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Gravitational Radiation from Post-Newtonian Sources and Inspiralling Compact Binaries

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Abstract

The article reviews the current status of a theoretical approach to the problem of the emission of gravitational waves by isolated systems in the context of general relativity. Part A of the article deals with general post-Newtonian sources. The exterior field of the source is investigated by means of a combination of analytic post-Minkowskian and multipolar approximations. The physical observables in the far-zone of the source are described by a specific set of radiative multipole moments. By matching the exterior solution to the metric of the post-Newtonian source in the near-zone we obtain the explicit expressions of the source multipole moments. The relationships between the radiative and source moments involve many non-linear multipole interactions, among them those associated with the tails (and tails-of-tails) of gravitational waves. Part B of the article is devoted to the application to compact binary systems. We present the equations of binary motion, and the associated Lagrangian and Hamiltonian, at the third post-Newtonian (3PN) order beyond the Newtonian acceleration. The gravitational-wave energy flux, taking consistently into account the relativistic corrections in the binary moments as well as the various tail effects, is derived through 3.5PN order with respect to the quadrupole formalism. The binary's orbital phase, whose prior knowledge is crucial for searching and analyzing the signals from inspiralling compact binaries, is deduced from an energy balance argument.

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Contents

1 Introduction

The theory of gravitational radiation from isolated sources, in the context of general relativity, is a fascinating science that can be explored by means of what was referred to in the French XVIIIth century as *l'analyse sublime*: the analytical (i.e. mathematical) method, and more specifically the resolution of partial differential equations. Indeed, the field equations of general relativity, when use is made of the harmonic-coordinate conditions, take the form of a quasi-linear hyperbolic differential system of equations, involving the famous wave operator or d'Alembertian (denoted $\Box$), invented by d'Alembert in his Traité de dynamique of 1743.

Nowadays, the importance of the field lies in the exciting possibility of comparing the theory with contemporary astrophysical observations, made by a new generation of detectors – large-scale optical interferometers LIGO, VIRGO, GEO and TAMA – that should routinely observe the gravitational waves produced by massive and rapidly evolving systems such as inspiralling compact binaries. To prepare these experiments, the required theoretical work consists of carrying out a sufficiently general solution of the Einstein field equations, valid for a large class of matter systems, and describing the physical processes of the emission and propagation of the waves from the source to the distant detector, as well as their back-reaction onto the source.

1.1 Gravitational-wave generation formalisms

The basic problem we face is to relate the asymptotic gravitational-wave form h_{ij} generated by some isolated source, at the location of some detector in the wave zone of the source, to the stress-energy tensor $T^{\alpha\beta}$ of the matter fields[1]. For general sources it is hopeless to solve the problem *via* a rigorous deduction within the exact theory of general relativity, and we have to resort to approximation methods, keeping in mind that, sadly, such methods are often not related in a very precise mathematical way to the first principles of the theory. Therefore, a general wave-generation formalism must solve the field equations, and the non-linearity therein, by imposing some suitable approximation series in one or several small physical parameters. Of ourse the ultimate aim of approximation methods is to extract from the theory some firm predictions for the outcome of experiments such as VIRGO and LIGO. Some important approximations that we shall use in this article are the post-Newtonian method (or non-linear $1/c$-expansion), the post-Minkowskian method or non-linear iteration (G-expansion), the multipole decomposition in irreducible representations of the rotation group (or equivalently a-expansion in the source radius), and the far-zone expansion ($1/R$-expansion in the distance). In particular, the post-Newtonian expansion has provided us in the past with our best insights into the problems of motion and radiation in general relativity. The most successful wave-generation formalisms make a gourmet cocktail of all these approximation methods. For reviews on analytic approximations and applications to the motion and the gravitational wave-generation see Refs. [211, 83, 84, 212, 218, 17, 22].

The post-Newtonian approximation is valid under the assumptions of a weak gravitational field inside the source (we shall see later how to model neutron stars and black holes), and of slow internal motions. The main problem with this approximation is its domain of validity, which is limited to the near zone of the source – the region surrounding the source that is of small extent with respect to the wavelength of waves. A serious consequence is the *a priori* inability of the post-Newtonian expansion to incorporate the boundary conditions at infinity, which determine the radiation reaction force in the source's local equations of motion. The post-Minkowskian expansion, by contrast, is uniformly valid, as soon as the source is weakly self-gravitating, over all space-time. In a sense, the post-Minkowskian method is more fundamental than the post-Newtonian one; it can be regarded as an "upstream" approximation with respect to the post-Newtonian expansion,

[1] In this article Greek indices take the values $0, 1, 2, 3$ and Latin $1, 2, 3$. Our signature is $+2$. G and c are Newton's constant and the speed of light.

because each coefficient of the post-Minkowskian series can in turn be re-expanded in a post-Newtonian fashion. Therefore, a way to take into account the boundary conditions at infinity in the post-Newtonian series is *first* to perform the post-Minkowskian expansion. Notice that the post-Minkowskian method is also upstream (in the previous sense) with respect to the multipole expansion, when considered outside the source, and with respect to the far-zone expansion, when considered far from the source.

The most "downstream" approximation that we shall use in this article is the post-Newtonian one; therefore this is the approximation that dictates the allowed physical properties of our matter source. We assume mainly that the source is at once *slowly moving* and *weakly stressed*, and we abbreviate this by saying that the source is *post-Newtonian.* For post-Newtonian sources, the parameter defined from the components of the matter stress-energy tensor $T^{\alpha\beta}$ and the source's Newtonian potential U by

$$\epsilon = \max\left\{\left|\frac{T^{0i}}{T^{00}}\right|, \left|\frac{T^{ij}}{T^{00}}\right|^{1/2}, \left|\frac{U}{c^2}\right|^{1/2}\right\}, \tag{1}$$

is much less than one. This parameter represents essentially a slow motion estimate $\epsilon \sim v/c$, where v denotes a typical internal velocity. By a slight abuse of notation, following Chandrasekhar et al. [66, 68, 67], we shall henceforth write $\epsilon \equiv 1/c$, even though ϵ is dimensionless whereas c has the dimension of a velocity. The small post-Newtonian remainders will be denoted $\mathcal{O}(1/c^n)$. Thus, $1/c \ll 1$ in the case of post-Newtonian sources. We have $|U/c^2|^{1/2} \ll 1/c$ for sources with negligible self-gravity, and whose dynamics are therefore driven by non-gravitational forces. However, we shall generally assume that the source is self-gravitating; in that case we see that it is necessarily *weakly* (but not negligibly) self-gravitating, i.e. $|U/c^2|^{1/2} = \mathcal{O}(1/c)$. Note that the adjective "slow-motion" is a bit clumsy because we shall in fact consider *very* relativistic sources such as inspiralling compact binaries, for which $1/c$ can be as large as 30% in the last rotations, and whose description necessitates the control of high post-Newtonian approximations.

The lowest-order wave generation formalism, in the Newtonian limit $1/c \to 0$, is the famous quadrupole formalism of Einstein [105] and Landau and Lifchitz [153]. This formalism can also be referred to as Newtonian because the evolution of the quadrupole moment of the source is computed using Newton's laws of gravity. It expresses the gravitational field h^{TT}_{ij} in a transverse and traceless (TT) coordinate system, covering the far zone of the source[2], as

$$h^{\mathrm{TT}}_{ij} = \frac{2G}{c^4 R}\mathcal{P}_{ijab}(\mathbf{N})\left\{\frac{d^2\mathrm{Q}_{ab}}{dT^2}(T-R/c) + \mathcal{O}\left(\frac{1}{c}\right)\right\} + \mathcal{O}\left(\frac{1}{R^2}\right), \tag{2}$$

where $R = |\mathbf{X}|$ is the distance to the source, $\mathbf{N} = \mathbf{X}/R$ is the unit direction from the source to the observer, and $\mathcal{P}_{ijab} = \mathcal{P}_{ia}\mathcal{P}_{jb} - \frac{1}{2}\delta_{ij}\mathcal{P}_{ij}\mathcal{P}_{ab}$ is the TT projection operator, with $\mathcal{P}_{ij} = \delta_{ij} - N_i N_j$ being the projector onto the plane orthogonal to $\mathbf{N}$. The source's quadrupole moment takes the familiar Newtonian form

$$\mathrm{Q}_{ij}(t) = \int_{\text{source}} d^3\mathbf{x}\,\rho(\mathbf{x},t)\left(x_i x_j - \frac{1}{3}\delta_{ij}\mathbf{x}^2\right), \tag{3}$$

where ρ is the Newtonian mass density. The total gravitational power emitted by the source in all directions is given by the Einstein quadrupole formula

$$\mathcal{L} = \frac{G}{5c^5}\left\{\frac{d^3\mathrm{Q}_{ab}}{dT^3}\frac{d^3\mathrm{Q}_{ab}}{dT^3} + \mathcal{O}\left(\frac{1}{c^2}\right)\right\}. \tag{4}$$

[2]The TT coordinate system can be extended to the near zone of the source as well; see for instance Ref. [151].

Our notation $\mathcal{L}$ stands for the total gravitational "luminosity" of the source. The cardinal virtues of the Einstein–Landau–Lifchitz quadrupole formalism are its generality – the only restrictions are that the source be Newtonian and bounded – its simplicity, as it necessitates only the computation of the time derivatives of the Newtonian quadrupole moment (using the Newtonian laws of motion), and, most importantly, its agreement with the observation of the dynamics of the Hulse-Taylor binary pulsar PSR 1913+16 [208, 209, 207]. Indeed the prediction of the quadrupole formalism for the waves emitted by the binary pulsar system comes from applying Equation (4) to a system of two point masses moving on an eccentric orbit (the classic reference is Peters and Mathews [178]; see also Refs. [108, 216]). Then, relying on the energy equation

$$\frac{dE}{dt} = -\mathcal{L}, \tag{5}$$

where E is the Newtonian binary's center-of-mass energy, we deduce from Kepler's third law the expression of the "observable", that is, the change in the orbital period P of the pulsar, or $\dot{P}$, as a function of P itself. From the binary pulsar test, we can say that the post-Newtonian corrections to the quadrupole formalism, which we shall compute in this article, have already received, in the case of compact binaries, strong observational support (in addition to having, as we shall demonstrate, a sound theoretical basis).

The multipole expansion is one of the most useful tools of physics, but its use in general relativity is difficult because of the non-linearity of the theory and the tensorial character of the gravitational interaction. In the stationary case, the multipole moments are determined by the expansion of the metric at spatial infinity [120, 129, 201], while, in the case of non-stationary fields, the moments, starting with the quadrupole, are defined at future null infinity. The multipole moments have been extensively studied in the linearized theory, which ignores the gravitational forces inside the source. Early studies have extended the formula (4) to include the current-quadrupole and mass-octupole moments [171, 170], and obtained the corresponding formulas for linear momentum [171, 170, 10, 186] and angular momentum [177, 75]. The general structure of the infinite multipole series in the linearized theory was investigated by several works [191, 192, 181, 210], from which it emerged that the expansion is characterized by two and only two sets of moments: mass-type and current-type moments. Below we shall use a particular multipole decomposition of the linearized (vacuum) metric, parametrized by symmetric and trace-free (STF) mass and current moments, as given by Thorne [210]. The explicit expressions of the multipole moments (for instance in STF guise) as integrals over the source, valid in the linearized theory but irrespective of a slow motion hypothesis, are completely known [159, 65, 64, 89].

In the full non-linear theory, the (radiative) multipole moments can be read off the coefficient of $1/R$ in the expansion of the metric when $R \to +\infty$, with a null coordinate $T - R/c = \mathrm{const}$. The solutions of the field equations in the form of a far-field expansion (power series in $1/R$) have been constructed, and their properties elucidated, by Bondi et al. [53] and Sachs [193]. The precise way under which such radiative space-times fall off asymptotically has been formulated geometrically by Penrose [175, 176] in the concept of an asymptotically simple space-time (see also Ref. [121]). The resulting Bondi–Sachs–Penrose approach is very powerful, but it can answer *a priori* only a part of the problem, because it gives information on the field only in the limit where $R \to +\infty$, which cannot be connected in a direct way to the actual behaviour of the source. In particular the multipole moments that one considers in this approach are those measured at infinity – we call them the *radiative* multipole moments. These moments are distinct, because of non-linearities, from some more natural *source* multipole moments, which are defined operationally by means of explicit integrals extending over the matter and gravitational fields.

An alternative way of defining the multipole expansion within the complete non-linear theory is that of Blanchet and Damour [26, 12], following pioneering work by Bonnor and collaborators [54, 55, 56, 130] and Thorne [210]. In this approach the basic multipole moments are the *source* moments, rather than the radiative ones. In a first stage, the moments are left unspecified, as being some arbitrary functions of time, supposed to describe an actual physical source. They are iterated by means of a post-Minkowskian expansion of the vacuum field equations (valid in the source's exterior). Technically, the post-Minkowskian approximation scheme is greatly simplified by the assumption of a multipolar expansion, as one can consider separately the iteration of the different multipole pieces composing the exterior field (whereas, the direct attack of the post-Minkowskian expansion, valid at once inside and outside the source, faces some calculational difficulties [215, 76]). In this "multipolar-post-Minkowskian" formalism, which is physically valid over the entire weak-field region outside the source, and in particular in the wave zone (up to future null infinity), the radiative multipole moments are obtained in the form of some non-linear functionals of the more basic source moments. *A priori*, the method is not limited to post-Newtonian sources, however we shall see that, in the current situation, the *closed-form* expressions of the source multipole moments can be established only in the case where the source is post-Newtonian [15, 20]. The reason is that in this case the domain of validity of the post-Newtonian iteration (viz. the near zone) overlaps the exterior weak-field region, so that there exists an intermediate zone in which the post-Newtonian and multipolar expansions can be matched together. This is a standard application of the method of matched asymptotic expansions in general relativity [63, 62].

To be more precise, we shall show how a systematic multipolar and post-Minkowskian iteration scheme for the vacuum Einstein field equations yields the most general physically admissible solution of these equations [26]. The solution is specified once we give two and only two sets of time-varying (source) multipole moments. Some general theorems about the near-zone and far-zone expansions of that general solution will be stated. Notably, we find [12] that the asymptotic behaviour of the solution at future null infinity is in agreement with the findings of the Bondi–Sachs–Penrose [53, 193, 175, 176, 121] approach to gravitational radiation. However, checking that the asymptotic structure of the radiative field is correct is not sufficient by itself, because the ultimate aim is to relate the far field to the properties of the source, and we are now obliged to ask: What are the multipole moments corresponding to a given stress-energy tensor $T^{\alpha\beta}$ describing the source? Only in the case of post-Newtonian sources has it been possible to answer this question. The general expression of the moments was obtained at the level of the second post-Newtonian (2PN) order in Ref. [15], and was subsequently proved to be in fact valid up to any post-Newtonian order in Ref. [20]. The source moments are given by some integrals extending over the post-Newtonian expansion of the total (pseudo) stress-energy tensor $\tau^{\alpha\beta}$, which is made of a matter part described by $T^{\alpha\beta}$ and a crucial non-linear gravitational source term $\Lambda^{\alpha\beta}$. These moments carry in front a particular operation of taking the finite part ($\mathcal{FP}$ as we call it below), which makes them mathematically well-defined despite the fact that the gravitational part $\Lambda^{\alpha\beta}$ has a spatially infinite support, which would have made the bound of the integral at spatial infinity singular (of course the finite part is not added *a posteriori* to restore the well-definiteness of the integral, but is *proved* to be actually present in this formalism). The expressions of the moments had been obtained earlier at the 1PN level, albeit in different forms, in Ref. [28] for the mass-type moments (strangely enough, the mass moments admit a compact-support expression at 1PN order), and in Ref. [90] for the current-type ones.

The wave-generation formalism resulting from matching the exterior multipolar and post-Minkowskian field [26, 12] to the post-Newtonian source [15, 20] is able to take into account, in principle, any post-Newtonian correction to both the source and radiative multipole moments (for any multipolarity of the moments). The relationships between the radiative and source moments include many non-linear multipole interactions, because the source moments mix with each other as they "propagate" from the source to the detector. Such multipole interactions include the famous effects of wave tails, corresponding to the coupling between the non-static moments with the total mass M of the source. The non-linear multipole interactions have been computed within the present wave-generation formalism up to the 3PN order in Refs. [29, 21, 19]. Furthermore, the back-reaction of the gravitational-wave emission onto the source, up to the 1.5PN order relative to the leading order of radiation reaction, has also been studied within this formalism [27, 14, 18]. Now, recall that the leading radiation reaction force, which is quadrupolar, occurs already at the 2.5PN order in the source's equations of motion. Therefore the 1.5PN "relative" order in the radiation reaction corresponds in fact to the 4PN order in the equations of motion, beyond the Newtonian acceleration. It has been shown that the gravitational wave tails enter the radiation reaction at precisely the 1.5PN *relative* order, which means 4PN "absolute" order [27]. A systematic post-Newtonian iteration scheme for the near-zone field, formally taking into account all radiation reaction effects, has been recently proposed, consistent with the present formalism [185, 41].

A different wave-generation formalism has been devised by Will and Wiseman [220] (see also Refs. [219, 173, 174]), after earlier attempts by Epstein and Wagoner [107] and Thorne [210]. This formalism has exactly the same scope as ours, i.e. it applies to any isolated post-Newtonian sources, but it differs in the definition of the source multipole moments and in many technical details when properly implemented [220]. In both formalisms, the moments are generated by the post-Newtonian expansion of the pseudo-tensor $\tau^{\alpha\beta}$, but in the Will–Wiseman formalism they are defined by some *compact-support* integrals terminating at some finite radius $\mathcal{R}$ enclosing the source, e.g., the radius of the near zone). By contrast, in our case [15, 20], the moments are given by some integrals covering the whole space and regularized by means of the finite part $\mathcal{FP}$. We shall prove the complete equivalence, at the most general level, between the two formalisms. What is interesting about both formalisms is that the source multipole moments, which involve a whole series of relativistic corrections, are coupled together, in the true non-linear solution, in a very complicated way. These multipole couplings give rise to the many tail and related non-linear effects, which form an integral part of the radiative moments at infinity and thereby of the observed signal.

Part A of this article is devoted to a presentation of the post-Newtonian wave generation formalism. We try to state the main results in a form that is simple enough to be understood without the full details, but at the same time we outline some of the proofs when they present some interest on their own. To emphasize the importance of some key results, we present them in the form of mathematical theorems.

1.2 Problem posed by compact binary systems

Inspiralling compact binaries, containing neutron stars and/or black holes, are promising sources of gravitational waves detectable by the detectors LIGO, VIRGO, GEO and TAMA. The two compact objects steadily lose their orbital binding energy by emission of gravitational radiation; as a result, the orbital separation between them decreases, and the orbital frequency increases. Thus, the frequency of the gravitational-wave signal, which equals twice the orbital frequency for the dominant harmonics, "chirps" in time (i.e. the signal becomes higher and higher pitched) until the two objects collide and merge.

The orbit of most inspiralling compact binaries can be considered to be circular, apart from the gradual inspiral, because the gravitational radiation reaction forces tend to circularize the motion rapidly. For instance, the eccentricity of the orbit of the Hulse–Taylor binary pulsar is presently $e_0 = 0.617$. At the time when the gravitational waves emitted by the binary system will become visible by the detectors, i.e. when the signal frequency reaches about 10 Hz (in a few hundred million years from now), the eccentricity will be $e = 5.3 \times 10^{-6}$ – a value calculated from the Peters [177] law, which is itself based on the quadrupole formula (2).

The main point about modelling the inspiralling compact binary is that a model made of two structureless point particles, characterized solely by two mass parameters m_1 and m_2 (and possibly two spins), is sufficient. Indeed, most of the non-gravitational effects usually plaguing the dynamics of binary star systems, such as the effects of a magnetic field, of an interstellar medium, and so on, are dominated by gravitational effects. However, the real justification for a model of point particles is that the effects due to the finite size of the compact bodies are small. Consider for instance the influence of the Newtonian quadrupole moments Q_1 and Q_2 induced by tidal interaction between two neutron stars. Let a_1 and a_2 be the radius of the stars, and L the distance between the two centers of mass. We have, for tidal moments,

$$\mathrm{Q}_1 = k_1 m_2 \frac{a_1^5}{L^3}, \qquad \mathrm{Q}_2 = k_2 m_1 \frac{a_2^5}{L^3}, \tag{6}$$

where k_1 and k_2 are the star's dimensionless (second) Love numbers [162], which depend on their internal structure, and are, typically, of the order unity. On the other hand, for compact objects, we can introduce their "compactness", defined by the dimensionless ratios

$$K_1 = \frac{G m_1}{a_1 c^2}, \qquad K_2 = \frac{G m_2}{a_2 c^2}, \tag{7}$$

which equal ~ 0.2 for neutron stars (depending on their equation of state). The quadrupoles Q_1 and Q_2 will affect both sides of Equation (5), i.e. the Newtonian binding energy E of the two bodies, and the emitted total gravitational flux $\mathcal{L}$ as computed using the Newtonian quadrupole formula (4). It is known that for inspiralling compact binaries the neutron stars are not co-rotating because the tidal synchronization time is much larger than the time left till the coalescence. As shown by Kochanek [147] the best models for the fluid motion inside the two neutron stars are the so-called Roche–Riemann ellipsoids, which have tidally locked figures (the quadrupole moments face each other at any instant during the inspiral), but for which the fluid motion has zero circulation in the inertial frame. In the Newtonian approximation we find that within such a model (in the case of two identical neutron stars) the orbital phase, deduced from Equation (5), reads

$$\phi^{\text{finite size}} - \phi_0 = -\frac{1}{8x^{5/2}} \left\{ 1 + \mathrm{const}\, k \left(\frac{x}{K}\right)^5 \right\}, \tag{8}$$

where $x = (Gm\omega/c^3)^{2/3}$ is a standard dimensionless post-Newtonian parameter $\sim 1/c^2$ (ω is the orbital frequency), and where k is the Love number and K is the compactness of the neutron star. The first term in the right-hand side of Equation (8) corresponds to the gravitational-wave damping of two point masses; the second term is the finite-size effect, which appears as a relative correction, proportional to $(x/K)^5$, to the latter radiation damping effect. Because the finite-size effect is purely Newtonian, its relative correction $\sim (x/K)^5$ should not depend on c; and indeed the factors $1/c^2$ cancel out in the ratio x/K. However, the compactness K of compact objects is by Equation (7) of the order unity (or, say, one half), therefore the $1/c^2$ it contains should not be taken into account numerically in this case, and so the real order of magnitude of the relative contribution of the finite-size effect in Equation (8) is given by x^5 alone. This means that for compact objects the finite-size effect should be comparable, numerically, to a post-Newtonian correction of magnitude $x^5 \sim 1/c^{10}$ namely 5PN order[3]. This is a much higher post-Newtonian order than the one at which we shall investigate the gravitational effects on the phasing formula. Using $k' \equiv \mathrm{const}\, k \sim 1$ and $K \sim 0.2$ for neutron stars (and the bandwidth of a VIRGO detector between 10 Hz and 1000 Hz), we find that the cumulative phase error due to the finite-size effect amounts to less that one orbital rotation over a total of $\sim 16,000$ produced by the gravitational-wave damping of point masses. The conclusion is that the finite-size effect can in general be neglected in comparison with purely gravitational-wave damping effects. But note that for non-compact or moderately compact objects (such as white dwarfs for instance) the Newtonian tidal interaction dominates over the radiation damping.

[3]See Ref. [81] for the proof of such an "effacement" principle in the context of relativistic equations of motion.

The inspiralling compact binaries are ideally suited for application of a high-order post-Newtonian wave generation formalism. The main reason is that these systems are very relativistic, with orbital velocities as high as $0.5c$ in the last rotations (as compared to $\sim 10^{-3}c$ for the binary pulsar), and it is not surprising that the quadrupole-moment formalism (2, 3, 4, 5) constitutes a poor description of the emitted gravitational waves, since many post-Newtonian corrections play a substantial role. This expectation has been confirmed in recent years by several measurement-analyses [77, 78, 111, 79, 203, 183, 184, 152, 92], which have demonstrated that the post-Newtonian precision needed to implement successively the optimal filtering technique in the LIGO/VIRGO detectors corresponds grossly, in the case of neutron-star binaries, to the 3PN approximation, or $1/c^6$ beyond the quadrupole moment approximation. Such a high precision is necessary because of the large number of orbital rotations that will be monitored in the detector's frequency bandwidth ($\sim 16,000$ in the case of neutron stars), giving the possibility of measuring very accurately the orbital phase of the binary. Thus, the 3PN order is required mostly to compute the time evolution of the orbital phase, which depends, *via* the energy equation (5), on the center-of-mass binding energy E and the total gravitational-wave energy flux $\mathcal{L}$.

In summary, the theoretical problem posed by inspiralling compact binaries is two-fold: On the one hand E, and on the other hand $\mathcal{L}$, are to be deduced from general relativity with the 3PN precision or better. To obtain E we must control the 3PN equations of motion of the binary in the case of general, not necessarily circular, orbits. As for $\mathcal{L}$ it necessitates the application of a 3PN wave generation formalism (actually, things are more complicated because the equations of motion are also needed during the computation of the flux). It is quite interesting that such a high order approximation as the 3PN one should be needed in preparation for LIGO and VIRGO data analysis. As we shall see, the signal from compact binaries contains at the 3PN order the signature of several non-linear effects which are specific to general relativity. Therefore, we have here the possibility of probing, experimentally, some aspects of the non-linear structure of Einstein's theory [47, 48].

1.3 Post-Newtonian equations of motion and radiation

By equations of motion we mean the explicit expression of the accelerations of the bodies in terms of the positions and velocities. In Newtonian gravity, writing the equations of motion for a system of N particles is trivial; in general relativity, even writing the equations in the case $N = 2$ is difficult. The first relativistic term, at the 1PN order, was derived by Lorentz and Droste [156]. Subsequently, Einstein, Infeld and Hoffmann [106] obtained the 1PN corrections by means of their famous "surface-integral" method, in which the equations of motion are deduced from the *vacuum* field equations, and which are therefore applicable to any compact objects (be they neutron stars, black holes, or, perhaps, naked singularities). The 1PN-accurate equations were also obtained, for the motion of the centers of mass of extended bodies, by Petrova [179] and Fock [112] (see also Ref. [169]).

The 2PN approximation was tackled by Ohta et al. [165, 167, 166], who considered the post-Newtonian iteration of the Hamiltonian of N point-particles. We refer here to the Hamiltonian as the Fokker-type Hamiltonian, which is obtained from the matter-plus-field Arnowitt–Deser–Misner (ADM) Hamiltonian by eliminating the field degrees of freedom. The result for the 2PN and even 2.5PN equations of binary motion in harmonic coordinates was obtained by Damour and Deruelle [86, 85, 104, 80, 81], building on a non-linear iteration of the metric of two particles initiated in Ref. [11]. The corresponding result for the ADM-Hamiltonian of two particles at the 2PN order was given in Ref. [98] (see also Refs. [195, 196]). Kopeikin [149] derived the 2.5PN equations of motion for two extended compact objects. The 2.5PN-accurate harmonic-coordinate equations as well as the complete gravitational field (namely the metric $g_{\alpha\beta}$) generated by two point masses were computed in Ref. [42], following a method based on previous work on wave generation [15].

Up to the 2PN level the equations of motion are conservative. Only at the 2.5PN order appears the first non-conservative effect, associated with the gravitational radiation reaction. The (harmonic-coordinate) equations of motion up to that level, as derived by Damour and Deruelle [86, 85, 104, 80, 81], have been used for the study of the radiation damping of the binary pulsar – its orbital $\dot{P}$ [81, 82, 102]. It is important to realize that the 2.5PN equations of motion have been proved to hold in the case of binary systems of strongly self-gravitating bodies [81]. This is *via* an "effacing" principle (in the terminology of Damour [81]) for the internal structure of the bodies. As a result, the equations depend only on the "Schwarzschild" masses, m_1 and m_2, of the compact objects. Notably their compactness parameters K_1 and K_2, defined by Equation (7), do not enter the equations of motion, as has been explicitly verified up to the 2.5PN order by Kopeikin et al. [149, 127], who made a "physical" computation, *à la* Fock, taking into account the internal structure of two self-gravitating extended bodies. The 2.5PN equations of motion have also been established by Itoh, Futamase and Asada [134, 135], who use a variant of the surface-integral approach of Einstein, Infeld and Hoffmann [106], that is valid for compact bodies, independently of the strength of the internal gravity.

The present state of the art is the 3PN approximation[4]. To this order the equations have been worked out independently by two groups, by means of different methods, and with equivalent results. On the one hand, Jaranowski and Schäfer [139, 140, 141], and Damour, Jaranowski, and Schäfer [95, 97, 96], following the line of research of Refs. [165, 167, 166, 98], employ the ADM-Hamiltonian formalism of general relativity; on the other hand, Blanchet and Faye [37, 38, 36, 39], and de Andrade, Blanchet, and Faye [103], founding their approach on the post-Newtonian iteration initiated in Ref. [42], compute directly the equations of motion (instead of a Hamiltonian) in harmonic coordinates. The end results have been shown [97, 103] to be physically equivalent in the sense that there exists a unique "contact" transformation of the dynamical variables that changes the harmonic-coordinates Lagrangian obtained in Ref. [103] into a new Lagrangian, whose Legendre transform coincides exactly with the Hamiltonian given in Ref. [95]. The 3PN equations of motion, however, depend on one unspecified numerical coefficient, ω_{static} in the ADM-Hamiltonian formalism and λ in the harmonic-coordinates approach, which is due to some incompleteness of the Hadamard self-field regularization method. This coefficient has been fixed by means of a *dimensional regularization*, both within the ADM-Hamiltonian formalism [96], and the harmonic-coordinates equations of motion [30]. The works [96, 30] have demonstrated the power of dimensional regularization and its perfect adequateness for the problem of the interaction between point masses in general relativity. Furthermore, an important work by Itoh and Futamase [133, 132] (using the same surface-integral method as in Refs. [134, 135]) succeeded in obtaining the complete 3PN equations of motion in harmonic coordinates directly, i.e. without ambiguity and containing the correct value for the parameter λ.

So far the status of the post-Newtonian equations of motion is quite satisfying. There is mutual agreement between all the results obtained by means of different approaches and techniques, whenever it is possible to compare them: point particles described by Dirac delta-functions, extended post-Newtonian fluids, surface-integrals methods, mixed post-Minkowskian and post-Newtonian expansions, direct post-Newtonian iteration and matching, harmonic coordinates versus ADM-type coordinates, and different processes or variants of the regularization of the self field of point particles. In Part B of this article, we shall present the complete results for the 3PN equations of motion, and for the associated Lagrangian and Hamiltonian formulations (from which we deduce the center-of-mass energy E).

The second sub-problem, that of the computation of the energy flux $\mathcal{L}$, has been carried out by

[4] Let us mention that the 3.5PN terms in the equations of motion are also known, both for point-particle binaries [136, 137, 138, 174, 148, 164] and extended fluid bodies [14, 18]; they correspond to 1PN "relative" corrections in the radiation reaction force. Known also is the contribution of wave tails in the equations of motion, which arises at the 4PN order and represents a 1.5PN modification of the gravitational radiation damping [27].

application of the wave-generation formalism described previously. Following earliest computations at the 1PN level [217, 49], at a time when the post-Newtonian corrections in $\mathcal{L}$ had a purely academic interest, the energy flux of inspiralling compact binaries was completed to the 2PN order by Blanchet, Damour and Iyer [33, 122], and, independently, by Will and Wiseman [220], using their own formalism (see Refs. [35, 46] for joint reports of these calculations). The preceding approximation, 1.5PN, which represents in fact the dominant contribution of tails in the wave zone, had been obtained in Refs. [221, 50] by application of the formula for tail integrals given in Ref. [29]. Higher-order tail effects at the 2.5PN and 3.5PN orders, as well as a crucial contribution of tails generated by the tails themselves (the so-called "tails of tails") at the 3PN order, were obtained by Blanchet [16, 19]. However, unlike the 1.5PN, 2.5PN, and 3.5PN orders that are entirely composed of tail terms, the 3PN approximation also involves, besides the tails of tails, many non-tail contributions coming from the relativistic corrections in the (source) multipole moments of the binary. These have been "almost" completed in Refs. [45, 40, 44], in the sense that the result still involves one unknown numerical coefficient, due to the use of the Hadamard regularization, which is a combination of the parameter λ in the equations of motion, and a new parameter θ coming from the computation of the 3PN quadrupole moment. The latter parameter is itself a linear combination of three unknown parameters, $\theta = \xi + 2\kappa + \zeta$. We shall review the computation of the three parameters ξ, κ, and ζ by means of dimensional regularization [31, 32]. In Part B of this article, we shall present the most up-to-date results for the 3.5PN energy flux and orbital phase, deduced from the energy balance equation (5), supposed to be valid at this order.

The post-Newtonian flux $\mathcal{L}$, which comes from a "standard" post-Newtonian calculation, is in complete agreement (up to the 3.5PN order) with the result given by the very different technique of linear black-hole perturbations, valid in the "test-mass" limit where the mass of one of the bodies tends to zero (limit $\nu \to 0$, where $\nu = \mu/m$). Linear black-hole perturbations, triggered by the geodesic motion of a small mass around the black hole, have been applied to this problem by Poisson [182] at the 1.5PN order (following the pioneering work of Galt'sov et al. [116]), and by Tagoshi and Nakamura [203], using a numerical code, up to the 4PN order. This technique has culminated with the beautiful analytical methods of Sasaki, Tagoshi and Tanaka [194, 205, 206] (see also Ref. [160]), who solved the problem up to the extremely high 5.5PN order.

Part A: Post-Newtonian Sources

2 Einstein's Field Equations

The field equations of general relativity form a system of ten second-order partial differential equations obeyed by the space-time metric $g_{\alpha\beta}$,

$$G^{\alpha\beta}[g,\partial g,\partial^2 g] = \frac{8\pi G}{c^4} T^{\alpha\beta}[g], \tag{9}$$

where the Einstein curvature tensor $G^{\alpha\beta} \equiv R^{\alpha\beta} - \frac{1}{2} R\, g^{\alpha\beta}$ is generated, through the gravitational coupling $\kappa = 8\pi G/c^4$, by the matter stress-energy tensor $T^{\alpha\beta}$. Among these ten equations, four govern, *via* the contracted Bianchi identity, the evolution of the matter system,

$$\nabla_\mu G^{\alpha\mu} \equiv 0 \quad \Longrightarrow \quad \nabla_\mu T^{\alpha\mu} = 0. \tag{10}$$

The space-time geometry is constrained by the six remaining equations, which place six independent constraints on the ten components of the metric $g_{\alpha\beta}$, leaving four of them to be fixed by a choice of a coordinate system.

In most of this paper we adopt the conditions of *harmonic*, or de Donder, coordinates. We define, as a basic variable, the gravitational-field amplitude

$$h^{\alpha\beta} = \sqrt{-g}\, g^{\alpha\beta} - \eta^{\alpha\beta}, \tag{11}$$

where $g^{\alpha\beta}$ denotes the contravariant metric (satisfying $g^{\alpha\mu} g_{\mu\beta} = \delta^\alpha_\beta$), where g is the determinant of the covariant metric, $g = \det(g_{\alpha\beta})$, and where $\eta^{\alpha\beta}$ represents an auxiliary Minkowskian metric. The harmonic-coordinate condition, which accounts exactly for the four equations (10) corresponding to the conservation of the matter tensor, reads

$$\partial_\mu h^{\alpha\mu} = 0. \tag{12}$$

Equations (11, 12) introduce into the definition of our coordinate system a preferred Minkowskian structure, with Minkowski metric $\eta_{\alpha\beta}$. Of course, this is not contrary to the spirit of general relativity, where there is only one physical metric $g_{\alpha\beta}$ without any flat prior geometry, because the coordinates are not governed by geometry (so to speak), but rather are chosen by researchers when studying physical phenomena and doing experiments. Actually, the coordinate condition (12) is especially useful when we view the gravitational waves as perturbations of space-time propagating on the fixed Minkowskian manifold with the background metric $\eta_{\alpha\beta}$. This view is perfectly legitimate and represents a fruitful and rigorous way to think of the problem when using approximation methods. Indeed, the metric $\eta_{\alpha\beta}$, originally introduced in the coordinate condition (12), does exist at any *finite* order of approximation (neglecting higher-order terms), and plays in a sense the role of some "prior" flat geometry.

The Einstein field equations in harmonic coordinates can be written in the form of inhomogeneous flat d'Alembertian equations,

$$\Box h^{\alpha\beta} = \frac{16\pi G}{c^4} \tau^{\alpha\beta}, \tag{13}$$

where $\Box \equiv \Box_\eta = \eta^{\mu\nu}\partial_\mu\partial_\nu$. The source term $\tau^{\alpha\beta}$ can rightly be interpreted as the stress-energy pseudo-tensor (actually, $\tau^{\alpha\beta}$ is a Lorentz tensor) of the matter fields, described by $T^{\alpha\beta}$, *and* the gravitational field, given by the gravitational source term $\Lambda^{\alpha\beta}$, i.e.

$$\tau^{\alpha\beta} = |g| T^{\alpha\beta} + \frac{c^4}{16\pi G} \Lambda^{\alpha\beta}. \tag{14}$$

The exact expression of $\Lambda^{\alpha\beta}$, including all non-linearities, reads[5]

$$\begin{aligned}\Lambda^{\alpha\beta} = &-h^{\mu\nu}\partial^2_{\mu\nu}h^{\alpha\beta} + \partial_\mu h^{\alpha\nu}\partial_\nu h^{\beta\mu} + \frac{1}{2}g^{\alpha\beta}g_{\mu\nu}\partial_\lambda h^{\mu\tau}\partial_\tau h^{\nu\lambda} \\ &-g^{\alpha\mu}g_{\nu\tau}\partial_\lambda h^{\beta\tau}\partial_\mu h^{\nu\lambda} - g^{\beta\mu}g_{\nu\tau}\partial_\lambda h^{\alpha\tau}\partial_\mu h^{\nu\lambda} + g_{\mu\nu}g^{\lambda\tau}\partial_\lambda h^{\alpha\mu}\partial_\tau h^{\beta\nu} \\ &+\frac{1}{8}(2g^{\alpha\mu}g^{\beta\nu} - g^{\alpha\beta}g^{\mu\nu})(2g_{\lambda\tau}g_{\epsilon\pi} - g_{\tau\epsilon}g_{\lambda\pi})\partial_\mu h^{\lambda\pi}\partial_\nu h^{\tau\epsilon}. \end{aligned} \tag{15}$$

As is clear from this expression, $\Lambda^{\alpha\beta}$ is made of terms at least quadratic in the gravitational-field strength h and its first and second space-time derivatives. In the following, for the highest post-Newtonian order that we consider (3PN), we need the quadratic, cubic and quartic pieces of $\Lambda^{\alpha\beta}$. With obvious notation, we can write them as

$$\Lambda^{\alpha\beta} = N^{\alpha\beta}[h,h] + M^{\alpha\beta}[h,h,h] + L^{\alpha\beta}[h,h,h,h] + \mathcal{O}(h^5). \tag{16}$$

These various terms can be straightforwardly computed from Equation (15); see Equations (3.8) in Ref. [38] for explicit expressions.

As said above, the condition (12) is equivalent to the matter equations of motion, in the sense of the conservation of the total pseudo-tensor $\tau^{\alpha\beta}$,

$$\partial_\mu \tau^{\alpha\mu} = 0 \quad \Longleftrightarrow \quad \nabla_\mu T^{\alpha\mu} = 0. \tag{17}$$

In this article, we look for the solutions of the field equations (13, 14, 15, 17) under the following four hypotheses:

1. The matter stress-energy tensor $T^{\alpha\beta}$ is of spatially compact support, i.e. can be enclosed into some time-like world tube, say $r \leq a$, where $r = |\mathbf{x}|$ is the harmonic-coordinate radial distance. Outside the domain of the source, when $r > a$, the gravitational source term, according to Equation (17), is divergence-free,

$$\partial_\mu \Lambda^{\alpha\mu} = 0 \qquad (\text{when } r > a). \tag{18}$$

2. The matter distribution inside the source is smooth[6]: $T^{\alpha\beta} \in C^\infty(\mathbb{R}^3)$. We have in mind a smooth hydrodynamical "fluid" system, without any singularities nor shocks (*a priori*), that is described by some Eulerian equations including high relativistic corrections. In particular, we exclude from the start any black holes (however we shall return to this question when we find a model for describing compact objects).

3. The source is post-Newtonian in the sense of the existence of the small parameter defined by Equation (1). For such a source we assume the legitimacy of the method of matched asymptotic expansions for identifying the inner post-Newtonian field and the outer multipolar decomposition in the source's exterior near zone.

4. The gravitational field has been independent of time (stationary) in some remote past, i.e. before some finite instant $-\mathcal{T}$ in the past, in the sense that

$$\frac{\partial}{\partial t}\left[h^{\alpha\beta}(\mathbf{x},t)\right] = 0 \qquad \text{when } t \leq -\mathcal{T}. \tag{19}$$

[5] See also Equation (140) for the expression in $d+1$ space-time dimensions.

[6] $\mathbb{N}$, $\mathbb{Z}$, $\mathbb{R}$, and $\mathbb{C}$ are the usual sets of non-negative integers, integers, real numbers, and complex numbers; $C^p(\Omega)$ is the set of p-times continuously differentiable functions on the open domain Ω ($p \leq +\infty$).

The latter condition is a means to impose, by brute force, the famous *no-incoming* radiation condition, ensuring that the matter source is isolated from the rest of the Universe and does not receive any radiation from infinity. Ideally, the no-incoming radiation condition should be imposed at past null infinity. We shall later argue (see Section 6) that our condition of stationarity in the past, Equation (19), although much weaker than the real no-incoming radiation condition, does not entail any physical restriction on the general validity of the formulas we derive.

Subject to the condition (19), the Einstein differential field equations (13) can be written equivalently into the form of the integro-differential equations

$$h^{\alpha\beta} = \frac{16\pi G}{c^4} \Box^{-1}_{\mathrm{ret}} \tau^{\alpha\beta}, \tag{20}$$

containing the usual retarded inverse d'Alembertian operator, given by

$$(\Box^{-1}_{\mathrm{ret}} f)(\mathbf{x}, t) \equiv -\frac{1}{4\pi} \iiint \frac{d^3\mathbf{x}'}{|\mathbf{x}-\mathbf{x}'|} f(\mathbf{x}', t - |\mathbf{x}-\mathbf{x}'|/c), \tag{21}$$

extending over the whole three-dimensional space $\mathbb{R}^3$.

3 Linearized Vacuum Equations

In what follows we solve the field equations (12, 13), in the *vacuum* region outside the compact-support source, in the form of a formal non-linearity or *post-Minkowskian* expansion, considering the field variable $h^{\alpha\beta}$ as a non-linear metric perturbation of Minkowski space-time. At the linearized level (or first-post-Minkowskian approximation), we write:

$$h^{\alpha\beta}_{\rm ext} = G h^{\alpha\beta}_1 + \mathcal{O}(G^2), \tag{22}$$

where the subscript "ext" reminds us that the solution is valid only in the exterior of the source, and where we have introduced Newton's constant G as a book-keeping parameter, enabling one to label very conveniently the successive post-Minkowskian approximations. Since $h^{\alpha\beta}$ is a dimensionless variable, with our convention the linear coefficient $h^{\alpha\beta}_1$ in Equation (22) has the dimension of the inverse of G – a mass squared in a system of units where $\hbar = c = 1$. In vacuum, the harmonic-coordinate metric coefficient $h^{\alpha\beta}_1$ satisfies

$$\Box h^{\alpha\beta}_1 = 0, \tag{23}$$

$$\partial_\mu h^{\alpha\mu}_1 = 0. \tag{24}$$

We want to solve those equations by means of an infinite multipolar series valid outside a time-like world tube containing the source. Indeed the multipole expansion is the correct method for describing the physics of the source as seen from its exterior ($r > a$). On the other hand, the post-Minkowskian series is physically valid in the weak-field region, which surely includes the exterior of any source, starting at a sufficiently large distance. For post-Newtonian sources the exterior weak-field region, where both multipole and post-Minkowskian expansions are valid, simply coincides with the exterior $r > a$. It is therefore quite natural, and even, one would say inescapable when considering general sources, to combine the post-Minkowskian approximation with the multipole decomposition. This is the original idea of the "double-expansion" series of Bonnor [54], which combines the G-expansion (or m-expansion in his notation) with the a-expansion (equivalent to the multipole expansion, since the lth order multipole moment scales like a^l with the source radius).

The multipolar-post-Minkowskian method will be implemented systematically, using STF-harmonics to describe the multipole expansion [210], and looking for a definite *algorithm* for the approximation scheme [26]. The solution of the system of equations (23, 24) takes the form of a series of retarded multipolar waves[7]

$$h^{\alpha\beta}_1 = \sum_{l=0}^{+\infty} \partial_L \left(\frac{K^{\alpha\beta}_L(t-r/c)}{r} \right), \tag{25}$$

where $r = |\mathbf{x}|$, and where the functions $K^{\alpha\beta}_L \equiv K^{\alpha\beta}_{i_1\dots i_l}$ are smooth functions of the retarded time $u \equiv t - r/c$ [$K_L(u) \in C^\infty(\mathbb{R})$], which become constant in the past, when $t \leq -\mathcal{T}$. It is evident, since a monopolar wave satisfies $\Box(K_L(u)/r) = 0$ and the d'Alembertian commutes

[7]Our notation is the following: $L = i_1 i_2 \dots i_l$ denotes a multi-index, made of l (spatial) indices. Similarly we write for instance $P = j_1 \dots j_p$ (in practice, we generally do not need to consider the carrier letter i or j), or $aL - 1 = a i_1 \dots i_{l-1}$. Always understood in expressions such as Equation (25) are l summations over the l indices $i_1, \dots, i_l$ ranging from 1 to 3. The derivative operator ∂_L is a short-hand for $\partial_{i_1} \dots \partial_{i_l}$. The function K_L is *symmetric and trace-free* (STF) with respect to the l indices composing L. This means that for any pair of indices $i_p, i_q \in L$, we have $K_{\dots i_p \dots i_q \dots} = K_{\dots i_q \dots i_p \dots}$ and that $\delta_{i_p i_q} K_{\dots i_p \dots i_q \dots} = 0$ (see Ref. [210] and Appendices A and B in Ref. [26] for reviews about the STF formalism). The STF projection is denoted with a hat, so $K_L \equiv \hat{K}_L$, or sometimes with carets around the indices, $K_L \equiv K_{\langle L \rangle}$. In particular, $\hat{n}_L = n_{\langle L \rangle}$ is the STF projection of the product of unit vectors $n_L = n_{i_1} \dots n_{i_l}$; an expansion into STF tensors $\hat{n}_L = \hat{n}_L(\theta, \phi)$ is equivalent to the usual expansion in spherical harmonics $\mathrm{Y}_{lm} = \mathrm{Y}_{lm}(\theta, \phi)$. Similarly, we denote $x_L = x_{i_1} \dots x_{i_l} = r^l n_L$ and $\hat{x}_L = x_{\langle L \rangle}$. Superscripts like (p) indicate p successive time-derivations.

with the multi-derivative ∂_L, that Equation (25) represents the most general solution of the wave equation (23) (see Section 2 in Ref. [26] for a proof based on the Euler–Poisson–Darboux equation). The gauge condition (24), however, is not fulfilled in general, and to satisfy it we must algebraically decompose the set of functions K_L^{00}, K_L^{0i}, K_L^{ij} into ten tensors which are STF with respect to all their indices, including the spatial indices i, ij. Imposing the condition (24) reduces the number of independent tensors to six, and we find that the solution takes an especially simple "canonical" form, parametrized by only two moments, plus some arbitrary linearized gauge transformation [210, 26].

Theorem 1 *The most general solution of the linearized field equations (23, 24), outside some time-like world tube enclosing the source ($r > a$), and stationary in the past (see Equation (19)), reads*

$$h_1^{\alpha\beta} = k_1^{\alpha\beta} + \partial^\alpha \varphi_1^\beta + \partial^\beta \varphi_1^\alpha - \eta^{\alpha\beta}\partial_\mu \varphi_1^\mu. \tag{26}$$

The first term depends on two STF-tensorial multipole moments, $\mathrm{I}_L(u)$ and $\mathrm{J}_L(u)$, which are arbitrary functions of time except for the laws of conservation of the monopole: $\mathrm{I} = \mathrm{const}$, and dipoles: $\mathrm{I}_i = \mathrm{const}$, $\mathrm{J}_i = \mathrm{const}$. It is given by

$$\begin{aligned}
k_1^{00} &= -\frac{4}{c^2}\sum_{l\geq 0}\frac{(-)^l}{l!}\partial_L\left(\frac{1}{r}\mathrm{I}_L(u)\right),\\
k_1^{0i} &= \frac{4}{c^3}\sum_{l\geq 1}\frac{(-)^l}{l!}\left\{\partial_{L-1}\left(\frac{1}{r}\mathrm{I}^{(1)}_{iL-1}(u)\right)+\frac{l}{l+1}\varepsilon_{iab}\partial_{aL-1}\left(\frac{1}{r}\mathrm{J}_{bL-1}(u)\right)\right\},\\
k_1^{ij} &= -\frac{4}{c^4}\sum_{l\geq 2}\frac{(-)^l}{l!}\left\{\partial_{L-2}\left(\frac{1}{r}\mathrm{I}^{(2)}_{ijL-2}(u)\right)+\frac{2l}{l+1}\partial_{aL-2}\left(\frac{1}{r}\varepsilon_{ab(i}\mathrm{J}^{(1)}_{j)bL-2}(u)\right)\right\}.
\end{aligned} \tag{27}$$

The other terms represent a linearized gauge transformation, with gauge vector φ_1^α of the type (25), and parametrized for four other multipole moments, say $\mathrm{W}_L(u)$, $\mathrm{X}_L(u)$, $\mathrm{Y}_L(u)$ and $\mathrm{Z}_L(u)$.

The conservation of the lowest-order moments gives the constancy of the total mass of the source, $\mathrm{M} \equiv \mathrm{I} = \mathrm{const}$, center-of-mass position[8], $\mathrm{X}_i \equiv \mathrm{I}_i/\mathrm{I} = \mathrm{const}$, total linear momentum $\mathrm{P}_i \equiv \mathrm{I}_i^{(1)} = 0$, and total angular momentum, $\mathrm{S}_i \equiv \mathrm{J}_i = \mathrm{const}$. It is always possible to achieve $\mathrm{X}_i = 0$ by translating the origin of our coordinates to the center of mass. The total mass M is the ADM mass of the Hamiltonian formulation of general relativity. Note that the quantities M, X_i, P_i and S_i include the contributions due to the waves emitted by the source. They describe the "initial" state of the source, before the emission of gravitational radiation.

The multipole functions $\mathrm{I}_L(u)$ and $\mathrm{J}_L(u)$, which thoroughly encode the physical properties of the source at the linearized level (because the other moments $\mathrm{W}_L, \dots, \mathrm{Z}_L$ parametrize a gauge transformation), will be referred to as the *mass-type* and *current-type* source multipole moments. Beware, however, that at this stage the moments are not specified in terms of the stress-energy tensor $T^{\alpha\beta}$ of the source: the above theorem follows merely from the algebraic and differential properties of the vacuum equations outside the source.

[8]The constancy of the center of mass X_i – rather than a linear variation with time – results from our assumption of stationarity before the date $-\mathcal{T}$. Hence, $\mathrm{P}_i = 0$.

For completeness, let us give the components of the gauge-vector φ_1^α entering Equation (26):

$$\begin{aligned}
\varphi_1^0 &= \frac{4}{c^3}\sum_{l\geq 0}\frac{(-)^l}{l!}\partial_L\left(\frac{1}{r}\mathrm{W}_L(u)\right),\\
\varphi_1^i &= -\frac{4}{c^4}\sum_{l\geq 0}\frac{(-)^l}{l!}\partial_{iL}\left(\frac{1}{r}\mathrm{X}_L(u)\right)\\
&\quad -\frac{4}{c^4}\sum_{l\geq 1}\frac{(-)^l}{l!}\left\{\partial_{L-1}\left(\frac{1}{r}\mathrm{Y}_{iL-1}(u)\right)+\frac{l}{l+1}\varepsilon_{iab}\partial_{aL-1}\left(\frac{1}{r}\mathrm{Z}_{bL-1}(u)\right)\right\}.
\end{aligned} \tag{28}$$

Because the theory is covariant with respect to non-linear diffeomorphisms and not merely with respect to linear gauge transformations, the moments $\mathrm{W}_L, \dots, \mathrm{Z}_L$ do play a physical role starting at the non-linear level, in the following sense. If one takes these moments equal to zero and continues the calculations one ends up with a metric depending on I_L and J_L only, but that metric will not describe the same physical source as the one constructed from the six moments $\mathrm{I}_L, \dots, \mathrm{Z}_L$. In other words, the two non-linear metrics associated with the sets of multipole moments $\{\mathrm{I}_L, \mathrm{J}_L, 0, \dots, 0\}$ and $\{\mathrm{I}_L, \mathrm{J}_L, \mathrm{W}_L, \dots, \mathrm{Z}_L\}$ are not isometric. We point out in Section 4.2 below that the full set of moments $\{\mathrm{I}_L, \mathrm{J}_L, \mathrm{W}_L, \dots, \mathrm{Z}_L\}$ is in fact physically equivalent to some reduced set $\{\mathrm{M}_L, \mathrm{S}_L, 0, \dots, 0\}$, but with some moments M_L, S_L that differ from I_L, J_L by non-linear corrections (see Equation (96)). All the multipole moments I_L, J_L, W_L, X_L, Y_L, Z_L will be computed in Section 5.

4 Non-linear Iteration of the Field Equations

By Theorem 1 we know the most general solution of the linearized equations in the exterior of the source. We then tackle the problem of the post-Minkowskian iteration of that solution. We consider the full post-Minkowskian series

$$h^{\alpha\beta}_{\rm ext} = \sum_{n=1}^{+\infty} G^n h^{\alpha\beta}_n, \tag{29}$$

where the first term is composed of the result given by Equations (26, 27, 28). In this article, we shall always understand the infinite sums such as the one in Equation (29) in the sense of *formal* power series, i.e. as an ordered collection of coefficients, e.g., $\left(h^{\alpha\beta}_n\right)_{n\in\mathbb{N}}$. We do not attempt to control the mathematical nature of the series and refer to the mathematical-physics literature for discussion (in the present context, see Refs. [72, 100, 187, 188, 189]).

4.1 The post-Minkowskian solution

We insert the ansatz (29) into the vacuum Einstein field equations (12, 13), i.e. with $\tau^{\alpha\beta} = c^4/(16\pi G)\Lambda^{\alpha\beta}$, and we equate term by term the factors of the successive powers of our book-keeping parameter G. We get an infinite set of equations for each of the $h^{\alpha\beta}_n$'s: $\forall n \geq 2$,

$$\Box h^{\alpha\beta}_n = \Lambda^{\alpha\beta}_n[h_1, h_2, \dots, h_{n-1}], \tag{30}$$

$$\partial_\mu h^{\alpha\mu}_n = 0. \tag{31}$$

The right-hand side of the wave equation (30) is obtained from inserting the previous iterations, up to the order $n-1$, into the gravitational source term. In more details, the series of equations (30) reads

$$\Box h^{\alpha\beta}_2 = N^{\alpha\beta}[h_1, h_1], \tag{32}$$

$$\Box h^{\alpha\beta}_3 = M^{\alpha\beta}[h_1, h_1, h_1] + N^{\alpha\beta}[h_1, h_2] + N^{\alpha\beta}[h_2, h_1], \tag{33}$$

$$\begin{aligned} \Box h^{\alpha\beta}_4 = {} & L^{\alpha\beta}[h_1, h_1, h_1, h_1] \\ & + M^{\alpha\beta}[h_1, h_1, h_2] + M^{\alpha\beta}[h_1, h_2, h_1] + M^{\alpha\beta}[h_2, h_1, h_1] \\ & + N^{\alpha\beta}[h_2, h_2] + N^{\alpha\beta}[h_1, h_3] + N^{\alpha\beta}[h_3, h_1] \\ & \vdots \end{aligned} \tag{34}$$

The quadratic, cubic and quartic pieces of $\Lambda^{\alpha\beta}$ are defined by Equation (16).

Let us now proceed by induction. Some n being given, we assume that we succeeded in constructing, from the linearized coefficient h_1, the sequence of post-Minkowskian coefficients $h_2, h_3, \dots, h_{n-1}$, and from this we want to infer the next coefficient h_n. The right-hand side of Equation (30), $\Lambda^{\alpha\beta}_n$, is known by induction hypothesis. Thus the problem is that of solving a wave equation whose source is given. The point is that this wave equation, instead of being valid everywhere in $\mathbb{R}^3$, is correct only outside the matter ($r > a$), and it makes no sense to solve it by means of the usual retarded integral. Technically speaking, the right-hand side of Equation (30) is composed of the product of many multipole expansions, which are singular at the origin of the spatial coordinates $r = 0$, and which make the retarded integral divergent at that point. This does not mean that there are no solutions to the wave equation, but simply that the retarded integral does not constitute the appropriate solution in that context.

What we need is a solution which takes the same structure as the source term $\Lambda^{\alpha\beta}_n$, i.e. is expanded into multipole contributions, with a singularity at $r = 0$, and satisfies the d'Alembertian equation as soon as $r > 0$. Such a particular solution can be obtained, following the suggestion in Ref. [26], by means of a mathematical trick in which one first "regularizes" the source term $\Lambda_n^{\alpha\beta}$ by multiplying it by the factor r^B, where $B \in \mathbb{C}$. Let us assume, for definiteness, that $\Lambda_n^{\alpha\beta}$ is composed of multipolar pieces with maximal multipolarity $l_{\max}$. This means that we start the iteration from the linearized metric (26, 27, 28) in which the multipolar sums are actually finite[9]. The divergences when $r \to 0$ of the source term are typically power-like, say $1/r^k$ (there are also powers of the logarithm of r), and with the previous assumption there will exist a maximal order of divergency, say $k_{\max}$. Thus, when the real part of B is large enough, i.e. $\Re(B) > k_{\max} - 3$, the "regularized" source term $r^B \Lambda_n^{\alpha\beta}$ is regular enough when $r \to 0$ so that one can perfectly apply the retarded integral operator. This defines the B-dependent retarded integral

$$I^{\alpha\beta}(B) \equiv \Box^{-1}_{\mathrm{ret}} \left[\widetilde{r}^B \Lambda_n^{\alpha\beta} \right], \tag{35}$$

where the symbol $\Box^{-1}_{\mathrm{ret}}$ stands for the retarded integral (21). It is convenient to introduce inside the regularizing factor some arbitrary constant length scale r_0 in order to make it dimensionless. Everywhere in this article we pose

$$\widetilde{r} \equiv \frac{r}{r_0}. \tag{36}$$

The fate of the constant r_0 in a detailed calculation will be interesting to follow, as we shall see, because it provides some check that the calculation is going well. Now the point for our purpose is that the function $I^{\alpha\beta}(B)$ on the complex plane, which was originally defined only when $\Re(B) > k_{\max} - 3$, admits a unique *analytic continuation* to all values of $B \in \mathbb{C}$ except at some integer values. Furthermore, the analytic continuation of $I^{\alpha\beta}(B)$ can be expanded, when $B \to 0$ (namely the limit of interest to us) into a Laurent expansion involving in general some multiple poles. The key idea, as we shall prove, is that the *finite part*, or the coefficient of the zeroth power of B in that expansion, represents the particular solution we are looking for. We write the Laurent expansion of $I^{\alpha\beta}(B)$, when $B \to 0$, in the form

$$I^{\alpha\beta}(B) = \sum_{p=p_0}^{+\infty} \iota_p^{\alpha\beta} B^p, \tag{37}$$

where $p \in \mathbb{Z}$, and the various coefficients $\iota_p^{\alpha\beta}$ are functions of the field point $(\mathbf{x}, t)$. When $p_0 \leq -1$ there are poles; $-p_0$, which depends on n, refers to the maximal order of the poles. By applying the box operator onto both sides of Equation (37), and equating the different powers of B, we arrive at

$$\begin{aligned} p_0 \leq p \leq -1 \quad &\Longrightarrow \quad \Box \iota_p^{\alpha\beta} = 0, \\ p \geq 0 \quad &\Longrightarrow \quad \Box \iota_p^{\alpha\beta} = \frac{(\ln r)^p}{p!} \Lambda_n^{\alpha\beta}. \end{aligned} \tag{38}$$

As we see, the case $p = 0$ shows that the finite-part coefficient in Equation (37), namely $\iota_0^{\alpha\beta}$, is a particular solution of the requested equation: $\Box \iota_0^{\alpha\beta} = \Lambda_n^{\alpha\beta}$. Furthermore, we can prove that this term, by its very construction, owns the same structure made of a multipolar expansion singular at $r = 0$.

Let us forget about the intermediate name $\iota_0^{\alpha\beta}$, and denote, from now on, the latter solution by $u_n^{\alpha\beta} \equiv \iota_0^{\alpha\beta}$, or, in more explicit terms,

$$u_n^{\alpha\beta} = \mathcal{FP}_{B=0} \, \Box^{-1}_{\mathrm{ret}} \left[\widetilde{r}^B \Lambda_n^{\alpha\beta} \right], \tag{39}$$

[9] This assumption is justified because we are ultimately interested in the radiation field at some given *finite* post-Newtonian precision like 3PN, and because only a finite number of multipole moments can contribute at any finite order of approximation. With a finite number of multipoles in the linearized metric (26, 27, 28), there is a maximal multipolarity $l_{\max}(n)$ at any post-Minkowskian order n, which grows linearly with n.

where the finite-part symbol $\mathcal{FP}_{B=0}$ means the previously detailed operations of considering the analytic continuation, taking the Laurent expansion, and picking up the finite-part coefficient when $B \to 0$. The story is not complete, however, because $u_n^{\alpha\beta}$ does not fulfill the constraint of harmonic coordinates (31); its divergence, say $w_n^\alpha = \partial_\mu u_n^{\alpha\mu}$, is different from zero in general. From the fact that the source term is divergence-free in vacuum, $\partial_\mu \Lambda_n^{\alpha\mu} = 0$ (see Equation (18)), we find instead

$$w_n^\alpha = \mathcal{FP}_{B=0}\, \Box_{\rm ret}^{-1} \left[B\, \widetilde{r}^B \frac{n_i}{r} \Lambda_n^{\alpha i} \right]. \tag{40}$$

The factor B comes from the differentiation of the regularization factor $\widetilde{r}^B$. So, w_n^α is zero only in the special case where the Laurent expansion of the retarded integral in Equation (40) does not develop any simple pole when $B \to 0$. Fortunately, when it does, the structure of the pole is quite easy to control. We find that it necessarily consists of a solution of the *source-free* d'Alembertian equation, and, what is more (from its stationarity in the past), the solution is a retarded one. Hence, taking into account the index structure of w_n^α, there must exist four STF-tensorial functions of the retarded time $u = t - r/c$, say $N_L(u)$, $P_L(u)$, $Q_L(u)$ and $R_L(u)$, such that

$$\begin{aligned} w_n^0 &= \sum_{l=0}^{+\infty} \partial_L \left[r^{-1} N_L(u) \right], \\ w_n^i &= \sum_{l=0}^{+\infty} \partial_{iL} \left[r^{-1} P_L(u) \right] + \sum_{l=1}^{+\infty} \left\{ \partial_{L-1} \left[r^{-1} Q_{iL-1}(u) \right] + \varepsilon_{iab} \partial_{aL-1} \left[r^{-1} R_{bL-1}(u) \right] \right\}. \end{aligned} \tag{41}$$

From that expression we are able to find a new object, say $v_n^{\alpha\beta}$, which takes the same structure as w_n^α (a retarded solution of the source-free wave equation) and, furthermore, whose divergence is exactly the opposite of the divergence of $u_n^{\alpha\beta}$, i.e. $\partial_\mu v_n^{\alpha\mu} = -w_n^\alpha$. Such a $v_n^{\alpha\beta}$ is not unique, but we shall see that it is simply necessary to make a choice for $v_n^{\alpha\beta}$ (the simplest one) in order to obtain the general solution. The formulas that we adopt are

$$\begin{aligned} v_n^{00} &= -r^{-1} N^{(-1)} + \partial_a \left[r^{-1} \left(-N_a^{(-1)} + C_a^{(-2)} - 3P_a \right) \right], \\ v_n^{0i} &= r^{-1} \left(-Q_i^{(-1)} + 3P_i^{(1)} \right) - \varepsilon_{iab} \partial_a \left[r^{-1} R_b^{(-1)} \right] - \sum_{l=2}^{+\infty} \partial_{L-1} \left[r^{-1} N_{iL-1} \right], \\ v_n^{ij} &= -\delta_{ij} r^{-1} P + \sum_{l=2}^{+\infty} \Big\{ 2\delta_{ij} \partial_{L-1} \left[r^{-1} P_{L-1} \right] - 6 \partial_{L-2(i} \left[r^{-1} P_{j)L-2} \right] \\ &\qquad + \partial_{L-2} \left[r^{-1} (N_{ijL-2}^{(1)} + 3P_{ijL-2}^{(2)} - Q_{ijL-2}) \right] - 2 \partial_{aL-2} \left[r^{-1} \varepsilon_{ab(i} R_{j)bL-2} \right] \Big\}. \end{aligned} \tag{42}$$

Notice the presence of anti-derivatives, denoted, e.g., by $N^{(-1)}(u) = \int_{-\infty}^u dv N(v)$; there is no problem with the limit $v \to -\infty$ since all the corresponding functions are zero when $t \leq -\mathcal{T}$. The choice made in Equations (42) is dictated by the fact that the 00 component involves only some monopolar and dipolar terms, and that the spatial trace ii is monopolar: $v_n^{ii} = -3r^{-1}P$. Finally, if we pose

$$h_n^{\alpha\beta} = u_n^{\alpha\beta} + v_n^{\alpha\beta}, \tag{43}$$

we see that we solve at once the d'Alembertian equation (30) *and* the coordinate condition (31). That is, we have succeeded in finding a solution of the field equations at the nth post-Minkowskian order. By induction the same method applies to *any* order n, and, therefore, we have constructed a complete post-Minkowskian series (29) based on the linearized approximation $h_1^{\alpha\beta}$ given by Equations (26, 27, 28). The previous procedure constitutes an *algorithm*, which could be implemented by an algebraic computer programme.

4.2 Generality of the solution

We have a solution, but is that a general solution? The answer, yes, is provided by the following result [26]:

Theorem 2 *The most general solution of the harmonic-coordinates Einstein field equations in the vacuum region outside an isolated source, admitting some post-Minkowskian and multipolar expansions, is given by the previous construction as* $h^{\alpha\beta} = \sum_{n=1}^{+\infty} G^n h_n^{\alpha\beta}[\mathrm{I}_L, \mathrm{J}_L, \dots, \mathrm{Z}_L]$*. It depends on two sets of arbitrary STF-tensorial functions of time* $\mathrm{I}_L(u)$ *and* $\mathrm{J}_L(u)$ *(satisfying the conservation laws) defined by Equations (27), and on four supplementary functions* $\mathrm{W}_L(u), \dots, \mathrm{Z}_L(u)$ *parametrizing the gauge vector (28).*

The proof is quite easy. With Equation (43) we obtained a *particular* solution of the system of equations (30, 31). To it we should add the most general solution of the corresponding *homogeneous* system of equations, which is obtained by setting $\Lambda_n^{\alpha\beta} = 0$ into Equations (30, 31). But this homogeneous system of equations is nothing but the *linearized* vacuum field equations (23, 24), for which we know the most general solution $h_1^{\alpha\beta}$ given by Equations (26, 27, 28). Thus, we must add to our "particular" solution $h_n^{\alpha\beta}$ a general homogeneous solution that is necessarily of the type $h_1^{\alpha\beta}[\delta\mathrm{I}_L, \dots, \delta\mathrm{Z}_L]$, where $\delta\mathrm{I}_L, \dots, \delta\mathrm{Z}_L$ denote some "corrections" to the multipole moments at the nth post-Minkowskian order. It is then clear, since precisely the linearized metric is a linear functional of all these moments, that the previous corrections to the moments can be absorbed into a re-definition of the original ones $\mathrm{I}_L, \dots, \mathrm{Z}_L$ by posing

$$\mathrm{I}_L^{\mathrm{new}} = \mathrm{I}_L + G^{n-1}\delta\mathrm{I}_L, \tag{44}$$

$$\vdots$$

$$\mathrm{Z}_L^{\mathrm{new}} = \mathrm{Z}_L + G^{n-1}\delta\mathrm{Z}_L. \tag{45}$$

After re-arranging the metric in terms of these new moments, taking into account the fact that the precision of the metric is limited to the nth post-Minkowskian order, and dropping the superscript "new", we find exactly the same solution as the one we had before (indeed, the moments are arbitrary functions of time) – hence the proof.

The six sets of multipole moments $\mathrm{I}_L(u), \dots, \mathrm{Z}_L(u)$ contain the physical information about *any* isolated source as seen in its exterior. However, as we now discuss, it is always possible to find *two*, and only two, sets of multipole moments, $\mathrm{M}_L(u)$ and $\mathrm{S}_L(u)$, for parametrizing the most general isolated source as well. The route for constructing such a general solution is to get rid of the moments $\mathrm{W}_L, \mathrm{X}_L, \mathrm{Y}_L, \mathrm{Z}_L$ at the linearized level by performing the linearized gauge transformation $\delta x^\alpha = \varphi_1^\alpha$, where φ_1^α is the gauge vector given by Equations (28). So, at the linearized level, we have only the two types of moments M_L and S_L, parametrizing $k_1^{\alpha\beta}$ by the same formulas as in Equations (27). We must be careful to denote these moments with some names different from I_L and J_L because they will ultimately correspond to a different physical source. Then we apply exactly the same post-Minkowskian algorithm, following the formulas (39, 40, 41, 42, 43) as we did above, but starting from the gauge-transformed linear metric $k_1^{\alpha\beta}$ instead of $h_1^{\alpha\beta}$. The result of the iteration is therefore some $k^{\alpha\beta} = \sum_{n=1}^{+\infty} G^n k_n^{\alpha\beta}[\mathrm{M}_L, \mathrm{S}_L]$. Obviously this post-Minkowskian algorithm yields some simpler calculations as we have only two multipole moments to iterate. The point is that one can show that the resulting metric $k^{\alpha\beta}[\mathrm{M}_L, \mathrm{S}_L]$ is *isometric* to the original one $h^{\alpha\beta}[\mathrm{I}_L, \mathrm{J}_L, \dots, \mathrm{Z}_L]$ if and only if M_L and S_L are related to the moments $\mathrm{I}_L, \mathrm{J}_L, \dots, \mathrm{Z}_L$ by some (quite involved) non-linear equations. Therefore, the most general solution of the field equations, modulo a coordinate transformation, can be obtained by starting from the linearized metric $k_1^{\alpha\beta}[\mathrm{M}_L, \mathrm{S}_L]$ instead of the more complicated $k_1^{\alpha\beta}[\mathrm{I}_L, \mathrm{J}_L] + \partial^\alpha\varphi_1^\beta + \partial^\beta\varphi_1^\alpha - \eta^{\alpha\beta}\partial_\mu\varphi_1^\mu$, and continuing the post-Minkowskian calculation.

So why not consider from the start that the best description of the isolated source is provided by only the two types of multipole moments, M_L and S_L, instead of the six, $\mathrm{I}_L, \mathrm{J}_L, \dots, \mathrm{Z}_L$? The reason is that we shall determine (in Theorem 6 below) the explicit closed-form expressions of the six moments $\mathrm{I}_L, \mathrm{J}_L, \dots, \mathrm{Z}_L$, but that, by contrast, it seems to be impossible to obtain some similar closed-form expressions for M_L and S_L. The only thing we can do is to write down the explicit non-linear algorithm that computes M_L, S_L starting from $\mathrm{I}_L, \mathrm{J}_L, \dots, \mathrm{Z}_L$. In consequence, it is better to view the moments $\mathrm{I}_L, \mathrm{J}_L, \dots, \mathrm{Z}_L$ as more "fundamental" than M_L and S_L, in the sense that they appear to be more tightly related to the description of the source, since they admit closed-form expressions as some explicit integrals over the source. Hence, we choose to refer collectively to the six moments $\mathrm{I}_L, \mathrm{J}_L, \dots, \mathrm{Z}_L$ as *the* multipole moments of the source. This being said, the moments M_L and S_L are often useful in practical computations because they yield a simpler post-Minkowskian iteration. Then, one can generally come back to the more fundamental source-rooted moments by using the fact that M_L and S_L differ from the corresponding I_L and J_L only by high-order post-Newtonian terms like 2.5PN; see Ref. [16] and Equation (96) below. Indeed, this is to be expected because the physical difference between both types of moments stems only from non-linearities.

4.3 Near-zone and far-zone structures

In our presentation of the post-Minkowskian algorithm (39, 40, 41, 42, 43) we have omitted a crucial recursive hypothesis, which is required in order to prove that at each post-Minkowskian order n, the inverse d'Alembertian operator can be applied in the way we did (and notably that the B-dependent retarded integral can be analytically continued down to a neighbourhood of $B = 0$). This hypothesis is that the "near-zone" expansion, i.e. when $r \to 0$, of each one of the post-Minkowskian coefficients $h_n^{\alpha\beta}$ has a certain structure. This hypothesis is established as a theorem once the mathematical induction succeeds.

Theorem 3 *The general structure of the expansion of the post-Minkowskian exterior metric in the near-zone (when $r \to 0$) is of the type: $\forall N \in \mathbb{N}$,*

$$h_n(\mathbf{x}, t) = \sum \hat{n}_L r^m (\ln r)^p F_{L,m,p,n}(t) + o(r^N), \tag{46}$$

where $m \in \mathbb{Z}$, with $m_0 \leq m \leq N$ (and m_0 becoming more and more negative as n grows), $p \in \mathbb{N}$ with $p \leq n-1$. The functions $F_{L,m,p,n}$ are multilinear functionals of the source multipole moments $\mathrm{I}_L, \dots, \mathrm{Z}_L$.

For the proof see Ref. [26][10]. As we see, the near-zone expansion involves, besides the simple powers of r, some powers of the logarithm of r, with a maximal power of $n-1$. As a corollary of that theorem, we find (by restoring all the powers of c in Equation (46) and using the fact that each r goes into the combination r/c), that the general structure of the post-Newtonian expansion ($c \to +\infty$) is necessarily of the type

$$h_n(c) \simeq \sum_{p,q \in \mathbb{N}} \frac{(\ln c)^p}{c^q}, \tag{47}$$

where $p \leq n-1$ (and $q \geq 2$). The post-Newtonian expansion proceeds not only with the normal powers of $1/c$ but also with powers of the logarithm of c [26].

Paralleling the structure of the near-zone expansion, we have a similar result concerning the structure of the *far-zone* expansion at Minkowskian future null infinity, i.e. when $r \to +\infty$ with

[10] The o and $\mathcal{O}$ Landau symbols for remainders have their standard meaning.

$u = t - r/c = \text{const}$: $\forall N \in \mathbb{N}$,

$$h_n(\mathbf{x}, t) = \sum \frac{\hat{n}_L (\ln r)^p}{r^k} G_{L,k,p,n}(u) + o\left(\frac{1}{r^N}\right), \tag{48}$$

where $k, p \in \mathbb{N}$, with $1 \leq k \leq N$, and where, likewise in the near-zone expansion (46), some powers of logarithms, such that $p \leq n-1$, appear. The appearance of logarithms in the far-zone expansion of the harmonic-coordinates metric has been known since the work of Fock [113]. One knows also that this is a coordinate effect, because the study of the "asymptotic" structure of space-time at future null infinity by Bondi et al. [53], Sachs [193], and Penrose [175, 176], has revealed the existence of other coordinate systems that avoid the appearance of any logarithms: the so-called *radiative* coordinates, in which the far-zone expansion of the metric proceeds with simple powers of the inverse radial distance. Hence, the logarithms are simply an artifact of the use of harmonic coordinates [131, 157]. The following theorem, proved in Ref. [12], shows that our general construction of the metric in the exterior of the source, when developed at future null infinity, is consistent with the Bondi–Sachs–Penrose [53, 193, 175, 176] approach to gravitational radiation.

Theorem 4 *The most general multipolar-post-Minkowskian solution, stationary in the past (see Equation (19)), admits some radiative coordinates* $(T, \mathbf{X})$*, for which the expansion at future null infinity,* $R \to +\infty$ *with* $U \equiv T - R/c = \text{const}$*, takes the form*

$$H_n(\mathbf{X}, T) = \sum \frac{\hat{N}_L}{R^k} K_{L,k,n}(U) + \mathcal{O}\left(\frac{1}{R^N}\right). \tag{49}$$

The functions $K_{L,k,n}$ *are computable functionals of the source multipole moments. In radiative coordinates the retarded time* $U = T - R/c$ *is a null coordinate in the asymptotic limit. The metric* $H^{\alpha\beta}_{\text{ext}} = \sum_{n\geq 1} G^n H^{\alpha\beta}_n$ *is asymptotically simple in the sense of Penrose [175, 176], perturbatively to any post-Minkowskian order.*

Proof: We introduce a linearized "radiative" metric by performing a gauge transformation of the harmonic-coordinates metric defined by Equations (26, 27, 28), namely

$$H^{\alpha\beta}_1 = h^{\alpha\beta}_1 + \partial^\alpha \xi^\beta_1 + \partial^\beta \xi^\alpha_1 - \eta^{\alpha\beta} \partial_\mu \xi^\mu_1, \tag{50}$$

where the gauge vector ξ^α_1 is

$$\xi^\alpha_1 = 2\mathrm{M}\, \eta^{0\alpha} \ln\left(\frac{r}{r_0}\right). \tag{51}$$

This gauge transformation is non-harmonic:

$$\partial_\mu H^{\alpha\mu}_1 = \Box \xi^\alpha_1 = \frac{2\mathrm{M}}{r^2} \eta^{0\alpha}. \tag{52}$$

Its effect is to "correct" for the well-known logarithmic deviation of the harmonic coordinates' retarded time with respect to the true space-time characteristic or light cones. After the change of gauge, the coordinate $u = t - r/c$ coincides with a null coordinate at the linearized level[11]. This is the requirement to be satisfied by a linearized metric so that it can constitute the linearized approximation to a full (post-Minkowskian) radiative field [157]. One can easily show that, at the dominant order when $r \to +\infty$,

$$k_\mu k_\nu H^{\mu\nu}_1 = \mathcal{O}\left(\frac{1}{r^2}\right), \tag{53}$$

[11] In this proof the coordinates are considered as dummy variables denoted (t, r). At the end, when we obtain the radiative metric, we shall denote the associated radiative coordinates by (T, R).

where $k^\alpha = (1, \mathbf{n})$ is the outgoing Minkowskian null vector. Given any $n \geq 2$, let us recursively assume that we have obtained all the previous radiative post-Minkowskian coefficients $H_m^{\alpha\beta}$, i.e. $\forall m \leq n-1$, and that all of them satisfy

$$k_\mu k_\nu H_m^{\mu\nu} = \mathcal{O}\left(\frac{1}{r^2}\right). \tag{54}$$

From this induction hypothesis one can prove that the nth post-Minkowskian source term $\Lambda_n^{\alpha\beta} = \Lambda_n^{\alpha\beta}(H_1, \dots, H_{n-1})$ is such that

$$\Lambda_n^{\alpha\beta} = \frac{k^\alpha k^\beta}{r^2} \sigma_n(u, \mathbf{n}) + \mathcal{O}\left(\frac{1}{r^3}\right). \tag{55}$$

To the leading order this term takes the classic form of the stress-energy tensor for a swarm of massless particles, with σ_n being related to the power in the waves. One can show that all the problems with the appearance of logarithms come from the retarded integral of the terms in Equation (55) that behave like $1/r^2$: See indeed the integration formula (109), which behaves like $\ln r/r$ at infinity. But now, thanks to the particular index structure of the term (55), we can correct for the effect by adjusting the gauge at the nth post-Minkowskian order. We pose, as a gauge vector,

$$\xi_n^\alpha = \mathcal{FP}\,\Box_{\text{ret}}^{-1}\left[\frac{k^\alpha}{2r^2}\int_{-\infty}^{u} dv\, \sigma_n(v, \mathbf{n})\right], \tag{56}$$

where $\mathcal{FP}$ refers to the same finite part operation as in Equation (39). This vector is such that the logarithms that will appear in the corresponding gauge terms *cancel out* the logarithms coming from the retarded integral of the source term (55); see Ref. [12] for the details. Hence, to the nth post-Minkowskian order, we define the radiative metric as

$$H_n^{\alpha\beta} = U_n^{\alpha\beta} + V_n^{\alpha\beta} + \partial^\alpha \xi_n^\beta + \partial^\beta \xi_n^\alpha - \eta^{\alpha\beta}\partial_\mu \xi_n^\mu. \tag{57}$$

Here $U_n^{\alpha\beta}$ and $V_n^{\alpha\beta}$ denote the quantities that are the analogues of $u_n^{\alpha\beta}$ and $v_n^{\alpha\beta}$, which were introduced into the harmonic-coordinates algorithm: See Equations (39, 40, 41, 42). In particular, these quantities are constructed in such a way that the sum $U_n^{\alpha\beta} + V_n^{\alpha\beta}$ is divergence-free, so we see that the radiative metric does not obey the harmonic-gauge condition:

$$\partial_\mu H_n^{\alpha\mu} = \Box \xi_n^\alpha = \frac{k^\alpha}{2r^2}\int_{-\infty}^{u} dv\, \sigma_n(v, \mathbf{n}). \tag{58}$$

The far-zone expansion of the latter metric is of the type (49), i.e. is free of any logarithms, and the retarded time in these coordinates tends asymptotically toward a null coordinate at infinity. The property of asymptotic simplicity, in the mathematical form given by Geroch and Horowitz [121], is proved by introducing the conformal factor $\Omega = 1/r$ in radiative coordinates (see Ref. [12]). Finally, it can be checked that the metric so constructed, which is a functional of the source multipole moments $\mathrm{I}_L, \dots, \mathrm{Z}_L$ (from the definition of the algorithm), is as general as the general harmonic-coordinate metric of Theorem 2, since it merely differs from it by a coordinate transformation $(t, \mathbf{x}) \longrightarrow (T, \mathbf{X})$, where $(t, \mathbf{x})$ are the harmonic coordinates and $(T, \mathbf{X})$ the radiative ones, together with a re-definition of the multipole moments.

4.4 The radiative multipole moments

The leading-order term $1/R$ of the metric in radiative coordinates, neglecting $\mathcal{O}(1/R^2)$, yields the operational definition of two sets of STF *radiative* multipole moments, mass-type $\mathrm{U}_L(U)$ and current-type $\mathrm{V}_L(U)$. *By definition*, we have

$$H_{ij}^{\mathrm{TT}}(U,\mathbf{N}) = \frac{4G}{c^2 R}\mathcal{P}_{ijab}(\mathbf{N})\sum_{l=2}^{+\infty}\frac{1}{c^l l!}\left\{N_{L-2}\mathrm{U}_{abL-2}(U) - \frac{2l}{c(l+1)}N_{cL-2}\varepsilon_{cd(a}\mathrm{V}_{b)dL-2}(U)\right\} + \mathcal{O}\left(\frac{1}{R^2}\right). \tag{59}$$

This multipole decomposition represents the generalization, up to any post-Newtonian order (witness the factors of $1/c$ in front of each of the multipolar pieces) of the quadrupole-moment formalism reviewed in Equation (2). The corresponding total gravitational flux reads

$$\mathcal{L}(U) = \sum_{l=2}^{+\infty}\frac{G}{c^{2l+1}}\left\{\frac{(l+1)(l+2)}{(l-1)ll!(2l+1)!!}\mathrm{U}_L^{(1)}(U)\mathrm{U}_L^{(1)}(U) + \frac{4l(l+2)}{c^2(l-1)(l+1)!(2l+1)!!}\mathrm{V}_L^{(1)}(U)\mathrm{V}_L^{(1)}(U)\right\}. \tag{60}$$

Notice that the meaning of such formulas is rather empty, because we do not know yet how the radiative moments are given in terms of the actual source parameters. Only at the Newtonian level do we know this relation, which from the comparison with the quadrupole formalism of Equations (2, 3, 4) reduces to

$$\mathrm{U}_{ij}(U) = \mathrm{Q}_{ij}^{(2)}(U) + \mathcal{O}\left(\frac{1}{c^2}\right), \tag{61}$$

where Q_{ij} is the Newtonian quadrupole given by Equation (3). Fortunately, we are not in such bad shape because we have learned from Theorem 4 the general method that permits us to compute the radiative multipole moments $\mathrm{U}_L, \mathrm{V}_L$ in terms of the source moments $\mathrm{I}_L, \mathrm{J}_L, \dots, \mathrm{Z}_L$. Therefore, what is missing is the explicit dependence of the *source* multipole moments as functions of the actual parameters of some isolated source. We come to grips with this question in the next Section 5.

5 Exterior Field of a Post-Newtonian Source

By Theorem 2 we control the most general class of solutions of the vacuum equations outside the source, in the form of non-linear functionals of the source multipole moments. For instance, these solutions include the Schwarzschild and Kerr solutions, as well as all their perturbations. By Theorem 4 we learned how to construct the radiative moments at infinity. We now want to understand how a specific choice of stress-energy tensor $T^{\alpha\beta}$ (i.e. a choice of some physical model describing the source) selects a particular physical exterior solution among our general class.

5.1 The matching equation

We shall provide the answer in the case of a post-Newtonian source for which the post-Newtonian parameter $1/c$ defined by Equation (1) is small. The fundamental fact that permits the connection of the exterior field to the inner field of the source is the existence of a "matching" region, in which both the multipole and the post-Newtonian expansions are valid. This region is nothing but the exterior near zone, such that $r > a$ (exterior) *and* $r \ll \lambda$ (near zone). It always exists around post-Newtonian sources.

Let us denote by $\mathcal{M}(h)$ the multipole expansion of h (for simplicity, we suppress the space-time indices). By $\mathcal{M}(h)$ we really mean the multipolar-post-Minkowskian exterior metric that we have constructed in Sections 3 and 4:

$$\mathcal{M}(h) \equiv h_{\text{ext}} = \sum_{n=1}^{+\infty} G^n h_n[\mathrm{I}_L, \dots, \mathrm{Z}_L]. \tag{62}$$

Of course, h agrees with its own multipole expansion in the exterior of the source,

$$r > a \quad \Longrightarrow \quad \mathcal{M}(h) = h. \tag{63}$$

By contrast, inside the source, h and $\mathcal{M}(h)$ disagree with each other because h is a fully-fledged solution of the field equations with matter source, while $\mathcal{M}(h)$ is a vacuum solution becoming singular at $r = 0$. Now let us denote by $\overline{h}$ the post-Newtonian expansion of h. We have already anticipated the general structure of this expansion as given in Equation (47). In the matching region, where both the multipolar and post-Newtonian expansions are valid, we write the numerical equality

$$a < r \ll \lambda \quad \Longrightarrow \quad \mathcal{M}(h) = \overline{h}. \tag{64}$$

This "numerical" equality is viewed here in a sense of formal expansions, as we do not control the convergence of the series. In fact, we should be aware that such an equality, though quite natural and even physically obvious, is probably not really justified within the approximation scheme (mathematically speaking), and we take it as part of our fundamental assumptions.

We now transform Equation (64) into a *matching equation*, by replacing in the left-hand side $\mathcal{M}(h)$ by its near-zone re-expansion $\overline{\mathcal{M}(h)}$, and in the right-hand side $\overline{h}$ by its multipole expansion $\mathcal{M}(\overline{h})$. The structure of the near-zone expansion ($r \to 0$) of the exterior multipolar field has been found in Equation (46). We denote the corresponding infinite series $\overline{\mathcal{M}(h)}$ with the same overbar as for the post-Newtonian expansion because it is really an expansion when $r/c \to 0$, equivalent to an expansion when $c \to \infty$. Concerning the multipole expansion of the post-Newtonian metric, $\mathcal{M}(\overline{h})$, we simply postulate its existence. Therefore, the matching equation is the statement that

$$\overline{\mathcal{M}(h)} = \mathcal{M}(\overline{h}), \tag{65}$$

by which we really mean an infinite set of *functional* identities, valid $\forall(\mathbf{x}, t) \in \mathbb{R}^3_* \times \mathbb{R}$, between the coefficients of the series in both sides of the equation. Note that such a meaning is somewhat different from that of a *numerical* equality like Equation (64), which is valid only when $\mathbf{x}$ belongs to some limited spatial domain. The matching equation (65) tells us that the formal *near-zone* expansion of the multipole decomposition is *identical*, term by term, to the multipole expansion of the post-Newtonian solution. However, the former expansion is nothing but the formal *far-zone* expansion, when $r \to \infty$, of each of the post-Newtonian coefficients. Most importantly, it is possible to write down, within the present formalism, the general structure of these identical expansions as a consequence of Theorem 3, Equation (46):

$$\overline{\mathcal{M}(h)} = \sum \hat{n}_L r^m (\ln r)^p F_{L,m,p}(t) = \mathcal{M}(\overline{h}), \tag{66}$$

where the functions $F_{L,m,p} = \sum_{n\geq 1} G^n F_{L,m,p,n}$. The latter expansion can be interpreted either as the singular re-expansion of the multipole decomposition when $r \to 0$ (first equality in Equation (66)), or the singular re-expansion of the post-Newtonian series when $r \to +\infty$ (second equality). We recognize the beauty of singular perturbation theory, where two asymptotic expansions, taken formally outside their respective domains of validity, are matched together. Of course, the method works because there exists, physically, an overlapping region in which the two approximation series are expected to be numerically close to the exact solution.

5.2 General expression of the multipole expansion

Theorem 5 *Under the hypothesis of matching, Equation (65), the multipole expansion of the solution of the Einstein field equation outside a post-Newtonian source reads*

$$\mathcal{M}(h^{\alpha\beta}) = \mathcal{FP}_{B=0}\,\Box^{-1}_{\mathrm{ret}}[\widetilde{r}^B \mathcal{M}(\Lambda^{\alpha\beta})] - \frac{4G}{c^4}\sum_{l=0}^{+\infty}\frac{(-)^l}{l!}\partial_L\left\{\frac{1}{r}\mathcal{H}^{\alpha\beta}_L(t-r/c)\right\}, \tag{67}$$

where the "multipole moments" are given by

$$\mathcal{H}^{\alpha\beta}_L(u) = \mathcal{FP}_{B=0}\int d^3\mathbf{x}\,|\widetilde{\mathbf{x}}|^B x_L\,\overline{\tau}^{\alpha\beta}(\mathbf{x},u). \tag{68}$$

Here, $\overline{\tau}^{\alpha\beta}$ denotes the post-Newtonian expansion of the stress-energy pseudo-tensor defined by Equation (14).

Proof [15, 20]: First notice where the physical restriction of considering a post-Newtonian source enters this theorem: the multipole moments (68) depend on the *post-Newtonian* expansion $\overline{\tau}^{\alpha\beta}$, rather than on $\tau^{\alpha\beta}$ itself. Consider $\Delta^{\alpha\beta}$, namely the difference between $h^{\alpha\beta}$, which is a solution of the field equations everywhere inside and outside the source, and the first term in Equation (67), namely the finite part of the retarded integral of the multipole expansion $\mathcal{M}(\Lambda^{\alpha\beta})$:

$$\Delta^{\alpha\beta} \equiv h^{\alpha\beta} - \mathcal{FP}\,\Box^{-1}_{\mathrm{ret}}[\mathcal{M}(\Lambda^{\alpha\beta})]. \tag{69}$$

From now on we shall generally abbreviate the symbols concerning the finite-part operation at $B = 0$ by a mere $\mathcal{FP}$. According to Equation (20), $h^{\alpha\beta}$ is given by the retarded integral of the pseudo-tensor $\tau^{\alpha\beta}$. So,

$$\Delta^{\alpha\beta} = \frac{16\pi G}{c^4}\Box^{-1}_{\mathrm{ret}}\tau^{\alpha\beta} - \mathcal{FP}\,\Box^{-1}_{\mathrm{ret}}\left[\mathcal{M}(\Lambda^{\alpha\beta})\right]. \tag{70}$$

In the second term the finite part plays a crucial role because the multipole expansion $\mathcal{M}(\Lambda^{\alpha\beta})$ is singular at $r = 0$. By contrast, the first term in Equation (70), as it stands, is well-defined because we are considering only some smooth field distributions: $\tau^{\alpha\beta} \in C^\infty(\mathbb{R}^4)$. There is no need to include a finite part $\mathcal{FP}$ in the first term, but *a contrario* there is no harm to add one in front of it, because for convergent integrals the finite part simply gives back the value of the integral. The advantage of adding "artificially" the $\mathcal{FP}$ in the first term is that we can re-write Equation (70) into the much more interesting form

$$\Delta^{\alpha\beta} = \frac{16\pi G}{c^4} \mathcal{FP}\, \Box^{-1}_{\mathrm{ret}} \left[\tau^{\alpha\beta} - \mathcal{M}(\tau^{\alpha\beta})\right], \tag{71}$$

in which we have also used the fact that $\mathcal{M}(\Lambda^{\alpha\beta}) = 16\pi G/c^4 \cdot \mathcal{M}(\tau^{\alpha\beta})$ because $T^{\alpha\beta}$ has a compact support. The interesting point about Equation (71) is that $\Delta^{\alpha\beta}$ appears now to be the (finite part of a) retarded integral of a source with spatially *compact* support. This follows from the fact that the pseudo-tensor agrees numerically with its own multipole expansion when $r > a$ (same equation as (63)). Therefore, $\mathcal{M}(\Delta^{\alpha\beta})$ can be obtained from the known formula for the multipole expansion of the retarded solution of a wave equation with compact-support source. This formula, given in Appendix B of Ref. [28], yields the second term in Equation (67),

$$\mathcal{M}(\Delta^{\alpha\beta}) = -\frac{4G}{c^4} \sum_{l=0}^{+\infty} \frac{(-)^l}{l!} \partial_L \left\{ \frac{1}{r} \mathcal{H}^{\alpha\beta}_L(u) \right\}, \tag{72}$$

but in which the moments do not yet match the result (68); instead,

$$\mathcal{H}^{\alpha\beta}_L = \mathcal{FP} \int d^3\mathbf{x}\, x_L \left[\tau^{\alpha\beta} - \mathcal{M}(\tau^{\alpha\beta})\right]. \tag{73}$$

The reason is that we have not yet applied the assumption of a post-Newtonian source. Such sources are entirely covered by their own near zone (i.e. $a \ll \lambda$), and, in addition, the integral (73) has a compact support limited to the domain of the source. In consequence, we can replace the integrand in Equation (73) by its post-Newtonian expansion, valid over all the near zone, i.e.

$$\mathcal{H}^{\alpha\beta}_L = \mathcal{FP} \int d^3\mathbf{x}\, x_L \left[\overline{\tau}^{\alpha\beta} - \overline{\mathcal{M}(\tau^{\alpha\beta})}\right]. \tag{74}$$

Strangely enough, we do not get the expected result because of the presence of the second term in Equation (74). Actually, this term is a bit curious, because the object $\overline{\mathcal{M}(\tau^{\alpha\beta})}$ it contains is only known in the form of the formal series whose structure is given by the first equality in Equation (66) (indeed τ and h have the same type of structure). Happily (because we would not know what to do with this term in applications), we are now going to prove that the second term in Equation (74) is in fact *identically zero*. The proof is based on the properties of the analytic continuation as applied to the formal structure (66) of $\overline{\mathcal{M}(\tau^{\alpha\beta})}$. Each term of this series yields a contribution to Equation (74) that takes the form, after performing the angular integration, of the integral $\mathcal{FP}_{B=0} \int_0^{+\infty} dr\, r^{B+b} (\ln r)^p$, and multiplied by some function of time. We want to prove that the radial integral $\int_0^{+\infty} dr\, r^{B+b} (\ln r)^p$ is zero by analytic continuation ($\forall B \in \mathbb{C}$). First we can get rid of the logarithms by considering some repeated differentiations with respect to B; thus we need only to consider the simpler integral $\int_0^{+\infty} dr\, r^{B+b}$. We split the integral into a "near-zone" integral $\int_0^{\mathcal{R}} dr\, r^{B+b}$ and a "far-zone" one $\int_{\mathcal{R}}^{+\infty} dr\, r^{B+b}$, where $\mathcal{R}$ is some constant radius. When $\Re(B)$ is a large enough *positive* number, the value of the near-zone integral is $\mathcal{R}^{B+b+1}/(B+b+1)$, while when $\Re(B)$ is a large *negative* number, the far-zone integral reads the opposite, $-\mathcal{R}^{B+b+1}/(B+b+1)$. Both obtained values represent the unique analytic continuations of the near-zone and far-zone integrals for any $B \in \mathbb{C}$ except $-b-1$. The complete integral $\int_0^{+\infty} dr\, r^{B+b}$ is equal to the sum of these analytic continuations, and is therefore identically zero ($\forall B \in \mathbb{C}$, including the value $-b-1$).

At last we have completed the proof of Theorem 5:

$$\mathcal{H}_L^{\alpha\beta} = \mathcal{FP} \int d^3\mathbf{x}\, x_L \overline{\tau}^{\alpha\beta}. \tag{75}$$

The latter proof makes it clear how crucial the analytic-continuation finite part $\mathcal{FP}$ is, which we recall is the same as in our iteration of the exterior post-Minkowskian field (see Equation (39)). Without a finite part, the multipole moment (75) would be strongly divergent, because the pseudo-tensor $\overline{\tau}^{\alpha\beta}$ has a non-compact support owing to the contribution of the gravitational field, and the multipolar factor x_L behaves like r^l when $r \to +\infty$. In applications (Part B of this article) we must carefully follow the rules for handling the $\mathcal{FP}$ operator.

The two terms in the right-hand side of Equation (67) depend separately on the length scale r_0 that we have introduced into the definition of the finite part, through the analytic-continuation factor $\widetilde{r}^B = (r/r_0)^B$ (see Equation (36)). However, the sum of these two terms, i.e. the exterior multipolar field $\mathcal{M}(h)$ itself, is independent of r_0. To see this, the simplest way is to differentiate formally $\mathcal{M}(h)$ with respect to r_0. The independence of the field upon r_0 is quite useful in applications, since in general many intermediate calculations do depend on r_0, and only in the final stage does the cancellation of the r_0's occur. For instance, we shall see that the source quadrupole moment depends on r_0 starting from the 3PN level [45], but that this r_0 is compensated by another r_0 coming from the non-linear "tails of tails" at the 3PN order.

5.3 Equivalence with the Will–Wiseman formalism

Recently, Will and Wiseman [220] (see also Refs. [219, 173]), extending previous work of Epstein and Wagoner [107] and Thorne [210], have obtained a different-looking multipole decomposition, with different definitions for the multipole moments of a post-Newtonian source. They find, instead of our multipole decomposition given by Equation (67),

$$\mathcal{M}(h^{\alpha\beta}) = \Box^{-1}_{\text{ret}}[\mathcal{M}(\Lambda^{\alpha\beta})]\big|_{\mathcal{R}} - \frac{4G}{c^4}\sum_{l=0}^{+\infty}\frac{(-)^l}{l!}\partial_L\left\{\frac{1}{r}\mathcal{W}_L^{\alpha\beta}(t-r/c)\right\}. \tag{76}$$

There is no $\mathcal{FP}$ operation in the first term, but instead the retarded integral is *truncated*, as indicated by the subscript $\mathcal{R}$, to extend only in the "far zone": i.e. $|\mathbf{x}'| > \mathcal{R}$ in the notation of Equation (21), where $\mathcal{R}$ is a constant radius enclosing the source ($\mathcal{R} > a$). The near-zone part of the retarded integral is thereby removed, and there is no problem with the singularity of the multipole expansion $\mathcal{M}(\Lambda^{\alpha\beta})$ at the origin. The multipole moments $\mathcal{W}_L$ are then given, in contrast with our result (68), by an integral extending over the "near zone" only:

$$\mathcal{W}_L^{\alpha\beta}(u) = \int_{|\mathbf{x}|<\mathcal{R}} d^3\mathbf{x}\; x_L\, \overline{\tau}^{\alpha\beta}(\mathbf{x},u). \tag{77}$$

Since the integrand is compact-supported there is no problem with the bound at infinity and the integral is well-defined (no need of a $\mathcal{FP}$).

Let us show that the two different formalisms are equivalent. We compute the difference between our moment $\mathcal{H}_L$, defined by Equation (68), and the Will–Wiseman moment $\mathcal{W}_L$, given by Equation (77). For the comparison we split $\mathcal{H}_L$ into far-zone and near-zone integrals corresponding to the radius $\mathcal{R}$. Since the finite part $\mathcal{FP}$ present in $\mathcal{H}_L$ deals only with the bound at infinity, it can be removed from the near-zone integral, which is then seen to be exactly equal to $\mathcal{W}_L$. So the difference between the two moments is simply given by the far-zone integral:

$$\mathcal{H}_L^{\alpha\beta}(u) - \mathcal{W}_L^{\alpha\beta}(u) = \mathcal{FP}\int_{|\mathbf{x}|>\mathcal{R}} d^3\mathbf{x}\, x_L \overline{\tau}^{\alpha\beta}(\mathbf{x},u). \tag{78}$$

Next, we transform this expression. Successively we write $\overline{\tau}^{\alpha\beta} = \mathcal{M}(\overline{\tau}^{\alpha\beta})$ because we are outside the source, and $\mathcal{M}(\overline{\tau}^{\alpha\beta}) = \overline{\mathcal{M}(\tau^{\alpha\beta})}$ from the matching equation (65). At this stage, we recall from our reasoning right after Equation (74) that the finite part of an integral over the whole space $\mathbb{R}^3$ of a quantity having the same structure as $\overline{\mathcal{M}(\tau^{\alpha\beta})}$ is identically zero by analytic continuation. The main trick of the proof is made possible by this fact, as it allows us to transform the far-zone integration $|\mathbf{x}| > \mathcal{R}$ in Equation (78) into a *near-zone* one $|\mathbf{x}| < \mathcal{R}$, at the price of changing the overall sign in front of the integral. So,

$$\mathcal{H}_L^{\alpha\beta}(u) - \mathcal{W}_L^{\alpha\beta}(u) = -\mathcal{FP} \int_{|\mathbf{x}|<\mathcal{R}} d^3\mathbf{x}\, x_L \overline{\mathcal{M}(\tau^{\alpha\beta})}(\mathbf{x}, u). \tag{79}$$

Finally, it is straightforward to check that the right-hand side of this equation, when summed up over all multipolarities l, accounts exactly for the near-zone part that was removed from the retarded integral of $\mathcal{M}(\Lambda^{\alpha\beta})$ (first term in Equation (76)), so that the "complete" retarded integral as given by the first term in our own definition (67) is exactly reconstituted. In conclusion, the formalism of Ref. [220] is equivalent to the one of Refs. [15, 20].

5.4 The source multipole moments

In principle the bridge between the exterior gravitational field generated by the post-Newtonian source and its inner field is provided by Theorem 5; however, we still have to make the connection with the explicit construction of the general multipolar and post-Minkowskian metric in Sections 3 and 4. Namely, we must find the expressions of the six STF source multipole moments $\mathrm{I}_L, \mathrm{J}_L, \dots, \mathrm{Z}_L$ parametrizing the linearized metric (26, 27, 28) at the basis of that construction[12].

To do this we first find the equivalent of the multipole expansion given in Theorem 5, which was parametrized by non-trace-free multipole functions $\mathcal{H}_L^{\alpha\beta}(u)$, in terms of new multipole functions $\mathcal{F}_L^{\alpha\beta}(u)$ that are STF in all their indices L. The result (which follows from Equation (B.14a) in [28]) is

$$\mathcal{M}(h^{\alpha\beta}) = \mathcal{FP}\, \Box^{-1}_{\mathrm{ret}}[\mathcal{M}(\Lambda^{\alpha\beta})] - \frac{4G}{c^4} \sum_{l=0}^{+\infty} \frac{(-)^l}{l!} \partial_L \left\{ \frac{1}{r} \mathcal{F}_L^{\alpha\beta}(t - r/c) \right\}, \tag{80}$$

where the STF multipole functions (witness the multipolar factor $\hat{x}_L \equiv \mathrm{STF}[x_L]$) read

$$\mathcal{F}_L^{\alpha\beta}(u) = \mathcal{FP} \int d^3\mathbf{x}\, \hat{x}_L \int_{-1}^{1} dz\, \delta_l(z)\, \overline{\tau}^{\alpha\beta}(\mathbf{x}, u + z|\mathbf{x}|/c). \tag{81}$$

Notice the presence of an extra integration variable z, ranging from -1 to 1. The z-integration involves the weighting function[13]

$$\delta_l(z) = \frac{(2l+1)!!}{2^{l+1} l!} (1 - z^2)^l, \tag{82}$$

which is normalized in such a way that

$$\int_{-1}^{1} dz\, \delta_l(z) = 1. \tag{83}$$

[12] Recall that in actual applications we need mostly the mass-type moment I_L and current-type one J_L, because the other moments parametrize a linearized gauge transformation.

[13] This function approaches the Dirac delta-function (hence its name) in the limit of large multipoles: $\lim_{l\to+\infty} \delta_l(z) = \delta(z)$. Indeed the source looks more and more like a point mass as we increase the multipolar order l.

The next step is to impose the harmonic-gauge conditions (12) onto the multipole decomposition (80), and to decompose the multipole functions $\mathcal{F}_L^{\alpha\beta}(u)$ into STF irreducible pieces with respect to both L and their space-time indices $\alpha\beta$. This technical part of the calculation is identical to the one of the STF irreducible multipole moments of linearized gravity [89]. The formulas needed in this decomposition read

$$\begin{aligned}
\mathcal{F}_L^{00} &= R_L, \\
\mathcal{F}_L^{0i} &= {}^{(+)}T_{iL} + \varepsilon_{ai<i_l}{}^{(0)}T_{L-1>a} + \delta_{i<i_l}{}^{(-)}T_{L-1>}, \\
\mathcal{F}_L^{ij} &= {}^{(+2)}U_{ijL} + \underset{L}{STF}\,\underset{ij}{STF}\,[\varepsilon_{aii_l}{}^{(+1)}U_{ajL-1} + \delta_{ii_l}{}^{(0)}U_{jL-1} \\
&\quad + \delta_{ii_l}\varepsilon_{aji_{l-1}}{}^{(-1)}U_{aL-2} + \delta_{ii_l}\delta_{ji_{l-1}}{}^{(-2)}U_{L-2}] + \delta_{ij}V_L,
\end{aligned} \tag{84}$$

where the ten tensors $R_L, {}^{(+)}T_{L+1}, \dots, {}^{(-2)}U_{L-2}, V_L$ are STF, and are uniquely given in terms of the $\mathcal{F}_L^{\alpha\beta}$'s by some inverse formulas. Finally, the latter decompositions lead to the following theorem.

Theorem 6 *The STF multipole moments* I_L *and* J_L *of a post-Newtonian source are given, formally up to any post-Newtonian order, by* ($l \geq 2$)

$$\begin{aligned}
\mathrm{I}_L(u) &= \mathcal{FP}\int d^3\mathbf{x}\int_{-1}^{1} dz \Big\{ \delta_l \hat{x}_L \Sigma - \frac{4(2l+1)}{c^2(l+1)(2l+3)}\delta_{l+1}\hat{x}_{iL}\Sigma_i^{(1)} \\
&\quad + \frac{2(2l+1)}{c^4(l+1)(l+2)(2l+5)}\delta_{l+2}\hat{x}_{ijL}\Sigma_{ij}^{(2)} \Big\}(\mathbf{x}, u+z|\mathbf{x}|/c), \\
\mathrm{J}_L(u) &= \mathcal{FP}\int d^3\mathbf{x}\int_{-1}^{1} dz\, \varepsilon_{ab\langle i_l} \Big\{ \delta_l \hat{x}_{L-1\rangle a}\Sigma_b - \frac{2l+1}{c^2(l+2)(2l+3)}\delta_{l+1}\hat{x}_{L-1\rangle ac}\Sigma_{bc}^{(1)} \Big\}(\mathbf{x}, u+z|\mathbf{x}|/c).
\end{aligned} \tag{85}$$

These moments are the ones that are to be inserted into the linearized metric $h_1^{\alpha\beta}$ *that represents the lowest approximation to the post-Minkowskian field* $h_{\text{ext}}^{\alpha\beta} = \sum_{n\geq 1} G^n h_n^{\alpha\beta}$ *defined in Section 4.*

In these formulas the notation is as follows: Some convenient source densities are defined from the post-Newtonian expansion of the pseudo-tensor $\tau^{\alpha\beta}$ by

$$\begin{aligned}
\Sigma &= \frac{\overline{\tau}^{00} + \overline{\tau}^{ii}}{c^2}, \\
\Sigma_i &= \frac{\overline{\tau}^{0i}}{c}, \\
\Sigma_{ij} &= \overline{\tau}^{ij}
\end{aligned} \tag{86}$$

(where $\overline{\tau}^{ii} \equiv \delta_{ij}\overline{\tau}^{ij}$). As indicated in Equations (85) these quantities are to be evaluated at the spatial point $\mathbf{x}$ and at time $u + z|\mathbf{x}|/c$.

For completeness, we give also the formulas for the four auxiliary source moments $\mathrm{W}_L, \dots, \mathrm{Z}_L$, which parametrize the gauge vector φ_1^α as defined in Equations (28):

$$\mathrm{W}_L(u) = \mathcal{FP}\int d^3\mathbf{x}\int_{-1}^{1} dz \Big\{ \frac{2l+1}{(l+1)(2l+3)}\delta_{l+1}\hat{x}_{iL}\Sigma_i - \frac{2l+1}{2c^2(l+1)(l+2)(2l+5)}\delta_{l+2}\hat{x}_{ijL}\Sigma_{ij}^{(1)} \Big\}, \tag{87}$$

$$\mathrm{X}_L(u) = \mathcal{FP}\int d^3\mathbf{x}\int_{-1}^{1} dz\left\{\frac{2l+1}{2(l+1)(l+2)(2l+5)}\delta_{l+2}\hat{x}_{ijL}\Sigma_{ij}\right\}, \tag{88}$$

$$\mathrm{Y}_L(u) = \mathcal{FP}\int d^3\mathbf{x}\int_{-1}^{1} dz\left\{-\delta_l\hat{x}_L\Sigma_{ii} + \frac{3(2l+1)}{(l+1)(2l+3)}\delta_{l+1}\hat{x}_{iL}\Sigma_i^{(1)} - \frac{2(2l+1)}{c^2(l+1)(l+2)(2l+5)}\delta_{l+2}\hat{x}_{ijL}\Sigma_{ij}^{(2)}\right\}, \tag{89}$$

$$\mathrm{Z}_L(u) = \mathcal{FP}\int d^3\mathbf{x}\int_{-1}^{1} dz\,\varepsilon_{ab\langle i_l}\left\{-\frac{2l+1}{(l+2)(2l+3)}\delta_{l+1}\hat{x}_{L-1\rangle bc}\Sigma_{ac}\right\}. \tag{90}$$

As discussed in Section 4, one can always find two intermediate "packages" of multipole moments, M_L and S_L, which are some non-linear functionals of the source moments (85) and Equations (87, 88, 89, 90), and such that the exterior field depends only on them, modulo a change of coordinates (see, e.g., Equation (96) below).

In fact, all these source moments make sense only in the form of a post-Newtonian expansion, so in practice we need to know how to expand all the z-integrals as series when $c \to +\infty$. Here is the appropriate formula:

$$\int_{-1}^{1} dz\ \delta_l(z)\tau(\mathbf{x}, u + z|\mathbf{x}|/c) = \sum_{k=0}^{+\infty}\frac{(2l+1)!!}{2^k k!(2l+2k+1)!!}\left(\frac{|\mathbf{x}|}{c}\frac{\partial}{\partial u}\right)^{2k}\tau(\mathbf{x}, u). \tag{91}$$

Since the right-hand side involves only even powers of $1/c$, the same result holds equally well for the "advanced" variable $u + z|\mathbf{x}|/c$ or the "retarded" one $u - z|\mathbf{x}|/c$. Of course, in the Newtonian limit, the moments I_L and J_L (and also M_L, S_L) reduce to the standard expressions. For instance, we have

$$\mathrm{I}_L(u) = \mathrm{Q}_L(u) + \mathcal{O}\left(\frac{1}{c^2}\right), \tag{92}$$

where Q_L is the Newtonian mass-type multipole moment (see Equation (3)). (The moments $\mathrm{W}_L, \dots, \mathrm{Z}_L$ have also a Newtonian limit, but it is not particularly illuminating.)

Needless to say, the formalism becomes prohibitively difficult to apply at very high post-Newtonian approximations. Some post-Newtonian order being given, we must first compute the relevant relativistic corrections to the pseudo stress-energy-tensor $\tau^{\alpha\beta}$ (this necessitates solving the field equations inside the matter, see Section 5.5) before inserting them into the source moments (85, 86, 82, 83, 91, 87, 88, 89, 90). The formula (91) is used to express all the terms up to that post-Newtonian order by means of more tractable integrals extending over $\mathbb{R}^3$. Given a specific model for the matter source we then have to find a way to compute all these spatial integrals (we do it in Section 10 in the case of point-mass binaries). Next, we must substitute the source multipole moments into the linearized metric (26, 27, 28), and iterate them until all the necessary multipole interactions taking place in the radiative moments U_L and V_L are under control. In fact, we shall work out these multipole interactions for general sources in the next section up to the 3PN order. Only at this point does one have the physical radiation field at infinity, from which we can build the templates for the detection and analysis of gravitational waves. We advocate here that the complexity of the formalism reflects simply the complexity of the Einstein field equations. It is probably impossible to devise a different formalism, valid for general sources devoid of symmetries, that would be substantially simpler.

5.5 Post-Newtonian field in the near zone

Theorem 6 solves in principle the question of the generation of gravitational waves by extended post-Newtonian sources. However, note that this result has to be completed by the definition of an explicit algorithm for the post-Newtonian iteration, analogous to the post-Minkowskian algorithm we defined in Section 4, so that the source multipole moments, which contain the full post-Newtonian expansion of the pseudo-tensor $\tau^{\alpha\beta}$, can be completely specified. Such a systematic post-Newtonian iteration scheme, valid (formally) to any post-Newtonian order, has been implemented [185, 41] using matched asymptotic expansions. The solution of this problem yields, in particular, some general expression, valid up to any order, of the terms associated with the gravitational radiation reaction force inside the post-Newtonian source[14].

Before proceeding, let us recall that the "standard" post-Newtonian approximation, as it was used until, say, the early 1980's (see for instance Refs. [2, 142, 143, 172]), is plagued with some apparently inherent difficulties, which crop up at some high post-Newtonian order. The first problem is that in higher approximations some *divergent* Poisson-type integrals appear. Indeed the post-Newtonian expansion replaces the resolution of a hyperbolic-like d'Alembertian equation by a perturbatively equivalent hierarchy of elliptic-like Poisson equations. Rapidly it is found during the post-Newtonian iteration that the right-hand side of the Poisson equations acquires a non-compact support (it is distributed over all space), and that as a result the standard Poisson integral diverges at the bound of the integral at spatial infinity, i.e. $r \equiv |\mathbf{x}| \to +\infty$, with $t = \text{const.}$

The second problem is related with the *a priori* limitation of the approximation to the near zone, which is the region surrounding the source of small extent with respect to the wavelength of the emitted radiation: $r \ll \lambda$. The post-Newtonian expansion assumes from the start that all retardations r/c are small, so it can rightly be viewed as a formal *near-zone* expansion, when $r \to 0$. In particular, the fact which makes the Poisson integrals to become typically divergent, namely that the coefficients of the post-Newtonian series blow up at "spatial infinity", when $r \to +\infty$, has nothing to do with the actual behaviour of the field at infinity. However, the serious consequence is that it is not possible, *a priori*, to implement within the post-Newtonian iteration the physical information that the matter system is isolated from the rest of the universe. Most importantly, the no-incoming radiation condition, imposed at past null infinity, cannot be taken into account, *a priori*, into the scheme. In a sense the post-Newtonian approximation is not "self-supporting", because it necessitates some information taken from outside its own domain of validity.

Here we present, following Refs. [185, 41], a solution of both problems, in the form of a general expression for the near-zone gravitational field, developed to any post-Newtonian order, which has been determined from implementing the matching equation (65). This solution is free of the divergences of Poisson-type integrals we mentionned above, and it incorporates the effects of gravitational radiation reaction appropriate to an isolated system.

Theorem 7 *The expression of the post-Newtonian field in the near zone of a post-Newtonian source, satisfying correct boundary conditions at infinity (no incoming radiation), reads*

$$\overline{h}^{\alpha\beta} = \frac{16\pi G}{c^4}\left[\mathcal{FP}\,\Box^{-1}_{\text{ret}}\left[\overline{\tau}^{\alpha\beta}\right] + \sum_{l=0}^{+\infty}\frac{(-)^l}{l!}\partial_L\left\{\frac{\mathcal{R}^{\alpha\beta}_L(t-r/c) - \mathcal{R}^{\alpha\beta}_L(t+r/c)}{2\,r}\right\}\right]. \tag{93}$$

The first term represents a particular solution of the hierarchy of post-Newtonian equations, while the second one is a homogeneous multipolar solution of the wave equation, of the "anti-symmetric" type that is regular at the origin $r = 0$ located in the source.

[14]An alternative approach to the problem of radiation reaction, besides the matching procedure, is to work only within a post-Minkowskian iteration scheme (which does not expand the retardations): see, e.g., Ref. [69].

More precisely, the flat retarded d'Alembertian operator in Equation (93) is given by the standard expression (21) but with all retardations expanded ($r/c \to 0$), and with the finite part $\mathcal{FP}$ procedure involved for dealing with the bound at infinity of the Poisson-type integrals (so that all the integrals are well-defined at any order of approximation),

$$\mathcal{FP}\,\Box^{-1}_{\mathrm{ret}}\left[\overline{\tau}^{\alpha\beta}\right] \equiv -\frac{1}{4\pi}\sum_{n=0}^{+\infty}\frac{(-)^n}{n!}\left(\frac{\partial}{c\,\partial t}\right)^n \mathcal{FP}\int d^3\mathbf{x}'\,|\mathbf{x}-\mathbf{x}'|^{n-1}\,\overline{\tau}^{\alpha\beta}(\mathbf{x}',t). \tag{94}$$

The existence of the solution (94) shows that the problem of divergences of the post-Newtonian expansion is simply due to the fact that the standard Poisson integral does not constitute the correct solution of the Poisson equation in the context of post-Newtonian expansions. So the problem is purely of a technical nature, and is solved once we succeed in finding the appropriate solution to the Poisson equation.

Theorem 7 is furthermore to be completed by the information concerning the multipolar functions $\mathcal{R}^{\alpha\beta}_L(u)$ parametrizing the anti-symmetric homogeneus solution, the second term of Equation (93). Note that this homogeneous solution represents the unique one for which the matching equation (65) is satisfied. The result is

$$\mathcal{R}^{\alpha\beta}_L(u) = -\frac{1}{4\pi}\mathcal{FP}\int d^3\mathbf{x}'\,\hat{x}'_L\int_1^{+\infty} dz\,\gamma_l(z)\,\mathcal{M}(\tau^{\alpha\beta})(\mathbf{x}', u - z|\mathbf{x}'|/c), \tag{95}$$

where $\mathcal{M}(\tau^{\alpha\beta})$ denotes the multipole expansion of the pseudo-tensor (in the sense of Equation (62)), and where we denote $\gamma_l(z) = -2\delta_l(z)$, with $\delta_l(z)$ being given by Equation (82)[15].

Importantly, we find that the post-Newtonian expansion $\overline{h}^{\alpha\beta}$ given by Theorem 7 is a functional not only of the related expansion of the pseudo-tensor, $\overline{\tau}^{\alpha\beta}$, but also, by Equation (95), of its multipole expansion $\mathcal{M}(\tau^{\alpha\beta})$, which is valid in the exterior of the source, and in particular in the asymptotic regions far from the source. This can be understood by the fact that the post-Newtonian solution (93) depends on the boundary conditions imposed at infinity, that describe a matter system isolated from the rest of the universe.

Equation (93) is interesting for providing a practical recipe for performing the post-Newtonian iteration *ad infinitum.* Moreover, it gives some insights on the structure of radiation reaction terms. Recall that the anti-symmetric waves, regular in the source, are associated with radiation reaction effects. More precisely, it has been shown [185] that the specific anti-symmetric wave given by the second term of Equation (93) is linked with some *non-linear* contribution due to gravitational wave tails in the radiation reaction force. Such a contribution constitutes a generalization of the tail-transported radiation reaction term at the 4PN order, i.e. 1.5PN order relative to the dominant radiation reaction order, as determined in Ref. [27]. This term is in fact required by energy conservation and the presence of tails in the wave zone (see, e.g., Equation (97) below). Hence, the second term of Equation (93) is dominantly of order 4PN and can be neglected in computations of the radiation reaction up to 3.5PN order (as in Ref. [164]). The usual radiation reaction terms, up to 3.5PN order, which are *linear* in the source multipole moments (for instance the usual radiation reaction term at 2.5PN order), are contained in the first term of Equation (93), and are given by the terms with odd powers of $1/c$ in the post-Newtonian expansion (94). It can be shown [41] that such terms take also the form of some anti-symmetric multipolar wave, which turn out to be parametrized by the same moments as in the exterior field, namely the moments which are the STF analogues of Equations (68).

[15] Notice that the normalization $\int_1^{+\infty} dz\,\gamma_l(z) = 1$ holds as a consequence of the corresponding normalization (83) for $\delta_l(z)$, together with the fact that $\int_{-\infty}^{+\infty} dz\,\gamma_l(z) = 0$ by analytic continuation in the variable $l \in \mathbb{C}$.

6 Non-linear Multipole Interactions

We shall now show that the radiative mass-type quadrupole moment U_{ij} includes a quadratic tail at the relative 1.5PN order (or $1/c^3$), corresponding to the interaction of the mass M of the source and its quadrupole moment I_{ij}. This is due to the back-scattering of quadrupolar waves off the Schwarzschild curvature generated by M. Next, U_{ij} includes a so-called non-linear memory integral at the 2.5PN order, due to the quadrupolar radiation of the stress-energy distribution of linear quadrupole waves themselves, i.e. of multipole interactions $\mathrm{I}_{ij} \times \mathrm{I}_{kl}$. Finally, we have also a cubic tail, or "tail of tail", arising at the 3PN order, and associated with the multipole interaction $\mathrm{M}^2 \times \mathrm{I}_{ij}$. The result for U_{ij} is better expressed in terms of the intermediate quadrupole moment M_{ij} already discussed in Section 4.2. This moment reads [16]

$$\mathrm{M}_{ij} = \mathrm{I}_{ij} - \frac{4G}{c^5}\left[\mathrm{W}^{(2)}\mathrm{I}_{ij} - \mathrm{W}^{(1)}\mathrm{I}^{(1)}_{ij}\right] + \mathcal{O}\left(\frac{1}{c^7}\right), \tag{96}$$

where W means W_L as given by Equation (87) in the case $l = 0$ (of course, in Equation (96) we need only the Newtonian value of W). The difference between the two moments M_{ij} and I_{ij} is a small 2.5PN quantity. Henceforth, we shall express many of the results in terms of the mass moments M_L and the corresponding current ones S_L. The complete formula for the radiative quadrupole, valid through the 3PN order, reads [21, 19]

$$\begin{aligned}
\mathrm{U}_{ij}(U) = {}& \mathrm{M}^{(2)}_{ij}(U) + \frac{2G\mathrm{M}}{c^3}\int_0^{+\infty} d\tau\, \mathrm{M}^{(4)}_{ij}(U-\tau)\left[\ln\left(\frac{c\tau}{2r_0}\right) + \frac{11}{12}\right] \\
&+ \frac{G}{c^5}\left\{-\frac{2}{7}\int_0^{+\infty} d\tau\, \mathrm{M}^{(3)}_{a\langle i}(U-\tau)\mathrm{M}^{(3)}_{j\rangle a}(U-\tau)\right. \\
&\qquad \left. -\frac{2}{7}\mathrm{M}^{(3)}_{a\langle i}\mathrm{M}^{(2)}_{j\rangle a} - \frac{5}{7}\mathrm{M}^{(4)}_{a\langle i}\mathrm{M}^{(1)}_{j\rangle a} + \frac{1}{7}\mathrm{M}^{(5)}_{a\langle i}\mathrm{M}_{j\rangle a} + \frac{1}{3}\varepsilon_{ab\langle i}\mathrm{M}^{(4)}_{j\rangle a}\mathrm{S}_b\right\} \\
&+ \frac{2G^2\mathrm{M}^2}{c^6}\int_0^{+\infty} d\tau\, \mathrm{M}^{(5)}_{ij}(U-\tau)\left[\ln^2\left(\frac{c\tau}{2r_0}\right) + \frac{57}{70}\ln\left(\frac{c\tau}{2r_0}\right) + \frac{124627}{44100}\right] \\
&+ \mathcal{O}\left(\frac{1}{c^7}\right).
\end{aligned} \tag{97}$$

The retarded time in radiative coordinates is denoted $U = T - R/c$. The constant r_0 is the one that enters our definition of the finite-part operation $\mathcal{FP}$ (see Equation (36)). The "Newtonian" term in Equation (97) contains the Newtonian quadrupole moment Q_{ij} (see Equation (92)). The dominant radiation tail at the 1.5PN order was computed within the present formalism in Ref. [29]. The 2.5PN non-linear memory integral – the first term inside the coefficient of G/c^5 – has been obtained using both post-Newtonian methods [13, 222, 213, 29, 21] and rigorous studies of the field at future null infinity [71]. The other multipole interactions at the 2.5PN order can be found in Ref. [21]. Finally the "tail of tail" integral appearing at the 3PN order has been derived in this formalism in Ref. [19]. Be careful to note that the latter post-Newtonian orders correspond to "relative" orders when counted in the local radiation-reaction force, present in the equations of motion: For instance, the 1.5PN tail integral in Equation (97) is due to a 4PN radiative effect in the equations of motion [27]; similarly, the 3PN tail-of-tail integral is (presumably) associated with some radiation-reaction terms occuring at the 5.5PN order.

Notice that all the radiative multipole moments, for any l, get some tail-induced contributions.

They are computed at the 1.5PN level in Appendix C of Ref. [15]. We find

$$\begin{aligned}
\mathrm{U}_L(U) &= \mathrm{M}_L^{(l)}(U) + \frac{2G\mathrm{M}}{c^3}\int_0^{+\infty} d\tau\, \mathrm{M}_L^{(l+2)}(U-\tau)\left[\ln\left(\frac{c\tau}{2r_0}\right)+\kappa_l\right] + \mathcal{O}\left(\frac{1}{c^5}\right), \\
\mathrm{V}_L(U) &= \mathrm{S}_L^{(l)}(U) + \frac{2G\mathrm{M}}{c^3}\int_0^{+\infty} d\tau\, \mathrm{S}_L^{(l+2)}(U-\tau)\left[\ln\left(\frac{c\tau}{2r_0}\right)+\pi_l\right] + \mathcal{O}\left(\frac{1}{c^5}\right),
\end{aligned} \tag{98}$$

where the constants κ_l and π_l are given by

$$\begin{aligned}
\kappa_l &= \frac{2l^2+5l+4}{l(l+1)(l+2)} + \sum_{k=1}^{l-2}\frac{1}{k}, \\
\pi_l &= \frac{l-1}{l(l+1)} + \sum_{k=1}^{l-1}\frac{1}{k}.
\end{aligned} \tag{99}$$

Recall that the retarded time U in radiative coordinates is given by

$$U = t - \frac{r}{c} - \frac{2GM}{c^3}\ln\left(\frac{r}{r_0}\right) + \mathcal{O}\left(G^2\right), \tag{100}$$

where (t,r) are harmonic coordinates; recall the gauge vector ξ_1^α in Equation (51). Inserting U as given by Equation (100) into Equations (98) we obtain the radiative moments expressed in terms of source-rooted coordinates (t,r), e.g.,

$$\mathrm{U}_L = \mathrm{M}_L^{(l)}(t-r/c) + \frac{2G\mathrm{M}}{c^3}\int_0^{+\infty} d\tau\, \mathrm{M}_L^{(l+2)}(t-\tau-r/c)\left[\ln\left(\frac{c\tau}{2r}\right)+\kappa_l\right] + \mathcal{O}\left(\frac{1}{c^5}\right). \tag{101}$$

This expression no longer depends on the constant r_0 (i.e. the r_0 gets replaced by r)[16]. If we now change the harmonic coordinates (t,r) to some new ones, such as, for instance, some "Schwarzschild-like" coordinates (t',r') such that $t'=t$ and $r'=r+G\mathrm{M}/c^2$, we get

$$\mathrm{U}_L = \mathrm{M}_L^{(l)}(t'-r'/c) + \frac{2G\mathrm{M}}{c^3}\int_0^{+\infty} d\tau\, \mathrm{M}_L^{(l+2)}(t'-\tau-r'/c)\left[\ln\left(\frac{c\tau}{2r'}\right)+\kappa'_l\right] + \mathcal{O}\left(\frac{1}{c^5}\right), \tag{102}$$

where $\kappa'_l = \kappa_l + 1/2$. Therefore the constant κ_l (and π_l as well) depends on the choice of source-rooted coordinates (t,r): For instance, we have $\kappa_2 = 11/12$ in harmonic coordinates (see Equation (97)), but $\kappa'_2 = 17/12$ in Schwarzschild coordinates [50].

The tail integrals in Equations (97, 98) involve all the instants from $-\infty$ in the past up to the current time U. However, strictly speaking, the integrals must not extend up to minus infinity in the past, because we have assumed from the start that the metric is stationary before the date $-\mathcal{T}$; see Equation (19). The range of integration of the tails is therefore limited *a priori* to the time interval $[-\mathcal{T},\, U]$. But now, once we have derived the tail integrals, thanks in part to the technical assumption of stationarity in the past, we can argue that the results are in fact valid in more general situations for which the field has *never* been stationary. We have in mind the case of two bodies moving initially on some unbound (hyperbolic-like) orbit, and which capture each other, because of the loss of energy by gravitational radiation, to form a bound system at our current epoch. In this situation we can check, using a simple Newtonian model for the behaviour

[16] At the 3PN order (taking into account the tails of tails), we find that r_0 does not completely cancel out after the replacement of U by the right-hand side of Equation (100). The reason is that the moment M_L also depends on r_0 at the 3PN order. Considering also the latter dependence we can check that the 3PN radiative moment U_L is actually free of the unphysical constant r_0.

of the quadrupole moment $\mathrm{M}_{ij}(U-\tau)$ when $\tau \to +\infty$, that the tail integrals, when assumed to extend over the whole time interval $[-\infty, U]$, remain perfectly well-defined (i.e. convergent) at the integration bound $\tau = +\infty$. We regard this fact as a solid *a posteriori* justification (though not a proof) of our *a priori* too restrictive assumption of stationarity in the past. This assumption does not seem to yield any physical restriction on the applicability of the final formulas.

To obtain the result (97), we must implement in details the post-Minkows-kian algorithm presented in Section 4.1. Let us outline here this computation, limiting ourselves to the interaction between one or two masses $\mathrm{M} \equiv \mathrm{M}_{\mathrm{ADM}} \equiv \mathrm{I}$ and the time-varying quadrupole moment $\mathrm{M}_{ab}(u)$ (that is related to the source quadrupole $\mathrm{I}_{ab}(u)$ by Equation (96)). For these moments the linearized metric (26, 27, 28) reads

$$h_1^{\alpha\beta} = h_{(\mathrm{M})}^{\alpha\beta} + h_{(\mathrm{M}_{ab})}^{\alpha\beta}, \tag{103}$$

where the monopole part is nothing but the linearized piece of the Schwarzschild metric in harmonic coordinates,

$$\begin{aligned} h_{(\mathrm{M})}^{00} &= -4r^{-1}\mathrm{M}, \\ h_{(\mathrm{M})}^{0i} &= 0, \\ h_{(\mathrm{M})}^{ij} &= 0, \end{aligned} \tag{104}$$

and the quadrupole part is

$$\begin{aligned} h_{(\mathrm{M}_{ab})}^{00} &= -2\partial_{ab}\left[r^{-1}\mathrm{M}_{ab}(u)\right], \\ h_{(\mathrm{M}_{ab})}^{0i} &= 2\partial_a\left[r^{-1}\mathrm{M}_{ai}^{(1)}(u)\right], \\ h_{(\mathrm{M}_{ab})}^{ij} &= -2r^{-1}\mathrm{M}_{ij}^{(2)}(u). \end{aligned} \tag{105}$$

(We pose $c = 1$ until the end of this section.) Consider next the quadratically non-linear metric $h_2^{\alpha\beta}$ generated by these moments. Evidently it involves a term proportional to M^2, the mixed term corresponding to the interaction $\mathrm{M} \times \mathrm{M}_{ab}$, and the self-interaction term of M_{ab}. Say,

$$h_2^{\alpha\beta} = h_{(\mathrm{M}^2)}^{\alpha\beta} + h_{(\mathrm{M}\mathrm{M}_{ab})}^{\alpha\beta} + h_{(\mathrm{M}_{ab}\mathrm{M}_{cd})}^{\alpha\beta}. \tag{106}$$

The first term represents the quadratic piece of the Schwarzschild metric,

$$\begin{aligned} h_{(\mathrm{M}^2)}^{00} &= -7r^{-2}\mathrm{M}^2, \\ h_{(\mathrm{M}^2)}^{0i} &= 0, \\ h_{(\mathrm{M}^2)}^{ij} &= -n_{ij}r^{-2}\mathrm{M}^2. \end{aligned} \tag{107}$$

The second term in Equation (106) represents the dominant non-static multipole interaction, that is between the mass and the quadrupole moment, and that we now compute[17]. We apply Equations (39, 40, 41, 42, 43) in Section 4. First we obtain the source for this term, viz.

$$\Lambda_{(\mathrm{M}\mathrm{M}_{ab})}^{\alpha\beta} = N^{\alpha\beta}[h_{(\mathrm{M})}, h_{(\mathrm{M}_{ab})}] + N^{\alpha\beta}[h_{(\mathrm{M}_{ab})}, h_{(\mathrm{M})}], \tag{108}$$

where $N^{\alpha\beta}(h,h)$ denotes the quadratic-order part of the gravitational source, as defined by Equation (16). To integrate this term we need some explicit formulas for the retarded integral of an extended (non-compact-support) source having some definite multipolarity l. A thorough account of the technical formulas necessary for handling the quadratic and cubic interactions is given in

[17] The computation of the third term in Equation (106), which corresponds to the interaction between two quadrupoles, $\mathrm{M}_{ab} \times \mathrm{M}_{cd}$, can be found in Ref. [21].

the appendices of Refs. [21] and [19]. For the present computation the crucial formula corresponds to a source term behaving like $1/r^2$:

$$\Box^{-1}_{\mathrm{ret}}\left[\frac{\hat{n}_L}{r^2}F(t-r)\right] = -\hat{n}_L\int_1^{+\infty} dx\, Q_l(x)F(t-rx), \tag{109}$$

where Q_l is the Legendre function of the second kind[18]. With the help of this and other formulas we obtain the object $u_2^{\alpha\beta}$ given by Equation (39). Next we compute the divergence $w_2^\alpha = \partial_\mu u_2^{\alpha\mu}$, and obtain the supplementary term $v_2^{\alpha\beta}$ by applying Equations (42). Actually, we find for this particular interaction $w_2^\alpha = 0$ and thus also $v_2^{\alpha\beta} = 0$. Following Equation (43), the result is the sum of $u_2^{\alpha\beta}$ and $v_2^{\alpha\beta}$, and we get

$$\begin{aligned}
\mathrm{M}^{-1}h^{00}_{(\mathrm{MM}_{ab})} &= n_{ab}r^{-4}\left[-21\mathrm{M}_{ab} - 21r\mathrm{M}^{(1)}_{ab} + 7r^2\mathrm{M}^{(2)}_{ab} + 10r^3\mathrm{M}^{(3)}_{ab}\right] \\
&\quad + 8n_{ab}\int_1^{+\infty} dx\, Q_2(x)\mathrm{M}^{(4)}_{ab}(t-rx), \\
\mathrm{M}^{-1}h^{0i}_{(\mathrm{MM}_{ab})} &= n_{iab}r^{-3}\left[-\mathrm{M}^{(1)}_{ab} - r\mathrm{M}^{(2)}_{ab} - \frac{1}{3}r^2\mathrm{M}^{(3)}_{ab}\right] \\
&\quad + n_a r^{-3}\left[-5\mathrm{M}^{(1)}_{ai} - 5r\mathrm{M}^{(2)}_{ai} + \frac{19}{3}r^2\mathrm{M}^{(3)}_{ai}\right] \\
&\quad + 8n_a\int_1^{+\infty} dx\, Q_1(x)\mathrm{M}^{(4)}_{ai}(t-rx), \\
\mathrm{M}^{-1}h^{ij}_{(\mathrm{MM}_{ab})} &= n_{ijab}r^{-4}\left[-\frac{15}{2}\mathrm{M}_{ab} - \frac{15}{2}r\mathrm{M}^{(1)}_{ab} - 3r^2\mathrm{M}^{(2)}_{ab} - \frac{1}{2}r^3\mathrm{M}^{(3)}_{ab}\right] \\
&\quad + \delta_{ij}n_{ab}r^{-4}\left[-\frac{1}{2}\mathrm{M}_{ab} - \frac{1}{2}r\mathrm{M}^{(1)}_{ab} - 2r^2\mathrm{M}^{(2)}_{ab} - \frac{11}{6}r^3\mathrm{M}^{(3)}_{ab}\right] \\
&\quad + n_{a(i}r^{-4}\left[6\mathrm{M}_{j)a} + 6r\mathrm{M}^{(1)}_{j)a} + 6r^2\mathrm{M}^{(2)}_{j)a} + 4r^3\mathrm{M}^{(3)}_{j)a}\right] \\
&\quad + r^{-4}\left[-\mathrm{M}_{ij} - r\mathrm{M}^{(1)}_{ij} - 4r^2\mathrm{M}^{(2)}_{ij} - \frac{11}{3}r^3\mathrm{M}^{(3)}_{ij}\right] \\
&\quad + 8\int_1^{+\infty} dx\, Q_0(x)\mathrm{M}^{(4)}_{ij}(t-rx).
\end{aligned} \tag{110}$$

The metric is composed of two types of terms: "instantaneous" ones depending on the values of the quadrupole moment at the retarded time $u = t-r$, and "non-local" or tail integrals, depending on all previous instants $t - rx \leq u$.

Let us investigate now the cubic interaction between *two* mass monopoles M with the quadrupole M_{ab}. Obviously, the source term corresponding to this interaction reads

$$\Lambda^{\alpha\beta}_{(\mathrm{M}^2\mathrm{M}_{ab})} = N^{\alpha\beta}[h_{(\mathrm{M})}, h_{(\mathrm{MM}_{ab})}] + N^{\alpha\beta}[h_{(\mathrm{MM}_{ab})}, h_{(\mathrm{M})}] + N^{\alpha\beta}[h_{(\mathrm{M}^2)}, h_{(\mathrm{M}_{ab})}] + N^{\alpha\beta}[h_{(\mathrm{M}_{ab})}, h_{(\mathrm{M}^2)}]$$

[18] The function Q_l is given in terms of the Legendre polynomial P_l by

$$Q_l(x) = \frac{1}{2}\int_{-1}^{1}\frac{dz\, P_l(z)}{x-z} = \frac{1}{2}P_l(x)\ln\left(\frac{x+1}{x-1}\right) - \sum_{j=1}^{l}\frac{1}{j}P_{l-j}(x)P_{j-1}(x).$$

In the complex plane there is a branch cut from $-\infty$ to 1. The first equality is known as the Neumann formula for the Legendre function.

$$+ M^{\alpha\beta}[h_{(\mathrm{M})}, h_{(\mathrm{M})}, h_{(\mathrm{M}_{ab})}] + M^{\alpha\beta}[h_{(\mathrm{M})}, h_{(\mathrm{M}_{ab})}, h_{(\mathrm{M})}] + M^{\alpha\beta}[h_{(\mathrm{M}_{ab})}, h_{(\mathrm{M})}, h_{(\mathrm{M})}] \quad (111)$$

(see Equation (33)). Notably, the N-terms in Equation (111) involve the interaction between a linearized metric, $h_{(\mathrm{M})}$ or $h_{(\mathrm{M}_{ab})}$, and a quadratic one, $h_{(\mathrm{M}^2)}$ or $h_{(\mathrm{MM}_{ab})}$. So, included into these terms are the tails present in the quadratic metric $h_{(\mathrm{MM}_{ab})}$ computed previously with the result (110). These tails will produce in turn some "tails of tails" in the cubic metric $h_{(\mathrm{M}^2\mathrm{M}_{ab})}$. The rather involved computation will not be detailed here (see Ref. [19]). Let us just mention the most difficult of the needed integration formulas[19]:

$$\mathcal{FP}\,\Box^{-1}_{\mathrm{ret}}\left[\frac{\hat{n}_L}{r}\int_1^{+\infty} dx\, Q_m(x) F(t-rx)\right] = \hat{n}_L \int_1^{+\infty} dy\, F^{(-1)}(t-ry) \times \left\{ Q_l(y)\int_1^y dx\, Q_m(x)\frac{dP_l}{dx}(x) + P_l(y)\int_y^{+\infty} dx\, Q_m(x)\frac{dQ_l}{dx}(x)\right\}, \quad (112)$$

where $F^{(-1)}$ is the time anti-derivative of F. With this formula and others given in Ref. [19] we are able to obtain the closed algebraic form of the metric $h^{\alpha\beta}_{(\mathrm{M}^2\mathrm{M}_{ab})}$, at the leading order in the distance to the source. The net result is

$$\begin{aligned}
\mathrm{M}^{-2} h^{00}_{(\mathrm{M}^2\mathrm{M}_{ab})} &= \frac{n_{ab}}{r}\int_0^{+\infty} d\tau\, \mathrm{M}^{(5)}_{ab}\left[-4\ln^2\left(\frac{\tau}{2r}\right) - 4\ln\left(\frac{\tau}{2r}\right) + \frac{116}{21}\ln\left(\frac{\tau}{2r_0}\right) - \frac{7136}{2205}\right] \\
&\quad + \mathcal{O}\left(\frac{1}{r^2}\right), \\
\mathrm{M}^{-2} h^{0i}_{(\mathrm{M}^2\mathrm{M}_{ab})} &= \frac{\hat{n}_{iab}}{r}\int_0^{+\infty} d\tau\, \mathrm{M}^{(5)}_{ab}\left[-\frac{2}{3}\ln\left(\frac{\tau}{2r}\right) - \frac{4}{105}\ln\left(\frac{\tau}{2r_0}\right) - \frac{716}{1225}\right] \\
&\quad + \frac{n_a}{r}\int_0^{+\infty} d\tau\, \mathrm{M}^{(5)}_{ai}\left[-4\ln^2\left(\frac{\tau}{2r}\right) - \frac{18}{5}\ln\left(\frac{\tau}{2r}\right) + \frac{416}{75}\ln\left(\frac{\tau}{2r_0}\right) - \frac{22724}{7875}\right] \\
&\quad + \mathcal{O}\left(\frac{1}{r^2}\right), \\
\mathrm{M}^{-2} h^{ij}_{(\mathrm{M}^2\mathrm{M}_{ab})} &= \frac{\hat{n}_{ijab}}{r}\int_0^{+\infty} d\tau\, \mathrm{M}^{(5)}_{ab}\left[-\ln\left(\frac{\tau}{2r}\right) - \frac{191}{210}\right] \\
&\quad + \frac{\delta_{ij} n_{ab}}{r}\int_0^{+\infty} d\tau\, \mathrm{M}^{(5)}_{ab}\left[-\frac{80}{21}\ln\left(\frac{\tau}{2r}\right) - \frac{32}{21}\ln\left(\frac{\tau}{2r_0}\right) - \frac{296}{35}\right] \\
&\quad + \frac{\hat{n}_{a(i}}{r}\int_0^{+\infty} d\tau\, \mathrm{M}^{(5)}_{j)a}\left[\frac{52}{7}\ln\left(\frac{\tau}{2r}\right) + \frac{104}{35}\ln\left(\frac{\tau}{2r_0}\right) + \frac{8812}{525}\right] \\
&\quad + \frac{1}{r}\int_0^{+\infty} d\tau\, \mathrm{M}^{(5)}_{ij}\left[-4\ln^2\left(\frac{\tau}{2r}\right) - \frac{24}{5}\ln\left(\frac{\tau}{2r}\right) + \frac{76}{15}\ln\left(\frac{\tau}{2r_0}\right) - \frac{198}{35}\right] \\
&\quad + \mathcal{O}\left(\frac{1}{r^2}\right),
\end{aligned} \quad (113)$$

where all the moments M_{ab} are evaluated at the instant $t-r-\tau$ (recall that $c=1$). Notice that some of the logarithms in Equations (113) contain the ratio τ/r while others involve τ/r_0. The

[19] Equation (112) has been obtained using a not so well known mathematical relation between the Legendre functions and polynomials:

$$\frac{1}{2}\int_{-1}^{1} \frac{dz\, P_l(z)}{\sqrt{(xy-z)^2-(x^2-1)(y^2-1)}} = Q_l(x)P_l(y)$$

(where $1 \leq y < x$ is assumed). See Appendix A in Ref. [19] for the proof. This relation constitutes a generalization of the Neumann formula (see footnote after Equation (109)).

indicated remainders $\mathcal{O}(1/r^2)$ contain some logarithms of r; in fact they should be more accurately written as $o(r^{\epsilon-2})$ for some $\epsilon \ll 1$.

The presence of logarithms of r in Equations (113) is an artifact of the harmonic coordinates x^α, and we need to gauge them away by introducing the radiative coordinates X^α at future null infinity (see Theorem 4). As it turns out, it is sufficient for the present calculation to take into account the "linearized" logarithmic deviation of the light cones in harmonic coordinates so that $X^\alpha = x^\alpha + G\xi_1^\alpha + \mathcal{O}(G^2)$, where ξ_1^α is the gauge vector defined by Equation (51) (see also Equation (100)). With this coordinate change one removes all the logarithms of r in Equations (113). Hence, we obtain the radiative metric

$$\begin{aligned}
\mathrm{M}^{-2} H^{00}_{(\mathrm{M}^2 \mathrm{M}_{ab})} &= \frac{N_{ab}}{R} \int_0^{+\infty} d\tau\, \mathrm{M}^{(5)}_{ab} \left[-4 \ln^2\left(\frac{\tau}{2r_0}\right) + \frac{32}{21} \ln\left(\frac{\tau}{2r_0}\right) - \frac{7136}{2205} \right] \\
&\quad + \mathcal{O}\left(\frac{1}{R^2}\right), \\
\mathrm{M}^{-2} H^{0i}_{(\mathrm{M}^2 \mathrm{M}_{ab})} &= \frac{\hat{N}_{iab}}{R} \int_0^{+\infty} d\tau\, \mathrm{M}^{(5)}_{ab} \left[-\frac{74}{105} \ln\left(\frac{\tau}{2r_0}\right) - \frac{716}{1225} \right] \\
&\quad + \frac{N_a}{R} \int_0^{+\infty} d\tau\, \mathrm{M}^{(5)}_{ai} \left[-4 \ln^2\left(\frac{\tau}{2r_0}\right) + \frac{146}{75} \ln\left(\frac{\tau}{2r_0}\right) - \frac{22724}{7875} \right] \\
&\quad + \mathcal{O}\left(\frac{1}{R^2}\right), \\
\mathrm{M}^{-2} H^{ij}_{(\mathrm{M}^2 \mathrm{M}_{ab})} &= \frac{\hat{N}_{ijab}}{R} \int_0^{+\infty} d\tau\, \mathrm{M}^{(5)}_{ab} \left[-\ln\left(\frac{\tau}{2r_0}\right) - \frac{191}{210} \right] \\
&\quad + \frac{\delta_{ij} N_{ab}}{R} \int_0^{+\infty} d\tau\, \mathrm{M}^{(5)}_{ab} \left[-\frac{16}{3} \ln\left(\frac{\tau}{2r_0}\right) - \frac{296}{35} \right] \\
&\quad + \frac{\hat{N}_{a(i}}{R} \int_0^{+\infty} d\tau\, \mathrm{M}^{(5)}_{j)a} \left[\frac{52}{5} \ln\left(\frac{\tau}{2r_0}\right) + \frac{8812}{525} \right] \\
&\quad + \frac{1}{R} \int_0^{+\infty} d\tau\, \mathrm{M}^{(5)}_{ij} \left[-4 \ln^2\left(\frac{\tau}{2r_0}\right) + \frac{4}{15} \ln\left(\frac{\tau}{2r_0}\right) - \frac{198}{35} \right] \\
&\quad + \mathcal{O}\left(\frac{1}{R^2}\right),
\end{aligned} \tag{114}$$

where the moments are evaluated at time $U - \tau \equiv T - R - \tau$. It is trivial to compute the contribution of the radiative moments $\mathrm{U}_L(U)$ and $\mathrm{V}_L(U)$ corresponding to that metric. We find the "tail of tail" term reported in Equation (97).

7 The Third Post-Newtonian Metric

The detailed calculations that are called for in applications necessitate having at one's disposal some explicit expressions of the metric coefficients $g_{\alpha\beta}$, in harmonic coordinates, at the highest possible post-Newtonian order. The 3PN metric that we present below[20] is expressed by means of some particular retarded-type potentials, V, V_i, $\hat{W}_{ij}$, etc., whose main advantages are to somewhat minimize the number of terms, so that even at the 3PN order the metric is still tractable, and to delineate the different problems associated with the computation of different categories of terms. Of course, these potentials have no physical significance by themselves. The basic idea in our post-Newtonian iteration is to use whenever possible a "direct" integration, with the help of some formulas like $\Box^{-1}_{\mathrm{ret}}(\partial_\mu V \partial^\mu V + V \Box V) = V^2/2$. The 3PN harmonic-coordinates metric (issued from Ref. [38]) reads

$$
\begin{aligned}
g_{00} &= -1 + \frac{2}{c^2}V - \frac{2}{c^4}V^2 + \frac{8}{c^6}\left(\hat{X} + V_i V_i + \frac{V^3}{6}\right) + \frac{32}{c^8}\left(\hat{T} - \frac{1}{2}V\hat{X} + \hat{R}_i V_i - \frac{1}{2}V V_i V_i - \frac{1}{48}V^4\right) \\
&\quad + \mathcal{O}\left(\frac{1}{c^{10}}\right), \\
g_{0i} &= -\frac{4}{c^3}V_i - \frac{8}{c^5}\hat{R}_i - \frac{16}{c^7}\left(\hat{Y}_i + \frac{1}{2}\hat{W}_{ij}V_j + \frac{1}{2}V^2 V_i\right) + \mathcal{O}\left(\frac{1}{c^9}\right), \\
g_{ij} &= \delta_{ij}\left[1 + \frac{2}{c^2}V + \frac{2}{c^4}V^2 + \frac{8}{c^6}\left(\hat{X} + V_k V_k + \frac{V^3}{6}\right)\right] + \frac{4}{c^4}\hat{W}_{ij} + \frac{16}{c^6}\left(\hat{Z}_{ij} + \frac{1}{2}V\hat{W}_{ij} - V_i V_j\right) \\
&\quad + \mathcal{O}\left(\frac{1}{c^8}\right).
\end{aligned}
\tag{115}
$$

All the potentials are generated by the matter stress-energy tensor $T^{\alpha\beta}$ through the definitions (analogous to Equations (86))

$$
\begin{aligned}
\sigma &= \frac{T^{00} + T^{ii}}{c^2}, \\
\sigma_i &= \frac{T^{0i}}{c}, \\
\sigma_{ij} &= T^{ij}.
\end{aligned}
\tag{116}
$$

V and V_i represent some retarded versions of the Newtonian and gravitomagnetic potentials,

$$
\begin{aligned}
V &= \Box^{-1}_{\mathrm{ret}}\left[-4\pi G \sigma\right], \\
V_i &= \Box^{-1}_{\mathrm{ret}}\left[-4\pi G \sigma_i\right].
\end{aligned}
\tag{117}
$$

From the 2PN order we have the potentials

$$
\begin{aligned}
\hat{X} &= \Box^{-1}_{\mathrm{ret}}\left[-4\pi G V \sigma_{ii} + \hat{W}_{ij}\partial_{ij}V + 2V_i\partial_t\partial_i V + V\partial_t^2 V + \frac{3}{2}(\partial_t V)^2 - 2\partial_i V_j \partial_j V_i\right], \\
\hat{R}_i &= \Box^{-1}_{\mathrm{ret}}\left[-4\pi G(V\sigma_i - V_i\sigma) - 2\partial_k V \partial_i V_k - \frac{3}{2}\partial_t V \partial_i V\right], \\
\hat{W}_{ij} &= \Box^{-1}_{\mathrm{ret}}\left[-4\pi G(\sigma_{ij} - \delta_{ij}\sigma_{kk}) - \partial_i V \partial_j V\right].
\end{aligned}
\tag{118}
$$

[20] Actually, such a metric is valid up to 3.5PN order.

Some parts of these potentials are directly generated by compact-support matter terms, while other parts are made of non-compact-support products of V-type potentials. There exists also a very important cubically non-linear term generated by the coupling between $\hat{W}_{ij}$ and V, the second term in the $\hat{X}$-potential. At the 3PN level we have the most complicated of these potentials, namely

$$\begin{aligned}
\hat{T} &= \Box^{-1}_{\mathrm{ret}}\Bigg[-4\pi G\left(\frac{1}{4}\sigma_{ij}\hat{W}_{ij}+\frac{1}{2}V^2\sigma_{ii}+\sigma V_i V_i\right)+\hat{Z}_{ij}\partial_{ij}V+\hat{R}_i\partial_t\partial_i V-2\partial_i V_j\partial_j\hat{R}_i-\partial_i V_j\partial_t\hat{W}_{ij} \\
&\qquad + VV_i\partial_t\partial_i V+2V_i\partial_j V_i\partial_j V+\frac{3}{2}V_i\partial_t V\partial_i V+\frac{1}{2}V^2\partial_t^2 V+\frac{3}{2}V(\partial_t V)^2-\frac{1}{2}(\partial_t V_i)^2\Bigg], \\
\hat{Y}_i &= \Box^{-1}_{\mathrm{ret}}\Bigg[-4\pi G\left(-\sigma\hat{R}_i-\sigma VV_i+\frac{1}{2}\sigma_k\hat{W}_{ik}+\frac{1}{2}\sigma_{ik}V_k+\frac{1}{2}\sigma_{kk}V_i\right) \\
&\qquad + \hat{W}_{kl}\partial_{kl}V_i-\partial_t\hat{W}_{ik}\partial_k V+\partial_i\hat{W}_{kl}\partial_k V_l-\partial_k\hat{W}_{il}\partial_l V_k-2\partial_k V\partial_i\hat{R}_k-\frac{3}{2}V_k\partial_i V\partial_k V \\
&\qquad - \frac{3}{2}V\partial_t V\partial_i V-2V\partial_k V\partial_k V_i+V\partial_t^2 V_i+2V_k\partial_k\partial_t V_i\Bigg], \\
\hat{Z}_{ij} &= \Box^{-1}_{\mathrm{ret}}\Bigg[-4\pi GV\left(\sigma_{ij}-\delta_{ij}\sigma_{kk}\right)-2\partial_{(i}V\partial_t V_{j)}+\partial_i V_k\partial_j V_k+\partial_k V_i\partial_k V_j-2\partial_{(i}V_k\partial_k V_{j)} \\
&\qquad - \frac{3}{4}\delta_{ij}(\partial_t V)^2-\delta_{ij}\partial_k V_m(\partial_k V_m-\partial_m V_k)\Bigg],
\end{aligned} \tag{119}$$

which involve many types of compact-support contributions, as well as quadratic-order and cubic-order parts; but, surprisingly, there are *no* quartically non-linear terms[21].

The above potentials are not independent. They are linked together by some differential identities issued from the harmonic gauge conditions, which are equivalent, *via* the Bianchi identities, to the equations of motion of the matter fields (see Equation (17)). These identities read

$$\begin{aligned}
0 &= \partial_t\left\{V+\frac{1}{c^2}\left[\frac{1}{2}\hat{W}_{kk}+2V^2\right]+\frac{4}{c^4}\left[\hat{X}+\frac{1}{2}\hat{Z}_{kk}+\frac{1}{2}V\hat{W}_{kk}+\frac{2}{3}V^3\right]\right\} \\
&\quad + \partial_i\left\{V_i+\frac{2}{c^2}\left[\hat{R}_i+VV_i\right]+\frac{4}{c^4}\left[\hat{Y}_i-\frac{1}{2}\hat{W}_{ij}V_j+\frac{1}{2}\hat{W}_{kk}V_i+V\hat{R}_i+V^2V_i\right]\right\} \\
&\quad + \mathcal{O}\left(\frac{1}{c^6}\right), \\
0 &= \partial_t\left\{V_i+\frac{2}{c^2}\left[\hat{R}_i+VV_i\right]\right\}+\partial_j\left\{\hat{W}_{ij}-\frac{1}{2}\hat{W}_{kk}\delta_{ij}+\frac{4}{c^2}\left[\hat{Z}_{ij}-\frac{1}{2}\hat{Z}_{kk}\delta_{ij}\right]\right\} \\
&\quad + \mathcal{O}\left(\frac{1}{c^4}\right).
\end{aligned} \tag{120}$$

It is important to remark that the above 3PN metric represents the inner post-Newtonian field of an *isolated* system, because it contains, to this order, the correct radiation-reaction terms corresponding to outgoing radiation. These terms come from the expansions of the retardations in the retarded-type potentials (117, 118, 119).

[21] It has been possible to "integrate directly" all the quartic contributions in the 3PN metric. See the terms composed of V^4 and $V\hat{X}$ in the first of Equations (115).

Part B: Compact Binary Systems

The problem of the motion and gravitational radiation of compact objects in post-Newtonian approximations of general relativity is of crucial importance, for at least three reasons. First, the motion of N objects at the 1PN level ($1/c^2$), according to the Einstein–Infeld–Hoffmann equations [106], is routinely taken into account to describe the Solar System dynamics (see Ref. [163]). Second, the gravitational radiation-reaction force, which appears in the equations of motion at the 2.5PN order, has been experimentally verified, by the observation of the secular acceleration of the orbital motion of the binary pulsar PSR 1913+16 [208, 209, 207].

Last but not least, the forthcoming detection and analysis of gravitational waves emitted by inspiralling compact binaries – two neutron stars or black holes driven into coalescence by emission of gravitational radiation – will necessitate the prior knowledge of the equations of motion and radiation field up to high post-Newtonian order. As discussed in the introduction in Section 1 (see around Equations (6, 7, 8)), the appropriate theoretical description of inspiralling compact binaries is by two structureless point-particles, characterized solely by their masses m_1 and m_2 (and possibly their spins), and moving on a quasi-circular orbit. Strategies to detect and analyze the very weak signals from compact binary inspiral involve matched filtering of a set of accurate theoretical template waveforms against the output of the detectors. Several analyses [77, 78, 111, 79, 203, 183, 184, 152, 92, 93, 59, 58, 91, 1, 6] have shown that, in order to get sufficiently accurate theoretical templates, one must include post-Newtonian effects up to the 3PN level at least.

To date, the templates have been completed through 3.5PN order for the phase evolution [35, 40, 31], and 2.5PN order for the amplitude corrections [46, 4]. Spin effects are known for the dominant relativistic spin-orbit coupling term at 1.5PN order and the spin-spin coupling term at 2PN order [146, 3, 144, 119, 118, 117, 70], and also for the next-to-leading spin-orbit coupling at 2.5PN order [168, 204, 110, 25].

8 Regularization of the Field of Point Particles

Our aim is to compute the metric (and its gradient needed in the equations of motion) at the 3PN order for a system of two point-like particles. *A priori* one is not allowed to use directly the metric expressions (115), as they have been derived under the assumption of a continuous (smooth) matter distribution. Applying them to a system of point particles, we find that most of the integrals become divergent at the location of the particles, i.e. when $\mathbf{x} \to \mathbf{y}_1(t)$ or $\mathbf{y}_2(t)$, where $\mathbf{y}_1(t)$ and $\mathbf{y}_2(t)$ denote the two trajectories. Consequently, we must supplement the calculation by a prescription for how to remove the "infinite part" of these integrals. At this stage different choices for a "self-field" regularization (which will take care of the infinite self-field of point particles) are possible. In this section we review:

1. Hadamard's self-field regularization, which has proved to be very convenient for doing practical computations (in particular, by computer), but suffers from the important drawback of yielding some ambiguity parameters, which cannot be determined within this regularization, at the 3PN order;

2. Dimensional self-field regularization, an extremely powerful regularization which is free of any ambiguities (at least up to the 3PN level), and permits therefore to uniquely fix the values of the ambiguity parameters coming from Hadamard's regularization. However, dimensional regularization has not yet been implemented to the present problem in the general case (i.e. for an arbitrary space dimension $d \in \mathbb{C}$).

The why and how the final results are unique and independent of the employed self-field regularization (in agreement with the physical expectation) stems from the effacing principle of general relativity [81] – namely that the internal structure of the compact bodies makes a contribution only at the formal 5PN approximation. However, we shall review several alternative computations, independent of the self-field regularization, which confirm the end results.

8.1 Hadamard self-field regularization

In most practical computations we employ the Hadamard regularization [128, 199] (see Ref. [200] for an entry to the mathematical literature). Let us present here an account of this regularization, as well as a theory of generalized functions (or pseudo-functions) associated with it, following the investigations detailed in Refs. [36, 39].

Consider the class $\mathcal{F}$ of functions $F(\mathbf{x})$ which are smooth (C^∞) on $\mathbb{R}^3$ *except* for the two points $\mathbf{y}_1$ and $\mathbf{y}_2$, around which they admit a power-like singular expansion of the type[22]

$$\forall n \in \mathbb{N}, \qquad F(\mathbf{x}) = \sum_{a_0 \leq a \leq n} r_1^a \underset{1}{f}_a(\mathbf{n}_1) + o(r_1^n), \tag{121}$$

and similarly for the other point 2. Here $r_1 = |\mathbf{x} - \mathbf{y}_1| \to 0$, and the coefficients ${}_1f_a$ of the various powers of r_1 depend on the unit direction $\mathbf{n}_1 = (\mathbf{x} - \mathbf{y}_1)/r_1$ of approach to the singular point. The powers a of r_1 are real, range in discrete steps (i.e. $a \in (a_i)_{i\in\mathbb{N}}$), and are bounded from below ($a_0 \leq a$). The coefficients ${}_1f_a$ (and ${}_2f_a$) for which $a < 0$ can be referred to as the *singular* coefficients of F. If F and G belong to $\mathcal{F}$ so does the ordinary product FG, as well as the ordinary gradient $\partial_i F$. We define the Hadamard *partie finie* of F at the location of the point 1 where it is singular as

$$(F)_1 = \int \frac{d\Omega_1}{4\pi} \underset{1}{f}_0(\mathbf{n}_1), \tag{122}$$

[22] The function $F(\mathbf{x})$ depends also on time t, through for instance its dependence on the velocities $\mathbf{v}_1(t)$ and $\mathbf{v}_2(t)$, but the (coordinate) t time is purely "spectator" in the regularization process, and thus will not be indicated.

where $d\Omega_1 = d\Omega(\mathbf{n}_1)$ denotes the solid angle element centered on $\mathbf{y}_1$ and of direction $\mathbf{n}_1$. Notice that because of the angular integration in Equation (122), the Hadamard *partie finie* is "non-distributive" in the sense that

$$(FG)_1 \neq (F)_1(G)_1 \qquad \text{in general.} \tag{123}$$

The non-distributivity of Hadamard's partie finie is the main source of the appearance of ambiguity parameters at the 3PN order, as discussed in Section 8.2.

The second notion of Hadamard *partie finie* (Pf) concerns that of the integral $\int d^3\mathbf{x}\, F$, which is generically divergent at the location of the two singular points $\mathbf{y}_1$ and $\mathbf{y}_2$ (we assume that the integral converges at infinity). It is defined by

$$\mathrm{Pf}_{s_1 s_2} \int d^3\mathbf{x}\, F = \lim_{s\to 0} \left\{ \int_{\mathcal{S}(s)} d^3\mathbf{x}\, F + 4\pi \sum_{a+3<0} \frac{s^{a+3}}{a+3} \left(\frac{F}{r_1^a}\right)_1 + 4\pi \ln\left(\frac{s}{s_1}\right) \left(r_1^3 F\right)_1 + 1 \leftrightarrow 2 \right\}. \tag{124}$$

The first term integrates over a domain $\mathcal{S}(s)$ defined as $\mathbb{R}^3$ from which the two spherical balls $r_1 \leq s$ and $r_2 \leq s$ of radius s and centered on the two singularities, denoted $\mathcal{B}(\mathbf{y}_1, s)$ and $\mathcal{B}(\mathbf{y}_2, s)$, are excised: $\mathcal{S}(s) \equiv \mathbb{R}^3 \setminus \mathcal{B}(\mathbf{y}_1, s) \cup \mathcal{B}(\mathbf{y}_2, s)$. The other terms, where the value of a function at point 1 takes the meaning (122) are such that they cancel out the divergent part of the first term in the limit where $s \to 0$ (the symbol $1 \leftrightarrow 2$ means the same terms but corresponding to the other point 2). The Hadamard partie-finie integral depends on two strictly positive constants s_1 and s_2, associated with the logarithms present in Equation (124). These constants will ultimately yield some gauge-type constants, denoted by r'_1 and r'_2, in the 3PN equations of motion and radiation field. See Ref. [36] for alternative expressions of the partie-finie integral.

We now come to a specific variant of Hadamard's regularization called the extended Hadamard regularization and defined in Refs. [36, 39]. The basic idea is to associate to any $F \in \mathcal{F}$ a *pseudo-function*, called the *partie finie* pseudo-function $\mathrm{Pf}\, F$, namely a linear form acting on functions G of $\mathcal{F}$, and which is defined by the duality bracket

$$\forall G \in \mathcal{F}, \qquad \langle \mathrm{Pf}\, F, G \rangle = \mathrm{Pf} \int d^3\mathbf{x}\, FG. \tag{125}$$

When restricted to the set $\mathcal{D}$ of smooth functions (i.e. $C^\infty(\mathbb{R}^4)$) with compact support (obviously we have $\mathcal{D} \subset \mathcal{F}$), the pseudo-function $\mathrm{Pf}\, F$ is a distribution in the sense of Schwartz [199]. The product of pseudo-functions coincides, by definition, with the ordinary pointwise product, namely $\mathrm{Pf}\, F . \mathrm{Pf}\, G = \mathrm{Pf}(FG)$. In practical computations, we use an interesting pseudo-function, constructed on the basis of the Riesz delta function [190], which plays a role analogous to the Dirac measure in distribution theory, $\delta_1(\mathbf{x}) \equiv \delta(\mathbf{x} - \mathbf{y}_1)$. This is the so-called delta-pseudo-function $\mathrm{Pf}\, \delta_1$ defined by

$$\forall F \in \mathcal{F}, \qquad \langle \mathrm{Pf}\, \delta_1, F \rangle = \mathrm{Pf} \int d^3\mathbf{x}\, \delta_1 F = (F)_1, \tag{126}$$

where $(F)_1$ is the *partie finie* of F as given by Equation (122). From the product of $\mathrm{Pf}\, \delta_1$ with any $\mathrm{Pf}\, F$ we obtain the new pseudo-function $\mathrm{Pf}(F\delta_1)$, that is such that

$$\forall G \in \mathcal{F}, \qquad \langle \mathrm{Pf}(F\delta_1), G \rangle = (FG)_1. \tag{127}$$

As a general rule, we are not allowed, in consequence of the "non-distributivity" of the Hadamard partie finie, Equation (123), to replace F within the pseudo-function $\mathrm{Pf}(F\delta_1)$ by its regularized value: $\mathrm{Pf}(F\delta_1) \neq (F)_1\, \mathrm{Pf}\, \delta_1$ in general. It should be noticed that the object $\mathrm{Pf}(F\delta_1)$ has no equivalent in distribution theory.

Next, we treat the spatial derivative of a pseudo-function of the type $\mathrm{Pf}\, F$, namely $\partial_i(\mathrm{Pf}\, F)$. Essentially, we require (in Ref. [36]) that the so-called rule of integration by parts holds. By this we mean that we are allowed to freely operate by parts any duality bracket, with the all-integrated ("surface") terms always zero, as in the case of non-singular functions. This requirement is motivated by our will that a computation involving singular functions be as much as possible the same as if we were dealing with regular functions. Thus, by definition,

$$\forall F, G \in \mathcal{F}, \quad \langle \partial_i(\mathrm{Pf}\, F), G\rangle = -\langle \partial_i(\mathrm{Pf}\, G), F\rangle. \tag{128}$$

Furthermore, we assume that when all the singular coefficients of F vanish, the derivative of $\mathrm{Pf}\, F$ reduces to the ordinary derivative, i.e. $\partial_i(\mathrm{Pf}\, F) = \mathrm{Pf}(\partial_i F)$. Then it is trivial to check that the rule (128) contains as a particular case the standard definition of the distributional derivative [199]. Notably, we see that the integral of a gradient is always zero: $\langle \partial_i(\mathrm{Pf}\, F), 1\rangle = 0$. This should certainly be the case if we want to compute a quantity (e.g., a Hamiltonian density) which is defined only modulo a total divergence. We pose

$$\partial_i(\mathrm{Pf}\, F) = \mathrm{Pf}(\partial_i F) + \mathrm{D}_i[F], \tag{129}$$

where $\mathrm{Pf}(\partial_i F)$ represents the "ordinary" derivative and $\mathrm{D}_i[F]$ the distributional term. The following solution of the basic relation (128) was obtained in Ref. [36]:

$$\mathrm{D}_i[F] = 4\pi\ \mathrm{Pf}\left(n_1^i\left[\frac{1}{2}r_1 \underset{1}{f}_{-1} + \sum_{k\geq 0}\frac{1}{r_1^k}\underset{1}{f}_{-2-k}\right]\delta_1\right) + 1 \leftrightarrow 2, \tag{130}$$

where for simplicity we assume that the powers a in the expansion (121) of F are relative integers. The distributional term (130) is of the form $\mathrm{Pf}(G\delta_1)$ (plus $1 \leftrightarrow 2$). It is generated solely by the singular coefficients of F (the sum over k in Equation (130) is always finite since there is a maximal order a_0 of divergency in Equation (121)). The formula for the distributional term associated with the lth distributional derivative, i.e. $\mathrm{D}_L[F] = \partial_L \mathrm{Pf}\, F - \mathrm{Pf}\, \partial_L F$, where $L = i_1 i_2 \dots i_l$, reads

$$\mathrm{D}_L[F] = \sum_{k=1}^{l} \partial_{i_1\dots i_{k-1}} \mathrm{D}_{i_k}[\partial_{i_{k+1}\dots i_l} F]. \tag{131}$$

We refer to Theorem 4 in Ref. [36] for the definition of another derivative operator, representing the most general derivative satisfying the same properties as the one defined by Equation (130), and, in addition, the commutation of successive derivatives (or Schwarz lemma)[23].

The distributional derivative (129, 130, 131) does not satisfy the Leibniz rule for the derivation of a product, in accordance with a general result of Schwartz [198]. Rather, the investigation [36] suggests that, in order to construct a consistent theory (using the "ordinary" product for pseudo-functions), the Leibniz rule should be weakened, and replaced by the rule of integration by part, Equation (128), which is in fact nothing but an "integrated" version of the Leibniz rule. However, the loss of the Leibniz rule *stricto sensu* constitutes one of the reasons for the appearance of the ambiguity parameters at 3PN order.

The Hadamard regularization $(F)_1$ is defined by Equation (122) in a preferred spatial hypersurface $t = \mathrm{const}$ of a coordinate system, and consequently is not *a priori* compatible with the Lorentz invariance. Thus we expect that the equations of motion in harmonic coordinates (which manifestly preserve the global Lorentz invariance) should exhibit at some stage a violation of the Lorentz invariance due to the latter regularization. In fact this occurs exactly at the 3PN order. Up to the 2.5PN level, the use of the regularization $(F)_1$ is sufficient to get some unambiguous equations of motion which are Lorentz invariant [42]. To deal with the problem at 3PN order, a

[23] It was shown in Ref. [38] that using one or the other of these derivatives results in some equations of motion that differ by a mere coordinate transformation. This result indicates that the distributional derivatives introduced in Ref. [36] constitute merely some technical tools which are devoid of physical meaning.

Lorentz-invariant variant of the regularization, denoted $[F]_1$, was introduced in Ref. [39]. It consists of performing the Hadamard regularization within the spatial hypersurface that is geometrically orthogonal (in a Minkowskian sense) to the four-velocity of the particle. The regularization $[F]_1$ differs from the simpler regularization $(F)_1$ by relativistic corrections of order $1/c^2$ at least. See Ref. [39] for the formulas defining this regularization in the form of some infinite power series in $1/c^2$. The regularization $[F]_1$ plays a crucial role in obtaining the equations of motion at the 3PN order in Refs. [37, 38]. In particular, the use of the Lorentz-invariant regularization $[F]_1$ permits to obtain the value of the ambiguity parameter ω_{kinetic} in Equation (132) below.

8.2 Hadamard regularization ambiguities

The "standard" Hadamard regularization yields some ambiguous results for the computation of certain integrals at the 3PN order, as Jaranowski and Schäfer [139, 140, 141] first noticed in their computation of the equations of motion within the ADM-Hamiltonian formulation of general relativity. By standard Hadamard regularization we mean the regularization based solely on the definitions of the partie finie of a singular function, Equation (122), and the partie finie of a divergent integral, Equation (124) (i.e. without using a theory of pseudo-functions and generalized distributional derivatives as proposed in Refs. [36, 39]). It was shown in Refs. [139, 140, 141] that there are *two and only two* types of ambiguous terms in the 3PN Hamiltonian, which were then parametrized by two unknown numerical coefficients ω_{static} and ω_{kinetic}.

Motivated by the previous result, Blanchet and Faye [36, 39] introduced their "extended" Hadamard regularization, the one we outlined in Section 8.1. This new regularization is mathematically well-defined and free of ambiguities; in particular it yields unique results for the computation of any of the integrals occuring in the 3PN equations of motion. Unfortunately, the extended Hadamard regularization turned out to be in a sense incomplete, because it was found [37, 38] that the 3PN equations of motion involve *one and only one* unknown numerical constant, called λ, which cannot be determined within the method. The comparison of this result with the work of Jaranowski and Schäfer [139, 140], on the basis of the computation of the invariant energy of compact binaries moving on circular orbits, showed [37] that

$$\omega_{\text{kinetic}} = \frac{41}{24}, \tag{132}$$

$$\omega_{\text{static}} = -\frac{11}{3}\lambda - \frac{1987}{840}. \tag{133}$$

Therefore, the ambiguity ω_{kinetic} is fixed, while λ is equivalent to the other ambiguity ω_{static}. Notice that the value (132) for the kinetic ambiguity parameter ω_{kinetic}, which is in factor of some velocity dependent terms, is the only one for which the 3PN equations of motion are Lorentz invariant. Fixing up this value was possible because the extended Hadamard regularization [36, 39] was defined in such a way that it keeps the Lorentz invariance.

Damour, Jaranowski, and Schäfer [95] recovered the value of ω_{kinetic} given in Equation (132) by directly proving that this value is the unique one for which the global Poincaré invariance of the ADM-Hamiltonian formalism is verified. Since the coordinate conditions associated with the ADM formalism do not manifestly respect the Poincaré symmetry, they had to prove that the 3PN Hamiltonian is compatible with the existence of generators for the Poincaré algebra. By contrast, the harmonic-coordinate conditions preserve the Poincaré invariance, and therefore the associated equations of motion at 3PN order should be manifestly Lorentz-invariant, as was indeed found to be the case in Refs. [37, 38].

The appearance of one and only one physical unknown coefficient λ in the equations of motion constitutes a quite striking fact, that is related specifically with the use of a Hadamard-type regularization[24]. Technically speaking, the presence of the ambiguity parameter λ is associated with the non-distributivity of Hadamard's regularization, in the sense of Equation (123). Mathematically speaking, λ is probably related to the fact that it is impossible to construct a distributional derivative operator, such as Equations (129, 130, 131), satisfying the Leibniz rule for the derivation of the product [198]. The Einstein field equations can be written in many different forms, by shifting the derivatives and operating some terms by parts with the help of the Leibniz rule. All these forms are equivalent in the case of regular sources, but since the derivative operator (129, 130, 131) violates the Leibniz rule they become inequivalent for point particles. Finally, physically speaking, let us argue that λ has its root in the fact that in a complete computation of the equations of motion valid for two regular *extended* weakly self-gravitating bodies, many non-linear integrals, when taken *individually*, start depending, from the 3PN order, on the internal structure of the bodies, even in the "compact-body" limit where the radii tend to zero. However, when considering the full equations of motion, we expect that all the terms depending on the internal structure can be removed, in the compact-body limit, by a coordinate transformation (or by some appropriate shifts of the central world lines of the bodies), and that finally λ is given by a pure number, for instance a rational fraction, independent of the details of the internal structure of the compact bodies. From this argument (which could be justified by the effacing principle in general relativity) the value of λ is necessarily the one we compute below, Equation (135), and will be valid for any compact objects, for instance black holes.

The ambiguity parameter ω_{static}, which is in factor of some static, velocity-independent term, and hence cannot be derived by invoking Lorentz invariance, was computed by Damour, Jaranowski, and Schäfer [96] by means of *dimensional regularization*, instead of some Hadamard-type one, within the ADM-Hamiltonian formalism. Their result is

$$\omega_{\text{static}} = 0. \tag{134}$$

As Damour et al. [96] argue, clearing up the static ambiguity is made possible by the fact that dimensional regularization, contrary to Hadamard's regularization, respects all the basic properties of the algebraic and differential calculus of ordinary functions: associativity, commutativity and distributivity of point-wise addition and multiplication, Leibniz's rule, and the Schwarz lemma. In this respect, dimensional regularization is certainly better than Hadamard's one, which does not respect the distributivity of the product (recall Equation (123)) and unavoidably violates at some stage the Leibniz rule for the differentiation of a product.

The ambiguity parameter λ is fixed from the result (134) and the necessary link (133) provided by the equivalence between the harmonic-coordinates and ADM-Hamiltonian formalisms [37, 97]. However, λ was also been computed directly by Blanchet, Damour, and Esposito-Farèse [30] applying dimensional regularization to the 3PN equations of motion in harmonic coordinates (in the line of Refs. [37, 38]). The end result,

$$\lambda = -\frac{1987}{3080}, \tag{135}$$

is in full agreement with Equation (134)[25]. Besides the independent confirmation of the value of ω_{static} or λ, the work [30] provides also a confirmation of the *consistency* of dimensional regularization, because the explicit calculations are entirely different from the ones of Ref. [96]: Harmonic

[24] Note also that the harmonic-coordinates 3PN equations of motion as they have been obtained in Refs. [37, 38] depend, in addition to λ, on two arbitrary constants r'_1 and r'_2 parametrizing some logarithmic terms. These constants are closely related to the constants s_1 and s_2 in the *partie-finie* integral (124); see Ref. [38] for the precise definition. However, r'_1 and r'_2 are not "physical" in the sense that they can be removed by a coordinate transformation.

[25] One may wonder why the value of λ is a complicated rational fraction while ω_{static} is so simple. This is because ω_{static} was introduced precisely to measure the amount of ambiguities of certain integrals, while, by contrast, λ was introduced as an unknown constant entering the relation between the arbitrary scales r'_1, r'_2 on the one hand, and s_1, s_2 on the other hand, which has *a priori* nothing to do with ambiguities of integrals.

coordinates are used instead of ADM-type ones, the work is at the level of the equations of motion instead of the Hamiltonian, and a different form of Einstein's field equations is solved by a different iteration scheme.

Let us comment here that the use of a self-field regularization, be it dimensional or based on Hadamard's partie finie, signals a somewhat unsatisfactory situation on the physical point of view, because, ideally, we would like to perform a complete calculation valid for extended bodies, taking into account the details of the internal structure of the bodies (energy density, pressure, internal velocity field, etc.). By considering the limit where the radii of the objects tend to zero, one should recover the same result as obtained by means of the point-mass regularization. This would demonstrate the suitability of the regularization. This program was undertaken at the 2PN order by Kopeikin et al. [149, 127] who derived the equations of motion of two extended fluid balls, and obtained equations of motion depending only on the two masses m_1 and m_2 of the compact bodies[26]. At the 3PN order we expect that the extended-body program should give the value of the regularization parameter λ (maybe after some gauge transformation to remove the terms depending on the internal structure). Ideally, its value should be confirmed by independent and more physical methods (like those of Refs. [214, 150, 101]).

An important work, in several respects more physical than the formal use of regularizations, is the one of Itoh and Futamase [133, 132], following previous investigations in Refs. [134, 135]. These authors derived the 3PN equations of motion in harmonic coordinates by means of a particular variant of the famous "surface-integral" method introduced long ago by Einstein, Infeld, and Hoffmann [106]. The aim is to describe extended relativistic compact binary systems in the strong-field point particle limit defined in Ref. [115]. This approach is very interesting because it is based on the physical notion of extended compact bodies in general relativity, and is free of the problems of ambiguities due to the Hadamard self-field regularization. The end result of Refs. [133, 132] is in agreement with the 3PN harmonic coordinates equations of motion [37, 38] and, moreover, is unambiguous, as it does determine the ambiguity parameter λ to exactly the value (135).

We next consider the problem of the binary's radiation field, where the same phenomenon occurs, with the appearance of some Hadamard regularization ambiguity parameters at 3PN order. More precisely, Blanchet, Iyer, and Joguet [45], in their computation of the 3PN compact binary's *mass quadrupole moment* I_{ij}, found it necessary to introduce *three* Hadamard regularization constants ξ, κ, and ζ, which are additional to and independent of the equation-of-motion related constant λ. The total gravitational-wave flux at 3PN order, in the case of circular orbits, was found to depend on a single combination of the latter constants, $\theta = \xi + 2\kappa + \zeta$, and the binary's orbital phase, for circular orbits, involves only the linear combination of θ and λ given by $\hat{\theta} = \theta - 7\lambda/3$, as shown in [40].

Dimensional regularization (instead of Hadamard's) has next been applied by Blanchet, Damour, Esposito-Farèse, and Iyer [31, 32] to the computation of the 3PN radiation field of compact binaries, leading to the following unique values for the ambiguity parameters[27]:

$$\begin{aligned} \xi &= -\frac{9871}{9240}, \\ \kappa &= 0, \\ \zeta &= -\frac{7}{33}. \end{aligned} \tag{136}$$

These values represent the end result of dimensional regularization. However, several alternative calculations provide a check, independent of dimensional regularization, for all the param-

[26] See some comments on this work in Ref. [84], pp. 168 – 169.

[27] The result for ξ happens to be amazingly related to the one for λ by a cyclic permutation of digits; compare $3\xi = -9871/3080$ with $\lambda = -1987/3080$.

eters (136). Blanchet and Iyer [44] compute the 3PN binary's *mass dipole moment* I_i using Hadamard's regularization, and identify I_i with the 3PN *center of mass vector position* G_i, already known as a conserved integral associated with the Poincaré invariance of the 3PN equations of motion in harmonic coordinates [103]. This yields $\xi + \kappa = -9871/9240$ in agreement with Equation (136). Next, we consider [34] the limiting physical situation where the mass of one of the particles is exactly zero (say, $m_2 = 0$), and the other particle moves with uniform velocity. Technically, the 3PN quadrupole moment of a *boosted* Schwarzschild black hole is computed and compared with the result for I_{ij} in the limit $m_2 = 0$. The result is $\zeta = -7/33$, and represents a direct verification of the global Poincaré invariance of the wave generation formalism (the parameter ζ represents the analogue for the radiation field of the equation-of-motion related parameter $\omega_{\mathrm{kinetic}}$)[28]. Finally, $\kappa = 0$ is proven [32] by showing that there are no dangerously divergent "diagrams" corresponding to non-zero κ-values, where a diagram is meant here in the sense of Ref. [87].

The determination of the parameters (136) completes the problem of the general relativistic prediction for the templates of inspiralling compact binaries up to 3PN order (and actually up to 3.5PN order as the corresponding tail terms have already been determined [19]). The relevant combination of the parameters (136) entering the 3PN energy flux in the case of circular orbits is now fixed to be

$$\theta \equiv \xi + 2\kappa + \zeta = -\frac{11831}{9240}. \tag{137}$$

Numerically, $\theta \simeq -1.28041$. The orbital phase of compact binaries, in the adiabatic inspiral regime (i.e. evolving by radiation reaction), involves at 3PN order a combination of parameters which is determined as

$$\hat{\theta} \equiv \theta - \frac{7}{3}\lambda = \frac{1039}{4620}. \tag{138}$$

The fact that the numerical value of this parameter is quite small, $\hat{\theta} \simeq 0.22489$, indicates, following measurement-accuracy analyses [59, 58, 91], that the 3PN (or, even better, 3.5PN) order should provide an excellent approximation for both the on-line search and the subsequent off-line analysis of gravitational wave signals from inspiralling compact binaries in the LIGO and VIRGO detectors.

8.3 Dimensional regularization of the equations of motion

As reviewed in Section 8.2, work at 3PN order using Hadamard's self-field regularization showed the appearance of ambiguity parameters, due to an incompleteness of the Hadamard regularization employed for curing the infinite self field of point particles. We give here more details on the determination using *dimensional regularization* of the ambiguity parameter λ which appeared in the 3PN equations of motion (recall that λ is equivalent to the static ambiguity parameter ω_{static}, see Equation (133)).

Dimensional regularization was invented as a means to preserve the gauge symmetry of perturbative quantum field theories [202, 51, 57, 73]. Our basic problem here is to respect the gauge symmetry associated with the diffeomorphism invariance of the classical general relativistic description of interacting point masses. Hence, we use dimensional regularization not merely as a trick to compute some particular integrals which would otherwise be divergent, but as a powerful tool for solving in a consistent way the Einstein field equations with singular point-mass sources, while preserving its crucial symmetries. In particular, we shall prove that dimensional regularization determines the kinetic ambiguity parameter $\omega_{\mathrm{kinetic}}$ (and its radiation-field analogue ζ), and is

[28]The work [34] provided also some new expressions for the multipole moments of an isolated post-Newtonian source, alternative to those given by Theorem 6, in the form of *surface integrals* extending on the outer part of the source's near zone.

therefore able to correctly keep track of the global Lorentz–Poincaré invariance of the gravitational field of isolated systems.

The Einstein field equations in $d+1$ space-time dimensions, relaxed by the condition of harmonic coordinates $\partial_\mu h^{\alpha\mu} = 0$, take exactly the same form as given in Equations (9, 14). In particular $\Box$ denotes the flat space-time d'Alembertian operator in $d+1$ dimensions. The gravitational constant G is related to the usual three-dimensional Newton's constant G_N by

$$G = G_N\, \ell_0^{d-3}, \tag{139}$$

where ℓ_0 denotes an arbitrary length scale. The explicit expression of the gravitational source term $\Lambda^{\alpha\beta}$ involves some d-dependent coefficients, and is given by

$$\begin{aligned}\Lambda^{\alpha\beta} = &- h^{\mu\nu}\partial^2_{\mu\nu}h^{\alpha\beta} + \partial_\mu h^{\alpha\nu}\partial_\nu h^{\beta\mu} + \frac{1}{2}g^{\alpha\beta}g_{\mu\nu}\partial_\lambda h^{\mu\tau}\partial_\tau h^{\nu\lambda} \\ &- g^{\alpha\mu}g_{\nu\tau}\partial_\lambda h^{\beta\tau}\partial_\mu h^{\nu\lambda} - g^{\beta\mu}g_{\nu\tau}\partial_\lambda h^{\alpha\tau}\partial_\mu h^{\nu\lambda} + g_{\mu\nu}g^{\lambda\tau}\partial_\lambda h^{\alpha\mu}\partial_\tau h^{\beta\nu} \\ &+ \frac{1}{4}(2g^{\alpha\mu}g^{\beta\nu} - g^{\alpha\beta}g^{\mu\nu})\left(g_{\lambda\tau}g_{\epsilon\pi} - \frac{1}{d-1}g_{\tau\epsilon}g_{\lambda\pi}\right)\partial_\mu h^{\lambda\pi}\partial_\nu h^{\tau\epsilon}.\end{aligned} \tag{140}$$

When $d = 3$ we recover Equation (15). In the following we assume, as usual in dimensional regularization, that the dimension of space is a complex number, $d \in \mathbb{C}$, and prove many results by invoking complex analytic continuation in d. We shall pose $\varepsilon \equiv d - 3$.

We parametrize the 3PN metric in d dimensions by means of straightforward d-dimensional generalizations of the retarded potentials V, V_i, $\hat{W}_{ij}$, $\hat{R}_i$, and $\hat{X}$ of Section 7. Those are obtained by post-Newtonian iteration of the d-dimensional field equations, starting from the following definitions of matter source densities

$$\begin{aligned}\sigma &= \frac{2}{d-1}\frac{(d-2)T^{00} + T^{ii}}{c^2}, \\ \sigma_i &= \frac{T^{0i}}{c}, \\ \sigma_{ij} &= T^{ij},\end{aligned} \tag{141}$$

which generalize Equations (116). As a result all the expressions of Section 7 acquire some explicit d-dependent coefficients. For instance we find [30]

$$\begin{aligned}V &= \Box^{-1}_{\mathrm{ret}}\left[-4\pi G\sigma\right], \\ \hat{W}_{ij} &= \Box^{-1}_{\mathrm{ret}}\left[-4\pi G\left(\sigma_{ij} - \delta_{ij}\frac{\sigma_{kk}}{d-2}\right) - \frac{d-1}{2(d-2)}\partial_i V\partial_j V\right].\end{aligned} \tag{142}$$

Here $\Box^{-1}_{\mathrm{ret}}$ means the retarded integral in $d+1$ space-time dimensions, which admits, though, no simple expression in physical $(t, \mathbf{x})$ space.

As reviewed in Section 8.1, the generic functions we have to deal with in 3 dimensions, say $F(\mathbf{x})$, are smooth on $\mathbb{R}^3$ except at $\mathbf{y}_1$ and $\mathbf{y}_2$, around which they admit singular Laurent-type expansions in powers and inverse powers of $r_1 \equiv |\mathbf{x} - \mathbf{y}_1|$ and $r_2 \equiv |\mathbf{x} - \mathbf{y}_2|$, given by Equation (121). In d spatial dimensions, there is an analogue of the function F, which results from the post-Newtonian iteration process performed in d dimensions as we just outlined. Let us call this function $F^{(d)}(\mathbf{x})$, where $\mathbf{x} \in \mathbb{R}^d$. When $r_1 \to 0$ the function $F^{(d)}$ admits a singular expansion which is a little bit more complicated than in 3 dimensions, as it reads

$$F^{(d)}(\mathbf{x}) = \sum_{\substack{p_0 \le p \le N \\ q_0 \le q \le q_1}} r_1^{p+q\varepsilon} \mathop{f}_{1}{}^{(\varepsilon)}_{p,q}(\mathbf{n}_1) + o(r_1^N). \tag{143}$$

The coefficients $\underset{1}{f}{}^{(\varepsilon)}_{p,q}(\mathbf{n}_1)$ depend on $\varepsilon = d-3$, and the powers of r_1 involve the relative integers p and q whose values are limited by some p_0, q_0, and q_1 as indicated. Here we will be interested in functions $F^{(d)}(\mathbf{x})$ which have no poles as $\varepsilon \to 0$ (this will always be the case at 3PN order). Therefore, we can deduce from the fact that $F^{(d)}(\mathbf{x})$ is continuous at $d=3$ the constraint

$$\sum_{q=q_0}^{q_1} \underset{1}{f}{}^{(\varepsilon=0)}_{p,q}(\mathbf{n}_1) = \underset{1}{f}{}_p(\mathbf{n}_1). \tag{144}$$

For the problem at hand, we essentially have to deal with the regularization of Poisson integrals, or iterated Poisson integrals (and their gradients needed in the equations of motion), of the generic function $F^{(d)}$. The Poisson integral of $F^{(d)}$, in d dimensions, is given by the Green's function for the Laplace operator,

$$P^{(d)}(\mathbf{x}') = \Delta^{-1}\left[F^{(d)}(\mathbf{x})\right] \equiv -\frac{\tilde{k}}{4\pi}\int \frac{d^d\mathbf{x}}{|\mathbf{x}-\mathbf{x}'|^{d-2}} F^{(d)}(\mathbf{x}), \tag{145}$$

where $\tilde{k}$ is a constant related to the usual Eulerian Γ-function by[29]

$$\tilde{k} = \frac{\Gamma\left(\frac{d-2}{2}\right)}{\pi^{\frac{d-2}{2}}}. \tag{146}$$

We need to evaluate the Poisson integral at the point $\mathbf{x}' = \mathbf{y}_1$ where it is singular; this is quite easy in dimensional regularization, because the nice properties of analytic continuation allow simply to get $[P^{(d)}(\mathbf{x}')]_{\mathbf{x}'=\mathbf{y}_1}$ by replacing $\mathbf{x}'$ by $\mathbf{y}_1$ in the explicit integral form (145). So we simply have

$$P^{(d)}(\mathbf{y}_1) = -\frac{\tilde{k}}{4\pi}\int \frac{d^d\mathbf{x}}{r_1^{d-2}} F^{(d)}(\mathbf{x}). \tag{147}$$

It is not possible at present to compute the equations of motion in the general d-dimensional case, but only in the limit where $\varepsilon \to 0$ [96, 30]. The main technical step of our strategy consists of computing, in the limit $\varepsilon \to 0$, the *difference* between the d-dimensional Poisson potential (147), and its Hadamard 3-dimensional counterpart given by $(P)_1$, where the Hadamard partie finie is defined by Equation (122). Actually, we must be very precise when defining the Hadamard partie finie of a Poisson integral. Indeed, the definition (122) *stricto sensu* is applicable when the expansion of the function F, when $r_1 \to 0$, does not involve logarithms of r_1; see Equation (121). However, the Poisson integral $P(\mathbf{x}')$ of $F(\mathbf{x})$ will typically involve such logarithms at the 3PN order, namely some $\ln r'_1$ where $r'_1 \equiv |\mathbf{x}'-\mathbf{y}_1|$ formally tends to zero (hence $\ln r'_1$ is formally infinite). The proper way to define the Hadamard partie finie in this case is to include the $\ln r'_1$ into its definition, so we arrive at [36]

$$(P)_1 = -\frac{1}{4\pi}\,\mathrm{Pf}_{r'_1,s_2}\int \frac{d^3\mathbf{x}}{r_1} F(\mathbf{x}) - (r_1^2\,F)_1. \tag{148}$$

The first term follows from Hadamard's partie finie integral (124); the second one is given by Equation (122). Notice that in this result the constant s_1 entering the partie finie integral (124) has been "replaced" by r'_1, which plays the role of a new regularization constant (together with r'_2 for the other particle), and which ultimately parametrizes the final Hadamard regularized 3PN

[29] We have $\lim_{d\to 3}\tilde{k} = 1$. Notice that $\tilde{k}$ is closely linked to the volume Ω_{d-1} of the sphere with $d-1$ dimensions (i.e. embedded into Euclidean d-dimensional space):

$$\tilde{k}\,\Omega_{d-1} = \frac{4\pi}{d-2}.$$

equations of motion. It was shown that r'_1 and r'_2 are unphysical, in the sense that they can be removed by a coordinate transformation [37, 38]. On the other hand, the constant s_2 remaining in the result (148) is the source for the appearance of the physical ambiguity parameter λ, as it will be related to it by Equation (150). Denoting the difference between the dimensional and Hadamard regularizations by means of the script letter $\mathcal{D}$, we pose (for the result concerning the point 1)

$$\mathcal{D}P_1 \equiv P^{(d)}(\mathbf{y}_1) - (P)_1. \tag{149}$$

That is, $\mathcal{D}P_1$ is what we shall have to *add* to the Hadamard-regularization result in order to get the d-dimensional result. However, we shall only compute the first two terms of the Laurent expansion of $\mathcal{D}P_1$ when $\varepsilon \to 0$, say $a_{-1}\,\varepsilon^{-1} + a_0 + \mathcal{O}(\varepsilon)$. This is the information we need to clear up the ambiguity parameter. We insist that the difference $\mathcal{D}P_1$ comes exclusively from the contribution of terms developing some poles $\propto 1/\varepsilon$ in the d-dimensional calculation.

Next we outline the way we obtain, starting from the computation of the "difference", the 3PN equations of motion in dimensional regularization, and show how the ambiguity parameter λ is determined. By contrast to r'_1 and r'_2 which are pure gauge, λ is a genuine physical ambiguity, introduced in Refs. [36, 38] as the *single* unknown numerical constant parametrizing the ratio between s_2 and r'_2 (where s_2 is the constant left in Equation (148)) as

$$\ln\left(\frac{r'_2}{s_2}\right) = \frac{159}{308} + \lambda\frac{m_1+m_2}{m_2} \qquad (\text{and } 1 \leftrightarrow 2), \tag{150}$$

where m_1 and m_2 are the two masses. The terms corresponding to the λ-ambiguity in the acceleration $\mathbf{a}_1 = d\mathbf{v}_1/dt$ of particle 1 read simply

$$\Delta\mathbf{a}_1[\lambda] = -\frac{44\lambda}{3}\,\frac{G_N^4\,m_1\,m_2^2\,(m_1+m_2)}{r_{12}^5\,c^6}\,\mathbf{n}_{12}, \tag{151}$$

where the relative distance between particles is denoted $\mathbf{y}_1 - \mathbf{y}_2 \equiv r_{12}\,\mathbf{n}_{12}$ (with $\mathbf{n}_{12}$ being the unit vector pointing from particle 2 to particle 1). We start from the end result of Ref. [38] for the 3PN harmonic coordinates acceleration $\mathbf{a}_1$ in Hadamard's regularization, abbreviated as HR. Since the result was obtained by means of the specific extended variant of Hadamard's regularization (in short EHR, see Section 8.1) we write it as

$$\mathbf{a}_1^{(\mathrm{HR})} = \mathbf{a}_1^{(\mathrm{EHR})} + \Delta\mathbf{a}_1[\lambda], \tag{152}$$

where $\mathbf{a}_1^{(\mathrm{EHR})}$ is a fully determined functional of the masses m_1 and m_2, the relative distance $r_{12}\,\mathbf{n}_{12}$, the coordinate velocities $\mathbf{v}_1$ and $\mathbf{v}_2$, and also the gauge constants r'_1 and r'_2. The only ambiguous term is the second one and is given by Equation (151).

Our strategy is to express both the dimensional and Hadamard regularizations in terms of their common "core" part, obtained by applying the so-called "pure-Hadamard–Schwartz" (pHS) regularization. Following the definition of Ref. [30], the pHS regularization is a specific, minimal Hadamard-type regularization of integrals, based on the partie finie integral (124), together with a minimal treatment of "contact" terms, in which the definition (124) is applied separately to each of the elementary potentials $V, V_i, \dots$ (and gradients) that enter the post-Newtonian metric in the form given in Section 7. Furthermore, the regularization of a product of these potentials is assumed to be distributive, i.e. $(FG)_1 = (F)_1(G)_1$ in the case where F and G are given by such elementary potentials (this is in contrast with Equation (123)). The pHS regularization also assumes the use of standard Schwartz distributional derivatives [199]. The interest of the pHS regularization is that the dimensional regularization is equal to it plus the "difference"; see Equation (155).

To obtain the pHS-regularized acceleration we need to substract from the EHR result a series of contributions, which are specific consequences of the use of EHR [36, 39]. For instance, one of these contributions corresponds to the fact that in the EHR the distributional derivative is given by Equations (129, 130) which differs from the Schwartz distributional derivative in the pHS regularization. Hence we define

$$\mathbf{a}_1^{(\mathrm{pHS})} = \mathbf{a}_1^{(\mathrm{EHR})} - \sum_A \delta_A \mathbf{a}_1, \tag{153}$$

where the $\delta_A \mathbf{a}_1$'s denote the extra terms following from the EHR prescriptions. The pHS-regularized acceleration (153) constitutes essentially the result of the first stage of the calculation of $\mathbf{a}_1$, as reported in Ref. [109].

The next step consists of evaluating the Laurent expansion, in powers of $\varepsilon = d - 3$, of the difference between the dimensional regularization and the pHS (3-dimensional) computation. As we reviewed above, this difference makes a contribution only when a term generates a *pole* $\sim 1/\varepsilon$, in which case the dimensional regularization adds an extra contribution, made of the pole and the finite part associated with the pole (we consistently neglect all terms $\mathcal{O}(\varepsilon)$). One must then be especially wary of combinations of terms whose pole parts finally cancel ("cancelled poles") but whose dimensionally regularized finite parts generally do not, and must be evaluated with care. We denote the above defined difference by

$$\mathcal{D}\mathbf{a}_1 = \sum \mathcal{D}P_1. \tag{154}$$

It is made of the sum of all the individual differences of Poisson or Poisson-like integrals as computed in Equation (149). The total difference (154) depends on the Hadamard regularization scales r'_1 and s_2 (or equivalently on λ and r'_1, r'_2), and on the parameters associated with dimensional regularization, namely ε and the characteristic length scale ℓ_0 introduced in Equation (139). Finally, our main result is the explicit computation of the ε-expansion of the dimensional regularization (DR) acceleration as

$$\mathbf{a}_1^{(\mathrm{DR})} = \mathbf{a}_1^{(\mathrm{pHS})} + \mathcal{D}\mathbf{a}_1. \tag{155}$$

With this result we can prove two theorems [30]:

Theorem 8 *The pole part $\propto 1/\varepsilon$ of the DR acceleration (155) can be re-absorbed (i.e. renormalized) into some shifts of the two "bare" world-lines: $\mathbf{y}_1 \to \mathbf{y}_1 + \boldsymbol{\xi}_1$ and $\mathbf{y}_2 \to \mathbf{y}_2 + \boldsymbol{\xi}_2$, with, say, $\boldsymbol{\xi}_{1,2} \propto 1/\varepsilon$, so that the result, expressed in terms of the "dressed" quantities, is finite when $\varepsilon \to 0$.*

The situation in harmonic coordinates is to be contrasted with the calculation in ADM-type coordinates within the Hamiltonian formalism, where it was shown that all pole parts directly cancel out in the total 3PN Hamiltonian: No renormalization of the world-lines is needed [96]. A central result is then as follows:

Theorem 9 *The renormalized (finite) DR acceleration is physically equivalent to the Hadamard-regularized (HR) acceleration (end result of Ref. [38]), in the sense that*

$$\mathbf{a}_1^{(\mathrm{HR})} = \lim_{\varepsilon \to 0} \left[\mathbf{a}_1^{(\mathrm{DR})} + \delta_{\boldsymbol{\xi}}\, \mathbf{a}_1 \right], \tag{156}$$

where $\delta_{\boldsymbol{\xi}}\, \mathbf{a}_1$ denotes the effect of the shifts on the acceleration, if and only if the HR ambiguity parameter λ entering the harmonic-coordinates equations of motion takes the unique value (135).

The precise shifts $\boldsymbol{\xi}_1$ and $\boldsymbol{\xi}_2$ needed in Theorem 9 involve not only a pole contribution $\propto 1/\varepsilon$ (which would define a renormalization by minimal subtraction (MS)), but also a finite contribution when

$\varepsilon \to 0$. Their explicit expressions read[30]:

$$\boldsymbol{\xi}_1 = \frac{11}{3}\frac{G_N^2\, m_1^2}{c^6}\left[\frac{1}{\varepsilon} - 2\ln\left(\frac{r_1'\overline{q}^{1/2}}{\ell_0}\right) - \frac{327}{1540}\right]\mathbf{a}_{N1} \qquad \text{(together with } 1 \leftrightarrow 2\text{)}, \tag{157}$$

where G_N is Newton's constant, ℓ_0 is the characteristic length scale of dimensional regularization (cf. Equation (139)), $\mathbf{a}_{N1}$ is the Newtonian acceleration of the particle 1 in d dimensions, and $\overline{q} \equiv 4\pi e^C$ depends on Euler's constant $C = 0.577\cdots$.

8.4 Dimensional regularization of the radiation field

We now address the similar problem concerning the binary's radiation field (3PN beyond the Einstein quadrupole formalism), for which three ambiguity parameters, ξ, κ, ζ, have been shown to appear [45, 44] (see Section 8.2).

To apply dimensional regularization, we must use as in Section 8.3 the d-dimensional post-Newtonian iteration [leading to equations such as (142)]; and, crucially, we have to generalize to d dimensions some key results of the wave generation formalism of Part A. Essentially we need the d-dimensional analogues of the multipole moments of an isolated source I_L and J_L, Equations (85). The result we find in the case of the mass-type moments is

$$\mathrm{I}_L^{(d)}(t) = \frac{d-1}{2(d-2)}\,\mathcal{FP}\int d^d\mathbf{x}\left\{\hat{x}_L \underset{[l]}{\Sigma}(\mathbf{x},t) - \frac{4(d+2l-2)}{c^2(d+l-2)(d+2l)}\,\hat{x}_{aL} \underset{[l+1]}{\overset{(1)}{\Sigma}}{}_a(\mathbf{x},t) \right.$$
$$\left. + \frac{2(d+2l-2)}{c^4(d+l-1)(d+l-2)(d+2l+2)}\hat{x}_{abL} \underset{[l+2]}{\overset{(2)}{\Sigma}}{}_{ab}(\mathbf{x},t)\right\}, \tag{158}$$

where we denote (generalizing Equations (86))

$$\begin{aligned}
\Sigma &= \frac{2}{d-1}\frac{(d-2)\overline{\tau}^{00} + \overline{\tau}^{ii}}{c^2}, \\
\Sigma_i &= \frac{\overline{\tau}^{0i}}{c}, \\
\Sigma_{ij} &= \overline{\tau}^{ij},
\end{aligned} \tag{159}$$

and where for any source densities the underscript $[l]$ means the infinite series

$$\underset{[l]}{\Sigma}(\mathbf{x},t) = \sum_{k=0}^{+\infty}\frac{1}{2^{2k}k!}\frac{\Gamma\left(\frac{d}{2}+l\right)}{\Gamma\left(\frac{d}{2}+l+k\right)}\left(\frac{|\mathbf{x}|}{c}\frac{\partial}{\partial t}\right)^{2k}\Sigma(\mathbf{x},t). \tag{160}$$

The latter definition represents the d-dimensional version of the post-Newtonian expansion series (91). At Newtonian order, Equation (158) reduces to the standard result $\mathrm{I}_L^{(d)} = \int d^d\mathbf{x}\,\rho\,\hat{x}_L + \mathcal{O}(c^{-2})$ with $\rho = T^{00}/c^2$.

The ambiguity parameters ξ, κ, and ζ come from the Hadamard regularization of the mass quadrupole moment I_{ij} at the 3PN order. The terms corresponding to these ambiguities were found to be

$$\Delta\mathrm{I}_{ij}[\xi,\kappa,\zeta] = \frac{44}{3}\frac{G_N^2\, m_1^3}{c^6}\left[\left(\xi + \kappa\frac{m_1+m_2}{m_1}\right)y_1^{\langle i}a_1^{j\rangle} + \zeta\, v_1^{\langle i}v_1^{j\rangle}\right] + 1 \leftrightarrow 2, \tag{161}$$

[30]When working at the level of the equations of motion (not considering the metric outside the world-lines), the effect of shifts can be seen as being induced by a coordinate transformation of the bulk metric as in Ref. [38].

where $\mathbf{y}_1$, $\mathbf{v}_1$, and $\mathbf{a}_1$ denote the first particle's position, velocity, and acceleration. We recall that the brackets $\langle\rangle$ surrounding indices refer to the symmetric-trace-free (STF) projection. Like in Section 8.3, we express both the Hadamard and dimensional results in terms of the more basic pHS regularization. The first step of the calculation [44] is therefore to relate the Hadamard-regularized quadrupole moment $I_{ij}^{(\mathrm{HR})}$, for general orbits, to its pHS part:

$$\mathrm{I}_{ij}^{(\mathrm{HR})} = \mathrm{I}_{ij}^{(\mathrm{pHS})} + \Delta \mathrm{I}_{ij}\left[\xi + \frac{1}{22}, \kappa, \zeta + \frac{9}{110}\right]. \tag{162}$$

In the right-hand side we find both the pHS part, and the effect of adding the ambiguities, with some numerical shifts of the ambiguity parameters coming from the difference between the specific Hadamard-type regularization scheme used in Ref. [45] and the pHS one. The pHS part is free of ambiguities but depends on the gauge constants r'_1 and r'_2 introduced in the harmonic-coordinates equations of motion [37, 38].

We next use the d-dimensional moment (158) to compute the difference between the dimensional regularization (DR) result and the pHS one [31, 32]. As in the work on equations of motion, we find that the ambiguities arise solely from the terms in the integration regions near the particles (i.e. $r_1 = |\mathbf{x}-\mathbf{y}_1| \to 0$ or $r_2 = |\mathbf{x}-\mathbf{y}_2| \to 0$) that give rise to poles $\propto 1/\varepsilon$, corresponding to logarithmic ultra-violet (UV) divergences in 3 dimensions. The infra-red (IR) region at infinity (i.e. $|\mathrm{x}| \to +\infty$) does not contribute to the difference DR $-$ pHS. The compact-support terms in the integrand of Equation (158), proportional to the matter source densities σ, σ_a, and σ_{ab}, are also found not to contribute to the difference. We are therefore left with evaluating the difference linked with the computation of the *non-compact* terms in the expansion of the integrand in (158) near the singularities that produce poles in d dimensions.

Let $F^{(d)}(\mathbf{x})$ be the non-compact part of the integrand of the quadrupole moment (158) (with indices $L \equiv ij$), where $F^{(d)}$ includes the appropriate multipolar factors such as $\hat{x}_{ij}$, so that

$$\mathrm{I}_{ij}^{(d)} = \int d^d\mathbf{x}\, F^{(d)}(\mathbf{x}). \tag{163}$$

We do not indicate that we are considering here only the non-compact part of the moments. Near the singularities the function $F^{(d)}(\mathbf{x})$ admits a singular expansion of the type (143). In practice, the various coefficients ${}_1f_{p,q}^{(\varepsilon)}$ are computed by specializing the general expressions of the non-linear retarded potentials $V, V_a, \hat{W}_{ab}, \dots$ (valid for general extended sources) to the point particles case in d dimensions. On the other hand, the analogue of Equation (163) in 3 dimensions is

$$\mathrm{I}_{ij} = \mathrm{Pf} \int d^3\mathbf{x}\, F(\mathbf{x}), \tag{164}$$

where Pf refers to the Hadamard partie finie defined in Equation (124). The difference $\mathcal{D}\mathrm{I}$ between the DR evaluation of the d-dimensional integral (163), and its corresponding three-dimensional evaluation, i.e. the partie finie (164), reads then

$$\mathcal{D}\mathrm{I}_{ij} = \mathrm{I}_{ij}^{(d)} - \mathrm{I}_{ij}. \tag{165}$$

Such difference depends only on the UV behaviour of the integrands, and can therefore be computed "locally", i.e. in the vicinity of the particles, when $r_1 \to 0$ and $r_2 \to 0$. We find that Equation (165) depends on two constant scales s_1 and s_2 coming from Hadamard's partie finie (124), and on the constants belonging to dimensional regularization, which are $\varepsilon = d-3$ and the length scale ℓ_0 defined by Equation (139). The dimensional regularization of the 3PN quadrupole moment is then obtained as the sum of the pHS part, and of the difference computed according to Equation (165), namely

$$\mathrm{I}_{ij}^{(\mathrm{DR})} = \mathrm{I}_{ij}^{(\mathrm{pHS})} + \mathcal{D}\mathrm{I}_{ij}. \tag{166}$$

An important fact, hidden in our too-compact notation (166), is that the sum of the two terms in the right-hand side of Equation (166) does not depend on the Hadamard regularization scales s_1 and s_2. Therefore it is possible without changing the sum to re-express these two terms (separately) by means of the constants r'_1 and r'_2 instead of s_1 and s_2, where r'_1, r'_2 are the two fiducial scales entering the Hadamard-regularization result (162). This replacement being made the pHS term in Equation (166) is exactly the same as the one in Equation (162). At this stage all elements are in place to prove the following theorem [31, 32]:

Theorem 10 *The DR quadrupole moment (166) is physically equivalent to the Hadamard-regularized one (end result of Refs. [45, 44]), in the sense that*

$$\mathrm{I}_{ij}^{(\mathrm{HR})} = \lim_{\varepsilon \to 0} \left[\mathrm{I}_{ij}^{(\mathrm{DR})} + \delta_{\boldsymbol{\xi}} \mathrm{I}_{ij} \right] , \tag{167}$$

where $\delta_{\boldsymbol{\xi}} \mathrm{I}_{ij}$ denotes the effect of the same shifts as determined in Theorems 8 and 9, if and only if the HR ambiguity parameters ξ, κ, and ζ take the unique values (136). Moreover, the poles $1/\varepsilon$ separately present in the two terms in the brackets of Equation (167) cancel out, so that the physical ("dressed") DR quadrupole moment is finite and given by the limit when $\varepsilon \to 0$ as shown in Equation (167).

This theorem finally provides an unambiguous determination of the 3PN radiation field by dimensional regularization. Furthermore, as reviewed in Section 8.2, several checks of this calculation could be done, which provide, together with comparisons with alternative methods [96, 30, 133, 132], independent confirmations for the four ambiguity parameters λ, ξ, κ, and ζ, and confirm the consistency of dimensional regularization and its validity for describing the general-relativistic dynamics of compact bodies.

9 Newtonian-like Equations of Motion

9.1 The 3PN acceleration and energy

We present the acceleration of one of the particles, say the particle 1, at the 3PN order, as well as the 3PN energy of the binary, which is conserved in the absence of radiation reaction. To get this result we used essentially a "direct" post-Newtonian method (issued from Ref. [42]), which consists of reducing the 3PN metric of an extended regular source, worked out in Equations (115), to the case where the matter tensor is made of delta functions, and then curing the self-field divergences by means of the Hadamard regularization technique. The equations of motion are simply the geodesic equations associated with the regularized metric (see Ref. [39] for a proof). The Hadamard ambiguity parameter λ is computed from dimensional regularization in Section 8.3. We also add the 3.5PN terms which are known from Refs. [136, 137, 138, 174, 148, 164].

Though the successive post-Newtonian approximations are really a consequence of general relativity, the final equations of motion must be interpreted in a Newtonian-like fashion. That is, once a convenient general-relativistic (Cartesian) coordinate system is chosen, we should express the results in terms of the *coordinate* positions, velocities, and accelerations of the bodies, and view the trajectories of the particles as taking place in the absolute Euclidean space of Newton. But because the equations of motion are actually relativistic, they must

(i) stay manifestly invariant – at least in harmonic coordinates – when we perform a global post-Newtonian-expanded Lorentz transformation,

(ii) possess the correct "perturbative" limit, given by the geodesics of the (post-Newtonian-expanded) Schwarzschild metric, when one of the masses tends to zero, and

(iii) be conservative, i.e. to admit a Lagrangian or Hamiltonian formulation, when the gravitational radiation reaction is turned off.

We denote by $r_{12} = |\mathbf{y}_1(t) - \mathbf{y}_2(t)|$ the harmonic-coordinate distance between the two particles, with $\mathbf{y}_1 = (y_1^i)$ and $\mathbf{y}_2 = (y_2^i)$, by $n_{12}^i = (y_1^i - y_2^i)/r_{12}$ the corresponding unit direction, and by $v_1^i = dy_1^i/dt$ and $a_1^i = dv_1^i/dt$ the coordinate velocity and acceleration of the particle 1 (and *idem* for 2). Sometimes we pose $v_{12}^i = v_1^i - v_2^i$ for the relative velocity. The usual Euclidean scalar product of vectors is denoted with parentheses, e.g., $(n_{12}v_1) = \mathbf{n}_{12} \cdot \mathbf{v}_1$ and $(v_1v_2) = \mathbf{v}_1 \cdot \mathbf{v}_2$. The equations of the body 2 are obtained by exchanging all the particle labels $1 \leftrightarrow 2$ (remembering that n_{12}^i and v_{12}^i change sign in this operation):

$$
\begin{aligned}
a_1^i = & -\frac{Gm_2 n_{12}^i}{r_{12}^2} \\
& + \frac{1}{c^2}\left\{ \left[\frac{5G^2 m_1 m_2}{r_{12}^3} + \frac{4G^2 m_2^2}{r_{12}^3} + \frac{Gm_2}{r_{12}^2}\left(\frac{3}{2}(n_{12}v_2)^2 - v_1^2 + 4(v_1v_2) - 2v_2^2 \right) \right] n_{12}^i \right. \\
& \qquad \left. + \frac{Gm_2}{r_{12}^2}\left(4(n_{12}v_1) - 3(n_{12}v_2)\right) v_{12}^i \right\} \\
& + \frac{1}{c^4}\left\{ \left[-\frac{57G^3 m_1^2 m_2}{4r_{12}^4} - \frac{69G^3 m_1 m_2^2}{2r_{12}^4} - \frac{9G^3 m_2^3}{r_{12}^4} \right. \right. \\
& \qquad + \frac{Gm_2}{r_{12}^2}\left(-\frac{15}{8}(n_{12}v_2)^4 + \frac{3}{2}(n_{12}v_2)^2 v_1^2 - 6(n_{12}v_2)^2(v_1v_2) - 2(v_1v_2)^2 + \frac{9}{2}(n_{12}v_2)^2 v_2^2 \right. \\
& \qquad\qquad \left. + 4(v_1v_2)v_2^2 - 2v_2^4 \right)
\end{aligned}
$$

$$
\begin{aligned}
&+\frac{G^2 m_1 m_2}{r_{12}^3}\left(\frac{39}{2}(n_{12}v_1)^2 - 39(n_{12}v_1)(n_{12}v_2) + \frac{17}{2}(n_{12}v_2)^2 - \frac{15}{4}v_1^2 - \frac{5}{2}(v_1v_2) + \frac{5}{4}v_2^2\right)\\
&+\frac{G^2 m_2^2}{r_{12}^3}\left(2(n_{12}v_1)^2 - 4(n_{12}v_1)(n_{12}v_2) - 6(n_{12}v_2)^2 - 8(v_1v_2) + 4v_2^2\right)\Bigg] n_{12}^i\\
&+\Bigg[\frac{G^2 m_2^2}{r_{12}^3}\left(-2(n_{12}v_1) - 2(n_{12}v_2)\right) + \frac{G^2 m_1 m_2}{r_{12}^3}\left(-\frac{63}{4}(n_{12}v_1) + \frac{55}{4}(n_{12}v_2)\right)\\
&\quad + \frac{G m_2}{r_{12}^2}\Bigg(-6(n_{12}v_1)(n_{12}v_2)^2 + \frac{9}{2}(n_{12}v_2)^3 + (n_{12}v_2)v_1^2 - 4(n_{12}v_1)(v_1v_2)\\
&\qquad + 4(n_{12}v_2)(v_1v_2) + 4(n_{12}v_1)v_2^2 - 5(n_{12}v_2)v_2^2\Bigg)\Bigg] v_{12}^i \Bigg\}\\
+\frac{1}{c^5}\Bigg\{&\left[\frac{208 G^3 m_1 m_2^2}{15 r_{12}^4}(n_{12}v_{12}) - \frac{24 G^3 m_1^2 m_2}{5 r_{12}^4}(n_{12}v_{12}) + \frac{12 G^2 m_1 m_2}{5 r_{12}^3}(n_{12}v_{12})v_{12}^2\right] n_{12}^i\\
&+\left[\frac{8 G^3 m_1^2 m_2}{5 r_{12}^4} - \frac{32 G^3 m_1 m_2^2}{5 r_{12}^4} - \frac{4 G^2 m_1 m_2}{5 r_{12}^3}v_{12}^2\right] v_{12}^i \Bigg\}\\
+\frac{1}{c^6}\Bigg\{&\Bigg[\frac{G m_2}{r_{12}^2}\Bigg(\frac{35}{16}(n_{12}v_2)^6 - \frac{15}{8}(n_{12}v_2)^4 v_1^2 + \frac{15}{2}(n_{12}v_2)^4(v_1v_2) + 3(n_{12}v_2)^2(v_1v_2)^2\\
&\quad -\frac{15}{2}(n_{12}v_2)^4 v_2^2 + \frac{3}{2}(n_{12}v_2)^2 v_1^2 v_2^2 - 12(n_{12}v_2)^2(v_1v_2)v_2^2 - 2(v_1v_2)^2 v_2^2\\
&\quad +\frac{15}{2}(n_{12}v_2)^2 v_2^4 + 4(v_1v_2)v_2^4 - 2v_2^6\Bigg)\\
&+\frac{G^2 m_1 m_2}{r_{12}^3}\Bigg(-\frac{171}{8}(n_{12}v_1)^4 + \frac{171}{2}(n_{12}v_1)^3(n_{12}v_2) - \frac{723}{4}(n_{12}v_1)^2(n_{12}v_2)^2\\
&\qquad + \frac{383}{2}(n_{12}v_1)(n_{12}v_2)^3 - \frac{455}{8}(n_{12}v_2)^4 + \frac{229}{4}(n_{12}v_1)^2 v_1^2\\
&\qquad - \frac{205}{2}(n_{12}v_1)(n_{12}v_2)v_1^2 + \frac{191}{4}(n_{12}v_2)^2 v_1^2 - \frac{91}{8}v_1^4 - \frac{229}{2}(n_{12}v_1)^2(v_1v_2)\\
&\qquad + 244(n_{12}v_1)(n_{12}v_2)(v_1v_2) - \frac{225}{2}(n_{12}v_2)^2(v_1v_2) + \frac{91}{2}v_1^2(v_1v_2)\\
&\qquad - \frac{177}{4}(v_1v_2)^2 + \frac{229}{4}(n_{12}v_1)^2 v_2^2 - \frac{283}{2}(n_{12}v_1)(n_{12}v_2)v_2^2\\
&\qquad + \frac{259}{4}(n_{12}v_2)^2 v_2^2 - \frac{91}{4}v_1^2 v_2^2 + 43(v_1v_2)v_2^2 - \frac{81}{8}v_2^4\Bigg)\\
&+\frac{G^2 m_2^2}{r_{12}^3}\Bigg(-6(n_{12}v_1)^2(n_{12}v_2)^2 + 12(n_{12}v_1)(n_{12}v_2)^3 + 6(n_{12}v_2)^4\\
&\qquad + 4(n_{12}v_1)(n_{12}v_2)(v_1v_2) + 12(n_{12}v_2)^2(v_1v_2) + 4(v_1v_2)^2\\
&\qquad - 4(n_{12}v_1)(n_{12}v_2)v_2^2 - 12(n_{12}v_2)^2 v_2^2 - 8(v_1v_2)v_2^2 + 4v_2^4\Bigg)\\
&+\frac{G^3 m_2^3}{r_{12}^4}\left(-(n_{12}v_1)^2 + 2(n_{12}v_1)(n_{12}v_2) + \frac{43}{2}(n_{12}v_2)^2 + 18(v_1v_2) - 9v_2^2\right)\\
&+\frac{G^3 m_1 m_2^2}{r_{12}^4}\Bigg(\frac{415}{8}(n_{12}v_1)^2 - \frac{375}{4}(n_{12}v_1)(n_{12}v_2) + \frac{1113}{8}(n_{12}v_2)^2 - \frac{615}{64}(n_{12}v_{12})^2\pi^2\\
&\qquad + 18v_1^2 + \frac{123}{64}\pi^2 v_{12}^2 + 33(v_1v_2) - \frac{33}{2}v_2^2\Bigg)
\end{aligned}
$$

$$
\begin{aligned}
&+\frac{G^3m_1^2m_2}{r_{12}^4}\Bigg(-\frac{45887}{168}(n_{12}v_1)^2+\frac{24025}{42}(n_{12}v_1)(n_{12}v_2)-\frac{10469}{42}(n_{12}v_2)^2+\frac{48197}{840}v_1^2\\
&\qquad -\frac{36227}{420}(v_1v_2)+\frac{36227}{840}v_2^2+110(n_{12}v_{12})^2\ln\left(\frac{r_{12}}{r'_1}\right)-22v_{12}^2\ln\left(\frac{r_{12}}{r'_1}\right)\Bigg)\\
&+\frac{16G^4m_2^4}{r_{12}^5}+\frac{G^4m_1^2m_2^2}{r_{12}^5}\left(175-\frac{41}{16}\pi^2\right)+\frac{G^4m_1^3m_2}{r_{12}^5}\left(-\frac{3187}{1260}+\frac{44}{3}\ln\left(\frac{r_{12}}{r'_1}\right)\right)\\
&+\frac{G^4m_1m_2^3}{r_{12}^5}\left(\frac{110741}{630}-\frac{41}{16}\pi^2-\frac{44}{3}\ln\left(\frac{r_{12}}{r'_2}\right)\right)\Bigg]n_{12}^i\\
&+\Bigg[\frac{Gm_2}{r_{12}^2}\Bigg(\frac{15}{2}(n_{12}v_1)(n_{12}v_2)^4-\frac{45}{8}(n_{12}v_2)^5-\frac{3}{2}(n_{12}v_2)^3v_1^2+6(n_{12}v_1)(n_{12}v_2)^2(v_1v_2)\\
&\qquad -6(n_{12}v_2)^3(v_1v_2)-2(n_{12}v_2)(v_1v_2)^2-12(n_{12}v_1)(n_{12}v_2)^2v_2^2+12(n_{12}v_2)^3v_2^2\\
&\qquad +(n_{12}v_2)v_1^2v_2^2-4(n_{12}v_1)(v_1v_2)v_2^2+8(n_{12}v_2)(v_1v_2)v_2^2+4(n_{12}v_1)v_2^4\\
&\qquad -7(n_{12}v_2)v_2^4\Bigg)\\
&\quad+\frac{G^2m_2^2}{r_{12}^3}\Bigg(-2(n_{12}v_1)^2(n_{12}v_2)+8(n_{12}v_1)(n_{12}v_2)^2+2(n_{12}v_2)^3+2(n_{12}v_1)(v_1v_2)\\
&\qquad +4(n_{12}v_2)(v_1v_2)-2(n_{12}v_1)v_2^2-4(n_{12}v_2)v_2^2\Bigg)\\
&\quad+\frac{G^2m_1m_2}{r_{12}^3}\Bigg(-\frac{243}{4}(n_{12}v_1)^3+\frac{565}{4}(n_{12}v_1)^2(n_{12}v_2)-\frac{269}{4}(n_{12}v_1)(n_{12}v_2)^2\\
&\qquad -\frac{95}{12}(n_{12}v_2)^3+\frac{207}{8}(n_{12}v_1)v_1^2-\frac{137}{8}(n_{12}v_2)v_1^2-36(n_{12}v_1)(v_1v_2)\\
&\qquad +\frac{27}{4}(n_{12}v_2)(v_1v_2)+\frac{81}{8}(n_{12}v_1)v_2^2+\frac{83}{8}(n_{12}v_2)v_2^2\Bigg)\\
&\quad+\frac{G^3m_2^3}{r_{12}^4}\left(4(n_{12}v_1)+5(n_{12}v_2)\right)\\
&\quad+\frac{G^3m_1m_2^2}{r_{12}^4}\left(-\frac{307}{8}(n_{12}v_1)+\frac{479}{8}(n_{12}v_2)+\frac{123}{32}(n_{12}v_{12})\pi^2\right)\\
&\quad+\frac{G^3m_1^2m_2}{r_{12}^4}\left(\frac{31397}{420}(n_{12}v_1)-\frac{36227}{420}(n_{12}v_2)-44(n_{12}v_{12})\ln\left(\frac{r_{12}}{r'_1}\right)\right)\Bigg]v_{12}^i\Bigg\}\\
+\frac{1}{c^7}\Bigg\{&\Bigg[\frac{G^4m_1^3m_2}{r_{12}^5}\left(\frac{3992}{105}(n_{12}v_1)-\frac{4328}{105}(n_{12}v_2)\right)\\
&+\frac{G^4m_1^2m_2^2}{r_{12}^6}\left(-\frac{13576}{105}(n_{12}v_1)+\frac{2872}{21}(n_{12}v_2)\right)-\frac{3172}{21}\frac{G^4m_1m_2^3}{r_{12}^6}(n_{12}v_{12})\\
&+\frac{G^3m_1^2m_2}{r_{12}^4}\Bigg(48(n_{12}v_1)^3-\frac{696}{5}(n_{12}v_1)^2(n_{12}v_2)+\frac{744}{5}(n_{12}v_1)(n_{12}v_2)^2-\frac{288}{5}(n_{12}v_2)^3\\
&\qquad -\frac{4888}{105}(n_{12}v_1)v_1^2+\frac{5056}{105}(n_{12}v_2)v_1^2+\frac{2056}{21}(n_{12}v_1)(v_1v_2)\\
&\qquad -\frac{2224}{21}(n_{12}v_2)(v_1v_2)-\frac{1028}{21}(n_{12}v_1)v_2^2+\frac{5812}{105}(n_{12}v_2)v_2^2\Bigg)\\
&+\frac{G^3m_1m_2^2}{r_{12}^4}\Bigg(-\frac{582}{5}(n_{12}v_1)^3+\frac{1746}{5}(n_{12}v_1)^2(n_{12}v_2)-\frac{1954}{5}(n_{12}v_1)(n_{12}v_2)^2
\end{aligned}
$$

$$
\begin{aligned}
&\quad + 158(n_{12}v_2)^3 + \frac{3568}{105}(n_{12}v_{12})v_1^2 - \frac{2864}{35}(n_{12}v_1)(v_1v_2) \\
&\quad + \frac{10048}{105}(n_{12}v_2)(v_1v_2) + \frac{1432}{35}(n_{12}v_1)v_2^2 - \frac{5752}{105}(n_{12}v_2)v_2^2\Bigg) \\
&+\frac{G^2m_1m_2}{r_{12}^3}\Bigg(- 56(n_{12}v_{12})^5 + 60(n_{12}v_1)^3v_{12}^2 - 180(n_{12}v_1)^2(n_{12}v_2)v_{12}^2 \\
&\quad + 174(n_{12}v_1)(n_{12}v_2)^2v_{12}^2 - 54(n_{12}v_2)^3v_{12}^2 - \frac{246}{35}(n_{12}v_{12})v_1^4 \\
&\quad + \frac{1068}{35}(n_{12}v_1)v_1^2(v_1v_2) - \frac{984}{35}(n_{12}v_2)v_1^2(v_1v_2) - \frac{1068}{35}(n_{12}v_1)(v_1v_2)^2 \\
&\quad + \frac{180}{7}(n_{12}v_2)(v_1v_2)^2 - \frac{534}{35}(n_{12}v_1)v_1^2v_2^2 + \frac{90}{7}(n_{12}v_2)v_1^2v_2^2 \\
&\quad + \frac{984}{35}(n_{12}v_1)(v_1v_2)v_2^2 - \frac{732}{35}(n_{12}v_2)(v_1v_2)v_2^2 - \frac{204}{35}(n_{12}v_1)v_2^4 \\
&\quad + \frac{24}{7}(n_{12}v_2)v_2^4\Bigg)\Bigg]n_{12}^i \\
&+\Bigg[- \frac{184}{21}\frac{G^4m_1^3m_2}{r_{12}^5} + \frac{6224}{105}\frac{G^4m_1^2m_2^2}{r_{12}^6} + \frac{6388}{105}\frac{G^4m_1m_2^3}{r_{12}^6} \\
&\quad +\frac{G^3m_1^2m_2}{r_{12}^4}\Bigg(\frac{52}{15}(n_{12}v_1)^2 - \frac{56}{15}(n_{12}v_1)(n_{12}v_2) - \frac{44}{15}(n_{12}v_2)^2 - \frac{132}{35}v_1^2 + \frac{152}{35}(v_1v_2) \\
&\qquad -\frac{48}{35}v_2^2\Bigg) \\
&\quad +\frac{G^3m_1m_2^2}{r_{12}^4}\Bigg(\frac{454}{15}(n_{12}v_1)^2 - \frac{372}{5}(n_{12}v_1)(n_{12}v_2) + \frac{854}{15}(n_{12}v_2)^2 - \frac{152}{21}v_1^2 \\
&\qquad +\frac{2864}{105}(v_1v_2) - \frac{1768}{105}v_2^2\Bigg) \\
&\quad +\frac{G^2m_1m_2}{r_{12}^3}\Bigg(60(n_{12}v_{12})^4 - \frac{348}{5}(n_{12}v_1)^2v_{12}^2 + \frac{684}{5}(n_{12}v_1)(n_{12}v_2)v_{12}^2 \\
&\qquad -66(n_{12}v_2)^2v_{12}^2 + \frac{334}{35}v_1^4 - \frac{1336}{35}v_1^2(v_1v_2) + \frac{1308}{35}(v_1v_2)^2 + \frac{654}{35}v_1^2v_2^2 \\
&\qquad -\frac{1252}{35}(v_1v_2)v_2^2 + \frac{292}{35}v_2^4\Bigg)\Bigg]v_{12}^i\Bigg\} \\
&+\mathcal{O}\left(\frac{1}{c^8}\right). \qquad (168)
\end{aligned}
$$

The 2.5PN and 3.5PN terms are associated with gravitational radiation reaction. The 3PN harmonic-coordinates equations of motion depend on two arbitrary length scales r'_1 and r'_2 associated with the logarithms present at the 3PN order[31]. It has been proved in Ref. [38] that r'_1 and r'_2 are merely linked with the choice of coordinates – we can refer to r'_1 and r'_2 as "gauge constants". In our approach [37, 38], the harmonic coordinate system is not uniquely fixed by the coordinate condition $\partial_\mu h^{\alpha\mu} = 0$. In fact there are infinitely many harmonic coordinate systems that are local. For general smooth sources, as in the general formalism of Part A, we expect

[31] Notice also the dependence upon π^2. Technically, the π^2 terms arise from non-linear interactions involving some integrals such as

$$\frac{1}{\pi}\int \frac{d^3\mathbf{x}}{r_1^2r_2^2} = \frac{\pi^2}{r_{12}}.$$

the existence and uniqueness of a global harmonic coordinate system. But here we have some point-particles, with delta-function singularities, and in this case we do not have the notion of a global coordinate system. We can always change the harmonic coordinates by means of the gauge vector $\eta^\alpha = \delta x^\alpha$, satisfying $\Delta\eta^\alpha = 0$ except at the location of the two particles (we assume that the transformation is at the 3PN level, so we can consider simply a flat-space Laplace equation). More precisely, we can show that the logarithms appearing in Equation (168), together with the constants r'_1 and r'_2 therein, can be removed by the coordinate transformation associated with the 3PN gauge vector (with $r_1 = |\mathbf{x} - \mathbf{y}_1(t)|$ and $r_2 = |\mathbf{x} - \mathbf{y}_2(t)|$):

$$\eta^\alpha = -\frac{22}{3}\frac{G^2 m_1 m_2}{c^6}\partial^\alpha \left[\frac{Gm_1}{r_2}\ln\left(\frac{r_{12}}{r'_1}\right) + \frac{Gm_2}{r_1}\ln\left(\frac{r_{12}}{r'_2}\right)\right]. \tag{169}$$

Therefore, the "ambiguity" in the choice of the constants r'_1 and r'_2 is completely innocuous on the physical point of view, because the physical results must be gauge invariant. Indeed we shall verify that r'_1 and r'_2 cancel out in our final results.

When retaining the "even" relativistic corrections at the 1PN, 2PN and 3PN orders, and neglecting the "odd" radiation reaction terms at the 2.5PN and 3.5PN orders, we find that the equations of motion admit a conserved energy (and a Lagrangian, as we shall see), and that energy can be straightforwardly obtained by guess-work starting from Equation (168), with the result

$$\begin{aligned}
E = {} & \frac{m_1 v_1^2}{2} - \frac{G m_1 m_2}{2 r_{12}} \\
& + \frac{1}{c^2}\left\{\frac{G^2 m_1^2 m_2}{2 r_{12}^2} + \frac{3 m_1 v_1^4}{8} + \frac{G m_1 m_2}{r_{12}}\left(-\frac{1}{4}(n_{12}v_1)(n_{12}v_2) + \frac{3}{2}v_1^2 - \frac{7}{4}(v_1 v_2)\right)\right\} \\
& + \frac{1}{c^4}\Bigg\{-\frac{G^3 m_1^3 m_2}{2 r_{12}^3} - \frac{19 G^3 m_1^2 m_2^2}{8 r_{12}^3} + \frac{5 m_1 v_1^6}{16} \\
& \qquad + \frac{G m_1 m_2}{r_{12}}\bigg(\frac{3}{8}(n_{12}v_1)^3(n_{12}v_2) + \frac{3}{16}(n_{12}v_1)^2(n_{12}v_2)^2 - \frac{9}{8}(n_{12}v_1)(n_{12}v_2)v_1^2 \\
& \qquad\qquad - \frac{13}{8}(n_{12}v_2)^2 v_1^2 + \frac{21}{8}v_1^4 + \frac{13}{8}(n_{12}v_1)^2(v_1 v_2) + \frac{3}{4}(n_{12}v_1)(n_{12}v_2)(v_1 v_2) \\
& \qquad\qquad - \frac{55}{8}v_1^2(v_1 v_2) + \frac{17}{8}(v_1 v_2)^2 + \frac{31}{16}v_1^2 v_2^2\bigg) \\
& \qquad + \frac{G^2 m_1^2 m_2}{r_{12}^2}\left(\frac{29}{4}(n_{12}v_1)^2 - \frac{13}{4}(n_{12}v_1)(n_{12}v_2) + \frac{1}{2}(n_{12}v_2)^2 - \frac{3}{2}v_1^2 + \frac{7}{4}v_2^2\right)\Bigg\} \\
& + \frac{1}{c^6}\Bigg\{\frac{35 m_1 v_1^8}{128} \\
& \qquad + \frac{G m_1 m_2}{r_{12}}\bigg(-\frac{5}{16}(n_{12}v_1)^5(n_{12}v_2) - \frac{5}{16}(n_{12}v_1)^4(n_{12}v_2)^2 - \frac{5}{32}(n_{12}v_1)^3(n_{12}v_2)^3 \\
& \qquad\qquad + \frac{19}{16}(n_{12}v_1)^3(n_{12}v_2)v_1^2 + \frac{15}{16}(n_{12}v_1)^2(n_{12}v_2)^2 v_1^2 + \frac{3}{4}(n_{12}v_1)(n_{12}v_2)^3 v_1^2 \\
& \qquad\qquad + \frac{19}{16}(n_{12}v_2)^4 v_1^2 - \frac{21}{16}(n_{12}v_1)(n_{12}v_2)v_1^4 - 2(n_{12}v_2)^2 v_1^4 \\
& \qquad\qquad + \frac{55}{16}v_1^6 - \frac{19}{16}(n_{12}v_1)^4(v_1 v_2) - (n_{12}v_1)^3(n_{12}v_2)(v_1 v_2) \\
& \qquad\qquad - \frac{15}{32}(n_{12}v_1)^2(n_{12}v_2)^2(v_1 v_2) + \frac{45}{16}(n_{12}v_1)^2 v_1^2(v_1 v_2) \\
& \qquad\qquad + \frac{5}{4}(n_{12}v_1)(n_{12}v_2)v_1^2(v_1 v_2) + \frac{11}{4}(n_{12}v_2)^2 v_1^2(v_1 v_2) - \frac{139}{16}v_1^4(v_1 v_2)
\end{aligned}$$

$$
\begin{aligned}
& -\frac{3}{4}(n_{12}v_1)^2(v_1v_2)^2 + \frac{5}{16}(n_{12}v_1)(n_{12}v_2)(v_1v_2)^2 + \frac{41}{8}v_1^2(v_1v_2)^2 + \frac{1}{16}(v_1v_2)^3 \\
& -\frac{45}{16}(n_{12}v_1)^2v_1^2v_2^2 - \frac{23}{32}(n_{12}v_1)(n_{12}v_2)v_1^2v_2^2 + \frac{79}{16}v_1^4v_2^2 - \frac{161}{32}v_1^2(v_1v_2)v_2^2\Big) \\
+ \frac{G^2m_1^2m_2}{r_{12}^2}\Big(& -\frac{49}{8}(n_{12}v_1)^4 + \frac{75}{8}(n_{12}v_1)^3(n_{12}v_2) - \frac{187}{8}(n_{12}v_1)^2(n_{12}v_2)^2 \\
& + \frac{247}{24}(n_{12}v_1)(n_{12}v_2)^3 + \frac{49}{8}(n_{12}v_1)^2v_1^2 + \frac{81}{8}(n_{12}v_1)(n_{12}v_2)v_1^2 \\
& - \frac{21}{4}(n_{12}v_2)^2v_1^2 + \frac{11}{2}v_1^4 - \frac{15}{2}(n_{12}v_1)^2(v_1v_2) - \frac{3}{2}(n_{12}v_1)(n_{12}v_2)(v_1v_2) \\
& + \frac{21}{4}(n_{12}v_2)^2(v_1v_2) - 27v_1^2(v_1v_2) + \frac{55}{2}(v_1v_2)^2 + \frac{49}{4}(n_{12}v_1)^2v_2^2 \\
& - \frac{27}{2}(n_{12}v_1)(n_{12}v_2)v_2^2 + \frac{3}{4}(n_{12}v_2)^2v_2^2 + \frac{55}{4}v_1^2v_2^2 - 28(v_1v_2)v_2^2 + \frac{135}{16}v_2^4\Big) \\
+ \frac{3G^4m_1^4m_2}{8r_{12}^4} & + \frac{G^4m_1^3m_2^2}{r_{12}^4}\left(\frac{9707}{420} - \frac{22}{3}\ln\left(\frac{r_{12}}{r'_1}\right)\right) \\
+ \frac{G^3m_1^2m_2^2}{r_{12}^3}\Big(& \frac{547}{12}(n_{12}v_1)^2 - \frac{3115}{48}(n_{12}v_1)(n_{12}v_2) - \frac{123}{64}(n_{12}v_1)(n_{12}v_{12})\pi^2 - \frac{575}{18}v_1^2 \\
& + \frac{41}{64}\pi^2(v_1v_{12}) + \frac{4429}{144}(v_1v_2)\Big) \\
+ \frac{G^3m_1^3m_2}{r_{12}^3}\Big(& -\frac{44627}{840}(n_{12}v_1)^2 + \frac{32027}{840}(n_{12}v_1)(n_{12}v_2) + \frac{3}{2}(n_{12}v_2)^2 + \frac{24187}{2520}v_1^2 \\
& - \frac{27967}{2520}(v_1v_2) + \frac{5}{4}v_2^2 + 22(n_{12}v_1)(n_{12}v_{12})\ln\left(\frac{r_{12}}{r'_1}\right) - \frac{22}{3}(v_1v_{12})\ln\left(\frac{r_{12}}{r'_1}\right)\Big)\Bigg\} \\
+1 \leftrightarrow 2 + \mathcal{O}\left(\frac{1}{c^7}\right). & \qquad (170)
\end{aligned}
$$

To the terms given above, we must add the terms corresponding to the relabelling $1 \leftrightarrow 2$. Actually, this energy is not conserved because of the radiation reaction. Thus its time derivative, as computed by means of the 3PN equations of motion themselves (i.e. order-reducing all the accelerations), is purely equal to the 2.5PN effect,

$$
\begin{aligned}
\frac{dE}{dt} = \frac{4}{5}\frac{G^2m_1^2m_2}{c^5r_{12}^3}\Bigg[& (v_1v_{12})\left(-v_{12}^2 + 2\frac{Gm_1}{r_{12}} - 8\frac{Gm_2}{r_{12}}\right) + (n_{12}v_1)(n_{12}v_{12})\left(3v_{12}^2 - 6\frac{Gm_1}{r_{12}} + \frac{52}{3}\frac{Gm_2}{r_{12}}\right)\Bigg] \\
& +1 \leftrightarrow 2 + \mathcal{O}\left(\frac{1}{c^7}\right). \qquad (171)
\end{aligned}
$$

The resulting "balance equation" can be better expressed by transfering to the left-hand side certain 2.5PN terms so that the right-hand side takes the familiar form of a total energy flux. Posing

$$
\widetilde{E} = E + \frac{4G^2m_1^2m_2}{5c^5r_{12}^2}(n_{12}v_1)\left[v_{12}^2 - \frac{2G(m_1-m_2)}{r_{12}}\right] + 1 \leftrightarrow 2, \qquad (172)
$$

we find agreement with the standard Einstein quadrupole formula (4, 5):

$$
\frac{d\widetilde{E}}{dt} = -\frac{G}{5c^5}\frac{d^3\mathrm{Q}_{ij}}{dt^3}\frac{d^3\mathrm{Q}_{ij}}{dt^3} + \mathcal{O}\left(\frac{1}{c^7}\right), \qquad (173)
$$

where the Newtonian trace-free quadrupole moment is $\mathrm{Q}_{ij} = m_1(y_1^iy_1^j - \frac{1}{3}\delta^{ij}\mathbf{y}_1^2) + 1 \leftrightarrow 2$. We refer to Iyer and Will [136, 137] for the discussion of the energy balance equation at the next 3.5PN order. As we can see, the 3.5PN equations of motion (168) are highly relativistic when describing the *motion*, but concerning the *radiation* they are in fact 1PN, because they contain merely the radiation reaction force at the 2.5PN + 3.5PN orders.

9.2 Lagrangian and Hamiltonian formulations

The conservative part of the equations of motion in harmonic coordinates (168) is derivable from a *generalized* Lagrangian, depending not only on the positions and velocities of the bodies, but also on their accelerations: $a_1^i = dv_1^i/dt$ and $a_2^i = dv_2^i/dt$. As shown by Damour and Deruelle [85], the accelerations in the harmonic-coordinates Lagrangian occur already from the 2PN order. This fact is in accordance with a general result of Martin and Sanz [158] that N-body equations of motion cannot be derived from an ordinary Lagrangian beyond the 1PN level, provided that the gauge conditions preserve the Lorentz invariance. Note that we can always arrange for the dependence of the Lagrangian upon the accelerations to be *linear*, at the price of adding some so-called "multi-zero" terms to the Lagrangian, which do not modify the equations of motion (see, e.g., Ref. [98]). At the 3PN level, we find that the Lagrangian also depends on accelerations. It is notable that these accelerations are sufficient – there is no need to include derivatives of accelerations. Note also that the Lagrangian is not unique because we can always add to it a total time derivative dF/dt, where F depends on the positions and velocities, without changing the dynamics. We find [103]

$$
\begin{aligned}
L^{\rm harm} = {} & \frac{Gm_1m_2}{2r_{12}} + \frac{m_1v_1^2}{2} \\
& + \frac{1}{c^2}\left\{ -\frac{G^2m_1^2m_2}{2r_{12}^2} + \frac{m_1v_1^4}{8} + \frac{Gm_1m_2}{r_{12}}\left(-\frac{1}{4}(n_{12}v_1)(n_{12}v_2) + \frac{3}{2}v_1^2 - \frac{7}{4}(v_1v_2) \right) \right\} \\
& + \frac{1}{c^4}\left\{ \frac{G^3m_1^3m_2}{2r_{12}^3} + \frac{19G^3m_1^2m_2^2}{8r_{12}^3} \right. \\
& \quad + \frac{G^2m_1^2m_2}{r_{12}^2}\left(\frac{7}{2}(n_{12}v_1)^2 - \frac{7}{2}(n_{12}v_1)(n_{12}v_2) + \frac{1}{2}(n_{12}v_2)^2 + \frac{1}{4}v_1^2 - \frac{7}{4}(v_1v_2) + \frac{7}{4}v_2^2 \right) \\
& \quad + \frac{Gm_1m_2}{r_{12}}\left(\frac{3}{16}(n_{12}v_1)^2(n_{12}v_2)^2 - \frac{7}{8}(n_{12}v_2)^2v_1^2 + \frac{7}{8}v_1^4 + \frac{3}{4}(n_{12}v_1)(n_{12}v_2)(v_1v_2) \right. \\
& \qquad\qquad \left. -2v_1^2(v_1v_2) + \frac{1}{8}(v_1v_2)^2 + \frac{15}{16}v_1^2v_2^2 \right) + \frac{m_1v_1^6}{16} \\
& \quad \left. + Gm_1m_2\left(-\frac{7}{4}(a_1v_2)(n_{12}v_2) - \frac{1}{8}(n_{12}a_1)(n_{12}v_2)^2 + \frac{7}{8}(n_{12}a_1)v_2^2 \right) \right\} \\
& + \frac{1}{c^6}\left\{ \frac{G^2m_1^2m_2}{r_{12}^2}\left(\frac{13}{18}(n_{12}v_1)^4 + \frac{83}{18}(n_{12}v_1)^3(n_{12}v_2) - \frac{35}{6}(n_{12}v_1)^2(n_{12}v_2)^2 - \frac{245}{24}(n_{12}v_1)^2v_1^2 \right. \right. \\
& \qquad + \frac{179}{12}(n_{12}v_1)(n_{12}v_2)v_1^2 - \frac{235}{24}(n_{12}v_2)^2v_1^2 + \frac{373}{48}v_1^4 + \frac{529}{24}(n_{12}v_1)^2(v_1v_2) \\
& \qquad - \frac{97}{6}(n_{12}v_1)(n_{12}v_2)(v_1v_2) - \frac{719}{24}v_1^2(v_1v_2) + \frac{463}{24}(v_1v_2)^2 - \frac{7}{24}(n_{12}v_1)^2v_2^2 \\
& \qquad \left. - \frac{1}{2}(n_{12}v_1)(n_{12}v_2)v_2^2 + \frac{1}{4}(n_{12}v_2)^2v_2^2 + \frac{463}{48}v_1^2v_2^2 - \frac{19}{2}(v_1v_2)v_2^2 + \frac{45}{16}v_2^4 \right) \\
& \quad + \frac{5m_1v_1^8}{128} \\
& \quad + Gm_1m_2\left(\frac{3}{8}(a_1v_2)(n_{12}v_1)(n_{12}v_2)^2 + \frac{5}{12}(a_1v_2)(n_{12}v_2)^3 + \frac{1}{8}(n_{12}a_1)(n_{12}v_1)(n_{12}v_2)^3 \right.
\end{aligned}
$$

$$
\begin{aligned}
&+\frac{1}{16}(n_{12}a_1)(n_{12}v_2)^4+\frac{11}{4}(a_1v_1)(n_{12}v_2)v_1^2-(a_1v_2)(n_{12}v_2)v_1^2\\
&-2(a_1v_1)(n_{12}v_2)(v_1v_2)+\frac{1}{4}(a_1v_2)(n_{12}v_2)(v_1v_2)\\
&+\frac{3}{8}(n_{12}a_1)(n_{12}v_2)^2(v_1v_2)-\frac{5}{8}(n_{12}a_1)(n_{12}v_1)^2v_2^2+\frac{15}{8}(a_1v_1)(n_{12}v_2)v_2^2\\
&-\frac{15}{8}(a_1v_2)(n_{12}v_2)v_2^2-\frac{1}{2}(n_{12}a_1)(n_{12}v_1)(n_{12}v_2)v_2^2\\
&-\frac{5}{16}(n_{12}a_1)(n_{12}v_2)^2v_2^2\Bigg)\\
&+\frac{G^2m_1^2m_2}{r_{12}}\Bigg(-\frac{235}{24}(a_2v_1)(n_{12}v_1)-\frac{29}{24}(n_{12}a_2)(n_{12}v_1)^2-\frac{235}{24}(a_1v_2)(n_{12}v_2)\\
&\qquad-\frac{17}{6}(n_{12}a_1)(n_{12}v_2)^2+\frac{185}{16}(n_{12}a_1)v_1^2-\frac{235}{48}(n_{12}a_2)v_1^2\\
&\qquad-\frac{185}{8}(n_{12}a_1)(v_1v_2)+\frac{20}{3}(n_{12}a_1)v_2^2\Bigg)\\
&+\frac{Gm_1m_2}{r_{12}}\Bigg(-\frac{5}{32}(n_{12}v_1)^3(n_{12}v_2)^3+\frac{1}{8}(n_{12}v_1)(n_{12}v_2)^3v_1^2+\frac{5}{8}(n_{12}v_2)^4v_1^2\\
&\qquad-\frac{11}{16}(n_{12}v_1)(n_{12}v_2)v_1^4+\frac{1}{4}(n_{12}v_2)^2v_1^4+\frac{11}{16}v_1^6\\
&\qquad-\frac{15}{32}(n_{12}v_1)^2(n_{12}v_2)^2(v_1v_2)+(n_{12}v_1)(n_{12}v_2)v_1^2(v_1v_2)\\
&\qquad+\frac{3}{8}(n_{12}v_2)^2v_1^2(v_1v_2)-\frac{13}{16}v_1^4(v_1v_2)+\frac{5}{16}(n_{12}v_1)(n_{12}v_2)(v_1v_2)^2\\
&\qquad+\frac{1}{16}(v_1v_2)^3-\frac{5}{8}(n_{12}v_1)^2v_1^2v_2^2-\frac{23}{32}(n_{12}v_1)(n_{12}v_2)v_1^2v_2^2+\frac{1}{16}v_1^4v_2^2\\
&\qquad-\frac{1}{32}v_1^2(v_1v_2)v_2^2\Bigg)\\
&-\frac{3G^4m_1^4m_2}{8r_{12}^4}+\frac{G^4m_1^3m_2^2}{r_{12}^4}\left(-\frac{9707}{420}+\frac{22}{3}\ln\left(\frac{r_{12}}{r'_1}\right)\right)\\
&+\frac{G^3m_1^2m_2^2}{r_{12}^3}\Bigg(\frac{383}{24}(n_{12}v_1)^2-\frac{889}{48}(n_{12}v_1)(n_{12}v_2)-\frac{123}{64}(n_{12}v_1)(n_{12}v_{12})\pi^2-\frac{305}{72}v_1^2\\
&\qquad+\frac{41}{64}\pi^2(v_1v_{12})+\frac{439}{144}(v_1v_2)\Bigg)\\
&+\frac{G^3m_1^3m_2}{r_{12}^3}\Bigg(-\frac{8243}{210}(n_{12}v_1)^2+\frac{15541}{420}(n_{12}v_1)(n_{12}v_2)+\frac{3}{2}(n_{12}v_2)^2+\frac{15611}{1260}v_1^2\\
&\qquad-\frac{17501}{1260}(v_1v_2)+\frac{5}{4}v_2^2+22(n_{12}v_1)(n_{12}v_{12})\ln\left(\frac{r_{12}}{r'_1}\right)\\
&\qquad-\frac{22}{3}(v_1v_{12})\ln\left(\frac{r_{12}}{r'_1}\right)\Bigg)\Bigg\}\\
&+1\leftrightarrow 2+\mathcal{O}\left(\frac{1}{c^7}\right). \qquad (174)
\end{aligned}
$$

Witness the accelerations occuring at the 2PN and 3PN orders; see also the gauge-dependent logarithms of r_{12}/r'_1 and r_{12}/r'_2. We refer to [103] for the explicit expressions of the ten conserved quantities corresponding to the integrals of energy (also given in Equation (170)), linear and angular momenta, and center-of-mass position. Notice that while it is strictly forbidden to replace the accelerations by the equations of motion in the Lagrangian, this can and *should* be done in the final expressions of the conserved integrals derived from that Lagrangian.

Now we want to exhibit a transformation of the particles dynamical variables – or *contact* transformation, as it is called in the jargon – which transforms the 3PN harmonic-coordinates Lagrangian (174) into a new Lagrangian, valid in some ADM or ADM-like coordinate system, and such that the associated Hamiltonian coincides with the 3PN Hamiltonian that has been obtained by Damour, Jaranowski, and Schäfer [95]. In ADM coordinates the Lagrangian will be "ordinary", depending only on the positions and velocities of the bodies. Let this contact transformation be $Y_1^i(t) = y_1^i(t) + \delta y_1^i(t)$ and $1 \leftrightarrow 2$, where Y_1^i and y_1^i denote the trajectories in ADM and harmonic coordinates, respectively. For this transformation to be able to remove all the accelerations in the initial Lagrangian $L^{\rm harm}$ up to the 3PN order, we determine [103] it to be necessarily of the form

$$\delta y_1^i = \frac{1}{m_1}\left[\frac{\partial L^{\rm harm}}{\partial a_1^i} + \frac{\partial F}{\partial v_1^i} + \frac{1}{c^6}X_1^i\right] + \mathcal{O}\left(\frac{1}{c^8}\right) \tag{175}$$

(and *idem* $1 \leftrightarrow 2$), where F is a freely adjustable function of the positions and velocities, made of 2PN and 3PN terms, and where X_1^i represents a special correction term, that is purely of order 3PN. The point is that once the function F is specified there is a *unique* determination of the correction term X_1^i for the contact transformation to work (see Ref. [103] for the details). Thus, the freedom we have is entirely coded into the function F, and the work then consists in showing that there exists a unique choice of F for which our Lagrangian $L^{\rm harm}$ is physically equivalent, *via* the contact transformation (175), to the ADM Hamiltonian of Ref. [95]. An interesting point is that not only the transformation must remove all the accelerations in $L^{\rm harm}$, but it should also cancel out all the logarithms $\ln(r_{12}/r'_1)$ and $\ln(r_{12}/r'_2)$, because there are no logarithms in ADM coordinates. The result we find, which can be checked to be in full agreement with the expression of the gauge vector in Equation (169), is that F involves the logarithmic terms

$$F = \frac{22}{3}\frac{G^3 m_1 m_2}{c^6 r_{12}^2}\left[m_1^2 (n_{12}v_1)\ln\left(\frac{r_{12}}{r'_1}\right) - m_2^2 (n_{12}v_2)\ln\left(\frac{r_{12}}{r'_2}\right)\right] + \dots, \tag{176}$$

together with many other non-logarithmic terms (indicated by dots) that are entirely specified by the isometry of the harmonic and ADM descriptions of the motion. For this particular choice of F the ADM Lagrangian reads

$$L^{\rm ADM} = L^{\rm harm} + \frac{\delta L^{\rm harm}}{\delta y_1^i}\delta y_1^i + \frac{\delta L^{\rm harm}}{\delta y_2^i}\delta y_2^i + \frac{dF}{dt} + \mathcal{O}\left(\frac{1}{c^8}\right). \tag{177}$$

Inserting into this equation all our explicit expressions we find

$$\begin{aligned}
L^{\rm ADM} = {} & \frac{G m_1 m_2}{2R_{12}} + \frac{1}{2}m_1 V_1^2 \\
& + \frac{1}{c^2}\left\{-\frac{G^2 m_1^2 m_2}{2R_{12}^2} + \frac{1}{8}m_1 V_1^4 + \frac{G m_1 m_2}{R_{12}}\left(-\frac{1}{4}(N_{12}V_1)(N_{12}V_2) + \frac{3}{2}V_1^2 - \frac{7}{4}(V_1V_2)\right)\right\} \\
& + \frac{1}{c^4}\left\{\frac{G^3 m_1^3 m_2}{4R_{12}^3} + \frac{5G^3 m_1^2 m_2^2}{8R_{12}^3} + \frac{m_1 V_1^6}{16}\right. \\
& \qquad + \frac{G^2 m_1^2 m_2}{R_{12}^2}\left(\frac{15}{8}(N_{12}V_1)^2 + \frac{11}{8}V_1^2 - \frac{15}{4}(V_1V_2) + 2V_2^2\right) \\
& \qquad + \frac{G m_1 m_2}{R_{12}}\left(\frac{3}{16}(N_{12}V_1)^2(N_{12}V_2)^2 - \frac{1}{4}(N_{12}V_1)(N_{12}V_2)V_1^2 - \frac{5}{8}(N_{12}V_2)^2 V_1^2 + \frac{7}{8}V_1^4\right.
\end{aligned}$$

$$
\begin{aligned}
&\qquad\qquad + \frac{3}{4}(N_{12}V_1)(N_{12}V_2)(V_1V_2) - \frac{7}{4}V_1^2(V_1V_2) + \frac{1}{8}(V_1V_2)^2 + \frac{11}{16}V_1^2V_2^2\Bigg)\Bigg\} \\
&+ \frac{1}{c^6}\Bigg\{\frac{5m_1V_1^8}{128} - \frac{G^4m_1^4m_2}{8R_{12}^4} + \frac{G^4m_1^3m_2^2}{R_{12}^4}\left(-\frac{227}{24} + \frac{21}{32}\pi^2\right) \\
&\quad + \frac{Gm_1m_2}{R_{12}}\Bigg(-\frac{5}{32}(N_{12}V_1)^3(N_{12}V_2)^3 + \frac{3}{16}(N_{12}V_1)^2(N_{12}V_2)^2V_1^2 \\
&\qquad + \frac{9}{16}(N_{12}V_1)(N_{12}V_2)^3V_1^2 - \frac{3}{16}(N_{12}V_1)(N_{12}V_2)V_1^4 - \frac{5}{16}(N_{12}V_2)^2V_1^4 \\
&\qquad + \frac{11}{16}V_1^6 - \frac{15}{32}(N_{12}V_1)^2(N_{12}V_2)^2(V_1V_2) + \frac{3}{4}(N_{12}V_1)(N_{12}V_2)V_1^2(V_1V_2) \\
&\qquad - \frac{1}{16}(N_{12}V_2)^2V_1^2(V_1V_2) - \frac{21}{16}V_1^4(V_1V_2) + \frac{5}{16}(N_{12}V_1)(N_{12}V_2)(V_1V_2)^2 \\
&\qquad + \frac{1}{8}V_1^2(V_1V_2)^2 + \frac{1}{16}(V_1V_2)^3 - \frac{5}{16}(N_{12}V_1)^2V_1^2V_2^2 \\
&\qquad - \frac{9}{32}(N_{12}V_1)(N_{12}V_2)V_1^2V_2^2 + \frac{7}{8}V_1^4V_2^2 - \frac{15}{32}V_1^2(V_1V_2)V_2^2\Bigg) \\
&\quad + \frac{G^2m_1^2m_2}{R_{12}^2}\Bigg(-\frac{5}{12}(N_{12}V_1)^4 - \frac{13}{8}(N_{12}V_1)^3(N_{12}V_2) - \frac{23}{24}(N_{12}V_1)^2(N_{12}V_2)^2 \\
&\qquad + \frac{13}{16}(N_{12}V_1)^2V_1^2 + \frac{1}{4}(N_{12}V_1)(N_{12}V_2)V_1^2 + \frac{5}{6}(N_{12}V_2)^2V_1^2 + \frac{21}{16}V_1^4 \\
&\qquad - \frac{1}{2}(N_{12}V_1)^2(V_1V_2) + \frac{1}{3}(N_{12}V_1)(N_{12}V_2)(V_1V_2) - \frac{97}{16}V_1^2(V_1V_2) \\
&\qquad + \frac{341}{48}(V_1V_2)^2 + \frac{29}{24}(N_{12}V_1)^2V_2^2 - (N_{12}V_1)(N_{12}V_2)V_2^2 + \frac{43}{12}V_1^2V_2^2 \\
&\qquad - \frac{71}{8}(V_1V_2)V_2^2 + \frac{47}{16}V_2^4\Bigg) \\
&\quad + \frac{G^3m_1^2m_2^2}{R_{12}^3}\Bigg(\frac{73}{16}(N_{12}V_1)^2 - 11(N_{12}V_1)(N_{12}V_2) + \frac{3}{64}\pi^2(N_{12}V_1)(N_{12}V_{12}) \\
&\qquad - \frac{265}{48}V_1^2 - \frac{1}{64}\pi^2(V_1V_{12}) + \frac{59}{8}(V_1V_2)\Bigg) \\
&\quad + \frac{G^3m_1^3m_2}{R_{12}^3}\left(-5(N_{12}V_1)^2 - \frac{1}{8}(N_{12}V_1)(N_{12}V_2) + \frac{173}{48}V_1^2 - \frac{27}{8}(V_1V_2) + \frac{13}{8}V_2^2\right)\Bigg\} \\
&+ 1 \leftrightarrow 2 + \mathcal{O}\left(\frac{1}{c^7}\right). \qquad (178)
\end{aligned}
$$

The notation is the same as in Equation (174), except that we use upper-case letters to denote the ADM-coordinates positions and velocities; thus, for instance $\mathbf{N}_{12} = (\mathbf{Y}_1 - \mathbf{Y}_2)/R_{12}$ and $(N_{12}V_1) = \mathbf{N}_{12}\cdot\mathbf{V}_1$. The Hamiltonian is simply deduced from the latter Lagrangian by applying the usual Legendre transformation. Posing $P_1^i = \partial L^{\mathrm{ADM}}/\partial V_1^i$ and $1 \leftrightarrow 2$, we get [139, 140, 141, 95, 103][32]

$$
H^{\mathrm{ADM}} = -\frac{Gm_1m_2}{2R_{12}} + \frac{P_1^2}{2m_1}
$$

[32] Note that in the result published in Ref. [95] the following terms are missing:

$$
\frac{G^2}{c^6r_{12}^2}\left(-\frac{55}{12}m_1 - \frac{193}{48}m_2\right)\frac{(N_{12}P_2)^2P_1^2}{m_1m_2} + 1 \leftrightarrow 2.
$$

This misprint has been corrected in an Erratum [95].

$$
\begin{aligned}
&+\frac{1}{c^2}\left\{-\frac{P_1^4}{8m_1^3}+\frac{G^2m_1^2m_2}{2R_{12}^2}+\frac{Gm_1m_2}{R_{12}}\left(\frac{1}{4}\frac{(N_{12}P_1)(N_{12}P_2)}{m_1m_2}-\frac{3}{2}\frac{P_1^2}{m_1^2}+\frac{7}{4}\frac{(P_1P_2)}{m_1m_2}\right)\right\}\\
&+\frac{1}{c^4}\left\{\frac{P_1^6}{16m_1^5}-\frac{G^3m_1^3m_2}{4R_{12}^3}-\frac{5G^3m_1^2m_2^2}{8R_{12}^3}\right.\\
&\quad+\frac{G^2m_1^2m_2}{R_{12}^2}\left(-\frac{3}{2}\frac{(N_{12}P_1)(N_{12}P_2)}{m_1m_2}+\frac{19}{4}\frac{P_1^2}{m_1^2}-\frac{27}{4}\frac{(P_1P_2)}{m_1m_2}+\frac{5P_2^2}{2m_2^2}\right)\\
&\quad+\frac{Gm_1m_2}{R_{12}}\left(-\frac{3}{16}\frac{(N_{12}P_1)^2(N_{12}P_2)^2}{m_1^2m_2^2}+\frac{5}{8}\frac{(N_{12}P_2)^2P_1^2}{m_1^2m_2^2}\right.\\
&\qquad\left.\left.+\frac{5}{8}\frac{P_1^4}{m_1^4}-\frac{3}{4}\frac{(N_{12}P_1)(N_{12}P_2)(P_1P_2)}{m_1^2m_2^2}-\frac{1}{8}\frac{(P_1P_2)^2}{m_1^2m_2^2}-\frac{11}{16}\frac{P_1^2P_2^2}{m_1^2m_2^2}\right)\right\}\\
&+\frac{1}{c^6}\left\{-\frac{5P_1^8}{128m_1^7}+\frac{G^4m_1^4m_2}{8R_{12}^4}+\frac{G^4m_1^3m_2^2}{R_{12}^4}\left(\frac{227}{24}-\frac{21}{32}\pi^2\right)\right.\\
&\quad+\frac{G^3m_1^2m_2^2}{R_{12}^3}\left(-\frac{43}{16}\frac{(N_{12}P_1)^2}{m_1^2}+\frac{119}{16}\frac{(N_{12}P_1)(N_{12}P_2)}{m_1m_2}-\frac{3}{64}\pi^2\frac{(N_{12}P_1)^2}{m_1^2}\right.\\
&\qquad+\frac{3}{64}\pi^2\frac{(N_{12}P_1)(N_{12}P_2)}{m_1m_2}-\frac{473}{48}\frac{P_1^2}{m_1^2}+\frac{1}{64}\pi^2\frac{P_1^2}{m_1^2}+\frac{143}{16}\frac{(P_1P_2)}{m_1m_2}\\
&\qquad\left.-\frac{1}{64}\pi^2\frac{(P_1P_2)}{m_1m_2}\right)\\
&\quad+\frac{G^3m_1^3m_2}{R_{12}^3}\left(\frac{5}{4}\frac{(N_{12}P_1)^2}{m_1^2}+\frac{21}{8}\frac{(N_{12}P_1)(N_{12}P_2)}{m_1m_2}-\frac{425}{48}\frac{P_1^2}{m_1^2}+\frac{77}{8}\frac{(P_1P_2)}{m_1m_2}-\frac{25P_2^2}{8m_2^2}\right)\\
&\quad+\frac{G^2m_1^2m_2}{R_{12}^2}\left(\frac{5}{12}\frac{(N_{12}P_1)^4}{m_1^4}-\frac{3}{2}\frac{(N_{12}P_1)^3(N_{12}P_2)}{m_1^3m_2}+\frac{10}{3}\frac{(N_{12}P_1)^2(N_{12}P_2)^2}{m_1^2m_2^2}\right.\\
&\qquad+\frac{17}{16}\frac{(N_{12}P_1)^2P_1^2}{m_1^4}-\frac{15}{8}\frac{(N_{12}P_1)(N_{12}P_2)P_1^2}{m_1^3m_2}-\frac{55}{12}\frac{(N_{12}P_2)^2P_1^2}{m_1^2m_2^2}\\
&\qquad+\frac{P_1^4}{16m_1^4}-\frac{11}{8}\frac{(N_{12}P_1)^2(P_1P_2)}{m_1^3m_2}+\frac{125}{12}\frac{(N_{12}P_1)(N_{12}P_2)(P_1P_2)}{m_1^2m_2^2}\\
&\qquad-\frac{115}{16}\frac{P_1^2(P_1P_2)}{m_1^3m_2}+\frac{25}{48}\frac{(P_1P_2)^2}{m_1^2m_2^2}-\frac{193}{48}\frac{(N_{12}P_1)^2P_2^2}{m_1^2m_2^2}+\frac{371}{48}\frac{P_1^2P_2^2}{m_1^2m_2^2}\\
&\qquad\left.-\frac{27}{16}\frac{P_2^4}{m_2^4}\right)\\
&\quad+\frac{Gm_1m_2}{R_{12}}\left(\frac{5}{32}\frac{(N_{12}P_1)^3(N_{12}P_2)^3}{m_1^3m_2^3}+\frac{3}{16}\frac{(N_{12}P_1)^2(N_{12}P_2)^2P_1^2}{m_1^4m_2^2}\right.\\
&\qquad-\frac{9}{16}\frac{(N_{12}P_1)(N_{12}P_2)^3P_1^2}{m_1^3m_2^3}-\frac{5}{16}\frac{(N_{12}P_2)^2P_1^4}{m_1^4m_2^2}-\frac{7}{16}\frac{P_1^6}{m_1^6}\\
&\qquad+\frac{15}{32}\frac{(N_{12}P_1)^2(N_{12}P_2)^2(P_1P_2)}{m_1^3m_2^3}+\frac{3}{4}\frac{(N_{12}P_1)(N_{12}P_2)P_1^2(P_1P_2)}{m_1^4m_2^2}\\
&\qquad+\frac{1}{16}\frac{(N_{12}P_2)^2P_1^2(P_1P_2)}{m_1^3m_2^3}-\frac{5}{16}\frac{(N_{12}P_1)(N_{12}P_2)(P_1P_2)^2}{m_1^3m_2^3}\\
&\qquad+\frac{1}{8}\frac{P_1^2(P_1P_2)^2}{m_1^4m_2^2}-\frac{1}{16}\frac{(P_1P_2)^3}{m_1^3m_2^3}-\frac{5}{16}\frac{(N_{12}P_1)^2P_1^2P_2^2}{m_1^4m_2^2}
\end{aligned}
$$

$$+\frac{7}{32}\frac{(N_{12}P_1)(N_{12}P_2)P_1^2P_2^2}{m_1^3m_2^3}+\frac{1}{2}\frac{P_1^4P_2^2}{m_1^4m_2^2}+\frac{1}{32}\frac{P_1^2(P_1P_2)P_2^2}{m_1^3m_2^3}\Bigg)\Bigg\}$$

$$+1\leftrightarrow 2+\mathcal{O}\left(\frac{1}{c^7}\right). \tag{179}$$

Arguably, the results given by the ADM-Hamiltonian formalism (for the problem at hand) look simpler than their harmonic-coordinate counterparts. Indeed, the ADM Lagrangian is ordinary – no accelerations – and there are no logarithms nor associated gauge constants r'_1 and r'_2. But of course, one is free to describe the binary motion in whatever coordinates one likes, and the two formalisms, harmonic (174) and ADM (178, 179), describe rigorously the same physics. On the other hand, the higher complexity of the harmonic-coordinates Lagrangian (174) enables one to perform more tests of the computations, notably by inquiring about the future of the constants r'_1 and r'_2, that we know *must* disappear from physical quantities such as the center-of-mass energy and the total gravitational-wave flux.

9.3 Equations of motion in the center-of-mass frame

In this section we translate the origin of coordinates to the binary's center-of-mass by imposing that the binary's dipole $\mathrm{I}_i=0$ (notation of Part A). Actually the dipole moment is computed as the center-of-mass conserved integral associated with the boost symmetry of the 3PN equations of motion and Lagrangian [103, 43]. This condition results in the (3PN-accurate, say) relationship between the individual positions in the center-of-mass frame y_1^i and y_2^i, and the relative position $x^i\equiv y_1^i-y_2^i$ and velocity $v^i\equiv v_1^i-v_2^i=dx^i/dt$ (formerly denoted y_{12}^i and v_{12}^i). We shall also use the orbital separation $r\equiv|\mathbf{x}|$, together with $\mathbf{n}=\mathbf{x}/r$ and $\dot{r}\equiv\mathbf{n}\cdot\mathbf{v}$. Mass parameters are the total mass $m=m_1+m_2$ ($m\equiv\mathrm{M}$ in the notation of Part A), the mass difference $\delta m=m_1-m_2$, the reduced mass $\mu=m_1m_2/m$, and the very useful symmetric mass ratio

$$\nu\equiv\frac{\mu}{m}\equiv\frac{m_1m_2}{(m_1+m_2)^2}. \tag{180}$$

The usefulness of this ratio lies in its interesting range of variation: $0<\nu\le 1/4$, with $\nu=1/4$ in the case of equal masses, and $\nu\to 0$ in the "test-mass" limit for one of the bodies.

The 3PN and even 3.5PN center-of-mass equations of motion are obtained by replacing in the general-frame 3.5PN equations of motion (168) the positions and velocities by their center-of-mass expressions, applying as usual the order-reduction of all accelerations where necessary. We write the relative acceleration in the center-of-mass frame in the form

$$\frac{dv^i}{dt}=-\frac{Gm}{r^2}\left[(1+\mathcal{A})\,n^i+\mathcal{B}\,v^i\right]+\mathcal{O}\left(\frac{1}{c^8}\right), \tag{181}$$

and find [43] that the coefficients $\mathcal{A}$ and $\mathcal{B}$ are

$$\begin{aligned}\mathcal{A}=&\frac{1}{c^2}\left\{-\frac{3\dot{r}^2\nu}{2}+v^2+3\nu v^2-\frac{Gm}{r}(4+2\nu)\right\}\\&+\frac{1}{c^4}\left\{\frac{15\dot{r}^4\nu}{8}-\frac{45\dot{r}^4\nu^2}{8}-\frac{9\dot{r}^2\nu v^2}{2}+6\dot{r}^2\nu^2v^2+3\nu v^4-4\nu^2v^4\right.\\&\qquad\left.+\frac{Gm}{r}\left(-2\dot{r}^2-25\dot{r}^2\nu-2\dot{r}^2\nu^2-\frac{13\nu v^2}{2}+2\nu^2v^2\right)+\frac{G^2m^2}{r^2}\left(9+\frac{87\nu}{4}\right)\right\}\\&+\frac{1}{c^5}\left\{-\frac{24\dot{r}\nu v^2}{5}\frac{Gm}{r}-\frac{136\dot{r}\nu}{15}\frac{G^2m^2}{r^2}\right\}\end{aligned}$$

$$
\begin{aligned}
&+\frac{1}{c^6}\left\{-\frac{35\dot{r}^6\nu}{16}+\frac{175\dot{r}^6\nu^2}{16}-\frac{175\dot{r}^6\nu^3}{16}+\frac{15\dot{r}^4\nu v^2}{2}-\frac{135\dot{r}^4\nu^2v^2}{4}+\frac{255\dot{r}^4\nu^3v^2}{8}\right.\\
&\qquad-\frac{15\dot{r}^2\nu v^4}{2}+\frac{237\dot{r}^2\nu^2v^4}{8}-\frac{45\dot{r}^2\nu^3v^4}{2}+\frac{11\nu v^6}{4}-\frac{49\nu^2v^6}{4}+13\nu^3v^6\\
&\qquad+\frac{Gm}{r}\left(79\dot{r}^4\nu-\frac{69\dot{r}^4\nu^2}{2}-30\dot{r}^4\nu^3-121\dot{r}^2\nu v^2+16\dot{r}^2\nu^2v^2+20\dot{r}^2\nu^3v^2+\frac{75\nu v^4}{4}\right.\\
&\qquad\qquad\left.+8\nu^2v^4-10\nu^3v^4\right)\\
&\qquad+\frac{G^2m^2}{r^2}\left(\dot{r}^2+\frac{32573\dot{r}^2\nu}{168}+\frac{11\dot{r}^2\nu^2}{8}-7\dot{r}^2\nu^3+\frac{615\dot{r}^2\nu\pi^2}{64}-\frac{26987\nu v^2}{840}+\nu^3v^2\right.\\
&\qquad\qquad\left.-\frac{123\nu\pi^2v^2}{64}-110\dot{r}^2\nu\ln\left(\frac{r}{r'_0}\right)+22\nu v^2\ln\left(\frac{r}{r'_0}\right)\right)\\
&\qquad\left.+\frac{G^3m^3}{r^3}\left(-16-\frac{437\nu}{4}-\frac{71\nu^2}{2}+\frac{41\nu\pi^2}{16}\right)\right\}\\
&+\frac{1}{c^7}\left\{\frac{Gm}{r}\left(\frac{366}{35}\nu v^4+12\nu^2v^4-114v^2\nu\dot{r}^2-12\nu^2v^2\dot{r}^2+112\nu\dot{r}^4\right)\right.\\
&\qquad+\frac{G^2m^2}{r^2}\left(\frac{692}{35}\nu v^2-\frac{724}{15}v^2\nu^2+\frac{294}{5}\nu\dot{r}^2+\frac{376}{5}\nu^2\dot{r}^2\right)\\
&\qquad\left.+\frac{G^3m^3}{r^3}\left(\frac{3956}{35}\nu+\frac{184}{5}\nu^2\right)\right\},
\end{aligned}
\tag{182}
$$

$$
\begin{aligned}
\mathcal{B}=&\frac{1}{c^2}\left\{-4\dot{r}+2\dot{r}\nu\right\}\\
&+\frac{1}{c^4}\left\{\frac{9\dot{r}^3\nu}{2}+3\dot{r}^3\nu^2-\frac{15\dot{r}\nu v^2}{2}-2\dot{r}\nu^2v^2+\frac{Gm}{r}\left(2\dot{r}+\frac{41\dot{r}\nu}{2}+4\dot{r}\nu^2\right)\right\}\\
&+\frac{1}{c^5}\left\{\frac{8\nu v^2}{5}\frac{Gm}{r}+\frac{24\nu}{5}\frac{G^2m^2}{r^2}\right\}\\
&+\frac{1}{c^6}\left\{-\frac{45\dot{r}^5\nu}{8}+15\dot{r}^5\nu^2+\frac{15\dot{r}^5\nu^3}{4}+12\dot{r}^3\nu v^2-\frac{111\dot{r}^3\nu^2v^2}{4}-12\dot{r}^3\nu^3v^2-\frac{65\dot{r}\nu v^4}{8}\right.\\
&\qquad+19\dot{r}\nu^2v^4+6\dot{r}\nu^3v^4\\
&\qquad+\frac{Gm}{r}\left(\frac{329\dot{r}^3\nu}{6}+\frac{59\dot{r}^3\nu^2}{2}+18\dot{r}^3\nu^3-15\dot{r}\nu v^2-27\dot{r}\nu^2v^2-10\dot{r}\nu^3v^2\right)\\
&\qquad\left.+\frac{G^2m^2}{r^2}\left(-4\dot{r}-\frac{18169\dot{r}\nu}{840}+25\dot{r}\nu^2+8\dot{r}\nu^3-\frac{123\dot{r}\nu\pi^2}{32}+44\dot{r}\nu\ln\left(\frac{r}{r'_0}\right)\right)\right\}\\
&+\frac{1}{c^7}\left\{\frac{Gm}{r}\left(-\frac{626}{35}\nu v^4-\frac{12}{5}\nu^2v^4+\frac{678}{5}\nu v^2\dot{r}^2+\frac{12}{5}\nu^2v^2\dot{r}^2-120\nu\dot{r}^4\right)\right.\\
&\qquad+\frac{G^2m^2}{r^2}\left(\frac{164}{21}\nu v^2+\frac{148}{5}\nu^2v^2-\frac{82}{3}\nu\dot{r}^2-\frac{848}{15}\nu^2\dot{r}^2\right)\\
&\qquad\left.+\frac{G^3m^3}{r^3}\left(-\frac{1060}{21}\nu-\frac{104}{5}\nu^2\right)\right\}.
\end{aligned}
\tag{183}
$$

Up to the 2.5PN order the result agrees with the calculation of [155]. The 3.5PN term is issued from Refs. [136, 137, 138, 174, 148, 164]. At the 3PN order we have some gauge-dependent logarithms containing a constant r'_0 which is the "logarithmic barycenter" of the two constants r'_1 and r'_2:

$$\ln r'_0 = X_1 \ln r'_1 + X_2 \ln r'_2. \tag{184}$$

The logarithms in Equations (182, 183), together with the constant r'_0 therein, can be removed by applying the gauge transformation (169), while still staying within the class of harmonic coordinates. The resulting modification of the equations of motion will affect only the coefficients of the 3PN order in Equations (182, 183), let us denote them by $\mathcal{A}_{3\mathrm{PN}}$ and $\mathcal{B}_{3\mathrm{PN}}$. The new values of these coefficients, say $\mathcal{A}'_{3\mathrm{PN}}$ and $\mathcal{B}'_{3\mathrm{PN}}$, obtained after removal of the logarithms by the latter harmonic gauge transformation, are then [161]

$$\begin{aligned}
\mathcal{A}'_{3\mathrm{PN}} = \frac{1}{c^6}\Bigg\{ & -\frac{35\dot{r}^6\nu}{16} + \frac{175\dot{r}^6\nu^2}{16} - \frac{175\dot{r}^6\nu^3}{16} + \frac{15\dot{r}^4\nu v^2}{2} - \frac{135\dot{r}^4\nu^2 v^2}{4} + \frac{255\dot{r}^4\nu^3 v^2}{8} - \frac{15\dot{r}^2\nu v^4}{2} \\
& + \frac{237\dot{r}^2\nu^2 v^4}{8} - \frac{45\dot{r}^2\nu^3 v^4}{2} + \frac{11\nu v^6}{4} - \frac{49\nu^2 v^6}{4} + 13\nu^3 v^6 \\
& + \frac{Gm}{r}\left(79\dot{r}^4\nu - \frac{69\dot{r}^4\nu^2}{2} - 30\dot{r}^4\nu^3 - 121\dot{r}^2\nu v^2 + 16\dot{r}^2\nu^2 v^2 + 20\dot{r}^2\nu^3 v^2 + \frac{75\nu v^4}{4} \right.\\
& \qquad \left. + 8\nu^2 v^4 - 10\nu^3 v^4\right) \\
& + \frac{G^2m^2}{r^2}\left(\dot{r}^2 + \frac{22717\dot{r}^2\nu}{168} + \frac{11\dot{r}^2\nu^2}{8} - 7\dot{r}^2\nu^3 + \frac{615\dot{r}^2\nu\pi^2}{64} - \frac{20827\nu v^2}{840} + \nu^3 v^2 \right.\\
& \qquad \left. - \frac{123\nu\pi^2 v^2}{64}\right) \\
& + \frac{G^3m^3}{r^3}\left(-16 - \frac{1399\nu}{12} - \frac{71\nu^2}{2} + \frac{41\nu\pi^2}{16}\right)\Bigg\},
\end{aligned} \tag{185}$$

$$\begin{aligned}
\mathcal{B}'_{3\mathrm{PN}} = \frac{1}{c^6}\Bigg\{ & -\frac{45\dot{r}^5\nu}{8} + 15\dot{r}^5\nu^2 + \frac{15\dot{r}^5\nu^3}{4} + 12\dot{r}^3\nu v^2 - \frac{111\dot{r}^3\nu^2 v^2}{4} - 12\dot{r}^3\nu^3 v^2 - \frac{65\dot{r}\nu v^4}{8} \\
& + 19\dot{r}\nu^2 v^4 + 6\dot{r}\nu^3 v^4 \\
& + \frac{Gm}{r}\left(\frac{329\dot{r}^3\nu}{6} + \frac{59\dot{r}^3\nu^2}{2} + 18\dot{r}^3\nu^3 - 15\dot{r}\nu v^2 - 27\dot{r}\nu^2 v^2 - 10\dot{r}\nu^3 v^2\right) \\
& + \frac{G^2m^2}{r^2}\left(-4\dot{r} - \frac{5849\dot{r}\nu}{840} + 25\dot{r}\nu^2 + 8\dot{r}\nu^3 - \frac{123\dot{r}\nu\pi^2}{32}\right)\Bigg\}.
\end{aligned} \tag{186}$$

These gauge-transformed coefficients are useful because they do not yield the usual complications associated with logarithms. However, they must be handled with care in applications such as [161], since one must ensure that all other quantities in the problem (energy, angular momentum, gravitational-wave fluxes, etc.) are defined in the same specific harmonic gauge avoiding logarithms. In the following we shall no longer use the coordinate system leading to Equations (185, 186). Therefore all expressions we shall derive below, notably all those concerning the radiation field, are valid in the "standard" harmonic coordinate system in which the equations of motion are given by Equation (168) or (182, 183).

9.4 Equations of motion and energy for circular orbits

Most inspiralling compact binaries will have been circularized by the time they become visible by the detectors LIGO and VIRGO. In the case of orbits that are circular – apart from the gradual 2.5PN radiation-reaction inspiral – the complicated equations of motion simplify drastically, since we have $\dot{r} = (nv) = \mathcal{O}(1/c^5)$, and the remainder can always be neglected at the 3PN level. In the case of circular orbits, up to the 2.5PN order, the relation between center-of-mass variables and the relative ones reads [16][33]

$$\begin{aligned} m\, y_1^i &= x^i \left[m_2 + 3\gamma^2 \nu \delta m\right] - \frac{4}{5}\frac{G^2 m^2 \nu \delta m}{r c^5} v^i + \mathcal{O}\left(\frac{1}{c^6}\right), \\ m\, y_2^i &= x^i \left[-m_1 + 3\gamma^2 \nu \delta m\right] - \frac{4}{5}\frac{G^2 m^2 \nu \delta m}{r c^5} v^i + \mathcal{O}\left(\frac{1}{c^6}\right). \end{aligned} \tag{187}$$

To display conveniently the successive post-Newtonian corrections, we employ the post-Newtonian parameter

$$\gamma \equiv \frac{Gm}{rc^2} = \mathcal{O}\left(\frac{1}{c^2}\right). \tag{188}$$

Notice that there are no corrections of order 1PN in Equations (187) for circular orbits; the dominant term is of order 2PN, i.e. proportional to $\gamma^2 = \mathcal{O}(1/c^4)$.

The relative acceleration $a^i \equiv a_1^i - a_2^i$ of two bodies moving on a circular orbit at the 3PN order is then given by

$$a^i = -\omega^2 x^i - \frac{32}{5}\frac{G^3 m^3 \nu}{c^5 r^4} v^i + \mathcal{O}\left(\frac{1}{c^7}\right), \tag{189}$$

where $x^i \equiv y_1^i - y_2^i$ is the relative separation (in harmonic coordinates) and ω denotes the angular frequency of the circular motion. The second term in Equation (189), opposite to the velocity $v^i \equiv v_1^i - v_2^i$, is the 2.5PN radiation reaction force (we neglect here its 3.5PN extension), which comes from the reduction of the coefficient of $1/c^5$ in Equations (182, 183). The main content of the 3PN equations (189) is the relation between the frequency ω and the orbital separation r, that we find to be given by the generalized version of Kepler's third law [37, 38]:

$$\begin{aligned} \omega^2 = \frac{Gm}{r^3} \Bigg\{ & 1 + (-3+\nu)\gamma + \left(6 + \frac{41}{4}\nu + \nu^2\right)\gamma^2 \\ & + \left(-10 + \left[-\frac{75707}{840} + \frac{41}{64}\pi^2 + 22 \ln\left(\frac{r}{r'_0}\right)\right]\nu + \frac{19}{2}\nu^2 + \nu^3\right)\gamma^3 \Bigg\} \\ & + \mathcal{O}\left(\frac{1}{c^8}\right). \end{aligned} \tag{190}$$

The length scale r'_0 is given in terms of the two gauge-constants r'_1 and r'_2 by Equation (184). As for the energy, it is immediately obtained from the circular-orbit reduction of the general result (170). We have

$$\begin{aligned} E = -\frac{\mu c^2 \gamma}{2} \Bigg\{ & 1 + \left(-\frac{7}{4} + \frac{1}{4}\nu\right)\gamma + \left(-\frac{7}{8} + \frac{49}{8}\nu + \frac{1}{8}\nu^2\right)\gamma^2 \\ & + \left(-\frac{235}{64} + \left[\frac{46031}{2240} - \frac{123}{64}\pi^2 + \frac{22}{3}\ln\left(\frac{r}{r'_0}\right)\right]\nu + \frac{27}{32}\nu^2 + \frac{5}{64}\nu^3\right)\gamma^3 \Bigg\} \\ & + \mathcal{O}\left(\frac{1}{c^8}\right). \end{aligned} \tag{191}$$

This expression is that of a physical observable E; however, it depends on the choice of a coordinate system, as it involves the post-Newtonian parameter γ defined from the harmonic-coordinate

[33] Actually, in the present computation we do not need the radiation-reaction 2.5PN term in these relations; we give it only for completeness.

separation r_{12}. But the *numerical* value of E should not depend on the choice of a coordinate system, so E must admit a frame-invariant expression, the same in all coordinate systems. To find it we re-express E with the help of a frequency-related parameter x instead of the post-Newtonian parameter γ. Posing

$$x \equiv \left(\frac{G\,m\,\omega}{c^3}\right)^{2/3} = \mathcal{O}\left(\frac{1}{c^2}\right), \tag{192}$$

we readily obtain from Equation (190) the expression of γ in terms of x at 3PN order,

$$\begin{aligned}\gamma = x\biggl\{1 &+ \left(1-\frac{\nu}{3}\right)x + \left(1-\frac{65}{12}\nu\right)x^2 \\ &+ \left(1+\left[-\frac{2203}{2520}-\frac{41}{192}\pi^2-\frac{22}{3}\ln\left(\frac{r}{r'_0}\right)\right]\nu + \frac{229}{36}\nu^2+\frac{1}{81}\nu^3\right)x^3 \\ &+\mathcal{O}\left(\frac{1}{c^8}\right)\biggr\},\end{aligned} \tag{193}$$

that we substitute back into Equation (191), making all appropriate post-Newtonian re-expansions. As a result, we gladly discover that the logarithms together with their associated gauge constant r'_0 have cancelled out. Therefore, our result is

$$\begin{aligned}E = -\frac{\mu c^2 x}{2}\biggl\{1 &+ \left(-\frac{3}{4}-\frac{1}{12}\nu\right)x + \left(-\frac{27}{8}+\frac{19}{8}\nu-\frac{1}{24}\nu^2\right)x^2 \\ &+\left(-\frac{675}{64}+\left[\frac{34445}{576}-\frac{205}{96}\pi^2\right]\nu-\frac{155}{96}\nu^2-\frac{35}{5184}\nu^3\right)x^3\biggr\} \\ &+\mathcal{O}\left(\frac{1}{c^8}\right).\end{aligned} \tag{194}$$

For circular orbits one can check that there are no terms of order $x^{7/2}$ in Equation (194), so our result for E is actually valid up to the 3.5PN order.

9.5 The innermost circular orbit (ICO)

Having in hand the circular-orbit energy, we define the innermost circular orbit (ICO) as the minimum, when it exists, of the energy function $E(x)$. Notice that we do not define the ICO as a point of dynamical general-relativistic unstability. Hence, we prefer to call this point the ICO rather than, strictly speaking, an innermost stable circular orbit or ISCO. A study of the dynamical stability of circular binary orbits in the post-Newtonian approximation of general relativity can be found in Ref. [43].

The previous definition of the ICO is motivated by our comparison with the results of numerical relativity. Indeed we shall confront the prediction of the standard (Taylor-based) post-Newtonian approach with a recent result of numerical relativity by Gourgoulhon, Grandclément, and Bonazzola [123, 126]. These authors computed numerically the energy of binary black holes under the assumptions of conformal flatness for the spatial metric and of exactly circular orbits. The latter restriction is implemented by requiring the existence of an “helical” Killing vector, which is time-like inside the light cylinder associated with the circular motion, and space-like outside. In the numerical approach [123, 126] there are no gravitational waves, the field is periodic in time, and the gravitational potentials tend to zero at spatial infinity within a restricted model equivalent to solving five out of the ten Einstein field equations (the so-called Isenberg–Wilson–Mathews approximation; see Ref. [114] for a discussion). Considering an evolutionary sequence of equilibrium configurations Refs. [123, 126] obtained numerically the circular-orbit energy $E(\omega)$ and looked for the ICO of binary black holes (see also Refs. [52, 124, 154] for related calculations of binary neutron and strange quark stars).

Since the numerical calculation [123, 126] has been performed in the case of *corotating* black holes, which are spinning with the orbital angular velocity ω, we must for the comparison include within our post-Newtonian formalism the effects of spins appropriate to two Kerr black holes rotating at the orbital rate. The total relativistic mass of the Kerr black hole is given by[34]

$$M^2 = M_{\rm irr}^2 + \frac{S^2}{4M_{\rm irr}^2}, \tag{195}$$

where S is the spin, related to the usual Kerr parameter by $S = M a$, and $M_{\rm irr}$ is the irreducible mass given by $M_{\rm irr} = \sqrt{A/(4\pi)}$ (A is the hole's surface area). The angular velocity of the corotating black hole is $\omega = \partial M/\partial S$ hence, from Equation (195),

$$\omega = \frac{S}{2M^3\left[1+\sqrt{1-\frac{S^2}{M^4}}\right]}. \tag{196}$$

Physically this angular velocity is the one of the outgoing photons that remain for ever at the location of the light-like horizon. Combining Equations (195, 196) we obtain M and S as functions of $M_{\rm irr}$ and ω,

$$\begin{aligned} M &= \frac{M_{\rm irr}}{\sqrt{1-4M_{\rm irr}^2\,\omega^2}}, \\ S &= \frac{4M_{\rm irr}^3\omega}{\sqrt{1-4M_{\rm irr}^2\,\omega^2}}. \end{aligned} \tag{197}$$

This is the right thing to do since ω is the basic variable describing each equilibrium configuration calculated numerically, and because the irreducible masses are the ones which are held constant along the numerical evolutionary sequences in Refs. [123, 126]. In the limit of slow rotation we get

$$S = I\,\omega + \mathcal{O}\left(\omega^3\right), \tag{198}$$

where $I = 4M_{\rm irr}^3$ is the moment of inertia of the black hole. Next the total mass-energy is

$$M = M_{\rm irr} + \frac{1}{2}I\,\omega^2 + \mathcal{O}\left(\omega^4\right), \tag{199}$$

which involves, as we see, the usual kinetic energy of the spin.

To take into account the spin effects our first task is to replace all the masses entering the energy function (194) by their equivalent expressions in terms of ω and the two irreducible masses. It is clear that the leading contribution is that of the spin kinetic energy given by Equation (199), and it comes from the replacement of the rest mass-energy $m\,c^2$ (where $m = M_1 + M_2$). From Equation (199) this effect is of order ω^2 in the case of corotating binaries, which means by comparison with Equation (194) that it is equivalent to an "orbital" effect at the 2PN order (i.e. $\propto x^2$). Higher-order corrections in Equation (199), which behave at least like ω^4, will correspond to the orbital 5PN order at least and are negligible for the present purpose. In addition there will be a subdominant contribution, of the order of $\omega^{8/3}$ equivalent to 3PN order, which comes from the replacement of the masses into the "Newtonian" part, proportional to $x \propto \omega^{2/3}$, of the energy E (see Equation (194)). With the 3PN accuracy we do not need to replace the masses that enter into the post-Newtonian corrections in E, so in these terms the masses can be considered to be the irreducible ones.

[34] In this section we pose $G = 1 = c$, and the two individual black hole masses are denoted M_1 and M_2.

Our second task is to include the specific relativistic effects due to the spins, namely the spin-orbit (SO) interaction and the spin-spin (SS) one. In the case of spins S_1 and S_2 aligned parallel to the orbital angular momentum (and right-handed with respect to the sense of motion) the SO energy reads

$$E_{\mathrm{SO}} = -\mu\,(m\omega)^{5/3}\left[\left(\frac{4}{3}\frac{M_1^2}{m^2}+\nu\right)\frac{S_1}{M_1^2}+\left(\frac{4}{3}\frac{M_2^2}{m^2}+\nu\right)\frac{S_2}{M_2^2}\right]. \tag{200}$$

Here we are employing the formula given by Kidder et al. [146, 144] (based on seminal works of Barker and O'Connell [7, 8]) who have computed the SO contribution and expressed it by means of the orbital frequency ω. The derivation of Equation (200) in Ref. [146, 144] takes into account the fact that the relation between the orbital separation r (in the harmonic coordinate system) and the frequency ω depends on the spins. We immediately infer from Equation (200) that in the case of corotating black holes the SO effect is equivalent to a 3PN orbital effect and thus must be retained with the present accuracy (with this approximation, the masses in Equation (200) are the irreducible ones). As for the SS interaction (still in the case of spins aligned with the orbital angular momentum) it is given by

$$E_{\mathrm{SS}} = \mu\,\nu\,(m\omega)^2\frac{S_1\,S_2}{M_1^2\,M_2^2}. \tag{201}$$

The SS effect can be neglected here because it is of order 5PN for corotating systems. Summing up all the spin contributions we find that the suplementary energy due to the corotating spins is [23]

$$\Delta E^{\mathrm{corot}} = m\,c^2\,x\left\{(2-6\nu)x^2+\left(-6\nu+13\nu^2\right)x^3+\mathcal{O}(x^4)\right\}, \tag{202}$$

where $x=(m\omega)^{2/3}$. The complete 3PN energy of the corotating binary is finally given by the sum of Equations (194) and (202), in which we must now understand all the masses as being the irreducible ones (we no longer indicate the superscript "irr"), which for the comparison with the numerical calculation must be assumed to stay constant when the binary evolves.

The Figure 1 (issued from Ref. [23]) presents our results for E_{ICO} in the case of irrotational and corotational binaries. Since $\Delta E^{\mathrm{corot}}$, given by Equation (202), is at least of order 2PN, the result for $1\mathrm{PN}^{\mathrm{corot}}$ is the same as for 1PN in the irrotational case; then, obviously, $2\mathrm{PN}^{\mathrm{corot}}$ takes into account only the leading 2PN corotation effect (i.e. the spin kinetic energy given by Equation (199)), while $3\mathrm{PN}^{\mathrm{corot}}$ involves also, in particular, the corotational SO coupling at the 3PN order. In addition we present in Figure 1 the numerical point obtained by numerical relativity under the assumptions of conformal flatness and of helical symmetry [123, 126]. As we can see the 3PN points, and even the 2PN ones, are rather close to the numerical value. The fact that the 2PN and 3PN values are so close to each other is a good sign of the convergence of the expansion; we shall further comment this point in Section 9.6. In fact one might say that the role of the 3PN approximation is merely to "confirm" the value already given by the 2PN one (but of course, had we not computed the 3PN term, we would not be able to trust very much the 2PN value). As expected, the best agreement we obtain is for the 3PN approximation and in the case of corotation, i.e. the point $3\mathrm{PN}^{\mathrm{corot}}$. However, the 1PN approximation is clearly not precise enough, but this is not surprising in the highly relativistic regime of the ICO.

In conclusion, we find that the location of the ICO as computed by numerical relativity, under the helical-symmetry and conformal-flatness approximations, is in good agreement with the post-Newtonian prediction. See also Ref. [88] for the results calculated within the effective-one-body approach method [60, 61] at the 3PN order, which are close to the ones reported in Figure 1. This agreement constitutes an appreciable improvement of the previous situation, because the earlier estimates of the ICO in post-Newtonian theory [145] and numerical relativity [180, 9] strongly disagreed with each other, and do not match with the present 3PN results. The numerical calculation of quasi-equilibrium configurations has been since then redone and refined by a number of groups,

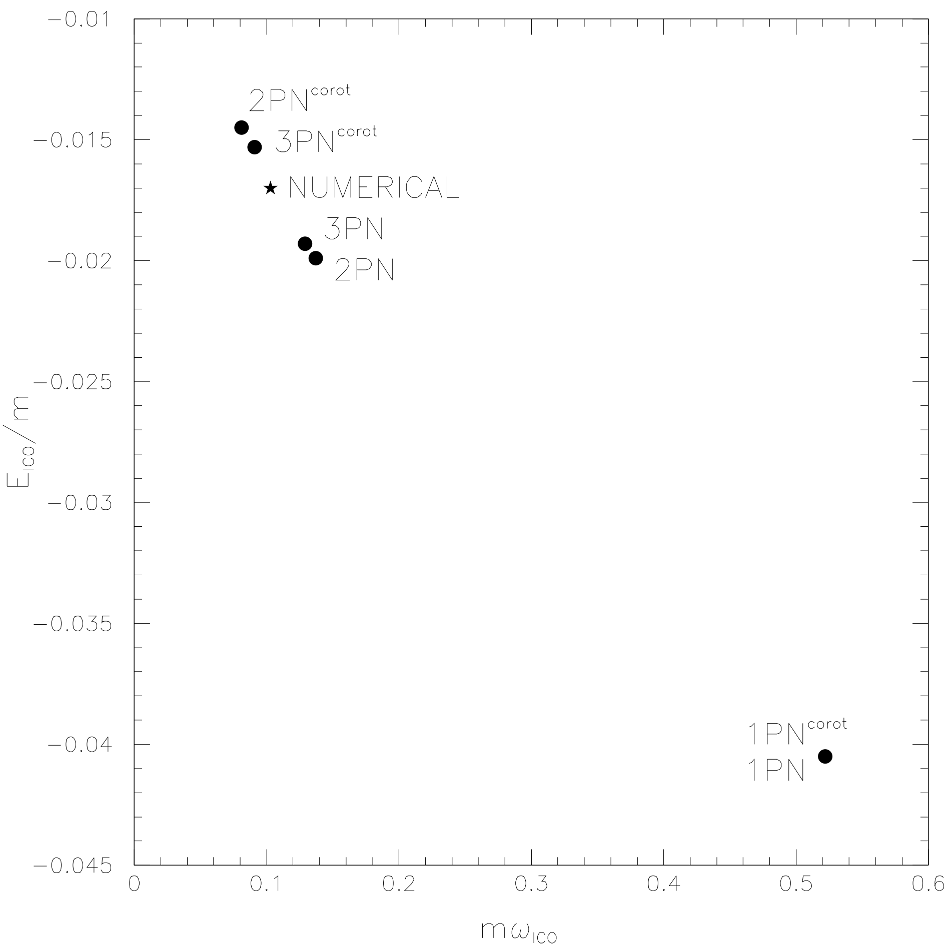

Figure 1: *Results for the binding energy* $E_{\rm ICO}$ *versus* $\omega_{\rm ICO}$ *in the equal-mass case* ($\nu = 1/4$)*. The asterisk marks the result calculated by numerical relativity. The points indicated by* 1PN, 2PN, *and* 3PN *are computed from the minimum of Equation (194), and correspond to irrotational binaries. The points denoted by* $1\mathrm{PN}^{\rm corot}$, $2\mathrm{PN}^{\rm corot}$, *and* $3\mathrm{PN}^{\rm corot}$ *come from the minimum of the sum of Equations (194) and (202), and describe corotational binaries.*

for both corotational and irrotational binaries (see in particular Ref. [74]). These works confirm the previous findings.

9.6 Accuracy of the post-Newtonian approximation

In this section we want to assess the validity of the post-Newtonian approximation, and, more precisely, to address, and to some extent to answer, the following questions: How accurate is the post-Newtonian expansion for describing the dynamics of binary black hole systems? Is the ICO of binary black holes, defined by the minimum of the energy function $E(\omega)$, accurately determined at the highest currently known post-Newtonian order? The latter question is pertinent because the ICO represents a point in the late stage of evolution of the binary which is very relativistic (orbital velocities of the order of 50% of the speed of light). How well does the 3PN approximation as compared with the prediction provided by numerical relativity (see Section 9.5)? What is the validity of the various post-Newtonian resummation techniques [92, 93, 60, 61] which aim at "boosting" the convergence of the standard post-Newtonian approximation?

The previous questions are interesting but difficult to settle down rigorously. Indeed the very essence of an approximation is to cope with our ignorance of the higher-order terms in some expansion, but the higher-order terms are precisely the ones which would be needed for a satisfying answer to these problems. So we shall be able to give only some educated guesses and/or plausible answers, that we cannot justify rigorously, but which seem very likely from the standard point of view on the post-Newtonian theory, in particular that the successive orders of approximation get smaller and smaller as they should (in average), with maybe only few accidents occuring at high orders where a particular approximation would be abnormally large with respect to the lower-order ones. Admittedly, in addition, our faith in the estimation we shall give regarding the accuracy of the 3PN order for instance, comes from the historical perspective, thanks to the many successes achieved in the past by the post-Newtonian approximation when confronting the theory and observations. It is indeed beyond question, from our past experience, that the post-Newtonian method does work.

Establishing the post-Newtonian expansion rigorously has been the subject of numerous mathematical oriented works, see, e.g., [187, 188, 189]. In the present section we shall simply look (much more modestly) at what can be said by inspection of the explicit post-Newtonian coefficients which have been computed so far. Basically, the point we would like to emphasize[35] is that the post-Newtonian approximation, in standard form (without using the resummation techniques advocated in Refs. [92, 60, 61]), is able to located the ICO of two black holes, in the case of *comparable masses* ($m_1 \simeq m_2$), with a very good accuracy. At first sight this statement is rather surprising, because the dynamics of two black holes at the point of the ICO is so relativistic. Indeed one sometimes hears about the "bad convergence", or the "fundamental breakdown", of the post-Newtonian series in the regime of the ICO. However our estimates do show that the 3PN approximation is good in this regime, for comparable masses, and we have already confirmed this by the remarkable agreement with the numerical calculations, as detailed in Section 9.5.

Let us center our discussion on the post-Newtonian expression of the circular-orbit energy (194), developed to the 3PN order, which is of the form

$$E(x) = -\frac{\mu\, c^2 x}{2} \left\{ 1 + a_1(\nu)\, x + a_2(\nu)\, x^2 + a_3(\nu)\, x^3 + \mathcal{O}(x^4) \right\}. \tag{203}$$

The first term, proportional to x, is the Newtonian term, and then we have many post-Newtonian

[35] We are following the discussion in Ref. [24]. Note that the arguments of this section are rather biased toward the author's own work [23, 24].

corrections, the coefficients of which are known up to 3PN order [139, 140, 95, 97, 37, 38, 103]:

$$\begin{aligned} a_1(\nu) &= -\frac{3}{4} - \frac{\nu}{12}, \\ a_2(\nu) &= -\frac{27}{8} + \frac{19}{8}\nu - \frac{\nu^2}{24}, \\ a_3(\nu) &= -\frac{675}{64} + \left[\frac{209323}{4032} - \frac{205}{96}\pi^2 - \frac{110}{9}\lambda\right]\nu - \frac{155}{96}\nu^2 - \frac{35}{5184}\nu^3. \end{aligned} \tag{204}$$

For the discussion it is helpful to keep the Hadamard regularization ambiguity parameter λ present in the 3PN coefficient $a_3(\nu)$. Recall from Section 8.2 that this parameter was introduced in Refs. [37, 38] and is equivalent to the parameter ω_{static} of Refs. [139, 140]. We already gave in Equation (133) the relation linking them,

$$\lambda = -\frac{3}{11}\omega_{\text{static}} - \frac{1987}{3080}. \tag{205}$$

Before its actual computation in general relativity, it has been argued in Ref. [94] that the numerical value of ω_{static} could be $\simeq -9$, because for such a value some different resummation techniques, when they are implemented at the 3PN order, give approximately the same result for the ICO. Even more, it was suggested [94] that ω_{static} might be precisely equal to ω^*_{static}, with

$$\omega^*_{\text{static}} = -\frac{47}{3} + \frac{41}{64}\pi^2 = -9.34\cdots. \tag{206}$$

However, as reviewed in Sections 8.2 and 8.3, the computations performed using dimensional regularization, within the ADM-Hamiltonian formalism [96] and harmonic-coordinate approaches [30], and the independent computation of Refs. [133, 132], have settled the value of this parameter in general relativity to be

$$\omega_{\text{static}} = 0 \qquad \Longleftrightarrow \qquad \lambda = -\frac{1987}{3080}. \tag{207}$$

We note that this result is quite different from ω^*_{static}, Equation (206). This already suggests that different resummation techniques, namely Padé approximants [92, 93, 94] and effective-one-body methods [60, 61, 94], which are designed to "accelerate" the convergence of the post-Newtonian series, do *not* in fact converge toward the same exact solution (or, at least, not as fast as expected).

In the limiting case $\nu \to 0$, the expression (203, 204) reduces to the 3PN approximation of the energy for a test particle in the Schwarzschild background,

$$E^{\text{Sch}}(x) = \mu\, c^2 \left[\frac{1-2x}{\sqrt{1-3x}} - 1\right]. \tag{208}$$

The minimum of that function or Schwarzschild ICO occurs at $x^{\text{Sch}}_{\text{ICO}} = 1/6$, and we have $E^{\text{Sch}}_{\text{ICO}} = \mu c^2\left(\sqrt{8/9} - 1\right)$. We know that the Schwarzschild ICO is also an innermost *stable* circular orbit or ISCO, i.e. it corresponds to a point of dynamical unstability. Another important feature of Equation (208) is the singularity at the value $x^{\text{Sch}}_{\text{light ring}} = 1/3$ which corresponds to the famous circular orbit of photons in the Schwarzschild metric ("light-ring" singularity). This orbit can also be viewed as the last *unstable* circular orbit. We can check that the post-Newtonian coefficients $a^{\text{Sch}}_n \equiv a_n(0)$ corresponding to Equation (208) are given by

$$a^{\text{Sch}}_n = -\frac{3^n(2n-1)!!(2n-1)}{2^n(n+1)!}. \tag{209}$$

They increase with n by roughly a factor 3 at each order. This is simply the consequence of the fact that the radius of convergence of the post-Newtonian series is given by the Schwarzschild light-ring singularity at the value 1/3. We may therefore recover the light-ring orbit by investigating the limit

$$\lim_{n\to+\infty} \frac{a_{n-1}^{\mathrm{Sch}}}{a_n^{\mathrm{Sch}}} = \frac{1}{3} = x_{\mathrm{light\,ring}}^{\mathrm{Sch}}. \tag{210}$$

Let us now discuss a few order-of-magnitude estimates. At the location of the ICO we have found (see Figure 1 in Section 9.5) that the frequency-related parameter x defined by Equation (192) is approximately of the order of $x \sim (0.1)^{2/3} \sim 20\%$ for equal masses. Therefore, we might *a priori* expect that the contribution of the 1PN approximation to the energy at the ICO should be of that order. For the present discussion we take the pessimistic view that the order of magnitude of an approximation represents also the order of magnitude of the higher-order terms which are neglected. We see that the 1PN approximation should yield a rather poor estimate of the "exact" result, but this is quite normal at this very relativistic point where the orbital velocity is $v/c \sim x^{1/2} \sim 50\%$. By the same argument we infer that the 2PN approximation should do much better, with fractional errors of the order of $x^2 \sim 5\%$, while 3PN will be even better, with the accuracy $x^3 \sim 1\%$.

Now the previous estimate makes sense only if the numerical values of the post-Newtonian coefficients in Equations (204) stay roughly of the order of one. If this is not the case, and if the coefficients increase dangerously with the post-Newtonian order n, one sees that the post-Newtonian approximation might in fact be very bad. It has often been emphasized in the litterature (see, e.g., Refs. [77, 183, 92]) that in the test-mass limit $\nu \to 0$ the post-Newtonian series converges slowly, so the post-Newtonian approximation is not very good in the regime of the ICO. Indeed we have seen that when $\nu = 0$ the radius of convergence of the series is 1/3 (not so far from $x_{\mathrm{ICO}}^{\mathrm{Sch}} = 1/6$), and that accordingly the post-Newtonian coefficients increase by a factor ~ 3 at each order. So it is perfectly correct to say that in the case of test particles in the Schwarzschild background the post-Newtonian approximation is to be carried out to a high order in order to locate the turning point of the ICO.

What happens when the two masses are comparable ($\nu = \frac{1}{4}$)? It is clear that the accuracy of the post-Newtonian approximation depends crucially on how rapidly the post-Newtonian coefficients increase with n. We have seen that in the case of the Schwarzschild metric the latter increase is in turn related to the existence of a light-ring orbit. For continuing the discussion we shall say that the relativistic interaction between two bodies of comparable masses is *"Schwarzschild-like"* if the post-Newtonian coefficients $a_n(\frac{1}{4})$ increase when $n \to +\infty$. If this is the case this signals the existence of something like a light-ring singularity which could be interpreted as the deformation, when the mass ratio ν is "turned on", of the Schwarzschild light-ring orbit. By analogy with Equation (210) we can estimate the location of this "pseudo-light-ring" orbit by

$$\frac{a_{n-1}(\nu)}{a_n(\nu)} \sim x_{\mathrm{light\,ring}}(\nu) \qquad \text{with } n = 3. \tag{211}$$

Here $n = 3$ is the highest known post-Newtonian order. If the two-body problem is "Schwarzschild-like" then the right-hand side of Equation (211) is small (say around 1/3), the post-Newtonian coefficients typically increase with n, and most likely it should be difficult to get a reliable estimate by post-Newtonian methods of the location of the ICO. So we ask: Is the gravitational interaction between two comparable masses Schwarzschild-like?

In Table 1 we present the values of the coefficients $a_n(\nu)$ in the test-mass limit $\nu = 0$ (see Equation (209) for their analytic expression), and in the equal-mass case $\nu = \frac{1}{4}$ when the ambiguity parameter takes the "uncorrect" value $\omega_{\mathrm{static}}^*$ defined by Equation (206), and the correct one $\omega_{\mathrm{static}} = 0$ predicted by general relativity. When $\nu = 0$ we clearly see the expected increase of the coefficients by roughly a factor 3 at each step. Now, when $\nu = \frac{1}{4}$ and $\omega_{\mathrm{static}} = \omega_{\mathrm{static}}^*$ we notice that the coefficients increase approximately in the same manner as in the test-mass case $\nu = 0$. This indicates that the gravitational interaction in the case of $\omega_{\mathrm{static}}^*$ looks like that in a one-body

		Newtonian	$a_1(\nu)$	$a_2(\nu)$	$a_3(\nu)$
$\nu = 0$		1	-0.75	-3.37	-10.55
$\nu = \frac{1}{4}$,	$\omega^*_{\text{static}} \simeq -9.34$	1	-0.77	-2.78	-8.75
$\nu = \frac{1}{4}$,	$\omega_{\text{static}} = 0$ (GR)	1	-0.77	-2.78	-0.97

Table 1: *Numerical values of the sequence of coefficients of the post-Newtonian series composing the energy function $E(x)$ as given by Equations (203, 204).*

problem. The associated pseudo-light-ring singularity is estimated using Equation (211) as

$$x_{\text{light ring}}\left(\frac{1}{4}, \omega^*_{\text{static}}\right) \sim 0.32. \tag{212}$$

The pseudo-light-ring orbit seems to be a very small deformation of the Schwarzschild light-ring orbit given by Equation (210). In this Schwarzschild-like situation, we should not expect the post-Newtonian series to be very accurate.

Now in the case $\nu = \frac{1}{4}$ but when the ambiguity parameter takes the correct value $\omega_{\text{static}} = 0$, we see that the 3PN coefficient $a_3(\frac{1}{4})$ is of the order of -1 instead of being ~ -10. This suggests, unless 3PN happens to be quite accidental, that the post-Newtonian coefficients in general relativity do not increase very much with n. This is an interesting finding because it indicates that the actual two-body interaction in general relativity is *not* Schwarzschild-like. There does not seem to exist something like a light-ring orbit which would be a deformation of the Schwarzschild one. Applying Equation (211) we obtain as an estimate of the "light ring",

$$x_{\text{light ring}}\left(\frac{1}{4}, \text{GR}\right) \sim 2.88. \tag{213}$$

It is clear that if we believe the correctness of this estimate we must conclude that there is in fact *no* notion of a light-ring orbit in the real two-body problem. Or, one might say (pictorially speaking) that the light-ring orbit gets hidden inside the horizon of the final black hole formed by coalescence. Furthermore, if we apply Equation (211) using the 2PN approximation $n = 2$ instead of the 3PN one $n = 3$, we get the value ~ 0.28 instead of Equation (213). So at the 2PN order the metric seems to admit a light ring, while at the 3PN order it apparently does not admit any. This erratic behaviour reinforces our idea that it is meaningless (with our present 3PN-based knowledge, and until fuller information is available) to assume the existence of a light-ring singularity when the masses are equal.

It is impossible of course to be thoroughly confident about the validity of the previous statement because we know only the coefficients up to 3PN order. Any tentative conclusion based on 3PN can be "falsified" when we obtain the next 4PN order. Nevertheless, we feel that the mere fact that $a_3(\frac{1}{4}) = -0.97$ in Table 1 is sufficient to motivate our conclusion that the gravitational field generated by two bodies is more complicated than the Schwarzschild space-time. This appraisal should look cogent to relativists and is in accordance with the author's respectfulness of the complexity of the Einstein field equations.

We want next to comment on a possible implication of our conclusion as regards the so-called post-Newtonian resummation techniques, i.e. Padé approximants [92, 93, 94], which aim at "boosting" the convergence of the post-Newtonian series in the pre-coalescence stage, and the effective-one-body (EOB) method [60, 61, 94], which attempts at describing the late stage of the coalescence of two black holes. These techniques are based on the idea that the gravitational two-body interaction is a "deformation" – with $\nu \leq \frac{1}{4}$ being the deformation parameter – of the Schwarzschild space-time. The Padé approximants are valuable tools for giving accurate representations of functions having some singularities. In the problem at hands they would be justified if the "exact" expression of the energy (whose 3PN expansion is given by Equations (203, 204)) would admit a singularity at some reasonable value of x (e.g., ≤ 0.5). In the Schwarzschild case, for which Equation (210) holds, the Padé series converges rapidly [92]: The Padé constructed only from the 2PN approximation of the energy – keeping only $a_1^{\rm Sch}$ and $a_2^{\rm Sch}$ – already coincide with the exact result given by Equation (208). On the other hand, the EOB method maps the post-Newtonian two-body dynamics (at the 2PN or 3PN orders) on the geodesic motion on some effective metric which happens to be a ν-deformation of the Schwarzschild space-time. In the EOB method the effective metric looks like Schwarzschild *by definition*, and we might of course expect the two-body interaction to own the main Schwarzschild-like features.

Our comment is that the validity of these post-Newtonian resummation techniques does not seem to be compatible with the value $\omega_{\rm static} = 0$, which suggests that the two-body interaction in general relativity is not Schwarzschild-like. This doubt is confirmed by the finding of Ref. [94] (already alluded to above) that in the case of the wrong ambiguity parameter $\omega^*_{\rm static} \simeq -9.34$ the Padé approximants and the EOB method at the 3PN order give the same result for the ICO. From the previous discussion we see that this agreement is to be expected because a deformed light-ring singularity seems to exist with that value $\omega^*_{\rm static}$. By contrast, in the case of general relativity, $\omega_{\rm static} = 0$, the Padé and EOB methods give quite different results (cf. the Figure 2 in [94]). Another confirmation comes from the light-ring singularity which is determined from the Padé approximants at the 2PN order (see Equation (3.22) in [92]) as

$$x_{\rm light\,ring}\left(\frac{1}{4}, \text{Padé}\right) \sim 0.44. \tag{214}$$

This value is rather close to Equation (212) but strongly disagrees with Equation (213). Our explanation is that the Padé series converges toward a theory having $\omega_{\rm static} \simeq \omega^*_{\rm static}$; such a theory is different from general relativity.

Finally we come to the good news that, if really the post-Newtonian coefficients when $\nu = \frac{1}{4}$ stay of the order of one (or minus one) as it seems to, this means that the *standard* post-Newtonian approach, based on the standard Taylor approximants, is probably very accurate. The post-Newtonian series is likely to "converge well", with a "convergence radius" of the order of one[36]. Hence the order-of-magnitude estimate we proposed at the beginning of this section is probably correct. In particular the 3PN order should be close to the "exact" solution for comparable masses even in the regime of the ICO.

[36] Actually, the post-Newtonian series could be only asymptotic (hence divergent), but nevertheless it should give excellent results provided that the series is truncated near some optimal order of approximation. In this discussion we assume that the 3PN order is not too far from that optimum.

10 Gravitational Waves from Compact Binaries

We pointed out that the 3PN equations of motion, Equations (189, 190), are merely Newtonian as regards the radiative aspects of the problem, because with that precision the radiation reaction force is at the lowest 2.5PN order. A solution would be to extend the precision of the equations of motion so as to include the full relative 3PN or 3.5PN precision into the radiation reaction force, but, needless to say, the equations of motion up to the 5.5PN or 6PN order are quite impossible to derive with the present technology. The much better alternative solution is to apply the wave-generation formalism described in Part A, and to determine by its means the work done by the radiation reaction force directly as a total energy flux at future null infinity. In this approach, we replace the knowledge of the higher-order radiation reaction force by the computation of the total flux $\mathcal{L}$, and we apply the energy balance equation as in the test of the $\dot{P}$ of the binary pulsar (see Equations (4, 5)):

$$\frac{dE}{dt} = -\mathcal{L}. \tag{215}$$

Therefore, the result (194) that we found for the 3.5PN binary's center-of-mass energy E constitutes only "half" of the solution of the problem. The second "half" consists of finding the rate of decrease dE/dt, which by the balance equation is nothing but finding the total gravitational-wave flux $\mathcal{L}$ at the 3.5PN order. Because the orbit of inspiralling binaries is circular, the balance equation for the energy is sufficient (no need of a balance equation for the angular momentum). This all sounds perfect, but it is important to realize that we shall use Equation (215) at the very high 3.5PN order, at which order there are no proofs (following from first principles in general relativity) that the equation is correct, despite its physically obvious character. Nevertheless, Equation (215) has been checked to be valid, both in the cases of point-particle binaries [136, 137] and extended weakly self-gravitating fluids [14, 18], at the 1PN order and even at 1.5PN (the 1.5PN approximation is especially important for this check because it contains the first wave tails).

Obtaining $\mathcal{L}$ can be divided into two equally important steps: (1) the computation of the *source* multipole moments I_L and J_L of the compact binary and (2) the control and determination of the tails and related non-linear effects occuring in the relation between the binary's source moments and the *radiative* ones U_L and V_L (cf. the general formalism of Part A).

10.1 The binary's multipole moments

The general expressions of the source multipole moments given by Theorem 6, Equations (85), are first to be worked out explicitly for general fluid systems at the 3PN order. For this computation one uses the formula (91), and we insert the 3PN metric coefficients (in harmonic coordinates) expressed in Equations (115) by means of the retarded-type elementary potentials (117, 118, 119). Then we specialize each of the (quite numerous) terms to the case of point-particle binaries by inserting, for the matter stress-energy tensor $T^{\alpha\beta}$, the standard expression made out of Dirac delta-functions. The infinite self-field of point-particles is removed by means of the Hadamard regularization; and dimensional regularization is used to compute the few ambiguity parameters (see Section 8). This computation has been performed in [49] at the 1PN order, and in [33] at the 2PN order; we report below the most accurate 3PN results obtained in Refs. [45, 44, 31, 32].

The difficult part of the analysis is to find the closed-form expressions, fully explicit in terms of the particle's positions and velocities, of many non-linear integrals. We refer to [45] for full details; nevertheless, let us give a few examples of the type of technical formulas that are employed in this calculation. Typically we have to compute some integrals like

$$\overset{(n,p)}{Y_L}(\mathbf{y}_1, \mathbf{y}_2) = -\frac{1}{2\pi}\mathcal{FP}\int d^3\mathbf{x}\,\hat{x}_L\, r_1^n r_2^p, \tag{216}$$

where $r_1 = |\mathbf{x} - \mathbf{y}_1|$ and $r_2 = |\mathbf{x} - \mathbf{y}_2|$. When $n > -3$ and $p > -3$, this integral is perfectly well-defined (recall that the finite part $\mathcal{FP}$ deals with the bound at infinity). When $n \leq -3$ or $p \leq -3$, our basic ansatz is that we apply the definition of the Hadamard *partie finie* provided by Equation (124). Two examples of closed-form formulas that we get, which do not necessitate the Hadamard *partie finie*, are (quadrupole case $l = 2$)

$$\begin{aligned}
\overset{(-1,-1)}{Y}_{ij} &= \frac{r_{12}}{3}\left[y_1^{\langle ij\rangle} + y_1^{\langle i}y_2^{j\rangle} + y_2^{\langle ij\rangle}\right], \\
\overset{(-2,-1)}{Y}_{ij} &= y_1^{\langle ij\rangle}\left[\frac{16}{15}\ln\left(\frac{r_{12}}{r_0}\right) - \frac{188}{225}\right] + y_1^{\langle i}y_2^{j\rangle}\left[\frac{8}{15}\ln\left(\frac{r_{12}}{r_0}\right) - \frac{4}{225}\right] + y_2^{\langle ij\rangle}\left[\frac{2}{5}\ln\left(\frac{r_{12}}{r_0}\right) - \frac{2}{25}\right].
\end{aligned} \tag{217}$$

We denote for example $y_1^{\langle ij\rangle} = y_1^{\langle i}y_1^{j\rangle}$ (and $r_{12} = r|\mathbf{y}_1 - \mathbf{y}_2|$); the constant r_0 is the one pertaining to the finite-part process (see Equation (36)). One example where the integral diverges at the location of the particle 1 is

$$\overset{(-3,0)}{Y}_{ij} = \left[2\ln\left(\frac{s_1}{r_0}\right) + \frac{16}{15}\right] y_1^{\langle ij\rangle}, \tag{218}$$

where s_1 is the Hadamard-regularization constant introduced in Equation (124)[37].

The crucial input of the computation of the flux at the 3PN order is the mass quadrupole moment I_{ij}, since this moment necessitates the full 3PN precision. The result of Ref. [45] for this moment (in the case of circular orbits) is

$$\mathrm{I}_{ij} = \mu\left(A\, x_{\langle ij\rangle} + B\,\frac{r_{12}^3}{Gm} v_{\langle ij\rangle} + \frac{48}{7}\,\frac{G^2m^2\nu}{c^5 r_{12}} x_{\langle i}v_{j\rangle}\right) + \mathcal{O}\left(\frac{1}{c^7}\right), \tag{219}$$

where we pose $x_i = x^i \equiv y_{12}^i$ and $v_i = v^i \equiv v_{12}^i$. The third term is the 2.5PN radiation-reaction term, which does not contribute to the energy flux for circular orbits. The two important coefficients are A and B, whose expressions through 3PN order are

$$\begin{aligned}
A &= 1 + \gamma\left(-\frac{1}{42} - \frac{13}{14}\nu\right) + \gamma^2\left(-\frac{461}{1512} - \frac{18395}{1512}\nu - \frac{241}{1512}\nu^2\right) \\
&\quad + \gamma^3\left\{\frac{395899}{13200} - \frac{428}{105}\ln\left(\frac{r_{12}}{r_0}\right) + \left[\frac{139675}{33264} - \frac{44}{3}\xi - \frac{88}{3}\kappa - \frac{44}{3}\ln\left(\frac{r_{12}}{r'_0}\right)\right]\nu \right.\\
&\qquad \left. + \frac{162539}{16632}\nu^2 + \frac{2351}{33264}\nu^3\right\}, \\
B &= \gamma\left(\frac{11}{21} - \frac{11}{7}\nu\right) + \gamma^2\left(\frac{1607}{378} - \frac{1681}{378}\nu + \frac{229}{378}\nu^2\right) \\
&\quad + \gamma^3\left(-\frac{357761}{19800} + \frac{428}{105}\ln\left(\frac{r_{12}}{r_0}\right) + \left[-\frac{75091}{5544} + \frac{44}{3}\zeta\right]\nu + \frac{35759}{924}\nu^2 + \frac{457}{5544}\nu^3\right).
\end{aligned} \tag{220}$$

These expressions are valid in harmonic coordinates *via* the post-Newtonian parameter γ given by Equation (188). As we see, there are two types of logarithms in the moment: One type involves the length scale r'_0 related by Equation (184) to the two gauge constants r'_1 and r'_2 present in the 3PN

[37] When computing the gravitational-wave flux in Ref. [45] we preferred to call the Hadamard-regularization constants u_1 and u_2, in order to distinguish them from the constants s_1 and s_2 that were used in our previous computation of the equations of motion in Ref. [38]. Indeed these regularization constants need not neccessarily be the same when employed in different contexts.

equations of motion; the other type contains the different length scale r_0 coming from the general formalism of Part A – indeed, recall that there is a $\mathcal{FP}$ operator in front of the source multipole moments in Theorem 6. As we know, that r'_0 is pure gauge; it will disappear from our physical results at the end. On the other hand, we have remarked that the multipole expansion outside a general post-Newtonian source is actually free of r_0, since the r_0's present in the two terms of Equation (67) cancel out. We shall indeed find that the constants r_0 present in Equations (220) are compensated by similar constants coming from the non-linear wave "tails of tails". Finally, the constants ξ, κ, and ζ are the Hadamard-regularization ambiguity parameters which take the values (136).

Besides the 3PN mass quadrupole (219, 220), we need also the mass octupole moment I_{ijk} and current quadrupole moment J_{ij}, both of them at the 2PN order; these are given by [45]

$$\begin{aligned}
\mathrm{I}_{ijk} &= \mu\frac{\delta m}{m}\hat{x}_{ijk}\left[-1+\gamma\nu+\gamma^2\left(\frac{139}{330}+\frac{11923}{660}\nu+\frac{29}{110}\nu^2\right)\right] \\
&\quad +\mu\frac{\delta m}{m}x_{\langle i}v_{jk\rangle}\frac{r_{12}^2}{c^2}\left[-1+2\nu+\gamma\left(-\frac{1066}{165}+\frac{1433}{330}\nu-\frac{21}{55}\nu^2\right)\right]+\mathcal{O}\left(\frac{1}{c^5}\right), \qquad (221)\\
\mathrm{J}_{ij} &= \mu\frac{\delta m}{m}\varepsilon_{ab\langle i}x_{j\rangle a}v_b\left[-1+\gamma\left(-\frac{67}{28}+\frac{2}{7}\nu\right)+\gamma^2\left(-\frac{13}{9}+\frac{4651}{252}\nu+\frac{1}{168}\nu^2\right)\right]+\mathcal{O}\left(\frac{1}{c^5}\right).
\end{aligned}$$

Also needed are the 1PN mass 2^4-pole, 1PN current 2^3-pole (octupole), Newtonian mass 2^5-pole and Newtonian current 2^4-pole:

$$\begin{aligned}
\mathrm{I}_{ijkl} &= \mu\,\hat{x}_{ijkl}\left[1-3\nu+\gamma\left(\frac{3}{110}-\frac{25}{22}\nu+\frac{69}{22}\nu^2\right)\right]+\frac{78}{55}\mu\,x_{\langle ij}v_{kl\rangle}\frac{r_{12}^2}{c^2}(1-5\nu+5\nu^2)+\mathcal{O}\left(\frac{1}{c^3}\right),\\
\mathrm{J}_{ijk} &= \mu\,\varepsilon_{ab\langle i}x_{jk\rangle a}v_b\left[1-3\nu+\gamma\left(\frac{181}{90}-\frac{109}{18}\nu+\frac{13}{18}\nu^2\right)\right]+\frac{7}{45}\mu\,(1-5\nu+5\nu^2)\varepsilon_{ab\langle i}v_{jk\rangle b}x_a\frac{r_{12}^2}{c^2}\\
&\quad +\mathcal{O}\left(\frac{1}{c^3}\right), \qquad (222)\\
\mathrm{I}_{ijklm} &= \mu\frac{\delta m}{m}(-1+2\nu)\hat{x}_{ijklm}+\mathcal{O}\left(\frac{1}{c}\right),\\
\mathrm{J}_{ijkl} &= \mu\frac{\delta m}{m}(-1+2\nu)\varepsilon_{ab\langle i}x_{jkl\rangle a}v_b+\mathcal{O}\left(\frac{1}{c}\right).
\end{aligned}$$

These results permit one to control what can be called the "instantaneous" part, say $\mathcal{L}_{\mathrm{inst}}$, of the total energy flux, by which we mean that part of the flux that is generated solely by the *source* multipole moments, i.e. not counting the "non-instantaneous" tail integrals. The instantaneous flux is defined by the replacement into the general expression of $\mathcal{L}$ given by Equation (60) of all the radiative moments U_L and V_L by the corresponding (lth time derivatives of the) source moments I_L and J_L. Actually, we prefer to define $\mathcal{L}_{\mathrm{inst}}$ by means of the intermediate moments M_L and S_L. Up to the 3.5PN order we have

$$\begin{aligned}
\mathcal{L}_{\mathrm{inst}} = \frac{G}{c^5}\Bigg\{&\frac{1}{5}\mathrm{M}^{(3)}_{ij}\mathrm{M}^{(3)}_{ij}+\frac{1}{c^2}\left[\frac{1}{189}\mathrm{M}^{(4)}_{ijk}\mathrm{M}^{(4)}_{ijk}+\frac{16}{45}\mathrm{S}^{(3)}_{ij}\mathrm{S}^{(3)}_{ij}\right]+\frac{1}{c^4}\left[\frac{1}{9072}\mathrm{M}^{(5)}_{ijkm}\mathrm{M}^{(5)}_{ijkm}+\frac{1}{84}\mathrm{S}^{(4)}_{ijk}\mathrm{S}^{(4)}_{ijk}\right]\\
&+\frac{1}{c^6}\left[\frac{1}{594000}\mathrm{M}^{(6)}_{ijkmn}\mathrm{M}^{(6)}_{ijkmn}+\frac{4}{14175}\mathrm{S}^{(5)}_{ijkm}\mathrm{S}^{(5)}_{ijkm}\right]+\mathcal{O}\left(\frac{1}{c^8}\right)\Bigg\}. \qquad (223)
\end{aligned}$$

The time derivatives of the source moments (219, 220, 221, 222) are computed by means of the circular-orbit equations of motion given by Equation (189, 190); then we substitute them into

Equation (223)[38]. The net result is

$$\begin{aligned}\mathcal{L}_{\text{inst}} = \frac{32c^5}{5G}\nu^2\gamma^5\Bigg\{1 &+ \left(-\frac{2927}{336} - \frac{5}{4}\nu\right)\gamma + \left(\frac{293383}{9072} + \frac{380}{9}\nu\right)\gamma^2 \\ &+ \left[\frac{53712289}{1108800} - \frac{1712}{105}\ln\left(\frac{r_{12}}{r_0}\right)\right. \\ &\quad \left. + \left(-\frac{50625}{112} + \frac{123}{64}\pi^2 + \frac{110}{3}\ln\left(\frac{r_{12}}{r'_0}\right)\right)\nu - \frac{383}{9}\nu^2\right]\gamma^3 \\ &+ \mathcal{O}\left(\frac{1}{c^8}\right)\Bigg\}. \end{aligned} \tag{224}$$

The Newtonian approximation, $\mathcal{L}_\text{N} = (32c^5/5G)\,\nu^2\gamma^5$, is the prediction of the Einstein quadrupole formula (4), as computed by Landau and Lifchitz [153]. In Equation (224), we have replaced the Hadamard regularization ambiguity parameters λ and θ arising at the 3PN order by their values (135) and (137).

10.2 Contribution of wave tails

To the "instantaneous" part of the flux, we must add the contribution of non-linear multipole interactions contained in the relationship between the source and radiative moments. The needed material has already been provided in Equations (97, 98). Up to the 3.5PN level we have the dominant quadratic-order tails, the cubic-order tails or tails of tails, and the non-linear memory integral. We shall see that the tails play a crucial role in the predicted signal of compact binaries. By contrast, the non-linear memory effect, given by the integral inside the 2.5PN term in Equation (97), does not contribute to the gravitational-wave energy flux before the 4PN order in the case of circular-orbit binaries (essentially because the memory integral is actually "instantaneous" in the flux), and therefore has rather poor observational consequences for future detections of inspiralling compact binaries. We split the energy flux into the different terms

$$\mathcal{L} = \mathcal{L}_{\text{inst}} + \mathcal{L}_{\text{tail}} + \mathcal{L}_{(\text{tail})^2} + \mathcal{L}_{\text{tail(tail)}}, \tag{225}$$

where $\mathcal{L}_{\text{inst}}$ has just been found in Equation (224); $\mathcal{L}_{\text{tail}}$ is made of the quadratic (multipolar) tail integrals in Equation (98); $\mathcal{L}_{(\text{tail})^2}$ is the square of the quadrupole tail in Equation (97); and $\mathcal{L}_{\text{tail(tail)}}$ is the quadrupole tail of tail in Equation (97). We find that $\mathcal{L}_{\text{tail}}$ contributes at the half-integer 1.5PN, 2.5PN, and 3.5PN orders, while both $\mathcal{L}_{(\text{tail})^2}$ and $\mathcal{L}_{\text{tail(tail)}}$ appear only at the 3PN order. It is quite remarkable that so small an effect as a "tail of tail" should be relevant to the present computation, which is aimed at preparing the ground for forthcoming experiments.

The results follow from the reduction to the case of circular compact binaries of the general formulas (97, 98), in which we make use of the explicit expressions for the source moments of compact binaries as found in Section 10.1. Without going into accessory details (see Ref. [19]), let us point out that following the general formalism of Part A, the total mass M in front of the tail integrals is the ADM mass of the binary which is given by the sum of the rest masses, $m = m_1 + m_2$ (which is the one appearing in the γ-parameter, Equation (188)), plus some relativistic corrections. At the 2PN relative order needed here to compute the tail integrals we have

$$\mathrm{M} = m\left[1 - \frac{\nu}{2}\gamma + \frac{\nu}{8}(7-\nu)\gamma^2 + \mathcal{O}\left(\frac{1}{c^6}\right)\right]. \tag{226}$$

[38] For circular orbits there is no difference at this order between I_L, J_L and M_L, S_L.

Let us give the two basic technical formulas needed when carrying out this reduction:

$$\begin{aligned}\int_0^{+\infty} d\tau \ln\tau\, e^{-\sigma\tau} &= -\frac{1}{\sigma}(C+\ln\sigma),\\ \int_0^{+\infty} d\tau \ln^2\tau\, e^{-\sigma\tau} &= \frac{1}{\sigma}\left[\frac{\pi^2}{6}+(C+\ln\sigma)^2\right],\end{aligned} \tag{227}$$

where $\sigma \in \mathbb{C}$ and $C = 0.577\cdots$ denotes the Euler constant [125]. The tail integrals are evaluated thanks to these formulas for a *fixed* (non-decaying) circular orbit. Indeed it can be shown [50] that the "remote-past" contribution to the tail integrals is negligible; the errors due to the fact that the orbit actually spirals in by gravitational radiation do not affect the signal before the 4PN order. We then find, for the quadratic tail term *stricto sensu*, the 1.5PN, 2.5PN, and 3.5PN contributions[39]

$$\mathcal{L}_{\text{tail}} = \frac{32c^5}{5G}\gamma^5\nu^2\left\{4\pi\gamma^{3/2} + \left(-\frac{25663}{672}-\frac{125}{8}\nu\right)\pi\gamma^{5/2} + \left(\frac{90205}{576}+\frac{505747}{1512}\nu+\frac{12809}{756}\nu^2\right)\pi\gamma^{7/2} + \mathcal{O}\left(\frac{1}{c^8}\right)\right\}. \tag{228}$$

For the sum of squared tails and cubic tails of tails at 3PN, we get

$$\mathcal{L}_{(\text{tail})^2+\text{tail(tail)}} = \frac{32c^5}{5G}\gamma^5\nu^2\left\{\left(-\frac{116761}{3675}+\frac{16}{3}\pi^2-\frac{1712}{105}C+\frac{1712}{105}\ln\left(\frac{r_{12}}{r_0}\right)-\frac{856}{105}\ln(16\gamma)\right)\gamma^3 + \mathcal{O}\left(\frac{1}{c^8}\right)\right\}. \tag{229}$$

By comparing Equations (224) and (229) we observe that the constants r_0 cleanly cancel out. Adding together all these contributions we obtain

$$\begin{aligned}\mathcal{L} = \frac{32c^5}{5G}\gamma^5\nu^2\Bigg\{1 &+ \left(-\frac{2927}{336}-\frac{5}{4}\nu\right)\gamma + 4\pi\gamma^{3/2} + \left(\frac{293383}{9072}+\frac{380}{9}\nu\right)\gamma^2 + \left(-\frac{25663}{672}-\frac{125}{8}\nu\right)\pi\gamma^{5/2}\\ &+ \Bigg[\frac{129386791}{7761600}+\frac{16\pi^2}{3}-\frac{1712}{105}C-\frac{856}{105}\ln(16\gamma)\\ &\quad + \left(-\frac{50625}{112}+\frac{110}{3}\ln\left(\frac{r_{12}}{r'_0}\right)+\frac{123\pi^2}{64}\right)\nu - \frac{383}{9}\nu^2\Bigg]\gamma^3\\ &+ \left(\frac{90205}{576}+\frac{505747}{1512}\nu+\frac{12809}{756}\nu^2\right)\pi\gamma^{7/2} + \mathcal{O}\left(\frac{1}{c^8}\right)\Bigg\}.\end{aligned} \tag{230}$$

The gauge constant r'_0 has not yet disappeared because the post-Newtonian expansion is still parametrized by γ instead of the frequency-related parameter x defined by Equation (192) – just as for E when it was given by Equation (191). After substituting the expression $\gamma(x)$ given by Equation (193), we find that r'_0 does cancel as well. Because the relation $\gamma(x)$ is issued from the equations of motion, the latter cancellation represents an interesting test of the consistency of the two computations, in harmonic coordinates, of the 3PN multipole moments and the 3PN equations of motion. At long last we obtain our end result:

$$\mathcal{L} = \frac{32c^5}{5G}\nu^2x^5\left\{1 + \left(-\frac{1247}{336}-\frac{35}{12}\nu\right)x + 4\pi x^{3/2} + \left(-\frac{44711}{9072}+\frac{9271}{504}\nu+\frac{65}{18}\nu^2\right)x^2\right.$$

[39] All formulas incorporate the changes in some equations following the published Errata (2005) to the works [16, 19, 45, 40, 4].

$$
\begin{aligned}
&+\left(-\frac{8191}{672}-\frac{583}{24}\nu\right)\pi x^{5/2}\\
&+\left[\frac{6643739519}{69854400}+\frac{16}{3}\pi^2-\frac{1712}{105}C-\frac{856}{105}\ln(16\,x)\right.\\
&\quad\left.+\left(-\frac{134543}{7776}+\frac{41}{48}\pi^2\right)\nu-\frac{94403}{3024}\nu^2-\frac{775}{324}\nu^3\right]x^3\\
&+\left(-\frac{16285}{504}+\frac{214745}{1728}\nu+\frac{193385}{3024}\nu^2\right)\pi x^{7/2}+\mathcal{O}\left(\frac{1}{c^8}\right)\Bigg\}.
\end{aligned}
\tag{231}
$$

In the test-mass limit $\nu \to 0$ for one of the bodies, we recover exactly the result following from linear black-hole perturbations obtained by Tagoshi and Sasaki [205]. In particular, the rational fraction 6643739519/69854400 comes out exactly the same as in black-hole perturbations. On the other hand, the ambiguity parameters λ and θ are part of the rational fraction $-134543/7776$, belonging to the coefficient of the term at 3PN order proportional to ν (hence this coefficient cannot be computed by linear black hole perturbations)[40].

10.3 Orbital phase evolution

We shall now deduce the laws of variation with time of the orbital frequency and phase of an inspiralling compact binary from the energy balance equation (215). The center-of-mass energy E is given by Equation (194) and the total flux $\mathcal{L}$ by Equation (231). For convenience we adopt the dimensionless time variable[41]

$$
\Theta \equiv \frac{\nu c^3}{5Gm}(t_\mathrm{c}-t),
\tag{232}
$$

where t_c denotes the instant of coalescence, at which the frequency tends to infinity (evidently, the post-Newtonian method breaks down well before this point). We transform the balance equation into an ordinary differential equation for the parameter x, which is immediately integrated with the result

$$
\begin{aligned}
x=\frac{1}{4}\Theta^{-1/4}\Bigg\{&1+\left(\frac{743}{4032}+\frac{11}{48}\nu\right)\Theta^{-1/4}-\frac{1}{5}\pi\Theta^{-3/8}+\left(\frac{19583}{254016}+\frac{24401}{193536}\nu+\frac{31}{288}\nu^2\right)\Theta^{-1/2}\\
&+\left(-\frac{11891}{53760}+\frac{109}{1920}\nu\right)\pi\Theta^{-5/8}\\
&+\left[-\frac{10052469856691}{6008596070400}+\frac{1}{6}\pi^2+\frac{107}{420}C-\frac{107}{3360}\ln\left(\frac{\Theta}{256}\right)\right.\\
&\quad\left.+\left(\frac{3147553127}{780337152}-\frac{451}{3072}\pi^2\right)\nu-\frac{15211}{442368}\nu^2+\frac{25565}{331776}\nu^3\right]\Theta^{-3/4}\\
&+\left(-\frac{113868647}{433520640}-\frac{31821}{143360}\nu+\frac{294941}{3870720}\nu^2\right)\pi\Theta^{-7/8}+\mathcal{O}\left(\frac{1}{c^8}\right)\Bigg\}.
\end{aligned}
\tag{233}
$$

The orbital phase is defined as the angle ϕ, oriented in the sense of the motion, between the separation of the two bodies and the direction of the ascending node $\mathcal{N}$ within the plane of the sky, namely the point on the orbit at which the bodies cross the plane of the sky moving toward

[40] Generalizing the flux formula (231) to point masses moving on *quasi elliptic* orbits dates back to the work of Peters and Mathews [178] at Newtonian order. The result was obtained in [217, 49] at 1PN order, and then further extended by Gopakumar and Iyer [122] up to 2PN order using an explicit quasi-Keplerian representation of the motion [99, 197]. No complete result at 3PN order is yet available.

[41] Notice the "strange" post-Newtonian order of this time variable: $\Theta = \mathcal{O}(c^{+8})$.

the detector. We have $d\phi/dt = \omega$, which translates, with our notation, into $d\phi/d\Theta = -5/\nu \cdot x^{3/2}$, from which we determine

$$\begin{aligned}\phi = -\frac{1}{\nu}\Theta^{5/8}\Bigg\{&1 + \left(\frac{3715}{8064} + \frac{55}{96}\nu\right)\Theta^{-1/4} - \frac{3}{4}\pi\Theta^{-3/8} + \left(\frac{9275495}{14450688} + \frac{284875}{258048}\nu + \frac{1855}{2048}\nu^2\right)\Theta^{-1/2} \\ &+ \left(-\frac{38645}{172032} + \frac{65}{2048}\nu\right)\pi\Theta^{-5/8}\ln\left(\frac{\Theta}{\Theta_0}\right) \\ &+ \left[\frac{831032450749357}{57682522275840} - \frac{53}{40}\pi^2 - \frac{107}{56}C + \frac{107}{448}\ln\left(\frac{\Theta}{256}\right) \right. \\ &\quad + \left(-\frac{126510089885}{4161798144} + \frac{2255}{2048}\pi^2\right)\nu \\ &\quad \left. + \frac{154565}{1835008}\nu^2 - \frac{1179625}{1769472}\nu^3\right]\Theta^{-3/4} \\ &+ \left(\frac{188516689}{173408256} + \frac{488825}{516096}\nu - \frac{141769}{516096}\nu^2\right)\pi\Theta^{-7/8} + \mathcal{O}\left(\frac{1}{c^8}\right)\Bigg\}, \end{aligned} \tag{234}$$

where Θ_0 is a constant of integration that can be fixed by the initial conditions when the wave frequency enters the detector's bandwidth. Finally we want also to dispose of the important expression of the phase in terms of the frequency x. For this we get

$$\begin{aligned}\phi = -\frac{x^{-5/2}}{32\nu}\Bigg\{&1 + \left(\frac{3715}{1008} + \frac{55}{12}\nu\right)x - 10\pi x^{3/2} + \left(\frac{15293365}{1016064} + \frac{27145}{1008}\nu + \frac{3085}{144}\nu^2\right)x^2 \\ &+ \left(\frac{38645}{1344} - \frac{65}{16}\nu\right)\pi x^{5/2}\ln\left(\frac{x}{x_0}\right) \\ &+ \left[\frac{12348611926451}{18776862720} - \frac{160}{3}\pi^2 - \frac{1712}{21}C - \frac{856}{21}\ln(16\,x) \right. \\ &\quad \left. + \left(-\frac{15737765635}{12192768} + \frac{2255}{48}\pi^2\right)\nu + \frac{76055}{6912}\nu^2 - \frac{127825}{5184}\nu^3\right]x^3 \\ &+ \left(\frac{77096675}{2032128} + \frac{378515}{12096}\nu - \frac{74045}{6048}\nu^2\right)\pi x^{7/2} + \mathcal{O}\left(\frac{1}{c^8}\right)\Bigg\}, \end{aligned} \tag{235}$$

where x_0 is another constant of integration. With the formula (235) the orbital phase is complete up to the 3.5PN order. The effects due to the spins of the particles, i.e. the spin-orbit (SO) coupling arising at the 1.5PN order for maximally rotating compact bodies and the spin-spin (SS) coupling at the 2PN order, can be added if necessary; they are known up to the 2.5PN order included [146, 144, 168, 204, 110, 25]. On the other hand, the contribution of the quadrupole moments of the compact objects, which are induced by tidal effects, is expected to come only at the 5PN order (see Equation (8)).

As a rough estimate of the relative importance of the various post-Newtonian terms, let us give in Table 2 their contributions to the accumulated number of gravitational-wave cycles $\mathcal{N}$ in the bandwidth of the LIGO and VIRGO detectors (see also Table I in Ref. [35] for the contributions of the SO and SS effects). Note that such an estimate is only indicative, because a full treatment would require the knowledge of the detector's power spectral density of noise, and a complete simulation of the parameter estimation using matched filtering [79, 184, 152]. We define $\mathcal{N}$ by

$$\mathcal{N} = \frac{1}{\pi}\left[\phi_{\text{ISCO}} - \phi_{\text{seismic}}\right]. \tag{236}$$

The frequency of the signal at the entrance of the bandwidth is the seismic cut-off frequency f_{seismic} of ground-based detectors; the terminal frequency f_{ISCO} is assumed for simplicity's sake to be given by the Schwarzschild innermost stable circular orbit. Here $f = \omega/\pi = 2/P$ is the signal frequency at the dominant harmonics (twice the orbital frequency). As we see in Table 2, with the 3PN or 3.5PN approximations we reach an acceptable level of, say, a few cycles, that roughly corresponds to the demand which was made by data-analysists in the case of neutron-star binaries [77, 78, 79, 183, 59, 58]. Indeed, the above estimation suggests that the neglected 4PN terms will yield some systematic errors that are, at most, of the same order of magnitude, i.e. a few cycles, and perhaps much less (see also the discussion in Section 9.6).

	$2 \times 1.4\,M_\odot$	$10\,M_\odot + 1.4\,M_\odot$	$2 \times 10\,M_\odot$
Newtonian order	16031	3576	602
1PN	441	213	59
1.5PN (dominant tail)	−211	−181	−51
2PN	9.9	9.8	4.1
2.5PN	−11.7	−20.0	−7.1
3PN	2.6	2.3	2.2
3.5PN	−0.9	−1.8	−0.8

Table 2: *Contributions of post-Newtonian orders to the accumulated number of gravitational-wave cycles $\mathcal{N}$ (defined by Equation (236)) in the bandwidth of VIRGO and LIGO detectors. Neutron stars have mass $1.4\,M_\odot$, and black holes $10\,M_\odot$. The entry frequency is $f_{\rm seismic} = 10$ Hz, and the terminal frequency is $f_{\rm ISCO} = c^3/(6^{3/2}\pi G m)$.*

10.4 The two polarization waveforms

The theoretical templates of the compact binary inspiral follow from insertion of the previous solutions for the 3.5PN-accurate orbital frequency and phase into the binary's two polarization waveforms h_+ and $h_\times$. We shall include in h_+ and $h_\times$ all the harmonics, besides the dominant one at twice the orbital frequency, up to the 2.5PN order, as they have been calculated in Refs. [46, 4]. The polarization waveforms are defined with respect to two polarization vectors $\mathbf{p} = (p_i)$ and $\mathbf{q} = (q_i)$,

$$\begin{aligned} h_+ &= \frac{1}{2}(p_i p_j - q_i q_j) h^{\rm TT}_{ij}, \\ h_\times &= \frac{1}{2}(p_i q_j + p_j q_i) h^{\rm TT}_{ij}, \end{aligned} \tag{237}$$

where $\mathbf{p}$ and $\mathbf{q}$ are chosen to lie along the major and minor axis, respectively, of the projection onto the plane of the sky of the circular orbit, with $\mathbf{p}$ oriented toward the ascending node $\mathcal{N}$. To the 2PN order we have

$$h_{+,\times} = \frac{2G\mu x}{c^2 R}\left\{ H^{(0)}_{+,\times} + x^{1/2} H^{(1/2)}_{+,\times} + x H^{(1)}_{+,\times} + x^{3/2} H^{(3/2)}_{+,\times} + x^2 H^{(2)}_{+,\times} + x^{5/2} H^{(5/2)}_{+,\times} + \mathcal{O}\left(\frac{1}{c^6}\right)\right\}. \tag{238}$$

The post-Newtonian terms are ordered by means of the frequency-related variable x. They depend on the binary's 3.5PN-accurate phase ϕ through the auxiliary phase variable

$$\psi = \phi - \frac{2G\mathrm{M}\omega}{c^3} \ln\left(\frac{\omega}{\omega_0}\right), \tag{239}$$

where $\mathrm{M} = m\left[1 - \nu\gamma/2 + \mathcal{O}\left(1/c^4\right)\right]$ is the ADM mass (cf. Equation (226)), and where ω_0 is a constant frequency that can conveniently be chosen to be the entry frequency of a laser-interferometric detector (say $\omega_0/\pi = 10$ Hz). For the plus polarization we have[42]

$$
\begin{aligned}
H_+^{(0)} &= -(1+c_i^2)\cos 2\psi, \\
H_+^{(1/2)} &= -\frac{s_i}{8}\frac{\delta m}{m}\left[(5+c_i^2)\cos\psi - 9(1+c_i^2)\cos 3\psi\right], \\
H_+^{(1)} &= \frac{1}{6}\left[19 + 9c_i^2 - 2c_i^4 - \nu(19 - 11c_i^2 - 6c_i^4)\right]\cos 2\psi - \frac{4}{3}s_i^2(1+c_i^2)(1-3\nu)\cos 4\psi, \\
H_+^{(3/2)} &= \frac{s_i}{192}\frac{\delta m}{m}\bigg\{\left[57 + 60c_i^2 - c_i^4 - 2\nu(49 - 12c_i^2 - c_i^4)\right]\cos\psi \\
&\qquad - \frac{27}{2}\left[73 + 40c_i^2 - 9c_i^4 - 2\nu(25 - 8c_i^2 - 9c_i^4)\right]\cos 3\psi \\
&\qquad + \frac{625}{2}(1-2\nu)s_i^2(1+c_i^2)\cos 5\psi\bigg\} - 2\pi(1+c_i^2)\cos 2\psi, \\
H_+^{(2)} &= \frac{1}{120}\bigg[22 + 396c_i^2 + 145c_i^4 - 5c_i^6 + \frac{5}{3}\nu(706 - 216c_i^2 - 251c_i^4 + 15c_i^6) \\
&\qquad - 5\nu^2(98 - 108c_i^2 + 7c_i^4 + 5c_i^6)\bigg]\cos 2\psi \\
&\quad + \frac{2}{15}s_i^2\left[59 + 35c_i^2 - 8c_i^4 - \frac{5}{3}\nu(131 + 59c_i^2 - 24c_i^4) + 5\nu^2(21 - 3c_i^2 - 8c_i^4)\right]\cos 4\psi \\
&\quad - \frac{81}{40}(1 - 5\nu + 5\nu^2)s_i^4(1+c_i^2)\cos 6\psi \\
&\quad + \frac{s_i}{40}\frac{\delta m}{m}\bigg\{\left[11 + 7c_i^2 + 10(5+c_i^2)\ln 2\right]\sin\psi - 5\pi(5+c_i^2)\cos\psi \\
&\qquad - 27\left[7 - 10\ln(3/2)\right](1+c_i^2)\sin 3\psi + 135\pi(1+c_i^2)\cos 3\psi\bigg\}.
\end{aligned}
$$

For the cross polarization, we have

$$
\begin{aligned}
H_\times^{(0)} &= -2c_i\sin 2\psi, \\
H_\times^{(1/2)} &= -\frac{3}{4}s_i c_i\frac{\delta m}{m}\left[\sin\psi - 3\sin 3\psi\right], \\
H_\times^{(1)} &= \frac{c_i}{3}\left[17 - 4c_i^2 - \nu(13 - 12c_i^2)\right]\sin 2\psi - \frac{8}{3}(1-3\nu)c_i s_i^2\sin 4\psi, \\
H_\times^{(3/2)} &= \frac{s_i c_i}{96}\frac{\delta m}{m}\bigg\{\left[63 - 5c_i^2 - 2\nu(23 - 5c_i^2)\right]\sin\psi - \frac{27}{2}\left[67 - 15c_i^2 - 2\nu(19 - 15c_i^2)\right]\sin 3\psi \\
&\qquad + \frac{625}{2}(1-2\nu)s_i^2\sin 5\psi\bigg\} - 4\pi c_i\sin 2\psi, \\
H_\times^{(2)} &= \frac{c_i}{60}\left[68 + 226c_i^2 - 15c_i^4 + \frac{5}{3}\nu(572 - 490c_i^2 + 45c_i^4) - 5\nu^2(56 - 70c_i^2 + 15c_i^4)\right]\sin 2\psi \\
&\quad + \frac{4}{15}c_i s_i^2\left[55 - 12c_i^2 - \frac{5}{3}\nu(119 - 36c_i^2) + 5\nu^2(17 - 12c_i^2)\right]\sin 4\psi \\
&\quad - \frac{81}{20}(1 - 5\nu + 5\nu^2)c_i s_i^4\sin 6\psi
\end{aligned}
$$

[42] We neglect the non-linear memory (DC) term present in the Newtonian plus polarization $H_+^{(0)}$. See Wiseman and Will [222] and Arun et al. [4] for the computation of this term.

$$-\frac{3}{20}s_ic_i\frac{\delta m}{m}\left\{[3+10\ln 2]\cos\psi+5\pi\sin\psi-9\left[7-10\ln(3/2)\right]\cos 3\psi-45\pi\sin 3\psi\right\}.$$

We use the shorthands $c_i = \cos i$ and $s_i = \sin i$ for the cosine and sine of the inclination angle i between the direction of the detector as seen from the binary's center-of-mass, and the normal to the orbital plane (we always suppose that the normal is right-handed with respect to the sense of motion, so that $0 \leq i \leq \pi$). Finally, the more recent calculation of the 2.5PN order in Ref. [4] is reported here:

$$\begin{aligned}
H_+^{(5/2)} = {} & s_i\frac{\delta m}{m}\cos\psi\left[\frac{1771}{5120}-\frac{1667}{5120}c_i^2+\frac{217}{9216}c_i^4-\frac{1}{9216}c_i^6\right.\\
&\qquad +\nu\left(\frac{681}{256}+\frac{13}{768}c_i^2-\frac{35}{768}c_i^4+\frac{1}{2304}c_i^6\right)\\
&\qquad \left.+\nu^2\left(-\frac{3451}{9216}+\frac{673}{3072}c_i^2-\frac{5}{9216}c_i^4-\frac{1}{3072}c_i^6\right)\right]\\
&+\pi\cos 2\psi\left[\frac{19}{3}+3c_i^2-\frac{2}{3}c_i^4+\nu\left(-\frac{16}{3}+\frac{14}{3}c_i^2+2c_i^4\right)\right]\\
&+s_i\frac{\delta m}{m}\cos 3\psi\left[\frac{3537}{1024}-\frac{22977}{5120}c_i^2-\frac{15309}{5120}c_i^4+\frac{729}{5120}c_i^6\right.\\
&\qquad +\nu\left(-\frac{23829}{1280}+\frac{5529}{1280}c_i^2+\frac{7749}{1280}c_i^4-\frac{729}{1280}c_i^6\right)\\
&\qquad \left.+\nu^2\left(\frac{29127}{5120}-\frac{27267}{5120}c_i^2-\frac{1647}{5120}c_i^4+\frac{2187}{5120}c_i^6\right)\right]\\
&+\cos 4\psi\left[-\frac{16\pi}{3}(1+c_i^2)s_i^2(1-3\nu)\right]\\
&+s_i\frac{\delta m}{m}\cos 5\psi\left[-\frac{108125}{9216}+\frac{40625}{9216}c_i^2+\frac{83125}{9216}c_i^4-\frac{15625}{9216}c_i^6\right.\\
&\qquad +\nu\left(\frac{8125}{256}-\frac{40625}{2304}c_i^2-\frac{48125}{2304}c_i^4+\frac{15625}{2304}c_i^6\right)\\
&\qquad \left.+\nu^2\left(-\frac{119375}{9216}+\frac{40625}{3072}c_i^2+\frac{44375}{9216}c_i^4-\frac{15625}{3072}c_i^6\right)\right]\\
&+\frac{\delta m}{m}\cos 7\psi\left[\frac{117649}{46080}s_i^5(1+c_i^2)(1-4\nu+3\nu^2)\right]\\
&+\sin 2\psi\left[-\frac{9}{5}+\frac{14}{5}c_i^2+\frac{7}{5}c_i^4+\nu\left(\frac{96}{5}-\frac{8}{5}c_i^2-\frac{28}{5}c_i^4\right)\right]\\
&+s_i^2(1+c_i^2)\sin 4\psi\left[\frac{56}{5}-\frac{32\ln 2}{3}-\nu\left(\frac{1193}{30}-32\ln 2\right)\right],
\end{aligned} \tag{240}$$

$$\begin{aligned}
H_\times^{(5/2)} = {} & \frac{6}{5}s_i^2c_i\nu\\
&+c_i\cos 2\psi\left[2-\frac{22}{5}c_i^2+\nu\left(-\frac{154}{5}+\frac{94}{5}c_i^2\right)\right]\\
&+c_is_i^2\cos 4\psi\left[-\frac{112}{5}+\frac{64}{3}\ln 2+\nu\left(\frac{1193}{15}-64\ln 2\right)\right]\\
&+s_ic_i\frac{\delta m}{m}\sin\psi\left[-\frac{913}{7680}+\frac{1891}{11520}c_i^2-\frac{7}{4608}c_i^4\right.
\end{aligned}$$

$$
\begin{aligned}
&\qquad\qquad + \nu\left(\frac{1165}{384} - \frac{235}{576}c_i^2 + \frac{7}{1152}c_i^4\right) \\
&\qquad\qquad + \nu^2\left(-\frac{1301}{4608} + \frac{301}{2304}c_i^2 - \frac{7}{1536}c_i^4\right)\Bigg] \\
&+\pi c_i \sin 2\psi\left[\frac{34}{3} - \frac{8}{3}c_i^2 - \nu\left(\frac{20}{3} - 8c_i^2\right)\right] \\
&+s_i c_i \frac{\delta m}{m}\sin 3\psi\Bigg[\frac{12501}{2560} - \frac{12069}{1280}c_i^2 + \frac{1701}{2560}c_i^4 \\
&\qquad\qquad +\nu\left(-\frac{19581}{640} + \frac{7821}{320}c_i^2 - \frac{1701}{640}c_i^4\right) \\
&\qquad\qquad +\nu^2\left(\frac{18903}{2560} - \frac{11403}{1280}c_i^2 + \frac{5103}{2560}c_i^4\right)\Bigg] \\
&+s_i^2 c_i \sin 4\psi\left[-\frac{32\pi}{3}(1-3\nu)\right] \\
&+\frac{\delta m}{m}s_i c_i \sin 5\psi\Bigg[-\frac{101875}{4608} + \frac{6875}{256}c_i^2 - \frac{21875}{4608}c_i^4 \\
&\qquad\qquad +\nu\left(\frac{66875}{1152} - \frac{44375}{576}c_i^2 + \frac{21875}{1152}c_i^4\right) \\
&\qquad\qquad +\nu^2\left(-\frac{100625}{4608} + \frac{83125}{2304}c_i^2 - \frac{21875}{1536}c_i^4\right)\Bigg] \\
&+\frac{\delta m}{m}s_i^5 c_i \sin 7\psi\left[\frac{117649}{23040}\left(1-4\nu+3\nu^2\right)\right].
\end{aligned} \tag{241}
$$

The practical implementation of the theoretical templates in the data analysis of detectors follows the standard matched filtering technique. The raw output of the detector $o(t)$ consists of the superposition of the real gravitational wave signal $h_{\mathrm{real}}(t)$ and of noise $n(t)$. The noise is assumed to be a stationary Gaussian random variable, with zero expectation value, and with (supposedly known) frequency-dependent power spectral density $S_n(\omega)$. The experimenters construct the correlation between $o(t)$ and a filter $q(t)$, i.e.

$$
c(t) = \int_{-\infty}^{+\infty} dt'\, o(t') q(t+t'), \tag{242}
$$

and divide $c(t)$ by the square root of its variance, or correlation noise. The expectation value of this ratio defines the filtered signal-to-noise ratio (SNR). Looking for the useful signal $h_{\mathrm{real}}(t)$ in the detector's output $o(t)$, the experimenters adopt for the filter

$$
\tilde{q}(\omega) = \frac{\tilde{h}(\omega)}{S_n(\omega)}, \tag{243}
$$

where $\tilde{q}(\omega)$ and $\tilde{h}(\omega)$ are the Fourier transforms of $q(t)$ and of the *theoretically computed* template $h(t)$. By the matched filtering theorem, the filter (243) maximizes the SNR if $h(t) = h_{\mathrm{real}}(t)$. The maximum SNR is then the best achievable with a linear filter. In practice, because of systematic errors in the theoretical modelling, the template $h(t)$ will not exactly match the real signal $h_{\mathrm{real}}(t)$, but if the template is to constitute a realistic representation of nature the errors will be small. This is of course the motivation for computing high order post-Newtonian templates, in order to reduce as much as possible the systematic errors due to the unknown post-Newtonian remainder.

To conclude, the use of theoretical templates based on the preceding 2.5PN wave forms, and having their frequency evolution built in *via* the 3.5PN phase evolution (234, 235), should yield some accurate detection and measurement of the binary signals. Interestingly, it should also permit some new tests of general relativity, because we have the possibility of checking that the observed signals do obey each of the terms of the phasing formulas (234, 235), e.g., those associated with the specific non-linear tails, exactly as they are predicted by Einstein's theory [47, 48, 5]. Indeed, we don't know of any other physical systems for which it would be possible to perform such tests.

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