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On the equations of motion of point-particle binaries at the third post-Newtonian order

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Abstract

We investigate the dynamics of two point-like particles through the third post-Newtonian (3PN) approximation of general relativity. The infinite self-field of each point-mass is regularized by means of Hadamard's concept of "partie finie". Distributional forms associated with the regularization are used systematically in the computation. We determine the stress-energy tensor of point-like particles compatible with the previous regularization. The Einstein field equations in harmonic coordinates are iterated to the 3PN order. The 3PN equations of motion are Lorentz-invariant and admit a conserved energy (neglecting the 2.5PN radiation reaction). They depend on an undetermined coefficient, in agreement with an earlier result of Jaranowski and Schäfer. This suggests an incompleteness of the formalism (in this stage of development) at the 3PN order. In this Letter we present the equations of motion in the center-of-mass frame and in the case of circular orbits. © 2000 Elsevier Science B.V. All rights reserved.

A cardinal problem in gravitational physics is that of the dynamics of binary systems of point particles. In general relativity, this problem is tackled by means of the post-Newtonian approximation, or formal expansion when the speed of light c goes to infinity. By definition, the n PN approximation refers to the terms in the equations of motion that are smaller than the Newtonian force by a factor of order $1/c^{2n}$. For the motion of two non-spinning point particles,

the 1PN approximation was obtained first by Lorentz and Droste [1,2]. Subsequently, Einstein, Infeld and Hoffmann [3] re-derived the 1PN order using their famous "surface-integral" method. In the eighties, Damour and Deruelle [4,5], starting from a "post-Minkowskian" iteration scheme developed by Bel et al. [6], were able to compute the equations of motion up to the 2.5PN order, at which the gravitational-radiation reaction effects first take place. The motivation was to firmly establishing the rate at which the orbit of the binary pulsar PSR 1913 + 16 decays because of gravitational-radiation emission. The 2.5PN approximation was then obtained by Schäfer [7,8] using an "ADM Hamiltonian" approach initi-

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ated by Ohta et al. [9,10]. Furthermore, Kopeikin et al. [11,12] derived the same result within their “extended-body” method (without any need of a regularization). More recently, the 2.5PN equations of motion as well as 2.5PN gravitational field were derived by Blanchet, Faye and Ponsot [13] applying a direct “post-Newtonian” iteration of the field equations. Finally, Jaranowski and Schäfer [14,15] investigated within the Hamiltonian approach the 3PN order and found some ambiguities linked to the regularization of the self-field of point masses. As for them, the 3.5PN terms in the equations of motion are well-known [16–18]; they are associated with higher-order radiation reaction effects.

The motivation for working out the 3PN equations of motion is not the timing of the binary pulsar anymore, but the detection of gravitational radiation by future experiments such as LIGO and VIRGO. Indeed, the 3PN equations are needed in particular to construct accurate 3.5PN templates for detecting and analyzing the waves generated by inspiralling compact binaries [19–23]. Currently, we know the complete templates up to the 2.5PN order [24–26], plus the contribution therein of non-linear effects to the 3.5PN order [27], plus all the terms in the vanishing mass-ratio limit to the 5.5PN order [28,29]. In this Letter, we outline our derivation of the 3PN equations of motion, based on the direct post-Newtonian approach of [13], and we present the result in the case, appropriate to inspiralling compact binaries, of circular orbits. We confirm by means of a well-defined regularization à la Hadamard the finding of Jaranowski and Schäfer [14,15] that there remains at the 3PN order an undetermined coefficient appearing in front of a quartically non-linear term (proportional to G^4).

Consider the class \mathcal{F} of functions $F(\mathbf{x})$ that are smooth on \mathbb{R}^3 except at two isolated points \mathbf{y}_1 and \mathbf{y}_2 , around which they admit a power-like singular expansion of the form

$$\forall n \in \mathbb{N}, \quad F(\mathbf{x}) = \sum_{a_0 \leq a \leq n} r_{11}^a f_a(\mathbf{n}_1) + o(r_1^n)$$

when $r_1 \rightarrow 0$, (1)

where $r_1 = |\mathbf{x} - \mathbf{y}_1|$, $\mathbf{n}_1 = (\mathbf{x} - \mathbf{y}_1)/r_1$, and where the powers a are supposed to be real, to range in dis-

crete steps: $a \in (a_i)_{i \in \mathbb{N}}$, and to be bounded from below: $a_0 \leq a$. The coefficients ${}_1f_a$ of the various powers of r_1 in this expansion are smooth functions of the unit vector \mathbf{n}_1 . We refer to the coefficients ${}_1f_a$ with $a < 0$ as the *singular* coefficients of F around 1; their number is always finite. Moreover, we have the same type of expansion around the other point (when $r_2 \rightarrow 0$). The Hadamard “partie finie” [30] of F at the location of the singular point 1 is equal to the angular average of the zeroth-order coefficient in (1), i.e.

$$(F)_1 = \int \frac{d\Omega_1}{4\pi} f_0(\mathbf{n}_1), \quad (2)$$

with $d\Omega_1 = d\Omega(\mathbf{n}_1)$ the usual solid angle element. The partie finie is “non-distributive” in the sense that $(FG)_1 \neq (F)(G)_1$ in general. Besides (2), we define also the partie finie (Pf) of the divergent integral $\int d^3\mathbf{x} F$, assuming that F decreases sufficiently rapidly when $|\mathbf{x}| \rightarrow +\infty$ so that the divergencies come only from the singular points 1 and 2. With full generality [30,31],

$$\begin{aligned} \text{Pf}_{s_1, s_2} \int d^3\mathbf{x} F &= \lim_{s \rightarrow 0} \left\{ \int_{\mathbb{R}^3 \setminus \mathcal{B}_1(s) \cup \mathcal{B}_2(s)} d^3\mathbf{x} F \right. \\ &\quad + 4\pi \sum_{a+3 < 0} \frac{s^{a+3}}{a+3} \left(\frac{F}{r_1^a} \right)_1 \\ &\quad \left. + 4\pi \ln \left(\frac{s}{s_1} \right) (r_1^3 F)_1 + 1 \leftrightarrow 2 \right\}. \end{aligned} \quad (3)$$

The first term is the finite integral over \mathbb{R}^3 deprived from the two spherical balls $\mathcal{B}_1(s)$ and $\mathcal{B}_2(s)$ with radius s and centred on the two singularities. The extra terms are such that they exactly cancel out the divergent part of the integral when $s \rightarrow 0$ (the notation $1 \leftrightarrow 2$ indicates the same extra terms but referring to the other singularity point). The logarithmic

terms depend on two arbitrary positive constants s_1 and s_2 that come from the arbitrariness in the choice of unit length for measuring s ; hence, the partie finie depends on both s_1 and s_2 (as indicated by the notation Pf_{s_1, s_2}). Applying (3) to the case of a gradient $\partial_i F$, we find [33]

$$\text{Pf} \int d^3 \mathbf{x} \partial_i F = -4\pi (n_1^i r_1^2 F)_1 + 1 \leftrightarrow 2. \quad (4)$$

In words, the integral of a gradient is equal to the sum of the surface integrals surrounding the two singularities, in the limit where the surface areas shrink to zero and following the regularization (2). Thus, the integral of a gradient is not zero in general, which shows that the ‘‘ordinary’’ derivative $\partial_i F$ is not adequate for applying to point-particles a formalism initially valid for continuous sources, since in the latter case the integral of a gradient does never contribute. To define a ‘‘better’’ notion of derivative, we must construct the distributional forms associated with the functions in the class \mathcal{F} .

For any $F \in \mathcal{F}$, we consider the ‘‘pseudo-function’’ $\text{Pf} F$ defined as the linear form on \mathcal{F} such that $\forall G \in \mathcal{F}$, $\langle \text{Pf} F, G \rangle = \text{Pf} \int d^3 \mathbf{x} FG$, the duality bracket denoting here the result of the action of $\text{Pf} F$ on the function G . The product of pseudo-functions is defined to be the ordinary pointwise product that we use in physics, i.e. $\text{Pf} F \cdot \text{Pf} G = \text{Pf}(FG)$. With the help of the Riesz [34] delta-function $\delta_\varepsilon(\mathbf{x} - \mathbf{y}_1) = [\varepsilon(1 - \varepsilon)/4\pi] r_1^{\varepsilon-3}$, which belongs to the class \mathcal{F} , we construct [33] the pseudo-function $\text{Pf} \delta_1$ (in the limit $\varepsilon \rightarrow 0$); by definition: $\forall F \in \mathcal{F}$, $\langle \text{Pf} \delta_1, F \rangle = (F)_1$. Clearly $\text{Pf} \delta_1$ generalizes the standard Dirac distribution $\delta_1 \equiv \delta(\mathbf{x} - \mathbf{y}_1)$ to the case of the Hadamard regularization of the functions in \mathcal{F} . Furthermore, consistently with the product of pseudo-functions, we construct the object $\text{Pf}(F\delta_1)$ which is such that $\forall G \in \mathcal{F}$, $\langle \text{Pf}(F\delta_1), G \rangle = (FG)_1$. A trivial consequence of the non-distributivity of the Hadamard partie finie is that $\text{Pf}(F\delta_1) \neq (F)_1 \text{Pf} \delta_1$ in general cases. The derivative of the pseudo-function $\text{Pf} F$ is then obtained from the requirements that (i) the ‘‘rule of integration by parts’’ is satisfied, i.e. $\forall F, G \in \mathcal{F}$, $\langle \partial_i(\text{Pf} F), G \rangle = -\langle \partial_i(\text{Pf} G), F \rangle$, (ii) the derivative reduces to the ‘‘ordinary’’ one in the case where all the singular coefficients of F vanish. These requirements imply in particular that $\langle \partial_i(\text{Pf} F), 1 \rangle = 0$, i.e. the integral of a gradient is

zero. A derivative operator satisfying (i) and (ii) is given by [33]

$$\partial_i(\text{Pf} F) = \text{Pf} \left(\partial_i F + 4\pi n_1^i \left[\frac{1}{2} r_{11} f_{-1} + \sum_{k \geq 0} \frac{1}{r_{11}^k} f_{-2-k} \right] \delta_1 + 1 \leftrightarrow 2 \right) \quad (5)$$

(assuming for simplicity that the ${}_1 f_a$'s have $a \in \mathbb{Z}$). This derivative reduces to the standard distributional derivative of Schwartz [31] when applied on smooth functions with compact support. We refer to [33] for the construction of the most general derivative operator satisfying (i), (ii) and, in addition, (iii) the rule of commutation of derivatives [not obeyed by (5)]. One can show however that it does not satisfy in general the Leibniz rule for the derivative of a product. The derivative (5) is sufficient in the derivation of the results below. See [33] for details about the Hadamard regularization and the associated pseudo-functions.

In the post-Newtonian application we are led to consider the partie finie, in the sense of (3), of the Poisson integral of F , i.e. $\text{Pf} \int d^3 \mathbf{x} F(\mathbf{x})/|\mathbf{x} - \mathbf{x}'|$; more specifically, we are interested in the regularized value, in the sense of (2), of the latter Poisson integral at the location of the singular point 1, i.e. when $r'_1 = |\mathbf{x}' - \mathbf{y}_1| \rightarrow 0$. We obtain [33]

$$\begin{aligned} & \left(\text{Pf}_{s_1, s_2} \int \frac{d^3 \mathbf{x}}{|\mathbf{x} - \mathbf{x}'|} F \right)_1 \\ &= \text{Pf}_{s_1, s_2} \int \frac{d^3 \mathbf{x}}{r_1} F - 4\pi \left[\ln \left(\frac{r'_1}{s_1} \right) - 1 \right] (r_1^2 F)_1 \\ &= 4\pi \ln \left(\frac{r_{12}}{r'_1} \right) (r_1^2 F)_1 + 4\pi \ln \left(\frac{r_{12}}{s_2} \right) \\ & \quad \times \left(\frac{r_2^3}{r_1} F \right)_2 + \dots \end{aligned} \quad (6a)$$

(with $r_{12} = |\mathbf{y}_1 - \mathbf{y}_2|$). The first term in (6a) represents simply what we get by replacing formally \mathbf{x}' by

\mathbf{y}_1 inside the integrand of the Poisson integral. The second term is due to the presence of some logarithms $\ln r'_1$ in the expansion of the integral. (An adaptation of the previous formalism, detailed in [33], is needed to take these logarithms into account, as well as the presence of the integrable singularity \mathbf{x}' .) As, at last, the $\ln r'_1$ can be gauged away, we regard it as a constant, taking some finite value (even though $r'_1 \rightarrow 0$). We check, on the other hand, that the true constant s_1 cancels out between the two terms of (6a), so that the result depends only on $\ln r'_1$ and $\ln s_2$. The complete dependence of the partie finie on these constants is shown in (6a), with the convention that the dots indicate the terms independent of the constants.

The Einstein field equations relaxed by the condition of harmonic coordinates [i.e. $\partial_\nu h^{\mu\nu} = 0$ with $h^{\mu\nu} = \sqrt{-g} g^{\mu\nu} - \eta^{\mu\nu}$; $g = \det g_{\mu\nu}$; $\eta^{\mu\nu} = \text{diag}(-1, 1, 1, 1)$] read as

$$\square h^{\mu\nu} = \frac{16\pi G}{c^4} (-g) T^{\mu\nu} + \Lambda^{\mu\nu} [h, \partial h, \partial^2 h], \quad (7)$$

where $T^{\mu\nu}$ is the matter stress-energy tensor and $\Lambda^{\mu\nu}$ a complicated functional of h which is at least of order $O(h^2)$ (and where $\square = \eta^{\mu\nu} \partial_\mu \partial_\nu$). We start by constructing a post-Newtonian solution of (7), initially valid in the case of a continuous (“fluid”) matter tensor $T^{\mu\nu}$, and parametrized by some appropriate potentials. We define a “Newtonian” potential $V = \square_R^{-1} [-4\pi G \sigma]$ where \square_R^{-1} denotes the standard retarded integral and $\sigma = (T^{00} + T^{ii})/c^2$; we also introduce a 1PN “gravitomagnetic” potential $V_i = \square_R^{-1} [-4\pi G \sigma_i]$ where $\sigma_i = T^{0i}/c$; some 2PN potentials \hat{X} , \hat{R}_i and \hat{W}_{ij} , e.g. $\hat{W}_{ij} = \square_R^{-1} [-4\pi G (\sigma_{ij} - \delta_{ij} \sigma_{kk}) - \partial_i V \partial_j V]$ where $\sigma_{ij} = T^{ij}$; and finally some 3PN potentials \hat{T} , \hat{Y}_i and \hat{Z}_{ij} . In particular, the potential \hat{W}_{ij} generates the non-linear term $\square_R^{-1} [\hat{W}_{ij} \partial_{ij} V]$, involving a cubic (G^3) contribution, which is part of the potential \hat{X} (many other cubic terms are contained in \hat{T} and \hat{Y}_i). With a specific choice of potentials we can arrange that all the quartic (G^4) terms in the metric appear in “all-integrated” form. Since V is dominantly Newtonian, it needs to be evaluated at the 3PN order but, for instance, the term $\square_R^{-1} [\hat{W}_{ij} \partial_{ij} V]$, inside the 2PN potential \hat{X} , needs only a relative 1PN precision.

The metric is expressed as a functional of all these potentials; and with our particular choice of potentials, it turns out not to be too complicated.

An important point is now to determine the expression of the matter stress-energy tensor $T^{\mu\nu}$ appropriate to the description of point-particles. We demand that the dynamics of point-masses follows from the variation, with respect to the metric, of the action

$$I_{\text{point-particle}} = -m_1 c \int_{-\infty}^{+\infty} dt \sqrt{-(g_{\mu\nu})_1 v_1^\mu v_1^\nu} + 1 \leftrightarrow 2, \quad (8)$$

where $v_1^\mu = (c, d\mathbf{y}_1/dt)$ is the coordinate velocity of particle 1. We can check that to the 3PN order all the metric coefficients $g_{\mu\nu}$ belong to \mathcal{F} (treating $\ln r'_1$ as a constant); so $(g_{\mu\nu})_1$ in (8) denotes the value of the metric at 1 in the sense of (2) [or, rather, in the sense of a Lorentz-covariant Hadamard regularization defined below]. The stationarity of the action with respect to a metric variation within the class \mathcal{F} (i.e. $\delta g_{\mu\nu} \in \mathcal{F}$) yields the stress-energy tensor

$$T_{\text{point-particle}}^{\mu\nu} = \frac{m_1 v_1^\mu v_1^\nu}{\sqrt{-(g_{\rho\sigma})_1 v_1^\rho v_1^\sigma / c^2}} \text{Pf} \left(\frac{\delta_1}{\sqrt{-g}} \right) + 1 \leftrightarrow 2, \quad (9)$$

where the pseudo-function $\text{Pf}((1/\sqrt{-g})\delta_1)$ is of the type $\text{Pf}(F\delta_1)$ defined before. [From the rule of multiplication of pseudo-functions we find that the matter source term in (7) involves the pseudo-function $\text{Pf}(\sqrt{-g}\delta_1)$.] To obtain the equations of motion of the particle 1 we integrate the matter equations of motion $\nabla_\nu T_{\text{point-particle}}^{\mu\nu} = 0$ over a volume surrounding 1 (exclusively), and use the properties of pseudo-functions. The equations turn out to have the same form as the geodesic equations, not with respect to some smooth background but with respect to the regularized metric generated by the two bodies. Namely,

$$\frac{d}{dt} \left(\frac{(g_{\mu\nu})_1 v_1^\nu}{\sqrt{-(g_{\rho\sigma})_1 v_1^\rho v_1^\sigma}} \right) = \frac{1}{2} \frac{(\partial_\mu g_{\nu\lambda})_1 v_1^\nu v_1^\lambda}{\sqrt{-(g_{\rho\sigma})_1 v_1^\rho v_1^\sigma}}, \quad (10)$$

where all the quantities at 1 are evaluated using the regularization. Let us emphasize that the equations of motion (10) are derived from the specific expression (9) of the stress-energy tensor; had we used another expression, e.g. by replacing $\text{Pf}((1/\sqrt{-g})\delta_1) \rightarrow (1/\sqrt{-g})_1 \text{Pf} \delta_1$ inside (9) (which is forbidden by the non-distributivity of Hadamard's partie finie), we would have obtained some different-looking, and a priori incorrect, equations.

The regularization (2) is defined *stricto sensu* within the spatial slice $t = \text{const.}$, and therefore should prevent, at some stage, the equations of motion from being Lorentz invariant (recall that the harmonic-gauge condition preserves the Lorentz invariance). It is known that regularizing within the slice $t = \text{const.}$ yields the correct, Lorentz-invariant, equations of motion up to the 2PN level [13]. We find that the breakdown of the Lorentz invariance due to the regularization (2) occurs precisely at the 3PN order. Therefore, starting at this order, we *must* in fact apply a Lorentz-covariant regularization in (8), (9), (10). Evidently, the good thing to do is to apply the Hadamard regularization in the frame at which the particle is instantaneously at rest. Let us consider the Lorentz boost $x'^{\mu} = \Lambda_{\nu}^{\mu}(\mathbf{V})x^{\nu}$, where \mathbf{V} denotes the constant boost velocity. We replace all the quantities in the original frame by their equivalent expressions, developed to the 3PN order, in the new frame. Notably, the trajectories $\mathbf{y}_1(t)$, $\mathbf{y}_2(t)$ and velocities $\mathbf{v}_1(t)$, $\mathbf{v}_2(t)$ are replaced by certain functionals of \mathbf{x}' and the new trajectories $\mathbf{y}'_1(t')$, $\mathbf{y}'_2(t')$ and velocities $\mathbf{v}'_1(t')$, $\mathbf{v}'_2(t')$ (where $t' = x'^0/c$). We apply the Hadamard regularization within the slice $t' = \text{const.}$, keeping \mathbf{V} as a constant ‘‘spectator’’ vector. Finally, we re-express all the quantities back into the original frame at the point 1 ($r_1 \rightarrow 0$), under the condition that $\mathbf{v}'_1(t') = 0$ and (equivalently) $\mathbf{V} = \mathbf{v}_1(t)$. This ensures that the new frame is indeed the rest frame of the particle 1 at the instant t . Now the 3PN equations of motion are Lorentz invariant.

All the potentials V , V_i , \hat{W}_{ij} , \dots and their gradients are computed at the point 1, using the regularization of Poisson-type integrals defined by formulas like (6a). All the derivatives appearing inside the non-linear sources of the potentials are considered as distributional and evaluated following the prescription (5). We carefully take into account the fact that the distributional derivative does not

obey the Leibniz rule (it does satisfy it only in an ‘‘integrated’’ sense, thanks to the rule of integration by parts). An important feature of the equations at the 3PN order is the occurrence of some logarithms. From (6a) we know that they are necessarily of the type $\ln(r_{12}/r'_1)$ and $\ln(r_{12}/s_2)$ in the equations of motion of body 1; interestingly, the $\ln(r_{12}/s_2)$ appears only in a quartic-interaction term proportional to $G^4 m_1 m_2^3$. Thus, at this stage, the 3PN equations of 1 depend on the constants $\ln r'_1$ and $\ln s_2$ (and idem for the equations of 2). Under the form we obtain them, the equations do not yet admit a conserved energy (of course we are speaking only about the conservative part of the acceleration, which excludes the radiation-reaction potential at 2.5PN order). However, we find that a conserved energy exists if and only if the logarithmic ratios $\ln(r'_2/s_2)$ and $\ln(r'_1/s_1)$ are adjusted in such a way that

$$\ln\left(\frac{r'_2}{s_2}\right) = \frac{159}{308} + \lambda \frac{m_1 + m_2}{m_2} \quad \text{and } 1 \leftrightarrow 2, \quad (11)$$

where λ is a single numerical constant. If (and only if) the condition (11) is realized, the equations admit an energy and, in fact, a Lagrangian formulation; in this case, they depend on some arbitrary constant λ . The dependence upon the masses in (11) is *a priori* allowed. Therefore, the formalism introduces at this point an undetermined constant λ . [Using (11), the equations of motion depend also on the constants $\ln r'_1$ and $\ln r'_2$, but it can be checked that the latter dependence is pure gauge.]

Finally, having in view the application to inspiralling compact binaries, we present the 3PN relative acceleration and center-of-mass energy in the case of circular orbits. The acceleration reads as

$$\frac{d\mathbf{v}_{12}}{dt} = -\omega^2 \mathbf{y}_{12} + \frac{1}{c^5} \mathbf{F}_{\text{reac}} + O\left(\frac{1}{c^7}\right), \quad (12)$$

where $\mathbf{y}_{12} = \mathbf{y}_1 - \mathbf{y}_2$ is the relative separation in harmonic coordinates, $\mathbf{v}_{12} = d\mathbf{y}_{12}/dt$ the relative velocity, and $\mathbf{F}_{\text{reac}} = -\frac{32}{5}(G^3 m^3 \nu / r_{12}^4) \mathbf{v}_{12}$ the standard radiation-reaction force at the 2.5PN order. The mass parameters are $m = m_1 + m_2$, $\mu = m_1 m_2 / m$ and $\nu = \mu / m$. The content of the 3PN approximation in (12)

lies in the relation between the orbital frequency ω and the coordinate distance $r_{12} = |\mathbf{y}_{12}|$. With $\gamma = Gm/r_{12}c^2$ denoting a small post-Newtonian parameter, we get

$$\begin{aligned} \omega^2 = & \frac{Gm}{r_{12}^3} \left\{ 1 + (-3 + \nu)\gamma + \left(6 + \frac{41}{4}\nu + \nu^2\right)\gamma^2 \right. \\ & + \left(-10 + \left[-\frac{67759}{840} + \frac{41}{64}\pi^2 + 22\ln\left(\frac{r_{12}}{r'_0}\right) \right. \right. \\ & \left. \left. + \frac{44}{3}\lambda \right] \nu + \frac{19}{2}\nu^2 + \nu^3 \right) \gamma^3 + O(\gamma^4) \left. \right\}. \quad (13) \end{aligned}$$

The logarithm at 3PN depends on a constant r'_0 defined as the ‘‘logarithmic’’ barycenter of the two constants r'_1 and r'_2 , namely $\ln r'_0 = (m_1/m)\ln r'_1 + (m_2/m)\ln r'_2$. The constant r'_0 can be eliminated by a change of coordinates. The center-of-mass energy E of the particles, such that $dE/dt = 0$ as a consequence of the conservative equations of motion (neglecting \mathbf{F}_{reac}), is obtained as

$$\begin{aligned} E = & -\frac{1}{2}\mu c^2 \gamma \left\{ 1 + \left(-\frac{7}{4} + \frac{1}{4}\nu\right)\gamma \right. \\ & + \left(-\frac{7}{8} + \frac{49}{8}\nu + \frac{1}{8}\nu^2\right)\gamma^2 + \left(-\frac{235}{64} \right. \\ & + \left[\frac{106301}{6720} - \frac{123}{64}\pi^2 + \frac{22}{3}\ln\left(\frac{r_{12}}{r'_0}\right) - \frac{22}{3}\lambda \right] \nu \\ & \left. + \frac{27}{32}\nu^2 + \frac{5}{64}\nu^3 \right) \gamma^3 + O(\gamma^4) \left. \right\}. \quad (14) \end{aligned}$$

At last, by substituting the expression of γ in terms of the orbital frequency ω following from the inverse of (13), we find that the 3PN energy in invariant form is given by

$$\begin{aligned} E = & -\frac{1}{2}\mu c^2 x \left\{ 1 + \left(-\frac{3}{4} - \frac{1}{12}\nu\right)x \right. \\ & + \left(-\frac{27}{8} + \frac{19}{8}\nu - \frac{1}{24}\nu^2\right)x^2 \\ & + \left(-\frac{675}{64} + \left[\frac{209323}{4032} - \frac{205}{96}\pi^2 - \frac{110}{9}\lambda \right] \nu \right. \\ & \left. - \frac{155}{96}\nu^2 - \frac{35}{5184}\nu^3 \right) x^3 + O(x^4) \left. \right\}, \quad (15) \end{aligned}$$

with $x = (Gm\omega/c^3)^{2/3}$. In the form (15) the logarithm $\ln(r_{12}/r'_0)$ cancels out. We can compare directly this result with the one obtained by Jaranowski and Schäfer [14,15] (see Eq. (5.13) in [32]). We find that there is perfect agreement provided that $\omega_{\text{static}} = -\frac{11}{3}\lambda - \frac{1987}{840}$ and $\omega_{\text{kinetic}} = \frac{41}{24}$, where ω_{static} and ω_{kinetic} are the two ‘‘ambiguous’’ parameters found by Jaranowski and Schäfer. Thus, our undetermined constant λ defined by (11) is related to the ambiguous parameter ω_{static} , while the other ambiguity ω_{kinetic} takes a unique value¹. Let us stress that in the present formalism we do not meet any ambiguity in the sense of Jaranowski and Schäfer. Rather, the formalism is well-defined thanks in particular to the rules we employ for handling the pseudo-functions associated with the Hadamard regularization [33]. All the integrals encountered in the problem have been given a precise mathematical sense, and are computed by means of a uniquely defined prescription. Yet, the appearance of the undetermined constant λ suggests that the present formalism might be physically incomplete, at least in this present stage of development. Notice that the constant λ enters only the term proportional to $G^4 m_1^2 m_2^2 (m_1 + m_2)$ in the 3PN energy, and that for general orbits, the energy contains also 164 other terms which are all uniquely determined. The details of these calculations will be published elsewhere.

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¹ Recently Damour et al. [35] were able to compute ω_{kinetic} by imposing the global Poincaré invariance of the ADM Hamiltonian. They checked that their result is in agreement with ours by comparing with the numerical value of ω_{kinetic} which was communicated to them by one of us (L.B.) in advance of the present publication and before the completion of their work.

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