of the term $A_i b_i$ we write the BRST action (3.13) in the following form:

$$S = \frac{1}{\pi}\int \partial\phi\bar{\partial}\phi - A_i V^i - \delta(b_i A_i). \tag{4.13}$$

The above form of the BRST action shows that the variation of $b_i A_i$ is the BRST-trivial term in the action. The entity $b_i A_i$ is also known as the gauge fermion.

Although we have only verified this relationship for the matter-sector anomalies so far, it is easy to see that it also holds for the ghost-sector anomalies. The path-integral ghost-sector anomalies have the same form as their counterparts in the matter-sector.[3] Unless this is true, there could be no possibility of a cancellation of anomalies between the two sectors. Similarly, the ghost-sector BRST-charge anomaly has the same form as its counterpart in the matter-sector. Therefore, one can easily extrapolate the result of the matter sector to the ghost-sector anomalies. The general form of the expression relating the path-integral anomaly $\mathcal{A}$ to the BRST-algebra anomaly [59] is

$$\mathcal{A} = \frac{1}{2\pi} \int \left\langle \langle J_B J_B \rangle_1, \Psi \right\rangle_1 , \tag{4.14}$$

where Ψ is the gauge fermion.

In the next chapter, we shall discuss why the BRST charge has failed to generate the transformation rules for the gauge fields. We shall then obtain a BRST charge which generates the transformation rules of all the fields. By formulating such a BRST charge, we shall also show that the relation proposed above, relating the path-integral anomaly to the BRST-charge anomaly, holds also under a more general gauge-fixing condition.

[3] We recall that the coefficients of the ghost-sector anomalies, after a zeta-function regularization, have opposite signs to those in the matter sector. This is obviously necessary for the cancellation of the anomalies.

Chapter 5

Derivative-gauge quantization of W_∞ gravity

In the last chapter, we discussed the anomalies from the point of view of the violation of the classical BRST algebra. We showed how the loss of nilpotence for the BRST charge Q at the quantum level (evaluated by taking a fully-contracted operator product Q^2) yields the anomalies. We then observed that the anomalous results obtained in this way did not have the same form as those obtained from the partition function. We suggested and confirmed by explicit calculations that the local functional expressing the anomaly in the partition function is given by application of the anomalous operator Q^2 to the "gauge fermion Ψ" of our theory. However, in the course of these discussions we encountered the following problem: the BRST charge that we derived from the Noether theorem did not generate the transformation rules of the gauge fields. This was because the standard BRST charge contains no operators whose products with the gauge fields are non-vanishing.

In this chapter, we discuss how an inappropriate choice of gauge-fixing condition can lead to a BRST charge which consequently fails to generate the transformation rules of the gauge fields.

Given any Lagrangian $\mathcal{L}(\varphi^\alpha)$ whose action is invariant under a continuous

symmetry

$$\delta\mathcal{L} = \partial_\mu K^\mu, \tag{5.1}$$

we can write the Noether charge Q as follows [12]:

$$Q = \int (\pi_\alpha \delta\varphi^\alpha - K^0), \tag{5.2}$$

where π_α are the momenta conjugate to the fields φ^α. In the generalized conformal gauge (i.e. for $A_i = A_i^{back}$) the momenta conjugate to the gauge fields vanish (i.e. $\pi_i = \partial\mathcal{L}/\partial(\bar{\partial}A_i) = 0$). Consequently, in deriving the BRST charge one loses the term containing π_i and hence the BRST charge fails to generate the transformation rules of the gauge fields.

To resolve the above difficulty, we shall instead impose derivative gauge conditions. The resulting gauge-fixed action will be shown to be classically invariant under a set of BRST transformations. These transformations can be derived as canonical transformations from the associated BRST charge which properly generates the transformations of all the fields, including the gauge fields.

In the derivative gauge, the ghost action is second-order in time derivatives. This action is replaced by our equivalent first-order form by introducing momenta as auxiliary fields. To derive the anomalies from the first-order form of the action one needs to introduce source terms for the BRST variations. The anomalies then result from the violation of the BRST Ward identities for the effective action Γ and are evaluated by considering anomalous one-particle-irreducible diagrams. In contrast, the BRST-charge anomalies result from the non-nilpotency of the BRST charge Q at the full quantum level, as they did before.

In this new formulation of the BRST quantization procedure the expression proposed in the last chapter, (4.14), relating the two versions of the anomalies, emerges in a natural way. We shall see that the local functional expressing the anomaly in the BRST Ward identity is given by the operator

product of the anomalous local operator Q^2 with the gauge fermion of our reformulated gauge-fixed theory.

5.1 Derivative gauge-fixing conditions

In this section, we shall discuss why some gauge-fixing conditions are preferable to others. We shall consider different gauge-fixing conditions first from the point of view of the boundry conditions and then from the point of view of the constraints imposed on a theory.

We take up the well-known example of Maxwell's electrodynamics to illustrate the various points concerning the gauge-fixing conditions. The action for the Maxwell theory is

$$S = -\int \frac{1}{4} F_{\mu\nu} F^{\mu\nu}, \tag{5.3}$$

where the signature is $(-1,1,1,1)$, $F_{\mu\nu} = \partial_\mu A_\nu - \partial_\nu A_\mu$, A_μ is the gauge field and $\mu = 0, i$ with i running from 1 to 3. The action can be put into its Hamiltonian form by using the conjugate momenta π_μ,

$$\pi_\mu = \frac{\delta \mathcal{L}}{\partial(\partial_0 A^\mu)}, \tag{5.4}$$

namely $\pi_0 = 0$, $\pi_i = F_{0i} = -E_i$. The Hamiltonian form of the action for the evolution from t_0 to t_f is

$$S = \int d^3x \int dt \Big(\pi_\mu \partial_0 A_\mu - \frac{1}{4} F_{ij} F^{ij} - \frac{1}{2} \pi_i \pi^i + A_0 \partial_i \pi^i \Big). \tag{5.5}$$

Under the gauge symmetries $\delta A_\mu = \partial_\mu \lambda$ and $\delta \pi_i = 0$, the above action transforms as

$$\delta S = \int d^3x \lambda(\vec{x}, t) \partial_i \pi^i \Big|_{t_0}^{t_f}. \tag{5.6}$$

Thus the invariance of the action for the general field configuration at t_f and t_0 requires that

$$\lambda(\vec{x}, t_0) = \lambda(\vec{x}, t_f) = 0. \tag{5.7}$$

Now suppose that for our gauge-fixing condition we choose the temporal gauge $A_0 = \epsilon$, then from this gauge we can reach the zero gauge $A_0 = 0$ by a gauge transformation of the form

$$\epsilon = \dot{\lambda}. \tag{5.8}$$

This differential equation should be solved subject to the boundry conditions (5.7). However, for a first-order differential equation (5.8), the imposition of two boundry conditions on the parameter λ overdetermines the problem, thus yielding no solution [33, 34, 81]. In other words, we cannot actually reach the configuration $A_0 = \epsilon$ by a gauge transformation. By contrast, in the covariant gauge $\partial_\mu A^\mu = 0$, the differential equation for the transformation parameter λ becomes of second order

$$\ddot{\lambda} = \epsilon, \tag{5.9}$$

which permits the imposition of two boundry conditions and so allows us to find a solution to move into the gauge $A_0 = \epsilon$.

The problem of choosing an acceptable gauge can also be discussed from the point of view of the constraints of the system. Varying the Lagrange multiplier A_0 in the canonical form of the action (5.5), we obtain the following equation of motion:

$$\nabla \cdot E = \partial_i \pi_i = 0. \tag{5.10}$$

This is the Gauss law in electrodynamics. Had we set $A_0 = 0$ before varying the action, we would have lost the Gauss law as an equation of motion. We can therefore define an acceptable gauge as one which can be imposed prior to or after varying the action in order to obtain the classical equations of motion [54]. Consequently, We need to choose a different gauge such as the Lorentz gauge $\partial^\mu A_\mu = 0$ as our gauge-fixing condition.

5.2 W_∞ gravity in the derivative gauge

Once again, we shall start from the gauge-invariant W_∞ action prior to fixing the gauge (3.1), but instead of imposing the conformal gauge, we complete the gauge fixing by choosing the derivative gauge condition $\bar{\partial} A_i = 0$. The gauge-fixed action then becomes

$$\begin{aligned} S &= \frac{1}{\pi}\int d^2z\Big\{\frac{1}{2}\partial\phi\bar{\partial}\phi + \sum_{i=0}^{\infty}\Big(- A_i V^i + \pi_i\bar{\partial}A_i \\ &\quad -b_i\bar{\partial}\Big[\bar{\partial}c_i + \sum_{l=0}^{\infty}\sum_{j=0}^{i+2l} f_{2l}^{ji-j+2l}(\partial_A,\partial_c)A_j c_{i-j+2l}\Big]\Big)\Big\}. \end{aligned} \tag{5.11}$$

As a result, the ghost action is now of second order in $\bar{\partial}$ derivative. To be able to use the canonical formalism, and also the operator-product-expansion techniques, we put the ghost action in the first-order form. This is achieved by introducing auxiliary fields as canonical momenta. From the action (5.11), one can see that the fields c_i and b_i are no longer conjugate, so we need to introduce conjugate momenta[1]

$$\begin{aligned} \pi_{c_i} &= \frac{\partial^l\mathcal{L}}{\partial(\bar{\partial}c_i)} = -\bar{\partial}b_i, \\ \pi_{b_i} &= \frac{\partial^l\mathcal{L}}{\partial(\bar{\partial}b_i)} = \bar{\partial}c_i + f_{2l}^{ji-j+2l}(\partial_A,\partial_c)A_j c_{i-j+2l}. \end{aligned} \tag{5.12}$$

We can write the second-order action (5.11) in first-order form as

$$S = \frac{1}{\pi}\sum_{i=0}^{\infty}\int\frac{1}{2}\partial\phi\bar{\partial}\phi - \pi_i\bar{\partial}A_i - \pi_{b_i}\pi_{c_i} - \pi_{c_i}\bar{\partial}c_i - \pi_{b_i}\bar{\partial}b_i - A_i(V^i + V^i_{gh}), \tag{5.13}$$

where

$$V^i_{gh} = \sum_{l=0}^{\infty}\sum_{j=0}^{i-2l} F_{2l}^{ij-i+2l}(\partial_{\pi_c},\partial_c)\pi_{c_j}c_{j-i+2l}. \tag{5.14}$$

[1]Note that we are treating the $\bar{\partial}$ derivative as the "evolution" derivative in this chapter.

In the above expression, F_{2l}^{ij} are the deformed structure constants of the W_∞ algebra:

$$F_{2l}^{ij}(m,n) = \frac{4^{-2l}\phi_{2l}^{ij}}{2(2l+1)!}\sum_{k=0}^{2l+1}\mathcal{M}_{2l,k}^{ji-j+2l}(-1)^k m^{2l+1-k}n^k, \tag{5.15}$$

where ϕ_{2l}^{ij} and $\mathcal{M}_{2l,k}^{ij}$ are given in (3.6) and (3.8).

The action (5.13) is invariant under the following BRST transformations:

$$\begin{aligned}
\delta\phi &= \lambda\sum_{i=0}^{\infty}c_i(\partial\phi)^{i+1} - \frac{\sqrt{\hbar}}{2}\partial c_0 + \cdots, \\
\delta A_i &= \lambda\pi_{b_i}, \\
\delta b_i &= \lambda\pi_i, \\
\delta\pi_i &= \delta\pi_{b_i} = 0, \\
\delta c_i &= \lambda\sum_{l=0}^{\infty}\sum_{j=0}^{i+2l} -\frac{1}{2}f_{2l}^{j,i-j+2l}(\partial_{c_j},\partial_{c_{i-j+2l}})c_j c_{i-j+2l}, \\
\delta\pi_{c_i} &= \lambda(V^i + V_{gh}^i),
\end{aligned} \tag{5.16}$$

where λ is the transformation parameter. These transformations are all nilpotent at the classical level and are generated by the BRST charge Q. The BRST charge can be obtained in exactly the same way as we derived the BRST charge in the last chapter (see Section 4.1). First we ensure the invariance of the action (5.13) under a global BRST parameter λ and then we allow λ to depend on $\bar{z}$. The BRST charge is then obtained by collecting the factors multiplying $\bar{\partial}\lambda$ in the expression for the variation of the action. The resulting expression for the BRST charge is

$$Q = \int\sum_{i=0}^{\infty}\Big(c_i\,(V^i + \frac{1}{2}V_{gh}^i) + \pi_i\,\pi_{b_i}\Big). \tag{5.17}$$

The following operator-product expansions:

$$c_i(z)\pi_{c_j}(w) \sim \frac{\delta_{ij}}{z-w},$$

$$
\begin{aligned}
b_i(z)\pi_{b_j}(w) &\sim \frac{\delta_{ij}}{z-w}, \\
\pi_i(z)A_j(w) &\sim \frac{\delta_{ij}}{z-w},
\end{aligned}
\tag{5.18}
$$

which are derived from the path-integral generating functional related to the action (5.13) can be used to verify that the BRST charge Q now correctly generates the canonical transformations (5.16) of all fields, including the gauge fields A_i.

The fact that the transformations (5.16) are canonical is one of the main benefits of the reformulated BRST quantization procedure. Using the canonical nature of the transformation, we can easily show that the action (5.13) is left invariant. Rewriting the action (5.13) as

$$
S = \frac{1}{\pi}\sum_{i=0}^{\infty}\int d^2z\Big(-\frac{1}{2}\bar{\partial}\phi\,\partial\phi + \pi_i\,\bar{\partial}A_i - \pi_{b_i}\,\bar{\partial}b_i - \pi_{c_i}\,\bar{\partial}c_i - \delta(A_i\,\pi_{c_i})\Big), \tag{5.19}
$$

we see that the Hamiltonian density $\mathcal{H} = \delta(A_i\,\pi_{c_i})$ is BRST trivial, and hence is invariant as a consequence of the nilpotence of the transformations (5.16). The remaining terms are also invariant because terms of the form $\pi_\chi\,\bar{\partial}\chi$ are invariant up to a total time derivative under arbitrary canonical transformations.[2]

Having a well-defined BRST formalism available, we now proceed to calculate the anomalies first from the BRST charge (5.17) and then from the BRST Ward identities.

5.3 Anomalies in the derivative gauge

With the aid of the operator-product-expansion techniques developed in the last chapter, we can evaluate the anomalies arising from the non-nilpotence of the BRST charge at the quantum level. Since the BRST charge

[2]Note that the matter-field kinetic term $\bar{\partial}\phi\partial\phi$ is already in first-order form with respect to the "evolution" $\bar{\partial}$ derivatives; for this reason we have not introduced conjugate momenta for the scalar field ϕ. One may easily verify that this term is also invariant.

(5.17) does not contain the fields conjugate to π_i and π_{b_i}, the last term in (5.17) which was missing in the generalized conformal gauge (4.3) does not contribute to the anomalies arising from Q^2. Also, because in evaluating Q^2 we are taking full contractions of all the fields, the expression for the BRST-charge anomalies does not contain any conjugate momenta, (they are all contracted). This implies that the BRST-charge anomalies in the derivative gauge are identical to the BRST-charge anomalies in the conformal gauge (4.3). Let us recall the form of the BRST-charge anomalies:

$$\begin{aligned} Q^2 &= \oint \frac{dz}{2\pi i} \langle J_B J_B \rangle_1(z); \\ \langle J_B J_B \rangle_1 &= \sum_{i=0}^{\infty} \Big(C_i c_i(z) \partial^{2i+3} c_i(z) + \alpha_i c_i(z) \partial^{2i+3} c_i(z) \Big), \end{aligned} \tag{5.20}$$

where the first term is the matter-sector and the second term is the ghost-sector anomaly. The coefficient of the ghost-sector anomaly α is equal to $-C_i$ (after zeta-function regularization). However, for the purpose of relating the path-integral and the BRST-charge anomalies, and also for the testing of the consistency of the anomalies (i.e. for seeing whether the anomalies satisfy the Wess-Zumino consistency conditions) we shall consider the ghost and matter sector anomalies separately prior to their cancelling against each other.

Next, we shall obtain an expression for the anomalous BRST Ward identities using a derivative gauge-fixing condition. The symmetries of the effective action are manifested in the Ward identities. The non-validity of the Ward identities under the symmetries of the classical action indicates the presence of anomalies. This method of anomaly derivation is basically of the same nature as the Fujikawa method. However, whereas in the Fujikawa scheme one derives the full expression for the anomalies from the measure of the partition function, in the Ward identity derivation of the anomalies one undertakes a diagram-by-diagram study of the one-particle irreducible Feynman diagrams is essential. Whether a theory is anomalous or not is independent of how

the anomalies are calculated. Also, because in the Fujikawa method and the Ward-identity methods of deriving anomalies the partition function is taken as the basic quantity, the results one obtains should be identical irrespective of the method used.

Before deriving the anomalous W_∞ Ward identities, we shall give a general review of the Ward-identity treatment of the anomalies [4, 41, 80]. In order to calculate correlation functions and derive the BRST form of the Ward identity, one has to introduce sources into the BRST-invariant action. The BRST Lagrangian $\mathcal{L}$ is extended to include two kinds of sources which are J_{χ^i} and K_{χ^i} for the fields χ^i and their variations $\delta\chi^i$ respectively [5]. The extended Lagrangian is then written as:

$$\mathcal{L}_{ext} = \mathcal{L} + J_{\chi^i}\chi^i + K_{\chi^i}\delta\chi^i. \tag{5.21}$$

The partition function is consequently given by [42]

$$\mathcal{Z}[J_{\chi^i}, K_{\chi^i}] = \int \mathcal{D}\chi e^{-\int \mathcal{L}_{ext}}. \tag{5.22}$$

The generating function $\mathcal{W}$ and the effective action functional Γ are defined respectively as

$$\mathcal{W}_{(J_{\chi^i}, K_{\chi^i})} = \ln\mathcal{Z}(J_{\chi^i}, K_{\chi^i}), \tag{5.23}$$

and

$$\Gamma(\chi^i, K_{\chi^i}) = \mathcal{W}(J^i_\chi, K^i_\chi) - \int d^2z J_{\chi^i}\chi^i. \tag{5.24}$$

The anomalies then originate from the non-invariance of the partition function under the BRST symmetries of the action (see Section 2.2). Under these symmetries, the partition function transforms as follows:

$$\begin{aligned} \mathcal{Z} \longrightarrow \mathcal{Z}' &= \int \mathcal{D}\chi \exp\left(\int \mathcal{L} + J_{\chi^i}\chi^i + K_{\chi^i}\delta\chi^i + \mathcal{A} + J_{\chi^i}\delta\chi^i\right) \\ &= \mathcal{Z} + \frac{\delta\Gamma}{\delta\chi^i}\frac{\delta\Gamma}{\delta K_{\chi^i}}\mathcal{Z} + \mathcal{A}\mathcal{Z}, \end{aligned} \tag{5.25}$$

where $\mathcal{A}$ is the anomaly. The invariance of the partition function is only achieved if

$$\frac{\delta\Gamma}{\delta\chi^i}\frac{\delta\Gamma}{\delta K_{\chi^i}} + \mathcal{A} = 0, \tag{5.26}$$

The above is the well known Ward-identity expression for the anomalies. Expanding the effective action and the anomaly loopwise in powers of $\hbar$, we have

$$\begin{aligned} \Gamma &= \Gamma_0 + \hbar\Gamma_1 + \hbar^2\Gamma_2 + \cdots, \\ \mathcal{A} &= \hbar\mathcal{A}_1 + \hbar^2\mathcal{A}_2 + \cdots. \end{aligned} \tag{5.27}$$

Returning to our specific case of chiral W_∞ gravity we may write the extended Lagrangian density $\mathcal{L}_{ext}$ as

$$\begin{aligned} \mathcal{L}_{ext} &= \sum_{i,l=0}^{\infty}\sum_{j=0}^{i+2l}\Big[\frac{1}{2}\partial\phi\bar{\partial}\phi + \pi_i\bar{\partial}A_i - \pi_{b_i}\pi_{c_i} - A_i(V^i + V^i_{gh}) \\ &\quad + J_\phi\phi + J_{A_i}A_i + J_{b_i}b_i + J_{\pi_{b_i}}\pi_{b_i} \\ &\quad + K_\phi(c_i(\partial\phi)^{i+1} - \sqrt{\hbar}\partial c_0 + \cdots) + K_{A_i}\pi_{b_i} + K_{b_i}\pi_{A_i} \\ &\quad + K_{c_i}(-\frac{1}{2}f^{ji-j+2l}_{2l}(\partial_{c_j}, \partial_{c_{i-j+2l}})c_j c_{i-j+2l}) \\ &\quad + K_{\pi_{c_i}}(V^i + V^i_{gh})\Big]. \end{aligned} \tag{5.28}$$

Consider the matter-sector effective action at the $i+1$-loop order:

$$\begin{aligned} \Gamma_{i+1} &= \frac{1}{2}\int d^2z\int d^2w\Big\langle (A_i - K_{\pi_{c_i}})(z)V^i(z), (A_j - K_{\pi_{c_j}})(w)V^j(w)\Big\rangle \\ &= \frac{C_i}{2}\int d^2z(A_i(z) - K_{\pi_{c_i}}(z))\frac{\partial^{2i+3}}{\bar{\partial}}(A_i(z) - K_{\pi_{c_i}}(z)). \end{aligned} \tag{5.29}$$

The anomaly at the order $i+1$ is given by

$$\mathcal{A}_{i+1} = \frac{\delta\Gamma_{i+1}}{\delta K_{\pi_{c_i}}}\frac{\delta\Gamma_0}{\delta\pi_{c_i}} + \frac{\delta\Gamma_{i+1}}{\delta A_i}\frac{\delta\Gamma_0}{\delta K_{A_i}} = C_i\int c_i\partial^{2i+3}(A_i - K_{\pi_{c_i}}). \tag{5.30}$$

In order to check the accuracy of the above expression for the anomaly, we need to verify that it satisfies the Wess-Zumino consistency condition. For

checking the BRST Ward identities the antibracket form of the Wess-Zumino consistency condition is more appropriate then the commutator form of this condition (see Section 3.4). This is given by

$$(\Gamma, (\Gamma, \Gamma)) = 0. \tag{5.31}$$

The above form of the consistency condition is a consequence of the Jacobi identity for the antibracket (.,.), which for arbitrary functionals A and B is defined as [40]

$$(A, B) = \frac{\delta A}{\delta \chi}\frac{\delta B}{\delta K_\chi} + \frac{\delta A}{\delta K_\chi}\frac{\delta B}{\delta \chi}. \tag{5.32}$$

The invariance of the extended classical action under the BRST symmetries can be expressed in the following antibracket form:

$$(\Gamma_0, \Gamma_0) = 0. \tag{5.33}$$

The antibracket form of the Wess-Zumino consistency condition can be tested for a given order of $\hbar$ only if one has already verified it at all the lower orders. This means that the consistency of the expression for the $i+1$-loop anomaly (5.30) cannot be tested without considering the various lower-order combinations that contribute to this anomaly.

The lowest-spin anomaly in the W_∞ gravity whose consistency we shall test is the Virasoro anomaly. The effective action at the two-loop level is given by

$$\Gamma_1 = -\frac{1}{24}\int (A_0 - K_{\pi_{c_0}})\frac{\partial^3}{\bar{\partial}}(A_0 - K_{\pi_{c_0}}). \tag{5.34}$$

The anomaly $\mathcal{A}_1$ can be obtained as follows:

$$\begin{aligned}\mathcal{A}_1 &= \mathcal{A}_{c_0 A_0} = (S_0, \Gamma_1) = \frac{\delta S_0}{\delta K_0}\frac{\delta \Gamma_1}{\delta A_0} + \frac{\delta S_0}{\delta \pi_{c_0}}\frac{\delta \Gamma_1}{\delta K_{\pi_{c_0}}} + \frac{\delta S_{1/2}}{\delta \phi}\frac{\delta S_{1/2}}{\delta K_\phi} \\ &= -\frac{1}{12\pi}\int c_0 \partial^3 (A_0 - K_{\pi_0}).\end{aligned} \tag{5.35}$$

The Wess-Zumino consistency condition for the order-$\hbar$ anomaly $\mathcal{A}_\hbar$ is obtained by taking an antibracket with S_0

$$(S_0, \mathcal{A}_1) = (S_0, (S_0, \Gamma_1)) + (S_0, (S_{1/2}, S_{1/2})) \tag{5.36}$$

and then using the Jacobi identity for the antibracket together with the relations

$$\begin{aligned} (S_0, S_0) &= 0, \\ (S_0, S_{1/2}) &= 0. \end{aligned} \tag{5.37}$$

Thus, we obtain the Wess-Zumino consistency condition at order of $\hbar$

$$(S_0, \mathcal{A}_1) = 0. \tag{5.38}$$

The anomaly $\mathcal{A}_1$ given in (5.35) does indeed satisfy the above Wess-Zumino consistency condition. This is shown explicitly below:

$$(S_0, \mathcal{A}_1) = -\frac{1}{6}\int(\pi_{b_0} - \bar{\partial}c_0 - c_0\partial A_0 + \partial c_0 A_0 + c_0\partial K_{\pi_{c_0}} - \partial c_0 K_{\pi_{c_0}}) = 0. \tag{5.39}$$

Next we shall consider the consistency of two-loop anomalies. We recall that at order $\hbar^2$, the off-diagonal anomalies contributed to the ghost-sector anomalies (see Section 3.3). The two-loop off-diagonal anomaly is of the form $c_0\partial^5 A_2 + c_2\partial^5 A_0$ (3.36). In Chapter 3 we used the commutator form of the Wess-Zumino consistency condition to show that the off-diagonal anomalies are not allowed and hence a zeta-function regularization scheme had to be adopted that rendered the coefficient of the off-diagonal anomalies equal to zero. In this section we shall use the antibracket form of the consistency condition to show that the diagonal part of the two-loop anomaly is consistent whereas the off-diagonal part of the anomaly violates the Wess-Zumino consistency condition. The effective action at the two-loop level is given by

$$\begin{aligned} \Gamma_2 \;=\; & -\frac{1}{240\pi}\int(A_1 - K_{\pi_{c_1}})\frac{\partial^5}{\bar{\partial}}(A_1 - K_{\pi_{c_1}}) \\ & +\alpha\int(A_0 - K_{\pi_{c_0}})\frac{\partial^5}{\bar{\partial}}(A_2 - K_{\pi_{c_2}}), \end{aligned} \tag{5.40}$$

where α is the coefficient of the off-diagonal anomalies.[3] The anomaly corresponding to the effective action (5.40) is given by

$$\mathcal{A}_2 = \mathcal{A}_{c_1 A_1} + \mathcal{A}_{c_0 A_2} + \mathcal{A}_{c_2 A_0}, \tag{5.41}$$

where

$$\begin{aligned} \mathcal{A}_{c_1 A_1} &= \frac{\delta S_0}{\pi_{c_1}} \frac{\delta \Gamma_2}{\delta K_{\pi_{c_1}}} + \frac{\delta S_0}{\delta K_{A_1}} \frac{\delta \Gamma_2}{\delta A_1} \\ &= \frac{1}{120\pi} \int c_1 \partial^5 (A_1 - K_{\pi_{c_1}}), \end{aligned} \tag{5.42}$$

$$\begin{aligned} \mathcal{A}_{c_0 A_2} + \mathcal{A}_{c_2 A_0} &= \frac{\delta S_0}{\pi_{c_0}} \frac{\delta \Gamma_2}{\delta K_{\pi_{c_0}}} + \frac{\delta S_0}{\delta K_{A_0}} \frac{\delta \Gamma_2}{\delta A_0} + \frac{\delta S_0}{\pi_{c_2}} \frac{\delta \Gamma_2}{\delta K_{\pi_{c_2}}} + \frac{\delta S_0}{\delta K_{A_2}} \frac{\delta \Gamma_2}{\delta A_2} \\ &= \alpha \int c_0 \partial^5 (A_2 - K_{\pi_{c_2}}) + c_2 \partial^5 (A_0 - K_{\pi_{c_0}}). \end{aligned} \tag{5.43}$$

The Wess-Zumino consistency condition at the order $\hbar^2$ is obtained in a similar manner to the order-$\hbar$ Wess-Zumino consistency condition (5.38). At this order the Wess-Zumino consistency condition also receives contributions from the one-loop anomaly and is given by

$$(S_0, \mathcal{A}_2) + (S_1, \mathcal{A}_1) = 0, \tag{5.44}$$

where S_1 is the order-$\hbar$ term in the action (5.13) and $\mathcal{A}_1$ is given by (5.35). Substituting for the anomalies from (5.35) in the above expression (5.44), we obtain the following Wess-Zumino consistency condition for the order-$\hbar^2$ W_∞ gravity anomalies:

$$(S_0, \mathcal{A}_{c_1 A_1}) + (S_1, \mathcal{A}_{c_0 A_0}) + (S_0, \mathcal{A}_{c_0 A_2}) + (S_0, \mathcal{A}_{c_2 A_0}) = 0. \tag{5.45}$$

[3]The explicit value of α, as mentioned in Chapters 2 and 3, depends on the method of summation that one adopts in writing the infinite coefficient of the ghost-sector anomalies. The explicit value of α is immaterial since we shall show that irrespective of the value α, the off-diagonal anomalies violate the Wess-Zumino consistency condition.

However, explicit calculations show that the above equation is not satisfied unless the off-diagonal anomaly $\mathcal{A}_{c_0A_2}+\mathcal{A}_{c_2A_0}$ is zero. The expression for the first two brackets are given below:

$$
\begin{aligned}
& (S_0,\mathcal{A}_{A_1c_1})+(S_1,\mathcal{A}_{c_0A_0}) = \\
- \;& \Big\{\Big((\pi_{b_1}-f_0^{j,1-j}(\partial_A,\partial_c)A_jc_{1-j}-\bar{\partial}c_1+f_0^{j,1-j}(\partial_k,\partial_c)K_{\pi_{c_j}}c_{1-j}\Big)\Big(\frac{1}{120\pi}\partial^5c_1\Big) \\
- \;& \frac{1}{120\pi}\Big(-\frac{1}{2}f_0^{j,1-j}(\partial_{c_j},\partial_{c_{1-j}})c_jc_{1-j}\Big)\Big(\partial^5A_1-\partial^5K_{\pi_{c_1}}\Big)-\Big(\frac{1}{120\pi}\pi_{b_1}\partial^5c_1\Big)\Big\} \\
- \;& \Big\{\Big(\pi_{b_0}-f_2^{j2-j}(\partial_A,\partial_c)A_jc_{2-j}-\bar{\partial}c_0+f_2^{j2-j}(\partial_k,\partial_cK_{\pi_{c_j}}c_{2-j}\Big)\Big(\frac{1}{6\pi}\partial^3c_0\Big) \\
+ \;& \frac{1}{6\pi}\Big(\frac{1}{2}f_2^{j2-j}(\partial_{c_j},\partial_{c_{2-j}})c_jc_{2-j}\Big)\Big(\partial^3A_0-\partial^3K_{\pi_{c_0}}\Big)+\Big(\frac{1}{6\pi}\pi_{b_0}\partial^3c_0\Big)\Big\}. \qquad (5.46)
\end{aligned}
$$

One can then show that the corresponding terms in the above first two brackets cancel. That is to say, the contribution to the Wess-Zumino consistency condition at order $\hbar^2$ from the one-loop anomaly $\mathcal{A}_{c_0A_0}$ cancels the contributions from the two-loop anomaly $\mathcal{A}_{c_1A_1}$. Similarly the term in the Wess-Zumino consistency condition arising from the off-diagonal anomalies is:

$$
\begin{aligned}
(S_0,\mathcal{A}_{02}) \;=\; & \alpha\int\Big\{\Big(-\frac{1}{2}f_0^{00}(\partial_c,\partial_c)c_0c_0\partial^5(K_{\pi_{c_2}}-A_2)\Big) \\
+ \;& \Big(-f_0^{j2-j}(\partial_A,\partial_c)A_jc_{2-j}+f_0^{j2-j}(\partial_K,\partial_c)K_{\pi_{c_j}}K_{\pi_{c_j}}c_{2-j}\Big)\partial^5c_0 \\
+ \;& \Big(-\frac{1}{2}f_0^{j2-j}(\partial_c,\partial_c)c_jc_{2-j}\partial^5(K_{\pi_{c_0}}-A_0)\Big) \\
+ \;& \Big(-f_0^{00}(\partial_A,\partial_c)A_0c_0+f_0^{00}(\partial_K,\partial_c)K_{\pi_{c_0}}K_{\pi_{c_0}}c_0\Big)\partial^5c_2\Big\}. \qquad (5.47)
\end{aligned}
$$

The above term does not vanish and hence the off-diagonal anomalies are not allowed by the Wess-Zumino consistency condition. Note that the diagonal anomalies satisfy the Wess-Zumino consistency condition because there are contributions from the one-loop anomaly to the Wess-Zumino consistency condition at order $\hbar^2$. The coefficient of the one and the two-loop diagonal anomalies are related (before using a zeta-function regularization scheme) in

such a way that the two together make a vanishing contribution to the Wess-Zumino consistency condition. However, the two-loop off-diagonal anomaly violates the consistency condition and unless the off-diagonal anomaly vanishes we do not have a consistent anomaly at order $\hbar^2$. The only way in which the off-diagonal two-loop anomaly could vanish is for its coefficient α to vanish. This can only be achieved if a proper zeta-function regularization is adopted. A zeta-function regularization scheme must therefore be selected that consistently regularizes the infinite coefficients of the diagonal anomalies and gives a vanishing coefficient for the off-diagonal anomalies. An example of such a scheme is discussed in Appendix B.

5.4 Relationship between the path-integral and the BRST-charge anomalies

In the preceding section, we derived an expression for anomalous Ward identities (5.30) and also an expression for the BRST-charge anomalies (5.20). In the preceding chapter we proposed an expression, (4.14), relating these two forms of the anomalies. Having derived the two forms of the anomalies in the derivative gauge, we now check the validity of the relationship (4.14) for these anomalies.

The Ward identity W_∞ anomalies $\mathcal{A}$ [the full expression for the Ward identity W_∞ anomaly is given by summing over the index i in (5.30)] in the derivative gauge can also be obtained by application of the operator Q^2 (5.20) to the gauge fermion Ψ. The gauge fermion in the derivative gauge is given by the BRST trivial terms in the action (5.28). This term can be easily found by rewriting (5.28) in the same form as (5.19). Thus the Ward-identity anomaly can be obtained as follows:

$$\frac{1}{2\pi}\int\Big\langle\langle J_B J_B\rangle_1,\Psi\Big\rangle_1 \;=\; \sum_{i,j=0}^{\infty}\int\frac{C_i}{2\pi}\Big\langle(c_i\partial^{2i+3}c_i),(\pi_{c_j}(A_j-K_{\pi_{c_j}}))\Big\rangle_1$$

$$
\begin{aligned}
&= \sum_{i=0}^{\infty} \int \frac{C_i}{\pi} c_i \partial^{2i+3} (A_i - K_{\pi_{c_i}}) \\
&= \mathcal{A}. \qquad (5.48)
\end{aligned}
$$

The fact that the relationship (4.14) is valid for both the conformal gauge and the derivative gauge in the W_∞ gravity, and has also been proved to be true for other theories, strongly indicates that this relationship might be independent of how one fixes the gauge.

Chapter 6

Conclusion

In this thesis we have studied the anomalies of the theories of w_∞ and W_∞ gravity in the context of path integral and BRST quantization. These theories contain anomalous diagrams at all loop orders. This special feature makes them particularly valuable for the study of anomalies. The main results of the present study are as follows.

The Fujikawa method of anomaly derivation

The Fujikawa method provides a natural method of anomaly derivation and is an alternative to the commonly-used Feynman diagram technique. In this scheme, the full expression for the one-loop anomalies can be derived from the measure of the path-integral without any need for a diagram-by-diagram search for the anomalies. In the Fujikawa scheme, different expressions for the anomalies are obtained upon choosing different regularizing operators. However, an appropriate choice of the regularizing operator can lead to anomalies that satisfy the Wess-Zumino consistency condition automatically. These operators arise in the Pauli-Villars regularization scheme and can be used within the framework of the Fujikawa regularization method to yield consistent anomalies. By using the Pauli-Villars regulators we have

derived all the one-loop matter-sector anomalies as well as the ghost-sector anomalies to all orders in $\hbar$.

Higher-loop anomalies

The Wess-Zumino consistency condition has played a central role in this thesis. Apart from restricting our choice of the Fujikawa regulators, the Wess-Zumino consistency condition allows us to obtain an expression for the higher-loop anomalies. The implication of this condition for W_∞ and w_∞ gravity anomalies is that they will be given by a polynomial whose first term is the Virasoro anomaly. Therefore, the Virasoro anomaly on its own does not satisfy the consistency condition in such theories and should be extended to include higher-loop anomalies. Here, the higher-order anomalies have been derived by requiring that the Virasoro anomaly satisfies the commutator form of the Wess-Zumino consistency condition.

Absence of off-diagonal anomalies

In the ghost sector, the Wess-Zumino consistency condition also plays a crucial role. The coefficients of the ghost-sector anomalies appear in the form of divergent sums. These sums can be regularized using a zeta-function regularization scheme. Ambiguities arise in the regularization procedure because the ghost-sector divergent sums of coefficients can be written in different ways, each of which yields a different result upon regularization. We have removed this ambiguity by using the Wess-Zumino consistency condition. This condition places a stringent requirement on the permissible methods of summation. More explicitly, it has been shown that of the two frequently used methods of summation, the methods of refs. [89] and [71], the former fails to comply with the Wess-Zumino consistency condition while the latter provides a vanishing coefficient for the off-diagonal anomalies and self-consistent

coefficients for all the diagonal ones.

Anomalies from the BRST charge

The BRST charge is classically nilpotent. However, in going from the classical to a quantum theory, the classical BRST algebra can be violated. The loss of nilpotence of the BRST charge at the quantum level will then lead to anomalies. A fully-contracted operator product of two BRST charges yields a local but non-vanishing anomalous result. This expression has a different form from that given by the anomalous Ward identities. Here, the path-integral anomalies correspond to Feynman diagrams with external gauge-field lines and hence the anomalous Ward identity is given by an expression containing gauge fields. On the other hand, the BRST charge, formed from the ghost and matter currents, contains no gauge fields. Consequently, the anomalies derived by taking the operator product of two BRST charges never include gauge fields.

Relationship between the path integral and the BRST anomalies

We have shown that the two forms of the anomalies can be related. The Ward-identity anomaly can be obtained from the BRST-charge anomaly via a construction involving the gauge fermion. The gauge fermion is extracted from the BRST-trivial terms in the action and has the gauge field and the field conjugate to the ghost as its basic constituents. We have seen that the anomaly in the Ward identity is derived by application of the anomalous operator given by the square of the BRST charge—calculated using operator products—to the gauge fermion. Although, there is still no proof for the general validity of this relationship, it has been verified in various theories [59]. Our results also confirms the validity of this relationship for the theory of W_∞ gravity.

The choice of gauge-fixing condition

The choice of gauge-fixing condition is of importance for all gauge theories. An acceptable gauge-fixing condition can be defined as one which can be imposed prior to or after varying the action in order to obtain the classical equations of motion. Part of the information contained in a theory can be lost if one imposes an inappropriate gauge-fixing condition on that theory as this may amount to a truncation of the theory's dynamical content. In the conformal gauge, the momenta conjugate to the gauge fields vanish. Consequently, the BRST charge does not contain these momenta and fails to generate the transformation rule of the gauge fields. To resolve this problem, we have replaced the conformal gauge-fixing condition by a derivative gauge-fixing condition. In the derivative gauge the BRST charge generates the transformation rules of all fields including the gauge fields. In this new gauge, we have also derived the Ward-identity anomalies and the BRST-charge anomalies. In this gauge, too, the Ward identity anomaly can be obtained by application of the anomalous operator given by square of the BRST charge—calculated using operator products— to the gauge fermion.

Appendix A

Pauli-Villars regularization

In this appendix, we shall discuss the Pauli-Villars scheme for the regularization of one-loop diagrams. We shall also consider how this method can be used to find consistent regulators in the Fujikawa scheme.

The Pauli-Villars method consists of introducing a field χ, with a large mass M for each field ϕ that gives divergent loops. The added Pauli-Villars fields give the same contributions to a loop as the original fields ϕ, but with a minus sign. For finite mass M, all loops are thus regularized and the final result may be obtained by taking the limit $M \to \infty$. Furthermore, in the Pauli-Villars regularization scheme the full measure including the original fields ϕ^i and the Pauli-Villars fields χ^i is invariant as the pair of contributions cancel identically [13, 14, 36, 39, 85].

In a path-integral framework, propagating field ϕ is supplemented with a Pauli-Villars field χ. The coupling of χ to other fields is exactly the same as the coupling of ϕ to other fields. There is also a Pauli-Villar mass term which is the only source of the anomaly in the Pauli-Villars scheme. To demonstrate this point we shall take a simple action as an example. Consider the action

$$S = \int \frac{1}{2}\phi^i R_{ij}\phi^j, \tag{A.1}$$

where R_{ij} is the differential operator depending on the fields coupling to the ϕ-loop. The Pauli-Villars action corresponding to (A.1) is:

$$S^{pv} = \int \mathcal{L}_p + \mathcal{L}_M, \tag{A.2}$$

where $\mathcal{L}_M$, the Pauli-Villars mass term, and $\mathcal{L}_p$ are:

$$\mathcal{L}_M = \frac{1}{2} M \chi^i \chi^i, \tag{A.3}$$

$$\mathcal{L}_p = \frac{1}{2} \chi^i R_{ij} \chi^j. \tag{A.4}$$

Suppose the total action, (A.1) and (A.2), excluding the Pauli-Villars mass term is invariant under the following symmetry transformations:

$$\delta\phi^i = F^i_j \phi^j, \tag{A.5}$$

$$\delta\chi^i = F^i_j \chi^j, \tag{A.6}$$

where F is the operator in the space of all fields (i.e. ϕ and the Pauli-Villars fields χ).

The corresponding partition function is

$$\mathcal{Z} = \int \mathcal{D}\phi \mathcal{D}\chi \exp -(S + S^{PV}). \tag{A.7}$$

Under the symmetries (A.5) and (A.6), the measure of the path-integral is invariant since the contribution to the Jacobian from the Pauli-Villars fields cancel those from their partners ϕ. Therefore, the anomaly in this scheme comes entirely from the mass term. Under the symmetry (A.6), the Pauli-Villars mass term transforms as follows:

$$\delta\mathcal{L}_M = M \chi^i F^i_j \chi^j. \tag{A.8}$$

With the aid of the two-point function from (A.7), the anomaly $\mathcal{A}^{pv}$ can be written as

$$\mathcal{A}^{pv} = Tr\Big[\frac{M F^i_j}{M + R_{ij}}\Big]$$

$$
\begin{aligned}
&= \mathrm{Tr}F\Big(\frac{1}{1+R/M}\Big) \\
&\sim \mathrm{Tr}Fe^{-R/M}, \qquad \text{(A.9)}
\end{aligned}
$$

where the trace operation includes integration over spacetime, and we have dropped the indices for simplicity.

A.1 Consistency of the Pauli-Villars anomaly

The anomalies arising in the Pauli-Villars scheme necessarily satisfy the consistency conditions. This is the reason why it is appropriate to use the regulators arising in the Pauli-Villars scheme in the Fujikawa regularization. In this section, we shall consider the simple example given in the preceding section to demonstrate why the Pauli-Villars regularization leads to consistent anomalies.

A consistent one-loop anomaly $\mathcal{A}$, by definition, satisfies the Wess-Zumino consistency condition,

$$(S_0, \mathcal{A}) = 0 \qquad \text{(A.10)}$$

(see Chapter 5). We shall show that the Pauli-Villars anomaly (A.9) can be written as a BRST variation itself and that, as a consequence of the nilpotence of the BRST variations, the Wess-Zumino consistency condition (A.10) is satisfied. We start from the master equation

$$(S, S) = 0. \qquad \text{(A.11)}$$

Differentiating this equation twice, we arrive at

$$\frac{\delta}{\delta\phi^i}\frac{\delta}{\delta\phi^j}(S, S) = 2RF + \delta R = 0, \qquad \text{(A.12)}$$

which, in turn, leads to

$$F = -\frac{1}{2}R^{-1}\delta R. \qquad \text{(A.13)}$$

Substitution of (A.13) in (A.9) results in

$$\begin{aligned}
\mathcal{A}^{pv} &= \mathrm{Tr}\Big[\frac{1}{2}R^{-1}\delta R\Big(\frac{1}{1-R/M}\Big)\Big] \\
&= \mathrm{Tr}\Big[-\frac{1}{2}(R^{-1}\delta R)\Big(1+\sum_{n=1}(\frac{R}{M})^n\Big)\Big] \\
&= -\frac{1}{2}\mathrm{Tr}\Big[R^{-1}\delta R+\frac{(R)^{n-1}}{M^n}\delta R\Big] \\
&= \delta\Big\{-\frac{1}{2}\mathrm{Tr}\Big[\ln R+\sum\frac{1}{n}(\frac{R}{M})^n\Big]\Big\}. \qquad (A.14)
\end{aligned}$$

Because the Pauli-Villars anomaly can be written as a δ variation, its BRST variation vanishes. We can therefore conclude that the Pauli-Villars regularization scheme satisfies the Wess-Zumino consistency condition at the one-loop level.

Although we have demonstrated this point here by only considering a simple example, the general proof of the consistency of the Pauli-Villars regularization follows from a similar argument and involves the same type of calculation [36, 85].

A.2 Relating Pauli-Villars regularization to Fujikawa regularization

In the Fujikawa, scheme anomalies arise when the measure of the path integral is not invariant under the symmetries of the action. For the theory represented by the action (A.1), the anomaly obtained from regularizing the Jacobian of the transformation (A.5) is

$$\mathcal{A}^f = \lim_{M\to\infty}\mathrm{Tr}Fe^{-H/M^2}. \qquad (A.15)$$

By comparing the Fujikawa anomaly (A.15) with the Pauli-Villars anomaly (A.9), we can see that the operator H is R, i.e. is the operator in the kinetic part of the action.

It is now clear how to construct a regulator[1] for the Fujikawa method from the Pauli-Villars field Lagrangian. The regulator to be used is a function of O/M where O is the operator in the kinetic part of the action, and M is the mass of the Pauli-Villars field. Note that if this operator is not already quadratic and Hermitian, then one frequently—but not always—takes the function $\exp(O/M)^2$ as the regulator.

To summarize, in the Pauli-Villars regularization scheme the anomalies do not arise from the measure. The Pauli-Villars fields χ^i are transformed in such a way as to cancel the Jacobian arising from the measure of the original fields ϕ^i. In this scheme the only source of an anomaly is the Pauli-Villars mass term. In the Fujikawa scheme, no Pauli-Villars fields appear and the only source of the anomaly is the measure. This contribution can be computed in such a way as to obtain the same value for the anomaly as that obtained from the Pauli-Villars mass term in the Pauli-Villars scheme. In this way one ensures that the Fujikawa anomalies satisfy the Wess-Zumino consistency condition.

In this section, we have chosen a simple example to illustrate the key points concerning the Pauli-Villars scheme and its relationship with the Fujikawa method. The generalization of the above ideas to more complicated BRST actions is also simple and proceeds along the same lines as those discussed above. In the next section we shall consider a more complicated theory by taking up the example of the w_∞ gravity.

A.3 Choosing the regulator for the w_∞-gravity anomaly

In Chapter 2, we saw that the anomalies in w_∞ gravity can be traced back to the measure of the path integral. The ill-defined Jacobian arising

[1] In this thesis, we have used the word regulator to denote e^{-H/M^2} or any other function of H/M^2.

from the measure can lead to consistent anomalies (i.e. those satisfying the Wess-Zumino consistency conditions) only if it is regularized in an appropriate manner. In this appendix, we derive and discuss the regulators that would lead to consistent w_∞ anomalies. These regulators, as we have already mentioned, arise in the context of the Pauli-Villars scheme.

The BRST action for w_∞ is as follows (see Chapter 2):

$$\begin{aligned} S &= \frac{1}{\pi}\int \frac{1}{2}\partial\phi\bar{\partial}\phi - A_i\frac{(\partial\phi)^{i+2}}{i+2} + \sum_{i=0}^{\infty}\Big(\pi_i(A_i - A_i^{back}) - b_i\bar{\partial}c_i \\ &\quad + \sum_{j=0}^{i}\Big[(j+1)b_i\partial c_{i-j}A_j - (i-j+1)b_i c_{i-j}\partial A_j\Big]\Big). \end{aligned} \tag{A.16}$$

To exhibit the operators in the quadratic part of the action, we write (A.16) in matrix form:

$$\begin{aligned} S &= \frac{1}{\pi}\sum_{l=0}^{\infty}\sum_{j=0}^{l}\int\Big[\phi(-\partial\bar{\partial} + \partial B_l\partial + B_l\partial^2)\phi + \pi_l(A_l - A_l^{back}) \\ &\quad + (c_{l-j} \quad b_l)\begin{pmatrix} 0 & D^\dagger \\ D & 0 \end{pmatrix}\begin{pmatrix} c_{l-j} \\ b_l \end{pmatrix}\Big], \end{aligned} \tag{A.17}$$

where B_l is $(l+1)A_l(\partial\phi)^l$ and $D = -\delta_{j0}\bar{\partial} + (j+1)A_j\partial - (l-j+1)\partial A_j$.[2] The action (A.16) is invariant under the following transformations:

$$\begin{aligned} \delta\phi &= \sum_{l=0}^{\infty} c_l(\partial\phi)^{l+1}, \\ \delta A_i &= \bar{\partial}c_i - \sum_{j=0}^{i}\Big((j+1)\partial c_{i-j}A_j + (i-j+1)c_{i-j}\partial A_j\Big), \\ \delta c_i &= \sum_{l=0}^{i}(l+1)c_l\partial c_{i-l}, \end{aligned}$$

[2]Hermitian conjugation and the inner product are defined by $(D\psi,\varphi) = (\psi, D^\dagger\varphi)$ and $(\psi,\varphi) = \int\bar{\psi}\varphi$, respectively.

$$
\begin{aligned}
\delta b_i &= \pi_i, \\
\delta \pi_i &= 0.
\end{aligned} \tag{A.18}
$$

The Pauli-Villars action can be written by introducing the Pauli-Villars scalar field χ and the ghosts c_i^{pv} and the antighosts b_i^{pv}. The Pauli-Villars action then assumes the form[3]

$$
\begin{aligned}
S_{pv} &= \lim_{M\to\infty} \frac{1}{\pi} \sum_{l=0}^{\infty} \int \chi(-\partial\bar{\partial} + \partial B_l \partial + B_l \partial^2)\chi \\
&\quad + (c_l^{pv} \quad b_l^{pv}) \begin{pmatrix} 0 & D^\dagger \\ D & 0 \end{pmatrix} \begin{pmatrix} c_l^{pv} \\ b_l^{pv} \end{pmatrix} \\
&\quad + (\chi \quad c_l^{pv} \quad b_l^{pv}) \begin{pmatrix} M & 0 & 0 \\ 0 & M_1 & 0 \\ 0 & 0 & M_2 \end{pmatrix} \begin{pmatrix} \chi \\ c_l^{pv} \\ b_l^{pv} \end{pmatrix}.
\end{aligned} \tag{A.19}
$$

The Pauli-Villars fields transform in such a way as to keep the measure of the path integral and the kinetic part of the action invariant. In this way the anomaly is shifted to the Pauli-Villars mass term. The Pauli-Villars fields thus transform as follows:[4]

$$
\begin{aligned}
\delta\chi &= \sum_{l=0}^{\infty} (l+1) c_l (\partial\phi)^l \partial\chi, \\
\delta b_i^{pv} &= \bar{\partial} c_i - \sum_{j=0}^{i} \Big((j+1)\partial c_{i-j} b_j^{pv} - (i-j+1) c_{i-j} \partial b_j^{pv} \Big), \\
\delta c_i^{pv} &= \sum_{l=0}^{i} (l+1) \Big(c_l^{pv} \partial c_{i-l} + c_l \partial c_{i-l}^{pv} \Big).
\end{aligned} \tag{A.20}
$$

In order to find the Fujikawa regulating operator, we shall use the expression (A.19). The operator in the scalar part of the action is already quadratic

[3]Note that all the scalar fields ϕ that appear in the Pauli-Villars action, and later on in the Pauli-Villars transformations, should be treated as constant fields. This can be achieved by expanding the original action about the mean field to obtain a Gaussian action and then reverting all the fields back to ϕ at the end of the calculation. However, we shall not demonstrate this explicitly, since we are not concerned with direct evaluation of the path integral; we are only concerned with finding the Pauli-Villars regulators.

[4]Note that the Pauli-Villars antighosts transform like the gauge fields.

and should be treated separately from that in the ghost part of the action [which is not quadratic in the operator D (A.19)].

We shall consider the anomalies arising from the Pauli-Villars matter part and the ghost part separately, since the matter and the ghost sectors are not coupled in our theory and contribute independently to the anomaly. The matter-sector w_∞ anomaly calculated from the Pauli-Villars scheme is given by

$$(\mathcal{A}^{pv})_{matt} = \lim_{M\to\infty} \sum_{j=0}^{\infty}\sum_{l=0}^{\infty} \mathrm{Tr}(j+1)c_j(\partial\phi)^j\partial\{-\exp\left[(\partial\bar{\partial}+\partial B_l\partial+B_l\partial^2)/M^2\right]\}. \tag{A.21}$$

The mass term of the Pauli-Villars action in the ghost sector gives rise to a similar anomaly:

$$\begin{aligned}(\mathcal{A}^{pv})_{gh} &= \lim_{M\to\infty} \sum_{l=0}^{\infty}\sum_{r=0}^{l} \mathrm{Tr}\Big[\Big(-(r+1)\partial c_{l-r}+(l-r+1)c_{l-r}\Big)\partial e^{-DD^\dagger/M_1M_2} \\ &\quad +\Big((r+1)\partial c_{l-r}-(l-r+1)c_{l-r}\Big)\partial e^{-D^\dagger D/M_1M_2}\Big]. \end{aligned} \tag{A.22}$$

Note that we have used the square of the operator in the quadratic part of the ghost action, i.e.

$$\begin{pmatrix} D^\dagger D & 0 \\ 0 & DD^\dagger \end{pmatrix}, \tag{A.23}$$

in order to obtain the Pauli-Villars ghost-sector anomaly.

The above considerations make it clear which regulator and which Jacobian should be used in the Fujikawa scheme. In the Fujikawa scheme, the anomaly arises from the measure of the path integral and is given by (see Chapter 2)

$$\mathcal{A} = \sum_{l=0}^{\infty}\sum_{r=0}^{l} \mathrm{Tr}\mathcal{N}^{lr}e^{-H^{lr}/M^2}, \tag{A.24}$$

where

$$\mathcal{N}^{lr} = \begin{pmatrix} \mathcal{N}^{l}_{\phi\phi} & 0 & \mathcal{N}^{lr}_{\phi c} \\ 0 & \mathcal{N}^{lr}_{gg} & \mathcal{N}^{lr}_{gc} \\ 0 & 0 & \mathcal{N}^{lr}_{cc} \end{pmatrix}, \tag{A.25}$$

and

$$
\begin{aligned}
\mathcal{N}^{l}_{\phi\phi} &= (l+1)c_l(\partial\phi)^l\partial, \\
\mathcal{N}^{lr}_{gg} &= -(r+1)\partial c_{l-r} + (l-r+1)c_{l-r}\partial, \\
\mathcal{N}^{lr}_{cc} &= (r+1)\partial c_{l-r} - (l-r+1)c_{l-r}\partial. \qquad \text{(A.26)}
\end{aligned}
$$

The matrix elements $\mathcal{N}^{lr}_{\phi c}$ and $\mathcal{N}^{lr}_{gc}$ have similar expressions. By using (A.21) and (A.22), we can easily find the regulator H that should be used in (A.24). This regulator is obtained by combining those in (A.21) and (A.22):

$$
H^{lr} = \begin{pmatrix} \mathcal{H}^{j}_{\phi\phi} & 0 & 0 \\ 0 & DD^\dagger & 0 \\ 0 & 0 & D^\dagger D \end{pmatrix}, \qquad \text{(A.27)}
$$

where the chiral parts of $\mathcal{H}_{\phi\phi}$, $D^\dagger D$ and $DD^\dagger$ are given explicitly below: [5]

$$
\begin{aligned}
\mathcal{H}_{\phi\phi} &= -\Big[\partial\bar{\partial} + \partial\Big((j+1)A_j(\partial\phi)^j\Big)\partial + \Big((j+1)A_j(\partial\phi)^j\Big)\partial^2\Big], \\
D^\dagger D &= -\partial\bar{\partial} + (r-2l)\partial A_{r-2l}\partial - (l+1)\partial^2 A_{r-l} + (r-l+1)A_{r-l}\partial^2, \\
DD^\dagger &= -\partial\bar{\partial} + (r-l+1)A_{r-l}\partial^2 - (l+1)\partial A_{r-l}\partial. \qquad \text{(A.28)}
\end{aligned}
$$

The regulators that we have used arise basically from the kinetic part of the action. For example, the regulator that regularizes the matter-loops is obtained from the operator in the quadratic part of the matter action. Similarly the operator that we have used to regularize the Jacobian associated with the ghost measure is constructed from the operator acting on the ghost in the ghost action. The gauge-field transformation also contributes to the Jacobian. But the gauge field in our theory is non-propagating and does not have a kinetic term in the action. The question then arises as to which regulator one should use for the regularization of the gauge-field Jacobian arising in the Fujikawa scheme. Here, the regulator that we have obtained

[5] Note that here we are only concerned with the chiral part of the anomaly, since we are only considering the chiral theory of w_∞ gravity.

from the Pauli-Villars scheme is the regulator for the antighosts. This is because in the Pauli-Villars scheme the gauge-field Jacobian cancels against that arising from the Pauli-Villars antighosts. Therefore the Pauli-Villars antighost fields and the gauge fields are partners. In the Fujikawa scheme, the antighost itself transforms into the Lagrange multipliers and does not contribute to the Jacobian. Accordingly, the operator acting on the antighost in the action is used to construct the regulator for the gauge fields.

It is not necessary to go through a full analysis of the Pauli-Villars scheme each time we need to obtain a consistent Fujikawa regulator. From the above discussion, it is clear that these regulators are constructed from the operators in the action. The construction of such regulators from the operators in the action is usually straightforward. If the operator in the action is already positive and contains a d'Alembertian, then it can be used without any change. Otherwise, we encounter a situation similar to that arising in the ghost sector: the operator in the kinetic part of the ghost action is not positive and does not contain a d'Alembertian. In such situations the regulator can normally be constructed by squaring the operator in the ghost action.

In theories where the gauge field is non-propagating, the Jacobian arising from the transformation of the gauge fields is regularized by using the operator acting on the antighost in the action. The regulator used in the gauge sector is the Hermitian conjugate of the regulator used for the regularization of the Jacobian associated with the ghost measure.

The above prescription for using the Fujikawa regulators works for the $\mathcal{W}$ gravities considered in this thesis, i.e. w_∞, W_∞ and W_3 gravity.

Appendix B

The zeta-function regularization scheme

The ghost-sector W_∞ anomalies have divergent coefficients. These coefficients, which appear in the form of divergent sums can be regularized using a zeta-function regularization scheme. However, the regularized values of these coefficients depend on how the infinite sums are written. In Chapters 2 and 3 we have discussed how the Wess-Zumino consistency condition restricts the allowed methods of summation to only those that give the same value of central charge in all spin sectors, as well as yielding a vanishing coefficient for the off-diagonal anomalies. In this appendix, we shall demonstrate two methods of summation adopted in the literature and show that one of these methods leads to a vanishing coefficient for the off-diagonal anomalies [61, 70, 71], whereas the other one fails to do so and as a result is incompatible with the requirements of the Wess-Zumino consistency condition [89]. These two methods are here illustrated for the diagonal anomalies at the spin-2 and spin-3 levels and for the spin-2/spin-4 off-diagonal anomaly.

B.1 The zeta-function regularization of the spin-2 universal ghost-sector anomaly

The spin-2 universal anomaly was derived in Chapter 2 and has the following form:

$$\mathcal{A}_2 = \frac{-\alpha_2}{12\pi} \int c_0 \partial^3 A_0, \tag{B.1}$$

where α_2 is the divergent coefficient at the spin-2 level and is given by

$$\begin{aligned} \alpha_2 &= \sum_{n=0}^{\infty} 6(n+1)^2 + 6(n+1) + 1 && \text{(B.2)} \\ &\longrightarrow 6\zeta(-2,1) + 6\zeta(-1,1) + \zeta(0,1) \\ &\longrightarrow -1, && \text{(B.3)} \end{aligned}$$

where ζ is the generalized Riemann zeta function [37] and the expressions on the right-hand sides of the arrow denote regularized values of the divergent sum. The sum (B.2) can be written in three essentially different ways. These different ways are determined by the symmetry of the ghost-antighost system [89]. In Chapter 2, we have seen that the regularized ghost-sector anomaly must remain invariant under the ghost-antighost spin exchange. That is to say, a ghost-antighost system (c_i, b_i) in which the ghost c_i has spin $-i-1$ and the antighost b_i has spin $i+2$, should remain symmetric under the exchange $i \leftrightarrow -i-3$. We have also shown (see Section 2.3) that as a result of this symmetry only the values $1, 2 \text{and} \frac{3}{2}$ of the parameter α are permissible in the ghost-sector infinite summations [these summations are basically in the form $\sum_{n=0}^{\infty}(a+n)^l$].

We shall now regularize the coefficient of spin-2 anomaly α_2 by using the various permissible values of the parameter a. In (B.3) we have already shown how the zeta-function regularization is performed using $a = 1$. Using $a = 2$, we obtain:

$$\begin{aligned}\alpha_2(2) &= \sum_{n=0}^{\infty} 6(n+2)^2 - 18(n+2) + 1 \\ &\longrightarrow 6\zeta(-2,2) - 18\zeta(-1,2) + \zeta(0,2) \\ &\longrightarrow -1. \end{aligned} \tag{B.4}$$

On the other hand if we use the value $\frac{3}{2}$ for writing the infinite sum (B.2) we obtain:

$$\alpha_2(3/2) = \sum_{n\geq 0}^{\infty} 6(n+\frac{3}{2})^2 - \frac{1}{2} \tag{B.5}$$

$$\begin{aligned} &\longrightarrow 6\zeta(-2,\frac{3}{2}) - \frac{1}{2}\zeta(0,\frac{3}{2}) \\ &\longrightarrow -1. \end{aligned} \tag{B.6}$$

One can see by comparing (B.2), (B.4) and (B.6) that the last method of summation, (B.5), has the special feature of only containing even powers. Later on, we shall show that this special feature is essential for the purpose of removing the off-diagonal anomalies and so the last method of summation is preferable to the other methods.

B.2 Zeta-function regularization of the spin-3 ghost-sector anomaly

In Chapter 3 we derived the spin-3 ghost-sector anomaly. This is given by

$$\mathcal{A}_{c_1 A_1} = \frac{-\alpha}{\pi} \int c_1 \partial^5 A_1, \tag{B.7}$$

where

$$\begin{aligned}\alpha_3 &= -\frac{1}{30}\sum_{n=0}^{\infty}\left[1 - \frac{3}{(2n+3)(2n+5)}\right]\Big[10(n+2)(2n+3)(n+1)^2 \\ &+15(n+1)^2(2n+3) + 30(n+1)^2 + 30(n+1) + 10(n+1)(2n+3) \\ &+10(n+2)(n+1)(2n+3) + 6\Big]. \end{aligned} \tag{B.8}$$

Each term of the above infinite sum can be written in the following manner:

$$\begin{aligned}
\alpha_3 \;=\; & -\frac{1}{30}\sum_{n=0}^{\infty}\Big\{20(n+1)^4+80(n+1)^3+105(n+1)^2+50(n+1)+6 \\
& -(\frac{3}{4})\Big[20(n+\frac{3}{2})^3-20(n+\frac{5}{2})^3+40(n+\frac{3}{2})^2 \\
& -40(n+\frac{5}{2})^2+15(n+\frac{3}{2})-15(n+\frac{5}{2})-5\Big] \\
& -\frac{3}{2}\Big(\frac{1}{n+3/2}-\frac{1}{n+5/2}\Big)\Big\}. \qquad \text{(B.9)}
\end{aligned}$$

This sum can now be regularized by using the zeta-function regularization scheme as follows:

$$\begin{aligned}
\alpha_3(1) \;\longrightarrow\; & -\frac{1}{30} \\
& \Big\{20\zeta(-4,1)+80\zeta(-3,1)+105\zeta(-2,1)+50\zeta(-1,1)+6\zeta(0,1) \\
& -\frac{3}{4}\Big[20\zeta(-3,3/2)-20\zeta(-3,5/2)+40\zeta(-2,3/2)+\zeta(-2,5/2) \\
& +15\zeta(-1,3/2)-15\zeta(-1,5/2)-5\zeta(0,3/2)-5\zeta(0,5/2)\Big] \\
& -(\frac{3}{2})(\frac{2}{3})\Big\} \\
\longrightarrow\; & \frac{1}{15}. \qquad \text{(B.10)}
\end{aligned}$$

Next, we shall demonstrate the regularization of the spin-3 ghost-sector coefficient using the alternative method of summation mentioned above. In this alternative method, one groups the terms in such a way that all the brackets have even powers. We have already seen an example of this scheme in the spin-2 level (B.6); a similar situation arises at the spin-3 level. The coefficient of the spin-3 anomaly, α_3, given in (B.9) can, as in the spin-2 case, be written as follows:

$$
\begin{aligned}
\alpha_3(2) &= -\frac{1}{30}\sum_{n=0}^{\infty}\Big\{20(n+2)^4 - 30(n+2)^2 + \frac{17}{2} \\
&\quad + \frac{9}{4}\Big(\frac{1}{(2n+3)} - \frac{1}{(2n+5)}\Big)\Big\} \qquad \text{(B.11)} \\
&\longrightarrow -\frac{1}{30}\Big\{20\zeta(-4,2) - 30\zeta(-2,2) + \frac{17}{2}\zeta(0,2) + \frac{3}{4}\Big\} \\
&\longrightarrow \frac{1}{15}. \qquad \text{(B.12)}
\end{aligned}
$$

It appears at this stage that both of these methods of summation give the same result upon regularization. However, the validity of these methods is tested when we consider the off-diagonal anomalies.

B.3 The zeta-function regularization of the off-diagonal anomalies

In Chapters 3 and 5, we showed that off-diagonal anomalies are forbidden by the Wess-Zumino consistency condition. We also pointed out that a proper zeta-function regularization should give a vanishing coefficient for the off-diagonal anomalies as well as the right central charge for the diagonal anomalies. To test which of the above methods of summation would yield a vanishing coefficient for the off-diagonal anomalies, we shall consider the simplest example of such anomalies. The simplest example of off-diagonal anomalies occurs at the spin-2/spin-4 level and is as follows:

$$
\mathcal{A}_{(c_0A_2+c_2A_0)} = \alpha_{2/4}\int(c_0\partial^5 A_2 + c_2\partial^5 A_0), \qquad \text{(B.13)}
$$

where the coefficient $\alpha_{2/4}$ is given by the divergent expression

$$
\begin{aligned}
\alpha_{2/4} &= \sum_{n=0}^{\infty}\Big\{(-\frac{1}{3})\Big[1-\frac{9}{5(2n+1)(2n+5)}\Big]\Big[16(n+1)(n+\frac{1}{2})(n+\frac{9}{2})n \\
&+16(n+\frac{1}{2})n+20n+80(n+1)(n+\frac{3}{4})+12\Big] \\
&-\frac{1}{3!}\Big[16(n+3)(n+\frac{5}{2})^2(n+2)-36(n+3)(n+2)\Big) \\
&-(\frac{1}{3!})\Big(16(n+1)(n+\frac{1}{2})^2n-36(n+1)n\Big]\Big\}. \qquad \text{(B.14)}
\end{aligned}
$$

We shall now demonstrate that the first scheme of summation [89] does not yield a vanishing coefficient for the off-diagonal anomaly. In this scheme the above sum is written as

$$
\begin{aligned}
\alpha_{2/4}(1) &= \sum_{n=0}^{\infty}\Big\{16(n+1)^4+32(n+1)^3+20(n+1)^2+4(n+1) \\
&-\frac{9}{40}\Big(16(n+\frac{1}{2})^3-16(n+\frac{5}{2})^3+64(n+\frac{1}{2})^2+64(n+\frac{5}{2})^2 \\
&+92(n+\frac{1}{2})-92(n+\frac{5}{2})+112+12\Big[\frac{1}{(n+1/2)}-\frac{1}{(n+5/2)}\Big]\Big) \\
&+\Big(16(n+1)^4+96(n+1)^3+176(n+1)^2+96(n+1)\Big)\Big\}. \qquad \text{(B.15)}
\end{aligned}
$$

A zeta-function regularization of the above sum then yields the value 77/45 and not zero. Recall that only the value zero for the coefficient of the off-diagonal anomalies is allowed by the Wess-Zumino consistency condition.

Now we demonstrate that the second method of regularization [71] does give a vanishing coefficient for the off-diagonal anomalies. The coefficient $\alpha_{2/4}$ (B.15) in the second scheme is written as follows:

$$
\begin{aligned}
\alpha_{2/4}(3/2) &= \sum_{n=0}^{\infty}\Big\{-\frac{1}{3}\Big[32(n+\frac{3}{2})^4+52(n+\frac{3}{2})^2\Big] \\
&+(\frac{9}{40})(\frac{1}{3})\Big[36(n+\frac{3}{2})^2+24\Big] \\
&-12\Big(\frac{1}{n+1/2}-\frac{1}{n+5/2}\Big)\Big\}. \qquad \text{(B.16)}
\end{aligned}
$$

The zeta-function regularization of the above infinite sum then yields:

$$\begin{aligned}
\alpha_{2/4}(3/2) \quad &\longrightarrow \quad -\frac{1}{3}\zeta(-4,3/2) + 52\zeta(-2,3/2) \\
&\qquad\quad +\frac{27}{10}\zeta(-2,3/2) + 24\zeta(0,3/2) - 12(\frac{8}{3}) \\
&\longrightarrow \quad 0. \qquad\qquad\qquad\

Appendix C

Chiral W_3 gravity

In this thesis, we have been concerned mainly with the theories of w_∞ and W_∞ gravity. Both of these theories have an underlying linear algebra. However, in recent years much attention has been paid to another type of $\mathcal{W}$ algebra, known as W_3 algebra [90]. Unlike the w_∞ and W_∞ algebras, W_3 is a nonlinear algebra. To arrive at this algebra, one starts from the Virasoro algebra which, in the form of the operator-product expansion, is

$$T(z)T(w) \sim \frac{\partial T}{z-w} + \frac{2T}{(z-w)^2} + \frac{c/2}{(z-w)^4}, \tag{C.1}$$

where T is the spin-2 energy-momentum tensor and c is the central charge of the Virasoro algebra. Next, one introduces a spin-3 current W which gives the following operator-product expansion with T:

$$T(z)W(w) \sim \frac{\partial W}{z-w} + \frac{3W}{(z-w)^2}. \tag{C.2}$$

The operator-product expansion of W with W then produces a spin-4 current which is interpreted as a composite operator. The W-W operator-product expansion is

$$
\begin{aligned}
W(z)W(w) \quad \sim \quad & \frac{16}{22+5c}\Big(\frac{\partial\Lambda}{z-w}+\frac{2\Lambda}{(z-w)^2}\Big) \\
& +\frac{1}{15}\frac{\partial^3 T}{z-w}+\frac{3}{10}\frac{\partial^2 T}{(z-w)^2}\frac{\partial T}{(z-w)^3}+\frac{2T}{(z-w)^4}+\frac{c/3}{(z-w)^6},
\end{aligned}
\tag{C.3}
$$

where Λ is a spin-4 composite:[7]

$$
\Lambda = (TT) - \frac{3}{10}\partial^2 T. \tag{C.4}
$$

Although nonlinear, the W_3 algebra does satisfy the Jacobi identities.

W_N algebras, which include generators of higher spins, are essentially similar to W_3. The detailed structures of these algebras are complicated, but the essential feature of the W_N algebra is that

$$
W^s(z)W^{s'}(w) \sim W^{s+s'-2}, W^{s+s'-4}, \cdots + c_s\frac{\delta^{ss'}}{(z-w)^{2s}}. \tag{C.5}
$$

All W_N algebras are nonlinear because at some point a higher-spin current must be expressed through a non-linear combination of lower-spin currents in order to close these algebras. However, if we take $N \to \infty$, then one never generates a current with spin that exceeds N. We might, therefore, expect to obtain a linear algebra when $N = \infty$. This has in fact been achieved and underlies the idea behind the formulation of the w_∞ and W_∞ algebras [61–71].

C.1 BRST action for W_3 gravity

The spin-2 and spin-3 currents of W_3 algebra have an explicit realization in terms of n scalar fields ϕ^i as given below [44, 46, 48, 50, 63]:[8]

$$
T_{matt} \quad = \quad -\frac{1}{2}\partial\phi^i\partial\phi^i,
$$

[7] In expression (C.4), the rounded brackets () denote normal ordering.

[8] In this appendix Einstein summation is implies everywhere.

$$W_{matt} = -\frac{1}{3}d_{ijk}\partial\phi^i\partial\phi^j\partial\phi^k, \tag{C.6}$$

where the ϕ^i satisfy the operator-product expansion

$$\partial\phi^i(z)\partial\phi^j(w) \sim \frac{-\delta^{ij}}{(z-w)^2}. \tag{C.7}$$

In (C.6), the structure constants d_{ijk} satisfy the relation

$$d^m_{(ij}d_{kl)m} = \frac{a}{2}\delta_{(ij}\delta_{kl)}, \tag{C.8}$$

where $a = 16(22+5C_{mat})^{-1}$ and C_{mat} is the central charge of the W_3 algebra for the realization (C.6).

The W_3 action

$$S = \frac{1}{\pi}\int d^2z\Big(-\frac{1}{2}\partial\phi^i\bar{\partial}\phi^i - hT_{mat} - BW_{mat}\Big) \tag{C.9}$$

is invariant under the following symmetries:

$$\begin{aligned}
\delta\phi^i &= \epsilon\partial\phi^i + d_{ijk}\lambda\partial\phi^j\partial\phi^k, \\
\delta h &= \bar{\partial}\epsilon + \epsilon\partial h - \partial\epsilon h - \frac{a}{2}(\lambda\partial B - \partial\lambda B)\partial\phi^i\partial\phi^i, \\
\delta B &= \bar{\partial}\lambda + \epsilon\partial B - 2\partial\epsilon B + 2\lambda\partial h - \partial\lambda h,
\end{aligned} \tag{C.10}$$

where ϵ and λ are the infinitesimal parameters for the Virasoro and the spin-3 transformations, respectively.

The conventional gauge-fixing conditions for the spin-2 and spin-3 gauge fields are $h = h_{back}$ and $B = B_{back}$, imposed by the Lagrange multipliers π_h and π_B [45]. This leads to the BRST action

$$\begin{aligned}
S_{BRST} = \frac{1}{\pi}\int d^2z\Big[&-\frac{1}{2}\partial\phi^i\bar{\partial}\phi^i - b\bar{\partial}c - \beta\bar{\partial}\gamma + \pi_h(h-h_{back}) + \pi_B(B-B_{back}) \\
&-h(T_{mat}+T_{gh}) - B(W_{mat}+W_{gh})\Big].
\end{aligned} \tag{C.11}$$

In the above expression, T_{mat} and W_{mat} are given in (C.6) and T_{gh} and W_{gh} are as follows:

$$\begin{aligned} T_{gh} &= -2b\partial c - \partial bc - 3\beta\partial\gamma - 2\partial\beta\gamma, \\ W_{gh} &= -\partial\beta c - 3\beta\partial c - a[\partial(b\gamma T_{mat}) + b\partial\gamma T_{mat}]. \end{aligned} \tag{C.12}$$

The ghost-antighost pairs (c, b) and (γ, β) correspond, respectively, to the T and W generators. They satisfy the following operator-product expansions

$$c(z)b(w) \sim \frac{1}{z-w}; \qquad \gamma(z)\beta(w) \sim \frac{1}{z-w}. \tag{C.13}$$

The transformation rules for the fields in (C.11) are given by

$$\begin{aligned} \delta\phi^i &= c\partial\phi^i + d_{ijk}\gamma\partial\phi^j\phi^k + ab\gamma\partial\gamma\partial\phi^i, \\ \delta h &= \bar{\partial}c + c\partial h - \partial ch - \frac{a}{2}(\gamma\partial B - \partial\gamma B)\partial\phi^i\partial\phi^i, \\ \delta B &= \bar{\partial}\gamma + c\partial B - 2\partial cB + 2\gamma\partial h - \partial\gamma h, \\ \delta c &= c\partial c - \frac{a}{2}\gamma\partial\gamma\partial\phi^i\partial\phi^i, \\ \delta\gamma &= c\partial\gamma - 2\partial c\gamma, \\ \delta b &= \pi_h, \quad \delta\beta = \pi_B \quad \delta\pi_h = 0 \quad \delta\pi_B = 0. \end{aligned} \tag{C.14}$$

By comparing (C.14) and (C.10), we can see that the BRST transformations of W_3 gravity have a structure that is not directly obtainable by the simple prescription of replacing the parameters of the classical W_3 gauge transformations by ghosts (as in w_∞ and W_∞). One normally expects to obtain the corresponding BRST transformation (C.14) simply by replacing ϵ in (C.10) with the spin-2 ghost c and λ in (C.10) with the spin-3 ghost γ. However, comparing the BRST transformation of ϕ^i given in (C.14) with the gauge transformation of the scalar field given in (C.10) one sees that the BRST transformation contains an unexpected term $b\gamma\partial\gamma\partial\phi$. Consequently, the classical action (C.9) is not invariant under the above BRST

transformations.[9]

In addition to the unusual extra term in the $\delta\phi$ transformation, there is another non-standard feature in the BRST transformations (C.14). Both the BRST variations of the scalar and the spin-2 gauge field h are only nilpotent on-shell, i.e. they are only nilpotent if one uses the equations of motion. This feature is not encountered in w_∞ and W_∞ gravity and is another consequence of the nonlinearity of W_3 algebra. This, however, does not prevent the BRST action (C.11) from being invariant under the BRST symmetries (C.14). The terms arising from the variation of h in $\delta(bh)$, due to the $\delta^2 h$ off-shell non-closure, are cancelled by the extra term $ab\gamma\partial\gamma\partial\phi^i$.

The problem of off-shell non-closure of the BRST symmetries can be resolved by choosing a derivative gauge-fixing condition. This is discussed in Section C.3 of the present appendix.

C.2 W_3 anomalies from the path-integral measure

The partition function corresponding to the BRST action (C.11) is

$$\mathcal{Z} = \int \mathcal{D}\phi\mathcal{D}h\mathcal{D}B\mathcal{D}C\mathcal{D}b\mathcal{D}\gamma\mathcal{D}\beta \exp S_{BRST}. \tag{C.15}$$

Under the BRST symmetries (C.14), the measure in (C.15) transforms through a Jacobian factor. This factor, when properly regularized, results in the anomaly $\mathcal{A}_1$. The expression for the W_3 one-loop anomaly derived from the path-integral measure is as follows:

$$\mathcal{A}_1 = \mathcal{A}_\phi + \mathcal{A}_{h,B} + \mathcal{A}_{c,\gamma}, \tag{C.16}$$

[9]Note that the invariance of the BRST action under the BRST symmetries at the classical level is guaranteed. This is because the ghost action for this nonlinear theory couples to the scalar field. As a result, the variation of the ghost action and the matter part of the action cannot, as for linear theories, be treated independently and indeed these variations cancel against each other. This feature is not encountered in w_∞ and W_∞ gravity and only arises as a consequence of the nonlinearity of W_3 algebra.

where

$$\mathcal{A}_\phi = \lim_{M\to\infty} \mathrm{Tr}\Big(c + 2d_{ilj}\gamma\partial\phi^l + ab\gamma\partial\gamma\delta^{ij}\Big)\partial e^{-H_{ji}/M^2}, \tag{C.17}$$

$$\begin{aligned}\mathcal{A}_{h,B} &= \lim_{M\to\infty} \mathrm{Tr}\begin{pmatrix} \frac{\partial\delta h}{\partial h} & \frac{\partial\delta h}{\partial B} \\ \frac{\partial\delta B}{\partial h} & \frac{\partial\delta B}{\partial B}\end{pmatrix} e^{-D_{h,B}/M^2} \\ &= \lim_{M\to\infty} \mathrm{Tr}\begin{pmatrix} c\partial - \partial c & \frac{a}{2}\partial\phi^i\partial\phi^i(\gamma\partial - \partial\gamma) \\ 2\gamma\partial - \partial\gamma & c\partial - 2\partial c\end{pmatrix} e^{-D_{h,B}/M^2}, \end{aligned} \tag{C.18}$$

$$\mathcal{A}_{c,\gamma} = \lim_{M\to\infty} \mathrm{Tr}\begin{pmatrix} \partial c - c\partial & \frac{a}{2}\partial\phi^i\partial\phi^i(\partial\gamma - \gamma\partial) \\ -2\gamma\partial + \partial\gamma & 2\partial c - c\partial\end{pmatrix} e^{-D_{c,\gamma}/M^2} \tag{C.19}$$

The regulators in the above expressions are obtained by using the prescription given in Appendix A. These regulators arise from the kinetic part of the action (C.11) which can be written as

$$S = \frac{1}{\pi}\int \phi^i H_{ji}\phi^j + bD_1c + \beta D_2c + bD_3\gamma + \beta D_4\gamma, \tag{C.20}$$

where the explicit expressions for the operators in (C.20) are

$$\begin{aligned} H_{ji} &= -\partial\bar\partial + \partial[(h - a(Bb\partial\gamma - \partial Bb\gamma) + 2d_{imj}B\partial\phi^m)\partial], \\ D_1 &= -\bar\partial + h\partial - \partial h, \\ D_2 &= 2B\partial - \partial B, \\ D_3 &= -\frac{a}{2}\partial B\partial\phi^i\partial\phi^i + \frac{a}{2}B\partial\phi^i\partial\phi^i\partial, \\ D_4 &= -\bar\partial + h\partial - 2\partial h. \end{aligned} \tag{C.21, C.22}$$

The regulator for the anomaly coming from the measure of the scalar field is H_{ji} which is already positive and contains the d'Alembertian ($\bar\partial\partial$). The operator acting on the space $\{h, B, c, \gamma\}$ can be obtained from (C.20) and is

$$D = \begin{pmatrix} 0 & 0 & D_1 & D_3 \\ 0 & 0 & D_2 & D_4 \\ D_1^\dagger & D_2^\dagger & 0 & 0 \\ D_3^\dagger & D_4^\dagger & 0 & 0 \end{pmatrix}. \tag{C.23}$$

The operator D^2 is then used to regularize the Jacobian arising from the measure of the ghosts and gauge fields (see Appendix A). Writing D^2 in block-diagonal form, we arrive at the operators $D_{h,B}$ and $D_{c,\gamma}$ which we have been using in (C.18) and (C.19) for the regularization of the gauge and ghost measures. These regulators are obtained from the following expression:

$$D^2 = \begin{pmatrix} D_{h,B} & 0 \\ 0 & D_{c,\gamma} \end{pmatrix}. \tag{C.24}$$

With the aid of the above regulators, the anomaly can then be evaluated in a very similar manner to the evaluation of w_∞ and W_∞ gravity (see Chapters 2 and 3). One expands the trace in a plane-wave basis and then discards the infinite parts of the resulting expression. The resulting expressions for the local parts of the chiral W_3 anomaly are

$$\begin{aligned} \mathcal{A}_\phi &= \frac{1}{\pi}\int\Big[\frac{C_{mat}}{12}c\partial^3 h - \frac{d_{iij}\partial\phi^j}{6}(B\partial^3 c - \gamma\partial^3 h) - \frac{d_{ijk}d_{ljk}}{3}B\partial\phi^i\partial^3(\gamma\partial\phi^l) \\ &\quad + \frac{3caC_{mat}}{16}\partial^3(Bb\partial\gamma - \partial Bb\gamma) + \frac{aC_{mat}}{12}b\gamma\partial\gamma\partial^3 h \\ &\quad + \frac{3a^2C_{mat}}{16}b\gamma\partial\gamma\partial^3(Bb\partial\gamma - \partial Bb\gamma) + \frac{a}{6}b\gamma\partial\gamma d_{iij}\partial^3(B\partial\phi^j)\Big], \end{aligned} \tag{C.25}$$

and

$$\begin{aligned} \mathcal{A}_{c,\gamma} + \mathcal{A}_{h,B} &= \frac{1}{\pi}\int\Big[-\frac{100}{12}c\partial^3 h - \frac{a}{2}\partial\phi^i\partial\phi^i \\ &\quad \Big(2\partial B\partial^2\gamma - \frac{5}{6}B\partial^3\gamma - 2\partial\gamma\partial^2 B + \frac{5}{6}\gamma\partial^3 B\Big)\Big]. \end{aligned} \tag{C.26}$$

The above expressions constitute [44, 46, 48, 49] all the one-loop anomalies that exist in the theory of chiral W_3 gravity.

The two-loop universal anomaly that exists in the spin-3 sector of this theory can be obtained by a similar analysis to that discussed for W_∞ gravity in Chapter 3. First, one tries to remove all the matter-dependent anomalies

from this theory. This leads to a renormalization of the BRST transformations and the matter and ghost-sector currents. The renormalized transformation rule of the gauge field is then incorporated in the commutator form of the Wess-Zumino consistency condition.[10] The renormalized consistency condition (i.e. the condition involving the renormalized gauge algebra) is then imposed on the one-loop universal anomaly. The requirement for the consistent one-loop anomaly leads to an expression for the two-loop spin-3 anomaly. The two-loop anomaly derived in this way is

$$\mathcal{A}_2 = \frac{C_{matt} - 100}{360\pi} \int \gamma \partial^5 B. \tag{C.27}$$

C.3 W_3 gravity in the derivative gauge

In this section, we shall quantize chiral W_3 gravity by using derivative gauge conditions for the spin-2 and spin-3 gauge fields. The motivation behind this is to find a new formalism that achieves full off-shell closure of the BRST algebra.[11] We would also like to find a BRST charge that is the generator of all BRST transformations, including those of the gauge fields.[12] An additional motivation is to find a relationship between the Ward-identity anomaly and the BRST-charge anomaly (see Section 5.4).

The gauge conditions that we shall choose for chiral W_3 gravity are $\bar{\partial} h = 0$ and $\bar{\partial} B = 0$. We shall consider the fully-renormalized gauge-fixed W_3 action and its renormalized symmetries. The renormalization terms are introduced

[10]The derivation of the two-loop anomaly by this method is more complicated for W_3 because in this theory the gauge field transforms nonlinearly into the scalar fields. However, since we are only considering the universal anomalies, which do not depend on the scalar fields, this complication is immaterial.

[11]We recall that in the conformal gauge the BRST transformations of the gauge and scalar fields are only nilpotent upon using the equations of motion.

[12]In the conformal gauge, the BRST charge does not generate the transformation rule of the gauge fields. This problem has been discussed in detail for W_∞ gravity in Chapter 5. Here, a similar problem arises.

in order to cancel the matter-dependent anomalies given in expressions (C.25) and (C.26). The renormalized W_3 action in the derivative gauge is as follows:

$$
\begin{aligned}
S \;=\; & \frac{1}{\pi}\int d^2z\Big(-\frac{1}{2}\bar\partial\phi^i\,\partial\phi^i - h\,T^{\rm r}_{\rm mat} - B\,W^{\rm r}_{\rm mat} + \pi_h\,\bar\partial h \\
& -b\,\bar\partial[\bar\partial c + c\,\partial h - \partial c\,h - \frac{a}{2}(\gamma\,\partial B - \partial\gamma\,B)\partial\phi^i\partial\phi^i - a\sqrt{\hbar}(\gamma\,\partial B - \partial\gamma\,B)\alpha_i\,\partial^2\phi^i \\
& \qquad + \frac{1-17a}{30}\hbar(2\gamma\,\partial^3 B - 3\partial\gamma\,\partial^2 B + 3\partial^2\gamma\,\partial B - 2\partial^3\gamma\,B)] \\
& \qquad -\beta\,\bar\partial[\bar\partial\gamma + c\,\partial B - 2\partial c\,B + 2\gamma\,\partial h - \partial\gamma\,h]\Big),
\end{aligned}
\tag{C.28}
$$

where $T^{\rm r}_{\rm mat}$ and $W^{\rm r}_{\rm mat}$ are given below

$$
\begin{aligned}
T^{\rm r}_{\rm mat} &= -\frac{1}{2}\partial\phi^i\,\partial\phi^i - \sqrt{\hbar}\alpha_i\,\partial^2\phi^i, \\
W^{\rm r}_{\rm mat} &= -\frac{1}{3}d_{ijk}\,\partial\phi^i\,\partial\phi^j\,\partial\phi^k - \sqrt{\hbar}\,e_{ij}\,\partial\phi^i\,\partial^2\phi^j - \hbar\,f_i\,\partial^3\phi^i.
\end{aligned}
\tag{C.29}
$$

As in the W_∞ case, we now reduce this action to a first-order form by introducing momenta conjugate to $c,\ b,\ \gamma,\ \beta$:

$$
\begin{aligned}
\pi_c &= \frac{\partial\mathcal{L}}{\partial\bar\partial c} = -\bar\partial b, \\
\pi_b &= \frac{\partial\mathcal{L}}{\partial\bar\partial b} = \bar\partial c + c\,\partial h - \partial c\,h - \frac{a}{2}(\gamma\,\partial B - \partial\gamma\,B)\partial\phi^i\partial\phi^i \\
& \quad -a\sqrt{\hbar}(\gamma\,\partial B - \partial\gamma\,B)\alpha_i\,\partial^2\phi^i \\
& \quad + \frac{1-17a}{30}\hbar(2\gamma\,\partial^3 B - 3\partial\gamma\,\partial^2 B + 3\partial^2\gamma\,\partial B - 2\partial^3\gamma\,B), \\
\pi_\gamma &= \frac{\partial\mathcal{L}}{\partial\bar\partial\gamma} = -\bar\partial\beta, \\
\pi_\beta &= \frac{\partial\mathcal{L}}{\partial\bar\partial\beta} = \bar\partial\gamma + c\,\partial B - 2\partial c\,B + 2\gamma\,\partial h - \partial\gamma\,h.
\end{aligned}
\tag{C.30}
$$

Using these definitions, the gauge-fixed action (C.28) may be put into the first-order form:[13]

[13] As in W_∞ case, we consider the off-diagonal kinetic term $\bar\partial\phi^i\,\partial\phi^i$ to be of first order in $\bar\partial$ derivatives.

$$
\begin{aligned}
S &= \frac{1}{\pi}\int d^2z\Big(-\frac{1}{2}\bar\partial\phi^i\,\partial\phi^i + \pi_h\,\bar\partial h + \pi_B\,\bar\partial B - \pi_b\,\bar\partial b - \pi_c\,\bar\partial c - \pi_\beta\,\bar\partial\beta - \pi_\gamma\,\bar\partial\gamma \\
&\quad -\pi_b\,\pi_c - \pi_\beta\,\pi_\gamma - h\,(T^{\rm r}_{\rm mat} + T^{\rm r}_{\rm gh}) - B\,(W^{\rm r}_{\rm mat} + W^{\rm r}_{\rm gh})\Big), \qquad \text{(C.31)}
\end{aligned}
$$

where $T^{\rm r}_{\rm gh}$ and $W^{\rm r}_{\rm gh}$ are no longer given by (C.12) but now include the conjugate momenta,

$$
\begin{aligned}
T^{\rm r}_{\rm gh} &= -2\pi_c\,\partial c - \partial\pi_c\,c - 3\pi_\gamma\,\partial\gamma - 2\partial\pi_\gamma\,\gamma, \\
W^{\rm r}_{\rm gh} &= -\partial\pi_\gamma\,c - 3\pi_\gamma\,\partial c - a[\partial(\pi_c\,\gamma\,T_{\rm mat}) + \pi_c\,\partial\gamma\,T_{\rm mat}] \\
&\quad + \frac{(1-17a)}{30}\hbar(2\gamma\,\partial^3\pi_c + 9\partial\gamma\,\partial^2\pi_c + 15\partial^2\gamma\,\partial\pi_c + 10\partial^3\gamma\,\pi_c).
\end{aligned} \qquad \text{(C.32)}
$$

From the path-integral generating functional derived from (C.31), we obtain the following operator-product expansion relations:

$$
\begin{aligned}
\partial\phi^i(z)\partial\phi^i(w) &\sim \frac{-1}{(z-w)^2}, \\
\pi_h(z)h(w) &\sim \frac{1}{z-w}, \\
c(z)\pi_c(w) &\sim \frac{1}{z-w}, \\
b(z)\pi_b(w) &\sim \frac{1}{z-w}, \\
\pi_B(z)B(w) &\sim \frac{1}{z-w}, \\
\gamma(z)\pi_\gamma(w) &\sim \frac{1}{z-w}, \\
\beta(z)\pi_\beta(w) &\sim \frac{1}{z-w}.
\end{aligned} \qquad \text{(C.33)}
$$

The BRST transformations corresponding to the action (C.31) are

$$
\begin{aligned}
\delta\phi^i &= c\,\partial\phi^i + d_{ijk}\,\gamma\,\partial\phi^j\partial\phi^k + a\,\pi_c\,\gamma\,\partial\gamma\,\partial\phi^i + \sqrt{\hbar}\Big(- a\,\alpha_i\,\partial(\pi_c\,\gamma\,\partial\gamma) \\
&\quad -\alpha_i\,\partial c + (e_{ij} - e_{ji})\,\gamma\,\partial^2\varphi^j - e_{ji}\,\partial\gamma\,\partial\phi^j\Big) \\
&\quad +\hbar\,f_i\,\partial^2\gamma, \\
\delta h &= \pi_b, \\
\delta B &= \pi_\beta, \\
\delta c &= c\,\partial c - \frac{a}{2}\gamma\,\partial\gamma\,\partial\phi^i\partial\phi^i - a\sqrt{\hbar}\,\alpha_i\,\gamma\,\partial\gamma\,\partial^2\phi^i + \frac{1-17a}{30}\hbar(2\gamma\,\partial^3\gamma - 3\partial\gamma\,\partial^2\gamma), \\
\delta\gamma &= c\,\partial\gamma - 2\partial c\,\gamma, \\
\delta b &= \pi_h, \quad \delta\beta = \pi_B, \quad \delta\pi_c = T^{\mathrm{r}}_{\mathrm{mat}} + T^{\mathrm{r}}_{\mathrm{gh}}, \quad \delta\pi_\gamma = W^{\mathrm{r}}_{\mathrm{mat}} + W^{\mathrm{r}}_{\mathrm{gh}}, \\
\delta\pi_h &= 0, \quad \delta\pi_B = 0, \quad \delta\pi_b = 0, \quad \delta\pi_\beta = 0,
\end{aligned}
\tag{C.34}
$$

where the constants $\alpha_i, d_{ijk}, e_{ij}$ and f_i satisfy the following relations [72]:

$$
\begin{aligned}
& d_{ijj} - 6e_{ij}\,\alpha_j + 6f_i = 0, \\
& e_{(ij)} - d_{ijk}\,\alpha_k = 0, \\
& 3f_i - \alpha_j\,e_{ji} = 0, \\
& d_{ikl}\,d_{jkl} + 6d_{ijk}\,f_k - 3e_{ik}\,e_{jk} = \frac{1}{2}\delta_{ij}, \\
& d_{(ij}{}^m\,d_{kl)m} = \frac{1}{2}a\,\delta_{(ij}\delta_{kl)}, \\
& d_{ijk}(e_{lk} - e_{kl}) + 2e_{(i}{}^l\,d_{j)kl} = a\,\alpha_k\,\delta_{ij}.
\end{aligned}
\tag{C.35}
$$

Two useful consequences of this set of equations are

$$
\begin{aligned}
& e_{ii} + 12\alpha_i\,f_i = 0, \\
& C_{\mathrm{mat}} = -2d_{ijk}\,d_{ijk} - 18e_{ij}\,e_{ij} - 12e_{ij}\,e_{ji} - 360f_i^2.
\end{aligned}
\tag{C.36}
$$

The central charge C_{mat} is given for the realization (C.29) by

$$
C_{\mathrm{mat}} = n + 12\alpha_i\alpha_i. \tag{C.37}
$$

In the derivative gauge, all the BRST transformations (C.34) are off-shell nilpotent up to order $\hbar^{1/2}$.

The canonical generator of the transformations (C.34) is the BRST charge Q in the derivative gauge:

$$Q = \int dz \Big(c\,(T^{\rm r}_{\rm mat} + \frac{1}{2} T^{\rm r}_{\rm gh}) + \gamma\,(W^{\rm r}_{\rm mat} + \frac{1}{2} W^{\rm r}_{\rm gh}) + \pi_h\,\pi_b + \pi_B\,\pi_\beta \Big). \qquad \text{(C.38)}$$

This BRST charge now contains the final two terms, which would not be present in the BRST charge derived in the conformal gauge. These new terms generate the transformations of the spin-2 and spin-3 gauge fields. Therefore, the BRST charge is the true generator of all transformations and is now fully canonical.

C.4 The W_3 anomalies in derivative gauge

We shall derive the anomalies of W_3 gravity in the derivative gauge by using the standard Ward-identity formalism.[14] We formulate the BRST Ward identity of W_3 gravity (as we have done for W_∞ in Section 5.3) by introducing sources K_{χ^i} for the nonvanishing variations (C.34). Here we denote all fields χ^i and the sources of their variations K_{χ^i} as

$$\begin{aligned} \chi^i &= (\phi^i, h, \pi_h, B, \pi_B, b, \pi_b, c, \pi_c, \beta, \pi_\beta, \gamma, \pi_\gamma), \\ K_{\chi^i} &= (K_{\phi^i}, K_h, K_B, K_b, K_\beta, K_\gamma, K_c, K_{\pi_c}, K_{\pi_\gamma}). \end{aligned} \qquad \text{(C.39)}$$

The extended Lagrangian is the sum of (C.31) plus the source terms,

[14] Note that the one-loop anomaly in the derivative gauge can also be derived by using the Fujikawa method. Here, we are using the Ward-identity treatment of anomalies as a more appropriate method for an extensive derivation of the anomalies and for the use of the antibracket form of the Wess-Zumino consistency condition.

$$
\begin{aligned}
\mathcal{L}_{\rm ext} \;=\; & -\frac{1}{2}\bar{\partial}\phi^i\,\partial\phi^i + \pi_h\,\bar{\partial}h + \pi_B\,\bar{\partial}B - \pi_b\,\bar{\partial}b - \pi_c\,\bar{\partial}c - \pi_\beta\,\bar{\partial}\beta - \pi_\gamma\,\bar{\partial}\gamma \\
& -\pi_b\,\pi_c - \pi_\beta\,\pi_\gamma - h\,(T^{\rm r}_{\rm mat} + T^{\rm r}_{\rm gh}) - B\,(W^{\rm r}_{\rm mat} + W^{\rm r}_{\rm gh}) \\
& +K_{\phi^i}[c\,\partial\phi^i + d_{ijk}\,\gamma\,\partial\phi^j\partial\phi^k + a\,\pi_c\,\gamma\,\partial\gamma\,\partial\phi^i \\
& +\sqrt{\hbar}(-\alpha_i\,\partial c + (e_{ij} - e_{ji})\,\gamma\,\partial^2\phi^j - e_{ji}\,\partial\gamma\,\partial\phi^j - a\,\alpha_i\,\partial(\pi_c\,\gamma\,\partial\gamma)) + \hbar\,f_i\,\partial^2\gamma] \\
& +K_h\,\pi_b + K_B\,\pi_\beta + K_b\,\pi_h + K_\beta\,\pi_B + K_\gamma(c\,\partial\gamma - 2\partial c\,\gamma) \\
& +K_c[c\,\partial c - \frac{a}{2}\gamma\,\partial\gamma\,\partial\phi^i\partial\phi^i - a\sqrt{\hbar}\,\alpha_i\,\gamma\,\partial\gamma\,\partial^2\phi^i \\
& +\frac{1}{30}(1-17a)\hbar(2\gamma\,\partial^3\gamma - 3\partial\gamma\,\partial^2\gamma)] \\
& +K_{\pi_c}(T^{\rm r}_{\rm mat} + T^{\rm r}_{\rm gh}) + K_{\pi_\gamma}(W^{\rm r}_{\rm mat} + W^{\rm r}_{\rm gh}).
\end{aligned} \tag{C.40}
$$

The anomalous Ward identities at orders $\hbar$, $\hbar^{3/2}$ and $\hbar^2$ are as follows:

$$
\begin{aligned}
\mathcal{A}_1 \;&=\; \frac{\delta S_0}{\delta\phi^i}\frac{\delta\Gamma_1}{\delta K_{\phi^i}} + \frac{\delta S_0}{\delta K_{\phi^i}}\frac{\delta\Gamma_1}{\delta\phi^i} + \frac{\delta S_{\frac{1}{2}}}{\delta\phi^i}\frac{\delta S_{\frac{1}{2}}}{\delta K_{\phi^i}}, \\
\mathcal{A}_{\frac{3}{2}} \;&=\; \frac{\delta S_0}{\delta\phi^i}\frac{\delta\Gamma_{\frac{3}{2}}}{\delta K_{\phi^i}} + \frac{\delta S_0}{\delta K_{\phi^i}}\frac{\delta\Gamma_{\frac{3}{2}}}{\delta\phi^i} + \frac{\delta S_{\frac{1}{2}}}{\delta\phi^i}\frac{\delta\Gamma_1}{\delta K_{\phi^i}} + \frac{\delta S_{\frac{1}{2}}}{\delta K_{\phi^i}}\frac{\delta\Gamma_1}{\delta\phi^i}, \\
\mathcal{A}_2 + \mathcal{A}_{2,\rm nl} \;&=\; \frac{\delta S_0}{\delta\phi^i}\frac{\delta\Gamma_2}{\delta K_{\phi^i}} + \frac{\delta S_0}{\delta K_{\phi^i}}\frac{\delta\Gamma_2}{\delta\phi^i} + \frac{\delta S_{\frac{1}{2}}}{\delta\phi^i}\frac{\delta\Gamma_{\frac{3}{2}}}{\delta K_{\phi^i}} + \frac{\delta S_{\frac{1}{2}}}{\delta K_{\phi^i}}\frac{\delta\Gamma_{\frac{3}{2}}}{\delta\phi^i} + \frac{\delta\Gamma_1}{\delta\phi^i}\frac{\delta\Gamma_1}{\delta K_{\phi^i}},
\end{aligned} \tag{C.41}
$$

where $\mathcal{A}_n$ represents the total local anomaly at order $\hbar^n$.[15] The term $\mathcal{A}_{2,\rm nl}$ is a dressing of the order-$\hbar$ anomaly yielding a non-local order-$\hbar^2$ anomaly.

Using the extended Lagrangian (C.40) and the relations (C.35), we find the following expressions for the local anomalies:

$$
\begin{aligned}
\mathcal{A}_1 \;=\; & (\frac{16}{30\pi}(1-17a) - \frac{a}{12\pi}C_{\rm mat})\int d^2z\,\Big(\gamma\,\pi_c\,\partial\gamma\,(\partial^3 h - \partial^3 K_{\pi_c}) \\
& +\partial^3 c\,[\gamma\,K_c\,\partial\gamma - \pi_c\,(\partial\gamma\,B - \gamma\,\partial B - \partial\gamma\,K_{\pi_\gamma} + \gamma\,\partial K_{\pi_\gamma})]\Big) \\
& -\frac{1}{12\pi}(100 - C_{\rm mat})\int d^2z\,c\,(\partial^3 h - \partial^3 K_{\pi_c}),
\end{aligned} \tag{C.42}
$$

[15] The non-local part of the anomaly at order $\hbar$ vanishes upon using the equations of motion.

and

$$\mathcal{A}_2 \;=\; (-\frac{29}{50\pi}(1-17a)+\frac{C_{\rm mat}}{360\pi})\int d^2z\,\gamma\,(\partial^5 B-\partial^5 K_{\pi_c}). \qquad \text{(C.43)}$$

Note that the above results contain only universal, i.e. purely gauge-field-dependent and ghost-dependent, anomalies. We recall that this is because we have used the renormalized action (C.31) and symmetries (C.34) which ensure the cancellation of matter-dependent anomalies.

We now check the consistency of the above one and two-loop anomalies. The Wess-Zumino consistency condition at one-loop level is given by an expression identical to that given for the W_∞ gravity (5.38). It is

$$(s_0,\mathcal{A}_1)=0. \qquad \text{(C.44)}$$

One can show that the anomaly (C.42) satisfies this condition.

At the two-loop level, the Wess-Zumino consistency condition takes a non-standard form because of the nonlinearity of W_3 algebra. The one-loop anomaly (C.42) contains fields that can be dressed further in loops. For example, the $\gamma\pi_c$ in the first term of (C.42) can be contracted with the term containing $\pi_\gamma c$ in the action. These further contractions lead to the following non-local expression at two-loop level:

$$\begin{aligned}\mathcal{A}_{2,\rm nl} \;=\; & \Big(\frac{16}{30\pi}(1-17a)-\frac{a}{12\pi}C_{mat}\Big)\\ & \times\int d^2z\Big[\Big(\partial^3 h-\partial^3 K_{\pi_c}\Big)\Big(2\partial\gamma\frac{\partial^2}{\bar\partial}(K_{\pi_c}-B)-\frac{5}{6}\gamma\frac{\partial^3}{\bar\partial}(K_{\pi_\gamma})\Big)\\ & +\partial^3 c\Big(\frac{5}{6}(B-K_{\pi_\gamma})\frac{\partial^3}{\bar\partial}(K_{\pi_\gamma}-B)-2(\partial B-\partial K_{\pi_\gamma})\frac{\partial^2}{\bar\partial}(K_{\pi_\gamma}-B)\Big)\\ & -\frac{3}{10}\Big(\partial\gamma(B-K_{\pi_\gamma})-\gamma(\partial B-\partial K_{\pi_\gamma})\Big)\frac{\partial^5}{\bar\partial}(K_{\pi_c}-h)\Big]. \qquad \text{(C.45)}\end{aligned}$$

The above anomaly, although non-local, can lead to local expressions in the Wess-Zumino consistency condition.[16] The Wess-Zumino consistency condi-

[16]Let us emphasize that this feature is not encountered in W_∞ and arises in W_3 gravity as a result of the nonlinearity of W_3 algebra.

tion at two-loop level is

$$(S_0, \mathcal{A}_2) + (S_0, \mathcal{A}_{2,\mathrm{nl}})_{\mathrm{loc}} + (S_1, \mathcal{A}_1) = 0, \tag{C.46}$$

where the second term is restricted to local contributions after taking the antibracket. Insertion of the calculated results (C.42), (C.43) and (C.45) in (C.46) shows that the consistency condition indeed is satisfied at order $\hbar^2$.

C.5 The relationship between the Ward-identity and BRST-algebra anomalies

In Section (4.2), we have obtained a relationship (4.14) between the Ward identity and the BRST algebra anomalies. In this section we verify this relationship (4.14) for the W_3 chiral anomalies given in (C.42) and (C.43). In order to do this, we need to have an expression for the BRST algebra anomalies. This anomaly is obtained by taking the full contraction of two BRST charges (C.38) in the operator-product expansion, and is as follows:

$$\begin{aligned} Q^2 &= \frac{1}{2\pi i}\oint \langle J_B J_B\rangle_1; \\ \langle J_B J_B\rangle_1 &= \frac{1}{6}(100 - C_{\mathrm{mat}})\, c\, \partial^3 c \\ &\quad + \Big(-\frac{16}{15}(1-17a) + \frac{a}{6}C_{\mathrm{mat}}\Big)\, \gamma\, \pi_c\, \partial\gamma\, \partial^3 c \\ &\quad + \Big(\frac{29}{25}(1-17a) - \frac{C_{\mathrm{mat}}}{180}\Big)\, \gamma\, \partial^5\gamma. \end{aligned} \tag{C.47}$$

Using this, we can express the non-vanishing Ward-identity anomalies (C.42) and (C.43) as

$$\mathcal{A}_1 + \mathcal{A}_2 = -\frac{1}{2\pi}\int \Big\langle \langle J_B J_B\rangle_1(z), \Psi_{W_3}\Big\rangle_1, \tag{C.48}$$

where Ψ_{W_3}, the gauge fermion obtained from the extended action corresponding to (C.40), is as follows:[17]

$$\Psi_{W_3} = \Big(\pi_c(h - K_{\pi_c}) + \pi_\gamma(B - K_{\pi_\gamma}) - cK_c\Big). \tag{C.49}$$

The simplicity of the relation (C.48) and the fact that it is valid for W_∞ as well as W_3 gravity suggests a general result for the relation between the BRST-algebra anomalies and the anomalies occurring in BRST Ward identities.

[17]The W_3 gauge fermion can be obtained in a similar fashion to that derived for W_∞ gravity in Chapters 4 and 5. The W_3 action (C.40) is written in a similar form as (4.13) containing only canonical and BRST trivial terms and the gauge fermion is then extracted from the BRST-trivial term.

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