

ADVANCED
GENERAL RELATIVITY

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General relativity (Einstein 1915) is based on two fundamental principles: one, the principle of special relativity, is probably at the basis of all Physics and applies to any fundamental interaction (at least up to Planck scale), the other, the principle of equivalence, is specific to the gravitational interaction and we do not know if it is a fundamental principle.

Principle of special relativity

- Among reference frames there exists a privileged class of ref. frames $\{X^\alpha\}$ (where $\alpha=0, 1, 2, 3$ and we pose $X^0=cT$) such that bodies not submitted to any force have an inertial motion i.e.

$$\boxed{\frac{d^2 X^\alpha}{dp^2} = 0}$$

p is a parameter along the trajectory (called affine parameter since $T = ap + b$)

- The laws of nature take the same form in all inertial frames. Thus they are invariant when we change $\{X^\alpha\} \rightarrow \{X'^\alpha\}$

- The laws of transformation between inertial frames are the Lorentz-Poincaré transformations

$$\boxed{X'^\alpha = \Lambda_\beta^\alpha X^\beta + a^\alpha}$$

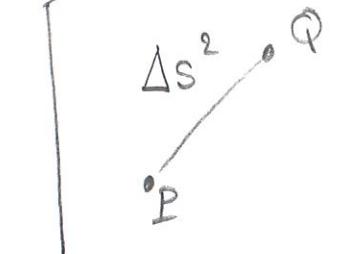
where Λ_β^α are the Lorentz matrices satisfying the fundamental relation

$$\boxed{\eta_{\gamma\delta} \Lambda_\alpha^\gamma \Lambda_\beta^\delta = \eta_{\alpha\beta}}$$

where $\eta_{\alpha\beta} = \text{diag}(-1, 1, 1, 1)$ is the Minkowski matrix (with signature -2)

The Lorentz - Poincaré transformations leave the interval between events P and Q invariant when we change $\{X^\beta\} \rightarrow \{X'^\beta\}$

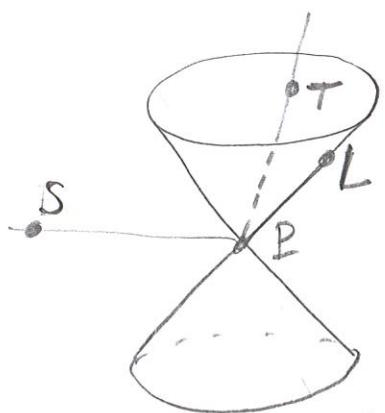
$$\boxed{\Delta s^2 = \eta_{\alpha\beta} \Delta X^\alpha \Delta X^\beta}$$



$$\Delta s'^2 = \eta_{\alpha\beta} \Delta X'^\alpha \Delta X'^\beta$$

$$= \eta_{\alpha\beta} \Lambda^\alpha_\gamma \Delta X^\gamma \Lambda^\beta_\delta \Delta X^\delta = \eta_{\alpha\beta} \Delta X^\alpha \Delta X^\beta$$

Thus the invariant structure of space-time (Minkowski space-time) is given by light cones $\mathcal{C}_P = \{ \text{events } L \text{ such that } \Delta s_{PL}^2 = 0 \}$



light-like separation

time-like

space-like

$$\Delta s_{PL}^2 = 0 \quad (\text{propagation of light signal from } P \text{ to } L)$$

$$\Delta s_{PT}^2 < 0$$

$$\Delta s_{PS}^2 > 0$$

One can always choose $\{X^\beta\}$ such that

$$\{X''^\beta\}$$

$$X''_T^i = X''_P^i \quad (T \text{ and } P \text{ occur at same position})$$

$$X''_S^0 = X''_P^0 \quad (S \text{ and } P \text{ are simultaneous})$$

conservation laws associated to invariances

Poincaré group has 10 parameters

a^0 time translation

1

a^i space translation

3

R^i_j spatial rotation

3

Euler angles

$(\delta_{kl} R^k_i R^l_j = \delta_{ij} \text{ is a particular case of relation of Lorentz matrices})$

$\Lambda^\alpha_\beta(v^i)$ Lorentz boost

3 components of \vec{v}

energy E 1

linear momentum P^i 3

angular momentum S^i 3

center-of-mass integral Z^i 3
(such that CM position is $G^i = P^i t + Z^i$)

Lorentz boost

$$\begin{aligned}\Lambda_0^0(\vec{v}) &= \gamma \\ \Lambda_0^i(\vec{v}) &= \Lambda_i^0(\vec{v}) = -\gamma \frac{v^i}{c} \\ \Lambda_j^i(\vec{v}) &= \delta_j^i + (\gamma - 1) \frac{v^i v_j}{c^2} \\ \text{where } \gamma &= \frac{1}{\sqrt{1 - \vec{v}^2/c^2}}\end{aligned}$$

If $\{X^\alpha\}$ is an inertial frame and a particle has velocity \vec{v} in that frame then the frame $\{X'^\alpha\}$ with $X'^\alpha = \Lambda_\beta^\alpha(\vec{v}) X^\beta$ is a rest frame for the particle.

Relativistic dynamics (particle with mass $m > 0$, time-like trajectory)

$$u^\alpha = \frac{dX^\alpha}{dT} \quad \text{where proper time defined by interval}$$

$$dT^2 = -\frac{ds^2}{c^2} \quad ds^2 = g_{\alpha\beta} dX^\alpha dX^\beta$$

along trajectory

$$\text{Thus } g_{\alpha\beta} u^\alpha u^\beta = -c^2$$

4-acceleration -

$$a^\alpha = \frac{du^\alpha}{dT} = u^\beta \frac{\partial u^\alpha}{\partial T} \quad \text{for fluid of particles}$$

(orthogonal to 4-velocity $g_{\alpha\beta} u^\alpha u^\beta = 0$, hence space-like)

$$u^\beta \partial_\beta = u^0 \left(\partial_t + v^i \partial_i \right) = u^0 \frac{d}{dT} \quad \text{"convective" derivative where } v^i = \frac{u^i}{u^0}$$

"coordinate velocity"

Linear momentum

$$p^\alpha = m u^\alpha$$

$$\begin{cases} E = p^0 c = mc u^0 = \frac{mc^2}{\sqrt{1 - \vec{v}^2/c^2}} & \text{energy} \\ p^i = m v^i = \frac{m v^i}{\sqrt{1 - \vec{v}^2/c^2}} & \text{linear momentum} \end{cases}$$

$$E^2 - \vec{p}^2 c^2 = m^2 c^4$$

$$E = mc^2 \quad \text{rest energy.}$$

Fundamental law of dynamics

$$\frac{dp^\alpha}{dT} = m a^\alpha = f^\alpha$$

The law is invariant under the Poincaré group

The force is defined from its expression $F^\alpha = (0, \vec{F})$ in the rest frame of the particle. Since $\eta_{\alpha\beta} u^\alpha u^\beta = 0$ we have necessarily $\eta_{\alpha\beta} u^\alpha F^\beta = 0$ and thus $F^0 = 0$ in rest frame. For instance, $\vec{F} = q\vec{E}$ is the electric force on a charged particle at rest. Then in a frame where the particle has velocity \vec{v} we have

$$f^\alpha = \Lambda_\beta^\alpha(\vec{v}) F^\beta = \Lambda_i^\alpha(\vec{v}) F^i$$

Principle of equivalence

inertial mass m_i body subjected to any force \vec{F} is accelerated
 $\vec{a} = \frac{1}{m_i} \vec{F}$

gravitational mass m_g body in gravitational field \vec{g} is subjected to gravitational force
 $\vec{F}_g = m_g \vec{g}$

Experimentally we observe $m_i = m_g$ for all test bodies
 (i.e. uncharged with sufficiently small extent)

at equilibrium

$$\vec{T} = \vec{F}_A + \vec{F}_B$$

$$\vec{C} = \vec{OA} \wedge \vec{F}_A + \vec{OB} \wedge \vec{F}_B$$

\vec{C} and \vec{T} aligned with direction of wire

$$\vec{C} = C \vec{m} \quad \vec{T} = T \vec{m}$$

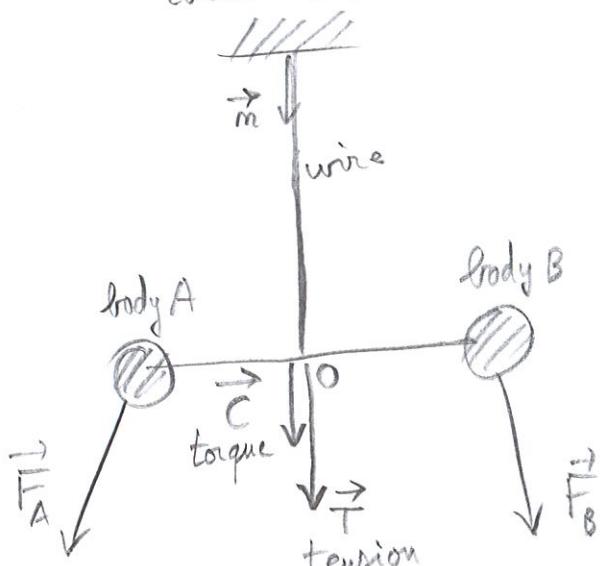
$$\vec{C} \cdot \vec{T} = CT = (\vec{OA} \wedge \vec{F}_A) \cdot \vec{F}_B + (\vec{OB} \wedge \vec{F}_B) \cdot \vec{F}_A$$

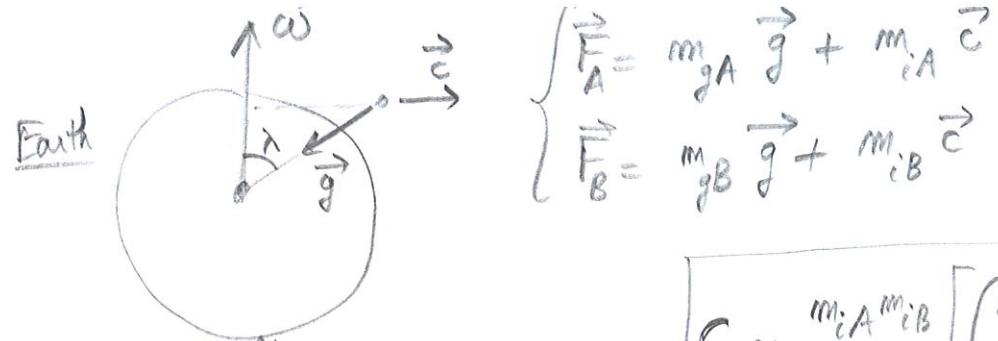
$$C = \frac{\vec{AB} \circ (\vec{F}_A \wedge \vec{F}_B)}{\vec{F}_A + \vec{F}_B}$$

Toque if \vec{F}_A and \vec{F}_B are not aligned

Eötvös experiment

torsion balance





\vec{g} = gravit. field
 \vec{c} = inertial field

$$C = R \omega^2 \sin^2 \theta$$

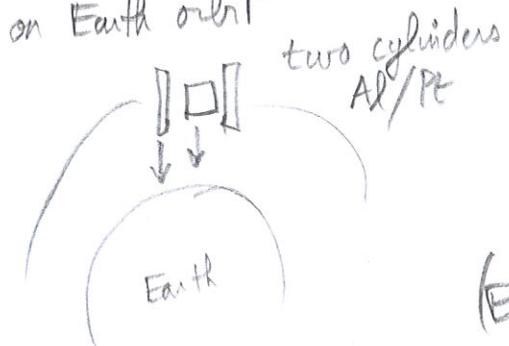
$$\begin{cases} \vec{F}_A = m_{gA} \vec{g} + m_{iA} \vec{c} \\ \vec{F}_B = m_{gB} \vec{g} + m_{iB} \vec{c} \end{cases}$$

$$C \approx \frac{m_{iA} m_{iB}}{m_{iA} + m_{iB}} \left[\left(\frac{m_g}{m_i} \right)_A - \left(\frac{m_g}{m_i} \right)_B \right] \frac{\vec{AB} \cdot (\vec{g} \wedge \vec{c})}{|\vec{g} + \vec{c}|^2}$$

torque is measurable

$$\left| \left(\frac{m_g}{m_i} \right)_{\text{pt}} - \left(\frac{m_g}{m_i} \right)_{\text{void}} \right| < 10^{-9}$$

Satellite Microscope (2017)
on Earth orbit



$$10^{-15}$$

$$a_{Al} = \left(\frac{m_g}{m_i} \right)_{Al} g \oplus$$

$$a_{Pt} = \left(\frac{m_g}{m_i} \right)_{Pt} g \oplus$$

(EEP)

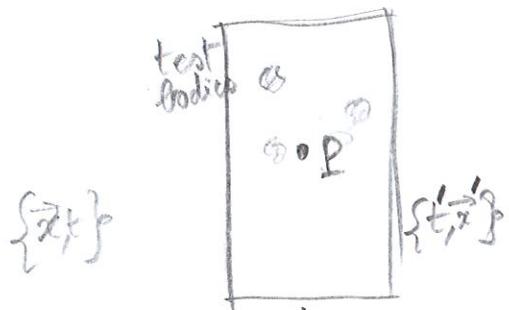
Einstein equivalence principle (1911)

- Weak equivalence principle (WEP): all test bodies fall with same acceleration $\frac{d^2 \vec{x}}{dt^2} = \vec{g}$ \forall body

We can thus define in a neighbourhood of any event a locally inertial frame in which test bodies are unaccelerated

In a sufficiently small region around some event P $\vec{g}(\vec{x}, t) \approx \vec{g}$ constant

Hence posing $\vec{x}' = \vec{x} - \frac{1}{2} \vec{g} t^2$ we have $\frac{d^2 \vec{x}'}{dt'^2} = 0$ for all bodies



- In local inertial frames the laws of special relativity are valid

In particular, the results of any non-gravitational experiment should be independent of

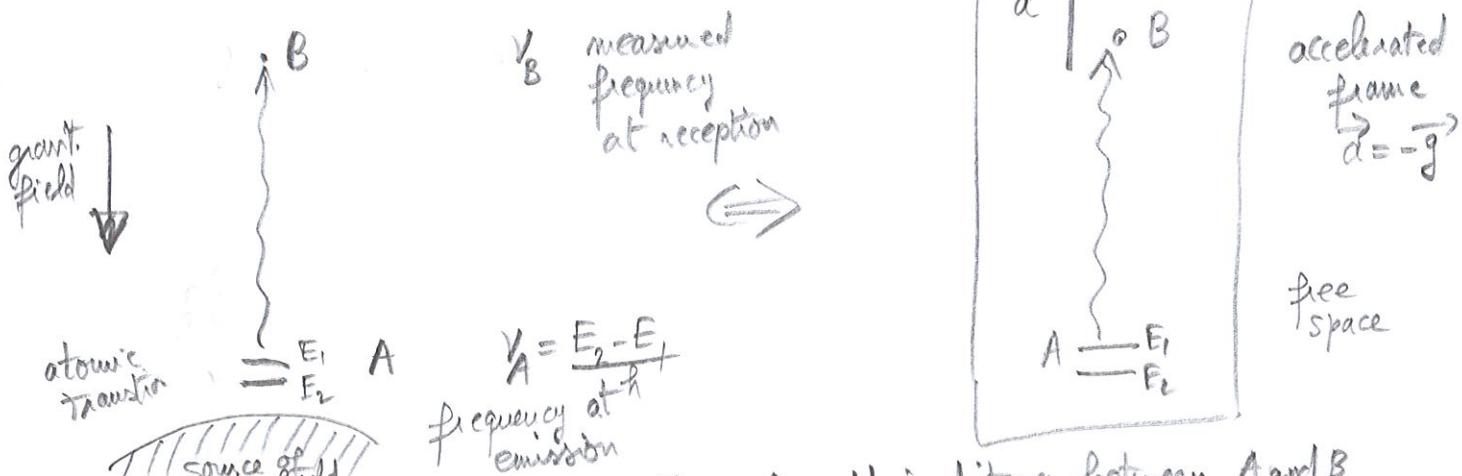
- the velocity of the (freely-falling) apparatus
this is called local Lorentz invariance (LLI)
- the position in space & time of the experiment
local position invariance (LPI)

In local inertial coordinates around $P \{X_P^\alpha\}$ we

$$\boxed{ds_P^2 = \eta_{\alpha\beta} dX_P^\alpha dX_P^\beta}$$

LLI checked by Michelson-Morley type experiments $\frac{\Delta c}{c} \approx 10^{-15}$

LPI checked by gravitational redshift exp. $\frac{\Delta V}{V} = \frac{U}{c^2} (10^{-4})$



During time of flight $T = \frac{H}{c}$ where H is distance between A and B
velocity of accel. frame increases by $\Delta V = aT$. Observer B sees
emitter A going away hence (1st order Doppler effect)

$$\frac{\gamma_B}{\gamma_A} = 1 - \frac{\Delta V}{c} = 1 - \frac{aH}{c^2}$$

$$\boxed{\frac{\gamma_B}{\gamma_A} = 1 - \frac{gH}{c^2}}$$

gravitational redshift
universal value, independent
of type of atom used for
the transition.

$$\frac{\gamma_B}{\gamma_A} = 1 - \frac{\Delta U}{c^2} \quad \text{where } \Delta U \text{ diff. of Newtonian potentials between A and B}$$

Strong equivalence principle (SEP)

What about the gravitational force itself?

SEP { In the freely falling frame of WEP, not only special relativity is valid, but one can also perform a gravitational experiment* and the result will be the same as in absence of ext. gravitational fields
 * like a Cavendish experiment

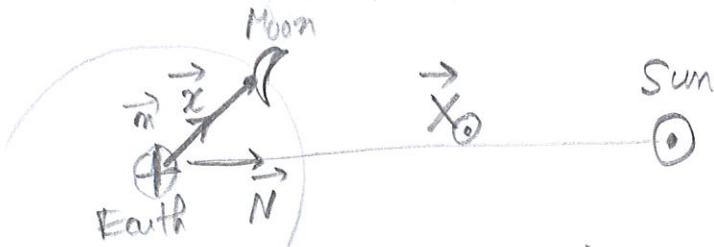
Nordtvedt effect: a test of SEP

Acceleration of planets toward the Sun should depend on their internal gravitational energy in case of violation of SEP

$$\left(\frac{m_g}{m_i}\right)_{\text{planet}} = 1 + \eta \frac{\Omega}{mc^2} \quad \Omega \text{ internal gravitational energy of planet } (\Omega < 0)$$

$$\Omega = -\frac{1}{2} \oint_{\text{planet}} U d^3x, \quad U = \text{usual Newtonian potential} \quad \Delta U = -4\pi G \rho \text{ Poisson equation}$$

Relative motion of Moon w.r.t. Earth in field of Sun



If $\left(\frac{m_g}{m_i}\right)_\oplus \neq \left(\frac{m_g}{m_i}\right)_\odot$ there is an abnormal acceleration

$$\delta a = \frac{GM_\odot}{R_\oplus^2} \left[\left(\frac{m_g}{m_i}\right)_\oplus - \left(\frac{m_g}{m_i}\right)_\odot \right]$$

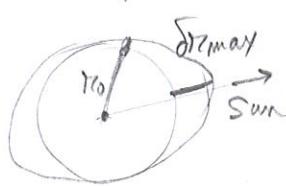
$$\boxed{\frac{d^2 \vec{x}}{dt^2} = -\frac{GM_\oplus \vec{r}}{r^2} - \delta a \vec{N}}$$

where we assume
 where $R_\oplus \gg r$

$M_\odot \gg m_\oplus$

and $\eta \ll 1$

Linearization of solutions around circular orbit gives
 $r(t) = r_0 + \delta r(t)$ where $r_0 = \text{const}$ where $\delta r(t)$ is a small perturbation. Obtain a polarization of Moon's orbit in Sun's direction



$$\boxed{\delta r(t) = -\frac{3\delta a}{2\omega_c \omega_0} \cos[(\omega_c - \omega_0)t]} \quad (\text{exercise})$$

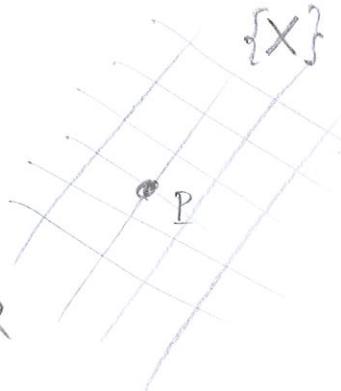
where
 $\omega_0 \ll \omega_c$

This effect is not observed using laser lunar ranging (LLR)
on the Moon hence test of SEP $\approx 10^{-12}$

③ GRAVITATIONAL FORCES

From EEP at any event P in space-time one can erect a locally inertial coord. syst $\{X^\alpha\}$ such that SR is valid in that frame and at P

$$\boxed{ds_P^2 = g_{\alpha\beta} dx^\alpha dx^\beta}$$



At point Q in a neighborhood of P we assume a deformation of interval of SR

$$ds^2 = G_{\alpha\beta}(x) dx^\alpha dx^\beta$$

$$\boxed{G_{\alpha\beta}(x) = g_{\alpha\beta} + \frac{1}{2} \frac{\partial^2 G_{\alpha\beta}}{\partial x^\gamma \partial x^\delta} \Big|_P (x^\gamma - x_P^\gamma)(x^\delta - x_P^\delta) + O((x-x_P)^3)}$$

second order deviation from SR

Now EEP is precisely implemented
Gravitational forces follow from basic differential calculus.

Perform change of coordinates $\{x^\alpha\}$ to some global coord. syst. $\{x^\mu\}$ at P.
Free particle is unaccelerated in loc. inertial frame $\{X^\alpha\}$ at P.

$$0 = \frac{d^2 X^\alpha}{dp^2} = \frac{d}{dp} \left(\frac{\partial X^\alpha}{\partial x^\mu} \frac{dx^\mu}{dp} \right) \quad (\text{calculation performed at P})$$

$$\left[\frac{d^2x^\mu}{dp^2} + \Gamma^\mu_{\nu\rho} \frac{dx^\nu}{dp} \frac{dx^\rho}{dp} = 0 \right] \quad \text{where } \Gamma^\mu_{\nu\rho} = \frac{\partial x^\mu}{\partial x^\nu} \frac{\partial^2 x^\alpha}{\partial x^\nu \partial x^\rho} \quad \begin{matrix} \text{Christoffel symbol} \\ \text{symmetric in } \nu\rho \end{matrix}$$

$$ds^2 = g_{\mu\nu}(x) dx^\mu dx^\nu \quad \text{where} \quad g_{\mu\nu}(x) = \frac{\partial x^\alpha}{\partial x^\mu} \frac{\partial x^\beta}{\partial x^\nu} g_{\alpha\beta}(x)$$

$$\text{At } P \quad g_{\mu\nu}|_P = \frac{\partial x^\alpha}{\partial x^\mu|_P} \frac{\partial x^\beta}{\partial x^\nu|_P} g_{\alpha\beta}$$

$$\partial_\lambda g_{\mu\nu}|_P = \left(\frac{\partial^2 x^\lambda}{\partial x^\mu \partial x^\nu} + \mu \leftrightarrow \nu \right)|_P h_{\lambda\mu\nu}$$

Hence we deduce

$$\boxed{\Gamma^\nu_{\nu\rho} = \frac{1}{2} g^{\mu\lambda} (\partial_\nu g_{\mu\rho} + \partial_\rho g_{\nu\lambda} - \partial_\lambda g_{\mu\nu})}$$

$$\text{Geodesic Lagrangian} \quad L(x, \dot{x}) = \frac{1}{2} g_{\mu\nu}(x) \dot{x}^\mu \dot{x}^\nu \quad \dot{x}^\mu = \frac{dx^\mu}{dp}$$

Geodesic equations

$$\boxed{\frac{d\pi_\mu}{dp} = \frac{\partial L}{\partial \dot{x}^\mu}}$$

$$\boxed{\pi_\mu = \frac{\partial L}{\partial \dot{x}^\mu} = g_{\mu\nu} \dot{x}^\nu} \quad \begin{matrix} \text{(conjugate)} \\ \text{momentum} \end{matrix}$$

$$\text{and} \quad \boxed{\frac{\partial L}{\partial x^\mu} = \partial_\nu g_{\nu\mu} \ddot{x}^\nu}$$

Associated "Hamiltonian"

$$H = \pi_\mu \dot{x}^\mu - L = \frac{1}{2} g_{\mu\nu} \dot{x}^\mu \dot{x}^\nu$$

$$\boxed{H(x, \pi) = \frac{1}{2} g^{\mu\nu}(x) \pi_\mu \pi_\nu}$$

$H = \text{const}$ "on-shell" hence

$$\boxed{g_{\mu\nu}(x) \dot{x}^\mu \dot{x}^\nu = e = \text{const}}$$

always consequence of geodesic equation

If metric does not depend on some coordinate x^λ (e.g. t for stationary system, or ϕ for axi-symmetric system)

The coordinate λ is said to be "ignorable".

$$\boxed{\pi_\lambda = \text{const}}$$

1st integral of motion associated with that symmetry.

Consider a system of particles moving on geodesics



$$u^\mu = \frac{dx^\mu}{d\tau} \quad \tau \text{ proper time} \quad d\tau^2 = -ds^2$$

$$g_{\mu\nu} dx^\mu dx^\nu = e dp^2 = -d\tau^2$$

$e < 0$ for time-like particles

Define acceleration a^μ and covariant derivative

$$a^\mu = u^\nu \nabla_\nu u^\mu = \frac{du^\mu}{d\tau} + \Gamma_{\nu\rho}^\mu u^\nu u^\rho = 0$$

for geodesic motion

Indeed all inertial effects (F) must be included in the definition of acceleration.

Principle of relativity has been "generalized" to any frame: all free particles (not subjected to any force) are inertial i.e. unaccelerated

Tensor variance $\begin{bmatrix} p \\ q \end{bmatrix}$ $T_{\rho\sigma\dots}^{\mu\nu\dots}(x)$

$$\{x\} \rightarrow \{x'\}$$

$$T'_{\rho\sigma\dots}^{\mu\nu\dots}(x') = \frac{\partial x'^\mu}{\partial x^\mu} \frac{\partial x''^\nu}{\partial x^\nu} \dots \frac{\partial x^\lambda}{\partial x^\rho} \frac{\partial x^\tau}{\partial x^\sigma} T_{\lambda\tau}^{\mu\nu\dots}(x)$$

Covariant derivative of tensor $\begin{bmatrix} p \\ q \end{bmatrix} \rightarrow$ tensor $\begin{bmatrix} p \\ q+1 \end{bmatrix}$

$$\nabla_\lambda T_{\rho\sigma\dots}^{\mu\nu\dots} = \frac{\partial}{\partial \lambda} T_{\rho\sigma\dots}^{\mu\nu\dots} + \Gamma_{\lambda\epsilon}^\mu T_{\rho\sigma\dots}^{\epsilon\nu\dots} + \Gamma_{\lambda\epsilon}^\nu T_{\rho\sigma\dots}^{\mu\epsilon\dots} + \dots$$

$$- \Gamma_{\lambda\rho}^\epsilon T_{\epsilon\sigma\dots}^{\mu\nu\dots} - \Gamma_{\lambda\sigma}^\epsilon T_{\rho\epsilon\dots}^{\mu\nu\dots}$$

p terms q terms

Scalar $\begin{bmatrix} 0 \\ 0 \end{bmatrix}$ $ds^2 = g_{\mu\nu} dx^\mu dx^\nu$

$\sqrt{-g} d^4x$ invariant volume element on space-time

$$g = \det(g_{\mu\nu})$$

Vector $\begin{bmatrix} 1 \\ 0 \end{bmatrix}$ u^μ , a^μ
velocity, acceleration

$$\begin{bmatrix} 0 \\ 1 \end{bmatrix} u_\mu = g_{\mu\nu} u^\nu$$

Tensor $\begin{bmatrix} 0 \\ 2 \end{bmatrix} g_{\mu\nu}$ metric

$$\begin{bmatrix} 2 \\ 0 \end{bmatrix} g^{\mu\nu}$$
 (inverse matrix)

$$\begin{bmatrix} 1 \\ 1 \end{bmatrix} \delta_\nu^\mu$$
 (Kronecker tensor)

Tensor $[0]$

$$\epsilon_{\mu\nu\rho} = \sqrt{-g} \epsilon_{\mu\nu\rho}$$

where $\epsilon_{\mu\nu\rho}$ is Levi-Civita symbol

(totally anti-symmetric with $\epsilon_{0123} = +1$)

$[4]$

$$\epsilon^{\mu\nu\rho} = \frac{1}{\sqrt{-g}} \epsilon^{\mu\nu\rho} \quad \text{avec } \epsilon^{\mu\nu\rho} = \epsilon_{\mu\nu\rho}$$

Θ is a tensor of any variance $[P]$.

$P_{\nu\rho}^{\mu}$ is zero in loc. inertial frame (since $\frac{d^2 X^\alpha}{dp^2} = 0$ in that frame) hence $P_{\nu\rho}^{\mu}$ is not a tensor

Ricci theorem

$$\boxed{\nabla_\lambda g_{\mu\nu} = 0}$$

one can raise and lower indices through cov. derivation with metric g

Indeed $\nabla_\lambda g_{\mu\nu}$ is a tensor which vanishes in inertial coord. syst. $\{X^\beta\}$ hence it is zero in any coord. $\{x^\beta\}$.

Riemann tensor $[1]$ $\begin{smallmatrix} 1 \\ 3 \end{smallmatrix}$ (or curvature tensor)

Commutator of covariant derivatives

$$\boxed{(\nabla_\mu \nabla_\nu - \nabla_\nu \nabla_\mu)V^\lambda = R^\lambda_{\mu\nu\rho} V^\rho}$$

(Ricci identity)

$$\boxed{R^\lambda_{\mu\nu\rho} = \partial_\mu H^\lambda_{\nu\rho} - \partial_\nu H^\lambda_{\mu\rho} + H^\varepsilon_{\mu\nu} H^\lambda_{\varepsilon\rho} - H^\varepsilon_{\mu\rho} H^\lambda_{\varepsilon\nu}}$$

Symmetries • $R_{\mu\nu\rho} = -R_{\nu\rho\mu} = R_{\nu\rho\mu}$

(antisym. on two pairs $\mu\nu, \rho\nu$)

• $R_{\mu\nu\rho} = R_{\rho\mu\nu}$

(symmetry pair exchange)

• $R^\lambda_{\mu\nu\rho} + R^\lambda_{\nu\rho\mu} + R^\lambda_{\rho\mu\nu} = 0$

(cyclic permutation)

20 independent components in 4 dimensions.

Bianchi identities (exercise, hint: use local inertial coordinates)

19

$$\boxed{\nabla_\mu R^\lambda_{\nu\rho\sigma} + \nabla_\nu R^\lambda_{\rho\mu\sigma} + \nabla_\rho R^\lambda_{\mu\nu\sigma} = 0}$$

Only one meaningful contraction of Bianchi

$$\nabla_\mu G^{\mu\nu} = 0 \quad \text{with} \quad \boxed{G^{\mu\nu} = R^{\mu\nu} - \frac{1}{2} R g^{\mu\nu}}$$

Einstein identity

Einstein tensor

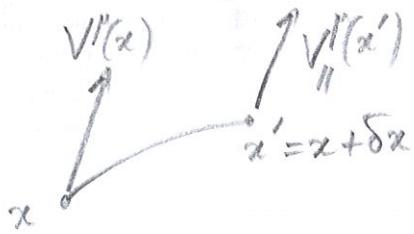
$$R_{\mu\nu} = R^\lambda_{\lambda\mu\nu} \quad \text{Ricci tensor (symmetric in } \mu\nu)$$

$$R = g^{\mu\nu} R_{\mu\nu} \quad \text{curvature scalar}$$

(Riemann 1854)

Various aspects of curvature

- exists global system of inertial coordinate $\Rightarrow R^\mu_{\nu\rho\sigma} = 0$
- Parallel transport of vector $V^\mu(x)$ from x to $x + \delta x$



$$V''_||^mu(x') = V''^mu(x) - \Gamma_{\nu\rho}^\mu V^\nu(x) \delta x^\rho$$

Vector parallelly transported along δx^ρ

$$\delta x^\rho \nabla_\nu V^\mu = 0$$

At some point one could erect some loc. inertial coord. $\{x^\alpha\}$ and define by parallel transport a new loc. in. frame, by // transport.

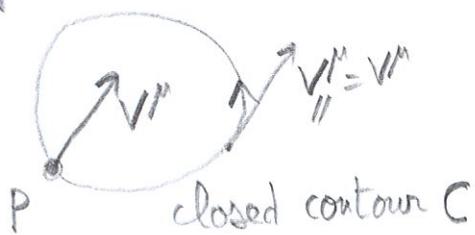
One would define the 4 vectors e_α^μ (for $\alpha = 0, 1, 2, 3$)

~~$\{x^\alpha\}$ $e_\alpha^\mu = \frac{\partial x^\mu}{\partial x^\alpha}$ and // transport them to a neighbouring point~~
In that way one could define a global in. cond. myt?

Impossible

Transport // vector V^μ along closed contour C

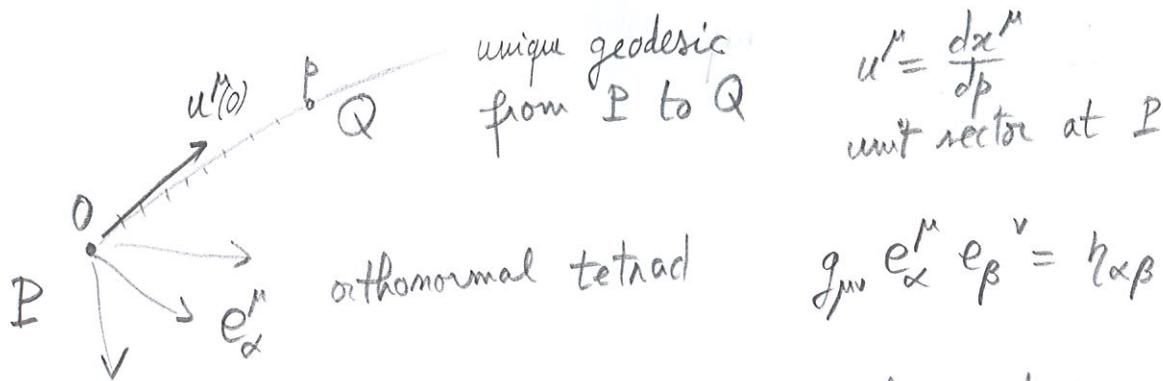
$$\Delta V^\mu = - \oint \Gamma_{\nu\rho}^\mu V^\nu d x^\rho \quad (\text{exercise})$$



$$\Delta V^\mu = - \frac{1}{2} R^\mu_{\nu\rho\sigma} \int_P \int_P V^\nu \int_C x^\rho dx^\sigma$$

= $\int d\lambda d\sigma$ of surface resting on C

• Gaussian normal coordinates



p: affine parameter of position of Q along that geodesic

$$u^\mu(0) = u^\alpha e_\alpha^\mu \quad \text{Gauss normal coordinates of event Q}$$

$$X^\alpha = p u^\alpha$$

Metric in normal coordinates
(exercise)

$$g_{\alpha\beta} = \eta_{\alpha\beta} + \frac{1}{3} R_{\alpha\gamma\beta\delta} X^\gamma X^\delta + O(3)$$

EINSTEIN FIELD EQUATIONS

They derive from the Einstein-Hilbert action (1915)

$$S = \frac{c^3}{16\pi G} \int d^4x \sqrt{-g} R + S_m[\psi, g]$$

↓

scalar curvature

invariant volume element

action for all non-gravitational fields

matter fields

supposed to be a scalar

To get the Einstein field eqs we vary the action w.r.t. the metric $g_{\mu\nu}$ assuming that

$$\delta g_{\mu\nu}(+\infty) = 0$$

i.e. when $|x^\lambda| \rightarrow \infty$
at the boundary of s.t.

In a first stage we assume also that the derivatives of the metric vanish on the boundary

$$\nabla_\lambda \delta g_{\mu\nu}(+\infty) = 0$$

but later we shall relax this assumption and shall discuss something interesting

We have to vary $R = g^{\mu\nu} R_{\mu\nu}$
 $\uparrow \mu\nu$ Ricci tensor

when $g_{\mu\nu} \rightarrow g_{\mu\nu} + \delta g_{\mu\nu}$

We consider that $\delta g_{\mu\nu}$ is a tensor field
on the background $g_{\mu\nu}$, and we use the covariant derivative
of $g_{\mu\nu}$ to express perturbations of Christoffel and Ricci.

Palatini formulas

$$\boxed{\begin{aligned}\delta T_{\nu\rho}^{\mu} &= \frac{1}{2} g^{\mu\lambda} (\nabla_{\nu} \delta g_{\rho\lambda} + \nabla_{\rho} \delta g_{\nu\lambda} - \nabla_{\lambda} \delta g_{\nu\rho}) \\ \delta R_{\mu\nu} &= \nabla_{\rho} \delta T_{\mu\nu}^{\rho} - \nabla_{\nu} \delta T_{\mu\rho}^{\rho}\end{aligned}}$$

Proofs: Tensorial relations that are true in loc. inatial coordinates
 $\{x^\alpha\}$ for which $\partial g = \Gamma = 0$ (but of course $\partial \Gamma \neq 0$)

$$\begin{aligned}\delta(\sqrt{-g} R) &= \delta(\sqrt{-g} g^{\mu\nu} R_{\mu\nu}) \\ &= \delta(\sqrt{-g} g^{\mu\nu}) R_{\mu\nu} + \sqrt{-g} g^{\mu\nu} \delta R_{\mu\nu}\end{aligned}$$

$$g = \det g_{\mu\nu} \quad \delta g = g g^{\mu\nu} \delta g_{\mu\nu} \quad \delta \sqrt{-g} = \frac{\sqrt{-g}}{2} g^{\mu\nu} \delta g_{\mu\nu}$$

$$\boxed{\delta(\sqrt{-g} R) = \sqrt{-g} \left(\delta g^{\mu\nu} G_{\mu\nu} + \nabla_\mu V^\mu \right)}$$

Einstein tensor $G^{\mu\nu} = R^{\mu\nu} - \frac{1}{2} g^{\mu\nu} R$

where $\boxed{V^\mu = g^{\rho\sigma} \delta T_{\rho\sigma}^{\mu} - g^{\mu\rho} \delta T_{\rho\sigma}^{\sigma}}$

Second term is a surface term

$$\sqrt{-g} \nabla_\mu V^\mu = \partial_\mu (\sqrt{-g} V^\mu)$$

$$\int d^4x \sqrt{-g} \nabla_\mu V^\mu = \int dS_\mu \sqrt{-g} V^\mu \quad \begin{matrix} \text{surface term} \\ \text{at infinity} \end{matrix}$$

Since δg and $\nabla \delta g = 0$ at infinity one can ignore this term
(see later for more complete treatment)

$$\delta S = \frac{c^3}{16\pi G} \int d^4x \sqrt{-g} \delta g^{\mu\nu} G_{\mu\nu} + \delta S_m$$

Stress-energy tensor of matter fields: $T^{\mu\nu}$

By definition

$$\delta S_m = \frac{1}{2} \int d^4x \sqrt{-g} T^{\mu\nu} \delta g_{\mu\nu}$$

Hence

$$T^{\mu\nu} = \frac{2}{\sqrt{-g}} \frac{\delta S_m}{\delta g^{\mu\nu}}$$

(with matter fields Ψ fixed)

By construction $T^{\mu\nu}$ is always symmetric.

$$\delta g^{\mu\nu} G_{\mu\nu} = -g^{\mu\rho} g^{\nu\sigma} \delta g_{\rho\sigma} G_{\mu\nu} = -\delta g_{\mu\nu} G^{\mu\nu} \text{ hence the EFE read}$$

$$G^{\mu\nu} = \frac{8\pi G}{c^4} T^{\mu\nu}$$

Cosmological constant

$$S = \frac{c^3}{16\pi G} \int d^4x \sqrt{-g} (R - 2\Lambda) + S_{\text{mat}}$$

$$G^{\mu\nu} + \Lambda g^{\mu\nu} = \frac{8\pi G}{c^4} T^{\mu\nu}$$

In standard cosmological model Λ -CDM

Dark energy
 $\approx 70\%$

Baryons + Dark matter (CDM)
5% 25%

Bianchi identity (or, rather, Einstein identity)

$$\nabla_\nu G^{\mu\nu} = 0 \Rightarrow \boxed{\nabla_\nu T^{\mu\nu} = 0}$$

(and $\nabla_\nu g^{\mu\nu} = 0$ from)
Ricci theorem

equation of motion
of matter fields

Let us find the relation to the equations of motion
of matter field.

We assume e.g. Ψ is a scalar field

$$S_m = \int d^4x L_m(\Psi, \partial_\mu \Psi, g_{\mu\nu})$$

Vary with $\delta\Psi(\infty) = 0$

$$\delta S_m = \int d^4x \left\{ \delta\Psi \underbrace{\frac{\partial L_m}{\partial \Psi}}_{\text{integrate by parts}} + \delta \partial_\mu \Psi \underbrace{\frac{\partial L_m}{\partial \partial_\mu \Psi}}_{\text{recognize definition of } T^{\mu\nu}} + \delta g_{\mu\nu} \frac{\partial L_m}{\partial g_{\mu\nu}} \right\}$$

$$\boxed{\delta S_m = \int d^4x \left\{ \delta\Psi \frac{\delta L_m}{\delta \Psi} + \frac{-g}{2} \delta g_{\mu\nu} T^{\mu\nu} \right\}}$$

Valid for any variation $\delta\Psi$ and $\delta g_{\mu\nu}$

If we vary only matter field $\delta\Psi$ with $g_{\mu\nu}$ fixed one
obtain the EoM of matter

$$\boxed{\frac{\delta L_m}{\delta \Psi} = \frac{\partial L_m}{\partial \Psi} - \partial^\mu \left(\frac{\partial L_m}{\partial \partial_\mu \Psi} \right) = 0}$$

To find the relation with $\nabla_\nu T^\mu$ we recall S_m is a scalar, must be invariant under a coordinate transformation $\{x\} \rightarrow \{x'\}$ 17

$$x'^\mu = x^\mu - \epsilon^\mu(x) \quad \text{for any } \epsilon^\mu$$

Thus the Lagrangian density \mathcal{L}_m must describe the same physics in $\{x\}$ and in $\{x'\}$

$$S_m = \int d^4x \mathcal{L}_m [\psi(x), g(x)] = \int d^4x' \underbrace{\mathcal{L}_m [\psi'(x'), g'(x')]}_{\substack{\text{same functional of} \\ \text{matter fields}}} \underbrace{\mathcal{L}_m [\psi'(x'), g'(x')]}_{\substack{\text{new components of} \\ \text{fields in new} \\ \text{coord. system}}}$$

Here x' is a dummy variable
can be replaced by x

$$0 = \int d^4x \left\{ \mathcal{L}_m [\psi'(x), g'(x)] - \mathcal{L}_m [\psi(x), g(x)] \right\}$$

ψ is a scalar [0] while $g_{\mu\nu}$ is a tensor [2]

$$\psi'(x') = \psi(x) \quad \psi'(x) = \psi(x) + \epsilon^\mu \partial_\mu \psi \quad \text{replace by covariant derivative}$$

Lie derivative $\boxed{\mathcal{L}_\epsilon \psi = \epsilon^\mu \nabla_\mu \psi}$ is a scalar [0]

$$g'_{\mu\nu}(x') = g_{\mu\nu}(x) \frac{\partial x^\rho}{\partial x'^\mu} \frac{\partial x^\sigma}{\partial x'^\nu} = g_{\mu\nu}(x) + 2 \partial_\mu \epsilon^\rho \partial_\nu \epsilon^\sigma$$

$$\mathcal{L}_\epsilon g_{\mu\nu} = \epsilon^\rho \partial_\rho g_{\mu\nu} + 2 \partial_\mu \epsilon^\rho \partial_\nu \epsilon^\sigma$$

Ricci theorem $\nabla_\rho g_{\mu\nu} = 0 \quad \partial_\rho g_{\mu\nu} = 2 \sum_{\rho} \epsilon^\rho g_{\mu\rho} g_{\nu\rho}$

$$\begin{aligned} \mathcal{L}_\epsilon g_{\mu\nu} &= 2 \partial_\mu \epsilon_\nu + \epsilon^\rho \partial_\nu \epsilon_\rho \\ &= 2 g_{\mu\nu} \nabla_\nu \epsilon^\rho \end{aligned}$$

$$\boxed{\mathcal{L}_\epsilon g_{\mu\nu} = 2 \nabla_\mu \epsilon_\nu}$$

tensor [2]

More generally Lie derivative of tensor $T^{\mu\nu\dots}_{\rho\dots}$ Lf/ 18

$$\mathcal{L}_\epsilon T^{\mu\nu\dots}_{\rho\dots} = \epsilon^\lambda \nabla_\lambda T^{\mu\nu\dots}_{\rho\dots} + \nabla_\rho \epsilon^\lambda T^{\mu\nu\dots}_{\lambda\dots} + \nabla_\lambda \epsilon^\rho T^{\mu\nu\dots}_{\lambda\dots} - \nabla_\lambda \epsilon^\mu T^{\lambda\nu\dots}_{\rho\dots} - \nabla_\lambda \epsilon^\nu T^{\mu\lambda\dots}_{\rho\dots} + \dots$$

Hence scalarity condition is (for any ϵ^μ such that $\epsilon^\mu(\infty) = 0$)

$$0 = \int d^4x \left\{ \mathcal{L}_\epsilon \psi \frac{\delta \mathcal{L}_m}{\delta \psi} + \frac{\sqrt{-g}}{2} \mathcal{L}_\epsilon g^{\mu\nu} T^{\mu\nu} \right\}$$

$$0 = \int d^4x \left\{ \epsilon^\mu \nabla_\mu \psi \frac{\delta \mathcal{L}_m}{\delta \psi} + \underbrace{\sqrt{-g} \nabla_\mu \epsilon^\mu T^{\mu\nu}}_{\partial_\mu (\sqrt{-g} \epsilon^\mu) T^{\mu\nu}} \right\}$$

$$0 = \int d^4x \epsilon^\mu \left\{ \nabla_\mu \psi \frac{\delta \mathcal{L}_m}{\delta \psi} - \sqrt{-g} \nabla_\mu T^\mu_\nu \right\} \quad \forall \epsilon^\mu \text{ with } \epsilon^\mu(\infty) = 0$$

$$\boxed{\sqrt{-g} \nabla_\mu T^\mu_\nu = \nabla_\mu \psi \frac{\delta \mathcal{L}_m}{\delta \psi} = 0}$$

Thus EFE contain correctly, via the Bianchi identities, the EOM of the matter fields.

This is a very important (and beautiful) property of GR.

Argument applied to S_g gives $\nabla_\mu G^{\mu\nu} = 0$. Thus Bianchi's identity guarantees that S_g is a covariant scalar.

Treatment of the surface term

We obtained

$$\delta(\int g R) = \int g (\delta g^{\mu\nu} G_{\mu\nu} + \nabla_\mu V^\mu)$$

$$\text{with } V^\mu = g^{\rho\sigma} \delta T^\mu_{\rho\sigma} - g^{\mu\rho} \delta T^\nu_{\rho\nu}$$

We can replace δT by its expression from Palatini's formula and get

$$\boxed{V_\mu = \nabla^\nu (\delta g_{\mu\nu} - g_{\mu\nu} g^{\rho\sigma} \delta g_{\rho\sigma})}$$

We have ignored this surface term assuming that $\delta g_{\mu\nu}$ and $\nabla \delta g_{\mu\nu}$ vanish on the boundary.

Let us do now the variation assuming only that

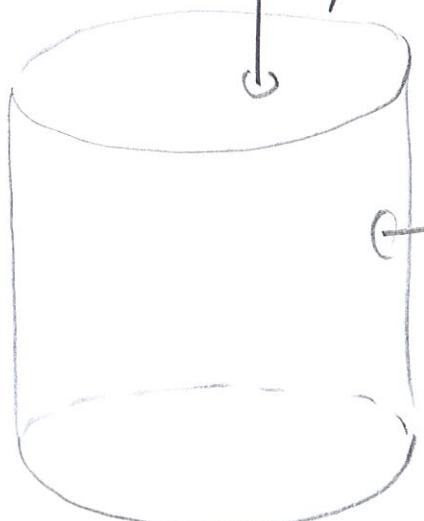
$$\delta g_{\mu\nu}(x) = 0$$

but with no conditions required on derivatives.

$$\int d^4x \int g \nabla_\mu V^\mu = \int d\Sigma_\mu V^\mu \quad \text{where } d\Sigma_\mu \text{ is surface vector on boundary}$$

$$\uparrow d\Sigma_\mu = m_\mu d\Sigma \text{ with } g_{\mu\nu} m^\mu m^\nu = -1 \text{ time-like}$$

$$(d\Sigma \sim d^3x)$$



$$d\Sigma_\mu = m_\mu d\Sigma \quad \text{with } g_{\mu\nu} m^\mu m^\nu = 1 \text{ space-like}$$

Induced metric on boundary

$$g_{\mu\nu} = g_{\mu\nu} + m_\mu m_\nu \quad \text{time-like}$$

$$g_{\mu\nu} = g_{\mu\nu} - m_\mu m_\nu \quad \text{space-like}$$

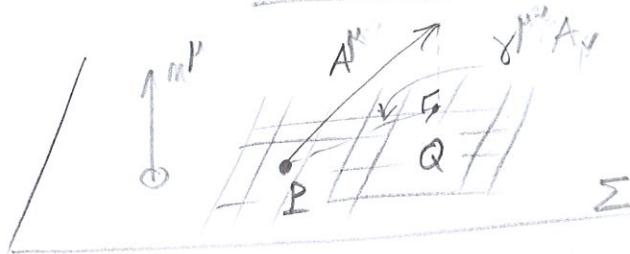
$$\begin{aligned} m^{\mu} V_{\mu} &= m^{\mu} \nabla^{\nu} \delta g_{\mu\nu} - m_{\nu} g^{\rho\sigma} \nabla^{\nu} \delta g_{\rho\sigma} \\ &= m^{\mu} g^{\nu\rho} \left(\underbrace{\nabla_{\rho} \delta g_{\mu\nu} - \nabla_{\mu} \delta g_{\nu\rho}}_{\text{anti-sym } \rho \leftrightarrow \mu} \right) \\ &= m^{\mu} \gamma^{\nu\rho} \left(\nabla_{\rho} \delta g_{\mu\nu} - \nabla_{\mu} \delta g_{\nu\rho} \right) \end{aligned}$$

where $\gamma^{\nu\rho} = g^{\nu\rho} \pm m^{\nu} m^{\rho}$

Since $\delta g_{\mu\nu} = 0$ on Σ we have

$$\boxed{\gamma^{\rho\sigma} \nabla_{\sigma} \delta g_{\mu\nu} = 0}$$

Proof: loc. inertial coord on Σ
 $\{x^{\alpha}\}$



$$\delta g_{\alpha\beta}|_Q = \delta g_{\alpha\beta}|_P + \underbrace{(X_Q^{\epsilon} - X_P^{\epsilon}) \frac{\partial \delta g_{\alpha\beta}}{\partial X^{\epsilon}}|_P}_{=0} \quad \text{and} \quad X_Q^{\epsilon} - X_P^{\epsilon} = \gamma^{\epsilon\delta} A_{\delta} \quad \text{for any } A_{\delta} \quad (\text{since } Q \text{ is arbitrary})$$

so $\gamma^{\epsilon\delta} \frac{\partial \delta g_{\alpha\beta}}{\partial X^{\epsilon}}|_P = 0$ in loc. inertial coord

So we get $m^{\mu} V_{\mu} = -m^{\mu} \gamma^{\nu\rho} \nabla_{\mu} \delta g_{\nu\rho}$ on Σ . This is given by the variation of the (trace of the) extrinsic curvature K on the surface

$$K = \underbrace{\gamma_{\nu}^{\mu} \nabla_{\mu} m^{\nu}}_{(\text{trace of the})} = (\gamma_{\nu}^{\mu} \pm g_{\nu\rho} m^{\mu} m^{\rho}) (\partial_{\mu} m^{\nu} + \Gamma_{\mu\sigma}^{\nu} m^{\sigma})$$

(see below for extensive study of $K_{\mu\nu}$)

$$\text{Since } \delta g = 0 \quad \delta K = \gamma_{\nu}^{\mu} \delta \Gamma_{\mu\sigma}^{\nu} m^{\sigma} = \frac{1}{2} \gamma^{\mu\lambda} \nabla_{\sigma} \delta g_{\mu\lambda} m^{\sigma}$$

$$\boxed{m^{\mu} V_{\mu} = -2 \delta K \quad \text{on surface}}$$

$$\delta S_g = \frac{c^3}{16\pi G} \int d^4x \sqrt{g} \delta g^{\mu\nu} G_{\mu\nu} - \frac{c^3}{8\pi G} \delta \int d\Sigma K$$

Hence the appropriate action when boundary terms are included is

$$S'_g = S_g + \frac{c^3}{8\pi G} \int d\Sigma K$$

Klein-Gordon field

$$\mathcal{L} = -\frac{\sqrt{-g}}{2} \left(g^{\mu\nu} \partial_\mu \varphi \partial_\nu \varphi + m^2 \varphi^2 \right)$$

$m \equiv \frac{mc}{\hbar}$
in SI units

$$\begin{aligned} \frac{\delta \mathcal{L}}{\delta \varphi} &= \frac{\partial \mathcal{L}}{\partial \varphi} - \partial^\mu \left(\frac{\partial \mathcal{L}}{\partial \partial_\mu \varphi} \right) \\ &= -m^2 \sqrt{-g} \varphi + \underbrace{\partial_\mu (\sqrt{-g} g^{\mu\nu} \partial_\nu \varphi)}_{\sqrt{-g} \nabla_\mu (g^{\mu\nu} \partial_\nu \varphi)} \equiv \sqrt{-g} \square \varphi \end{aligned}$$

Klein-Gordon equation $\boxed{\square \varphi - m^2 \varphi = 0}$

$\square = g^{\mu\nu} \partial_\mu \partial_\nu$ acting
on any tensor

$$\begin{aligned} \frac{\partial \mathcal{L}}{\partial g_{\mu\nu}} &= -\frac{\sqrt{-g}}{4} g^{\mu\nu} \left(g^{\rho\sigma} \partial_\rho \varphi \partial_\sigma \varphi + m^2 \varphi^2 \right) \\ &\quad - \frac{\sqrt{-g}}{2} \left(-g^{\mu\rho} g^{\nu\sigma} \partial_\rho \varphi \partial_\sigma \varphi \right) \end{aligned}$$

$$\boxed{T^{\mu\nu} = \frac{2}{\sqrt{-g}} \frac{\partial \mathcal{L}}{\partial g_{\mu\nu}} = \nabla^\mu \varphi \nabla^\nu \varphi - \frac{1}{2} g^{\mu\nu} \left(\nabla_\rho \varphi \nabla^\rho \varphi + m^2 \varphi^2 \right)}$$

check $\nabla_\nu T^{\mu\nu} = 0$ when KG equation
is satisfied.

Electromagnetic field

$$\mathcal{L} = -\frac{\sqrt{-g}}{4\mu_0} g^{\mu\rho} g^{\nu\sigma} F_{\mu\nu} F_{\rho\sigma} + \sqrt{-g} j^\mu e A_\mu \quad \left(\mu_0 = \text{diamagnetic constant related to dielectric constant } \epsilon_0 \text{ by } \mu_0 \epsilon_0 c^2 = 1 \right)$$

where Faraday tensor is

$$\begin{aligned} F_{\mu\nu} &= \partial_\mu A_\nu - \partial_\nu A_\mu \\ &= \underbrace{\partial_\mu A_\nu}_{\text{independent of metric}} - \underbrace{\partial_\nu A_\mu}_{\text{independent of metric}} \end{aligned}$$

A_μ vector potential is the independent d.o.f. of EM field
 j^μ charged current conserved in the sense $\boxed{\nabla_\mu j^\mu_e = 0}$

$$\frac{\delta \mathcal{L}}{\delta A_\mu} = \sqrt{-g} j^\mu_e$$

$$\frac{\delta \mathcal{L}}{\delta \partial_\nu A_\mu} = -\frac{\sqrt{-g}}{4\mu_0} g^{\mu\rho} g^{\nu\sigma} (2)(2)(-1) F_{\rho\sigma}$$

$$\frac{\delta \mathcal{L}}{\delta A_\mu} = \sqrt{-g} j^\mu_e + \underbrace{\frac{1}{\mu_0} \partial_\nu (\sqrt{-g} F^{\mu\nu})}_{\sqrt{-g} \nabla_\nu F^{\mu\nu} \text{ since } F^{\mu\nu} \text{ is antisymmetric}}$$

$$\boxed{\nabla_\nu F^{\mu\nu} = -\mu_0 j^\mu_e}$$

Maxwell's equations in GR

Compatibility with charge conservation.

$$\begin{aligned} \nabla_\mu \nabla_\nu F^{\mu\nu} &= \nabla_\nu \nabla_\mu F^{\mu\nu} + \underbrace{R^\mu_{\nu\epsilon\mu\nu} F^{\epsilon\nu}}_{R_{\nu\epsilon} F^{\epsilon\nu} = 0} + \underbrace{R^\nu_{\epsilon\mu\nu\mu} F^{\mu\epsilon}}_{-R_{\epsilon\mu} F^{\mu\epsilon} = 0} \\ &= 0 \end{aligned} \quad \text{since } F^{\mu\nu} \text{ is antisymmetric}$$

Hence charge conservation is implied by Maxwell's equations

Interesting form: introduce coordinate charged current

$$\boxed{j^\mu_e = \sqrt{-g} \overset{*}{j}{}^\mu_e} \quad \begin{matrix} \text{(not a tensor)} \\ \text{but a} \\ \text{tensor density} \end{matrix} \quad \boxed{\partial_\mu \overset{*}{j}{}^\mu_e = 0}$$

This $\overset{*}{j}{}^\mu_e$ represents
 the true d.o.f. of
 the charges particles

$$\boxed{\partial_\nu (\sqrt{-g} F^{\mu\nu}) = -\mu_0 \overset{*}{j}{}^\mu_e}$$

This coordinate current j_e^μ is independent of metric.

Thus the coupling term

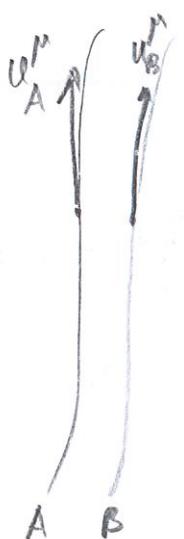
$$\sqrt{-g} j_e^\mu A_\mu = \underset{*}{j_e^\mu} A_\mu$$

The independent d.o.f. are thus $(A_\mu, \underset{*}{j_e^\mu}, g_{\mu\nu})$

does not contribute to the stress-energy tensor. Hence

$$T^{\mu\nu} = \frac{1}{\mu_0} \left(F_{,\rho}^\mu F^{\nu\rho} - \frac{1}{4} g^{\mu\nu} F^2 \right)$$

Point-particles



$$S = \sum_A m_A \int_{-\infty}^{+\infty} dt_A = \sum_A m_A \int \sqrt{-(g_{\mu\nu})_A dy_A^\mu dy_A^\nu}$$

$\underbrace{\hspace{1cm}}$
propagating
along particle's
world line

Vary w.r.t. metric

$$\delta S = \sum_A -\frac{m_A}{2} \int \frac{-(\delta g_{\mu\nu})_A dy_A^\mu dy_A^\nu}{\sqrt{-(g_{\mu\nu})_A dy_A^\mu dy_A^\nu}}$$

Hence

$$T^{\mu\nu} = \frac{1}{\sqrt{-g}} \sum_A m_A \int_{-\infty}^{+\infty} dt_A u_A^\mu u_A^\nu \delta_{(4)}(x - y_A)$$

$$u_A^\mu = \frac{dy_A^\mu}{dt_A}$$

Vary w.r.t. particles

$$y_A^\mu \int \delta y_A^\nu + \delta y_A^\mu$$

$$\delta S = \sum_A -m_A \int \frac{1}{2 dt_A} \left[-2(g_{\mu\nu})_A d\delta y_A^\mu d\delta y_A^\nu - \delta y_A^\mu (\partial_\mu g_{\rho\sigma})_A dy_A^\rho dy_A^\sigma \right]$$

$$= \sum_A m_A \int \left(g_{\mu\nu}^A \right) \partial \delta y_A^\mu u_A^\nu + \frac{1}{2} \delta y_A^\mu (\partial_\mu g_{\rho\sigma}^A) \delta y_A^\rho \delta y_A^\sigma$$

$$= \sum_A m_A \int dt \delta y_A^\mu \left[- \frac{d}{dt} \left(\left(g_{\mu\nu}^A \right) u_A^\nu \right) + \frac{1}{2} (\partial_\mu g_{\rho\sigma}^A) u_A^\rho u_A^\sigma \right]$$

$$\boxed{\frac{d}{dt} \left(\left(g_{\mu\nu}^A \right) u_A^\nu \right) = \frac{1}{2} (\partial_\mu g_{\rho\sigma}^A) u_A^\rho u_A^\sigma}$$

geodesic equation

Here this is the motion of N particle on a fixed background $g_{\mu\nu}$. A much more complicated problem is the motion of N particles under their mutual gravitational interactions.

One can show that the geodesic equation remains valid provided that one removes the self-field of the particles by means of a suitable regularization (dimensional regularization)

$$\left(g_{\mu\nu}^A \right) \rightarrow \text{Dim Reg } \left(g_{\mu\nu}^A \right)$$

where the space dimension d is considered a complex number and analytic continuation is performed on $d \in \mathbb{C}$.

Fluid of particles

For a fluid of particles w.o. interactions

$$S = \sum -m_A \int_{-\infty}^{+\infty} dt \sqrt{-\left(g_{\mu\nu}^A \right) v_A^\mu v_A^\nu} = - \int d^4x \rho \sqrt{-g_{\mu\nu} v^\mu v^\nu}$$

$v_A^\mu = \frac{dy_A^\mu}{dt}$ $y_A^0 = t$
 coordinate velocity $v_A^\mu = (c, v_A^i)$

$$\rho^*(\vec{x}, t) = \sum_A m_A \delta_3(\vec{x} - \vec{y}_A(t))$$

coordinate density

$$\partial_t \rho^* = \sum_A m_A (-v_A^i) \partial_i \delta_3(\vec{x} - \vec{y}_A) = -\partial_i \left[\sum_A m_A v_A^i \delta_3 \right]$$

$$\partial_t \rho^* + \partial_i (\rho^* v^i) = 0$$

$$\partial_\mu j^\mu = 0 \quad \text{where } j^\mu = \rho^* v^\mu = \sqrt{-g} j^\mu$$

coordinate current

$$j^\mu = \rho^* v^\mu = \sqrt{-g} \rho u^\mu$$

$$\rho^* = \sqrt{-g} \rho u^0$$

$$\nabla_\mu j^\mu = 0$$

ρ = proper density (i.e. density of particles in particle's rest frame)
is a scalar

ρ^* = coordinate density (number of particles in unit volume of coordinate grid) is not a scalar

$$u^0 = \frac{1}{\sqrt{-g_{\mu\nu} v^\mu v^\nu}} \quad u^\mu = u^0 v^\mu \quad \text{4-velocity}$$

Fluid of particles w/o interactions described by

$$S = - \int d^4x \sqrt{-g} \rho$$

$$T^{\mu\nu} = \rho u^\mu u^\nu$$

where ρ is the conserved scalar density $\nabla_\mu (\rho u^\mu) = 0$

E.O.M. of fluid is

$$0 = \nabla_\nu T^{\mu\nu} = \underbrace{\nabla_\nu (\rho u^\nu)}_{=0} u^\mu + \rho u^\nu \nabla_\nu u^\mu$$

$$a^\mu = 0$$

where acceleration is $a^\mu = \dot{u}^\nu \nabla_\nu u^\mu$
geodesic motion.

Perfect fluid

$$S = - \int d^4x \sqrt{-g} \epsilon$$

$$T^{\mu\nu} = (\epsilon + p) u^\mu u^\nu + p g^{\mu\nu}$$

(no heat flow included)
 no viscosity
 so-called perfect fluid

where $\epsilon = \rho(1 + \tau\tau)$

ϵ = proper energy density (i.e. measured in rest frame)
 p = pressure
 $\tau\tau$ = specific internal energy (i.e. per unit mass)
 ρ = proper conserved mass density
 $\nabla_\mu(\rho u^\mu) = 0$

Thermodynamical relation

$$dU = TdS - pdV \quad \text{where}$$

$$U = M\tau\tau$$

$$V = \frac{M}{\rho}$$

$$S = Ms$$

↑
specific entropy

$$d\tau\tau = Tds - p d\left(\frac{1}{\rho}\right)$$

Other form where we pose

$$h = \frac{\epsilon + p}{\rho} = 1 + \tau\tau + \frac{p}{\rho}$$

specific enthalpy

$$d\epsilon = h dp + \rho T ds$$

Independent d.o.f. of fluid

$j^\mu = \rho u^\mu$ conserved current
 $s = \text{specific enthalpy}$

E.O.M. $\nabla_\nu T^\mu{}^\nu = 0$ is equivalent to

$$\boxed{u^\mu \nabla_\mu s = 0 \quad \text{entropy is conserved along fluid lines} \quad |}$$

$$(E+p) u^\mu + (g^{\mu\nu} + u^\mu u^\nu) \nabla_\nu p = 0 \quad \begin{array}{l} \text{Euler equation} \\ (\text{non geodesic motion}) \end{array}$$

Actually these two equations are equivalent to

$$\boxed{u^\nu \left(\frac{\partial C_\mu}{\partial j^\mu} - \frac{\partial C_\nu}{\partial j^\mu} \right) = T^\mu_\nu s \quad |}$$

where $C_\mu = h u_\mu$ is the enthalpy current

This permits to make the link with the action

$$\mathcal{L} = \int g L \quad \text{with} \quad L(j^\mu, s) = -\varepsilon$$

Lagrange equations with fluid variables (exercise)

$$\boxed{j^\nu \left[\frac{\partial}{\partial j^\mu} \left(\frac{\partial L}{\partial j^\mu} \right) - \frac{\partial}{\partial j^\mu} \left(\frac{\partial L}{\partial j^\nu} \right) \right] = - \frac{\partial L}{\partial s} \frac{\partial s}{\partial j^\mu} \quad |}$$

$$\text{Here } dL = -d\varepsilon = -h dp + p T ds$$

$$\text{with } dp = \sqrt{-g^{\mu\nu} j^\mu j^\nu} = \frac{1}{2\rho} (-2 j^\mu d j^\mu) = -u_\mu d j^\mu$$

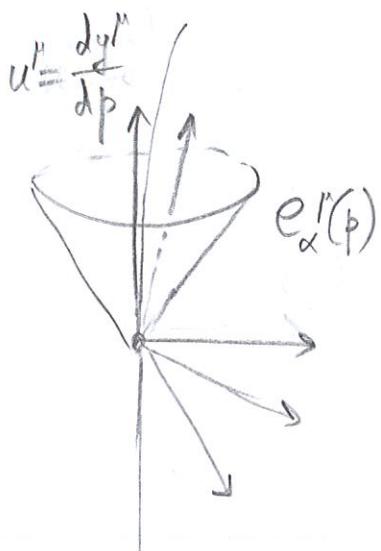
$$dL = h u_\mu d j^\mu + p T ds$$

$$\frac{\partial L}{\partial j^\mu} = h u_\mu = C_\mu \quad \frac{\partial L}{\partial s} = p T$$

in agreement with previous E.O.M.

Particle with spin, Mathisson-Papapetrou equation.

We consider a spinning point particle moving in a given background metric $g_{\mu\nu}$. Particle follows the world line $y^\mu(p)$ where p is an affine parameter



$$4\text{-velocity } u^\mu = \frac{dy^\mu}{dp}$$

We do not assume in a first stage that p is necessarily the proper time.

As we shall see the correct way to describe extra degrees of freedom associated with the spin of the particle is to introduce a moving orthonormal tetrad $e_{\alpha}^{\mu}(p)$ along the trajectory.

$$\boxed{g_{\mu\nu} e_{\alpha}^{\mu} e_{\beta}^{\nu} = \eta_{\alpha\beta}} *$$

Here μ, ν are spacetime indices, while α, β label the tetrad vectors ($\alpha, \beta, \dots = 0, 1, 2, 3$).

The rotation tensor $\Omega^{\mu\nu}$ associated with the tetrad is such that

$$\boxed{\frac{De_{\alpha}^{\mu}}{dp} = -\Omega^{\mu\nu} e_{\alpha\nu} \quad \text{or} \quad \Omega^{\mu\nu} = e^{\mu\rho} \frac{De_{\alpha}^{\nu}}{dp}}$$

where the covariant derivative is $\frac{D}{dp} = \dot{u}^{\nu} \nabla_{\nu}$. Because of

* The inverse tetrad is $e^{\alpha}_{\mu} = h^{\alpha\beta} g_{\mu\nu} e_{\beta}^{\nu}$, and satisfies $e_{\alpha}^{\mu} e^{\alpha}_{\nu} = \delta_{\nu}^{\mu}$ and $h_{\alpha\beta} e^{\alpha}_{\mu} e^{\beta}_{\nu} = g_{\mu\nu}$, as well as the "completeness" relation $e_{\alpha}^{\mu} e^{\beta}_{\mu} = \delta_{\alpha}^{\beta}$.

$$\boxed{-\Omega^{\mu\nu} = -\Omega^{\nu\mu}}$$

The action of the spinning particle (to be added to the Einstein-Hilbert action for gravity) will depend on dynamical variables, which are here the position y^μ (like for point particles without spin) and the tetrad e_α^μ .

Furthermore we shall restrict to a formalism called "pole-dipole" where the Lagrangian will only depend on the 4-velocity u^μ , the rotation tensor $\Omega^{\mu\nu}$ and the metric. Hence

$$\boxed{S[y^\mu, e_\alpha^\mu] = \int_{-\infty}^{+\infty} dp L(u^\mu, \Omega^{\mu\nu}, g_{\mu\nu})}$$

This formalism will be able to describe particles with spins, but we neglect any internal structure of the particle, which in this formalism will appear at quadratic order in the spins, for instance through a quadrupole deformation of the body induced by the spin rotation.

We shall now impose some very general requirements on this action, and show how it describes correctly a spinning particle.

We impose

1. The action should be a covariant scalar (w.r.t. indices μ, ν, \dots)
2. The action should be a Lorentz scalar (w.r.t. indices α, β, \dots)
3. The action is reparametrization invariant $p \rightarrow \lambda p$

We define

$$\boxed{p_\mu = \frac{\partial L}{\partial u^\mu}}$$

conjugate momentum of u^μ

$$S_{\mu\nu} = 2 \frac{\partial L}{\partial \Omega^{\mu\nu}}$$

conjugate momentum of tetrad
will be the spin antisymmetric tensor

1. Scalability under $\{x\} \rightarrow \{x'\}$ with say $x^\mu = x^\mu - \epsilon^\mu(x)$

$$\int dp L(u^\mu(x), \Omega^{\mu\nu}(x), g_{\mu\nu}(x)) = \int dp L(u^\mu(x'), \Omega^{\mu\nu}(x'), g'_{\mu\nu}(x'))$$

↑
same Lagrangian in $\{x\}$ and $\{x'\}$ for any ϵ^μ

Using laws of transformation of tensors we get (exercise)

$$\boxed{2 \frac{\partial L}{\partial g_{\mu\nu}} = p^\mu u^\nu + S^\mu_{\nu\rho} \Omega^{\rho\nu}}$$

Such dependence of L on the metric will ensure the general covariance of the action and therefore the scalability.

2. Since the Lagrangian depends only on u^μ , $\Omega^{\mu\nu}$ and $g_{\mu\nu}$ which are Lorentz scalars, invariant by tetrad changes

$$e_\alpha^\mu \rightarrow \Lambda^\beta_\alpha e_\beta^\mu$$

the Lagrangian is itself a Lorentz scalar.

3. Invariance by reparametrization $p \rightarrow \lambda p$

$$\lambda L\left(\frac{u^\mu}{\lambda}, \frac{\Omega^{\mu\nu}}{\lambda}, g_{\mu\nu}\right) = L$$

hence by Euler's theorem on homogeneous functions

$$L = u^\mu p_\mu + \frac{1}{2} \Omega^{\mu\nu} S_{\mu\nu}$$

We shall now vary the action w.r.t. the tetrad e_α^μ .

We must distinguish intrinsic changes in the tetrad from changes which are induced by a change of the metric. Changes in the metric are of the form

$$\delta g^{\mu\nu} = \delta(h^{\alpha\beta} e_\alpha^\mu e_\beta^\nu) = 2 e^{\alpha\mu} \delta e_\alpha^\nu$$

while intrinsic changes of the tetrad will correspond to

$$\delta \theta^{\mu\nu} = 2 e^{\alpha\mu} \delta e_\alpha^\nu$$

hence

$$\delta e_\alpha^\nu = e_{\alpha\mu} \left(\delta \theta^{\mu\nu} + \frac{1}{2} \delta g^{\mu\nu} \right)$$

↑ ↑
intrinsic change the metric

So we vary the tetrad assuming no change in the metric, $\delta g_{\mu\nu} = 0$.

$$\delta \Omega^{\mu\nu} = \delta \left(e^{\alpha\mu} \frac{D e_\alpha^\nu}{dp} \right) = \delta e^{\alpha\mu} \frac{D e_\alpha^\nu}{dp} + e^{\alpha\mu} \frac{D \delta e_\alpha^\nu}{dp}$$

$$\begin{aligned}
 \delta S &= \int dp \frac{1}{2} \delta \Omega^{\mu\nu} S_{\mu\nu} \\
 &= \frac{1}{2} \int dp \left(\delta e^{\alpha\mu} \frac{D e_\alpha^\nu}{dp} + e^{\alpha\mu} \frac{D \delta e_\alpha^\nu}{dp} \right) S_{\mu\nu} \\
 &\quad \text{integrate by parts} \\
 &= \frac{1}{2} \int dp \left(\delta e^{\alpha\mu} \frac{D e_\alpha^\nu}{dp} S_{\mu\nu} - \frac{D e^{\alpha\mu}}{dp} \delta e_\alpha^\nu S_{\mu\nu} - e^{\alpha\mu} \delta e_\alpha^\nu \frac{D S_{\mu\nu}}{dp} \right) \\
 &= \frac{1}{2} \int dp \delta e^{\alpha\mu} \left(2 \frac{D e_\alpha^\nu}{dp} S_{\mu\nu} + e_\alpha^\rho \frac{D S_{\mu\nu}}{dp} \right)
 \end{aligned}$$

But $\delta e^{\alpha\mu} = e^\alpha, \delta \theta^{\nu\mu}$ where $\delta \theta^{\nu\mu}$ is the arbitrary (antisymmetric) intrinsic tetrad change.

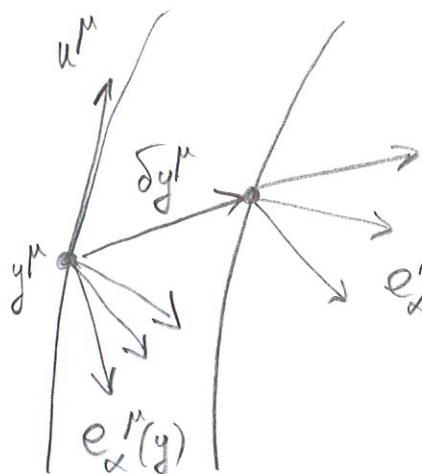
$$= \frac{1}{2} \int dp \delta \theta^{\nu\mu} \left(2 \Omega_\nu^{\mu\rho} S_{\mu\rho} + \frac{D S_{\mu\nu}}{dp} \right)$$

$$\frac{D S_{\mu\nu}}{dp} = 2 \Omega_{[\mu}^{\nu\rho} S_{\nu]\rho}$$

But using the scalability condition we have $0 = p^{[\mu} u^{\nu]} + S^{[\mu} \cdot p^{\nu]} \Omega^{u]p}$
hence we have obtained the spin precession equation

$$\boxed{\frac{D S_{\mu\nu}}{dp} = \Omega_\mu^{\nu\rho} S_{\nu\rho} - \Omega_{\nu}^{\mu\rho} S_{\nu\rho} = p_\mu u_\nu - p_\nu u_\mu}$$

We now vary w.r.t. the position y^μ keeping the tetrad "fixed".



By tetrad fixed we mean that we shall parallel transport it along the position change δy^μ

$$e_\alpha^\mu(y + \delta y) = e_\alpha^\mu(y)$$

$$\text{Hence } [\delta y^\nu \nabla_\nu e_\alpha^\mu] = 0$$

$$\text{or } \delta e_\alpha^\mu = -\delta y^\nu \Gamma_{\nu\rho}^\mu e_\alpha^\rho$$

We shall use loc. inertial coordinates hence $\Gamma = 0$ and $\delta e_\alpha^\mu = 0$

$$\begin{aligned} \delta \left(\frac{D e_\alpha^\mu}{d p} \right) &= \delta \left(\frac{d e_\alpha^\mu}{d p} + \Gamma_{\rho\sigma}^\mu u^\rho e_\alpha^\sigma \right) \\ &= \frac{d}{d p} \left(-\delta y^\nu \Gamma_{\nu\rho}^\mu e_\alpha^\rho \right) + \delta y^\nu \partial_\nu \Gamma_{\rho\sigma}^\mu u^\rho e_\alpha^\sigma \\ &= -\delta y^\nu u^\sigma \partial_\sigma \Gamma_{\nu\rho}^\mu e_\alpha^\rho + \delta y^\nu \partial_\nu \Gamma_{\rho\sigma}^\mu u^\rho e_\alpha^\sigma \end{aligned}$$

$$= \delta y^\nu \underbrace{\left(\partial_\nu \Gamma_{\rho\sigma}^\mu - \partial_\rho \Gamma_{\nu\sigma}^\mu \right)}_{\text{we recognize the curvature in loc. in. coord.}} u^\rho e_\alpha^\sigma$$

$$= \delta y^\nu R_{,\rho\sigma}^\mu u^\rho e_\alpha^\sigma$$

$$\begin{aligned}
 \delta S &= \int dp \left\{ \frac{D\delta y^\mu}{dp} p_\mu + \frac{1}{2} S_{\mu\nu} e^{\alpha\mu} \delta \left(\frac{D e^\nu}{dp} \right) \right\} \\
 &= \int dp \left\{ -\delta y^\mu \frac{D p_\mu}{dp} + \frac{1}{2} S_{\mu\nu} e^{\alpha\mu} \underbrace{\delta y^\rho R_{\nu\rho\sigma} u^\sigma e_\alpha}_g \right\} \\
 &= \int dp \left\{ -\delta y^\mu \frac{D p_\mu}{dp} + \frac{1}{2} S_{\mu\nu} \delta y^\rho R_{\nu\rho}^\mu \right\}
 \end{aligned}$$

Hence the Mathisson-Papapetrou equations of motion

$$\boxed{\frac{D p_\mu}{dp} = -\frac{1}{2} u^\nu R_{\mu\nu\rho\sigma} S^{\rho\sigma}}$$

By varying the action w.r.t. the metric one can show that the stress-energy-tensor of the spinning particle is

$$\boxed{T^{\mu\nu} = \int_{-\infty}^{+\infty} dp p^{(\mu} u^{\nu)} \frac{\delta_{(4)}(x-y)}{\sqrt{-g}} - \nabla_\rho \left(\int_{-\infty}^{+\infty} dp S^{\rho(\mu} u^{\nu)} \frac{\delta_{(4)}(x-y)}{\sqrt{-g}} \right)}$$

However this is yet not satisfying. The particle's spin has 3 independent components, while here we have described it by 6 indep. components, either the 4×4 tetrad subject to 10 constraints $g_{\mu\nu} e_\alpha^\mu e_\beta^\nu = \eta_{\alpha\beta}$, or the 6 components of the antisymmetric tensor $S^{\mu\nu}$.

To correctly account for the spin d.o.f. we must impose a spin supplementary condition (SSC)

$$\boxed{S^{\mu\nu} p_\nu = 0}$$

In the case of extended bodies the SSC corresponds to a choice of a central worldline w.r.t. which the spin angular momentum is defined.

We can then define a spin covariant vector S_μ by

$$S^\mu = \frac{1}{m} \epsilon^{\mu\nu\rho\sigma} p_\rho S_\sigma$$

where the mass is defined by $m^2 = -g^{\mu\nu} p_\mu p_\nu$. By contracting the spin precession equation one then obtain the relation between p_μ and u_μ

$$p_\mu (pu) + m^2 u_\mu = \underbrace{\frac{1}{2} u^\lambda R_{\cdot\lambda\rho\sigma} S^\rho S^\sigma}_{\text{order } O(S^2)}$$

We can show that the mass is conserved $\frac{dm}{dp} = 0$ and also the spin magnitude $s^2 = S_\mu S^\mu$, $\frac{ds}{dp} = 0$.

Neglecting $O(s^2)$ and restricting to terms linear in spins (so-called spin-orbit interaction) we obtain

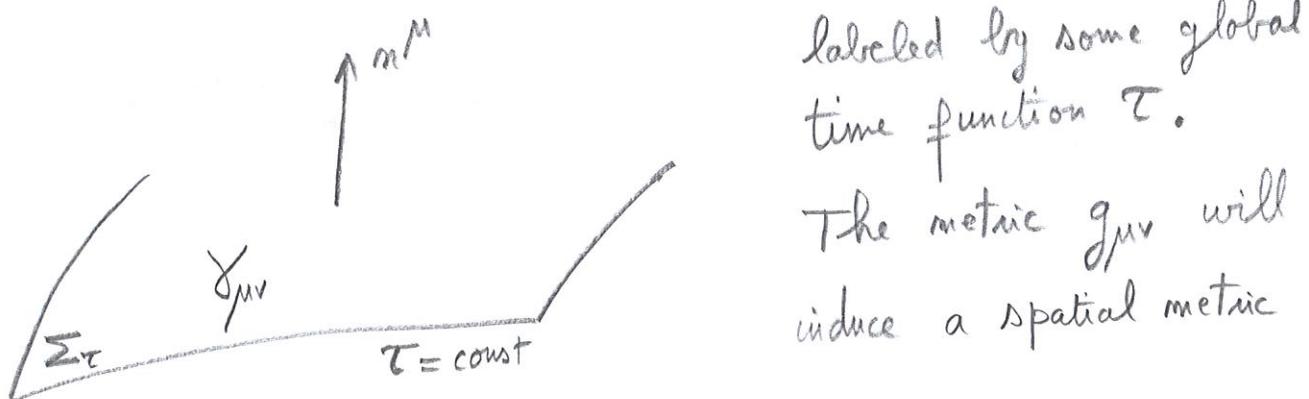
$$\boxed{p_\mu = m u_\mu + O(s^2) \quad \frac{DS_\mu}{dp} = O(s^2) \quad \left(\begin{array}{l} \text{spin vector is} \\ \text{parallelly} \\ \text{transported} \end{array} \right)}$$

Initial value problem for GR

In ordinary physics we are describing phenomena taking place in a given background space-time. We determine the time evolution of physical quantities from their initial values and time derivatives at $t=0$.

In GR we are solving for the space-time itself. What should be the quantities to be prescribed "initially" in order to determine the full space-time?

We shall view the space-time as foliated by three-dimensional space-like hypersurfaces Σ_τ (Cauchy surface)



labeled by some global time function τ .

The metric $g_{\mu\nu}$ will induce a spatial metric

$$\boxed{\gamma_{\mu\nu} = g_{\mu\nu} + n^\mu n^\nu} \quad (\text{or "1st fundamental form"})$$

on each of the surfaces, where n^μ is time-like unit vector field orthogonal to the surface

$$\boxed{g_{\mu\nu} n^\mu n^\nu = -1}$$

The induced metric $\gamma_{\mu\nu}$ can be used as a projector orthogonal to n^μ as $\boxed{n^\nu \gamma_{\mu\nu} = 0 \quad \text{and} \quad \gamma_\mu^\nu \gamma_\nu^\rho = \delta_\mu^\rho}$

We shall define as initial values for the complete space-time evolution the induced metric on some given hypersurface Σ_τ , and its "time derivative" orthogonal to the surface, i.e. in the direction of n

I.C. $\boxed{\gamma_{\mu\nu} ; \underbrace{K_{\mu\nu} = \frac{1}{2} \mathcal{L}_m \gamma_{\mu\nu}}_{\text{extrinsic curvature}}}$

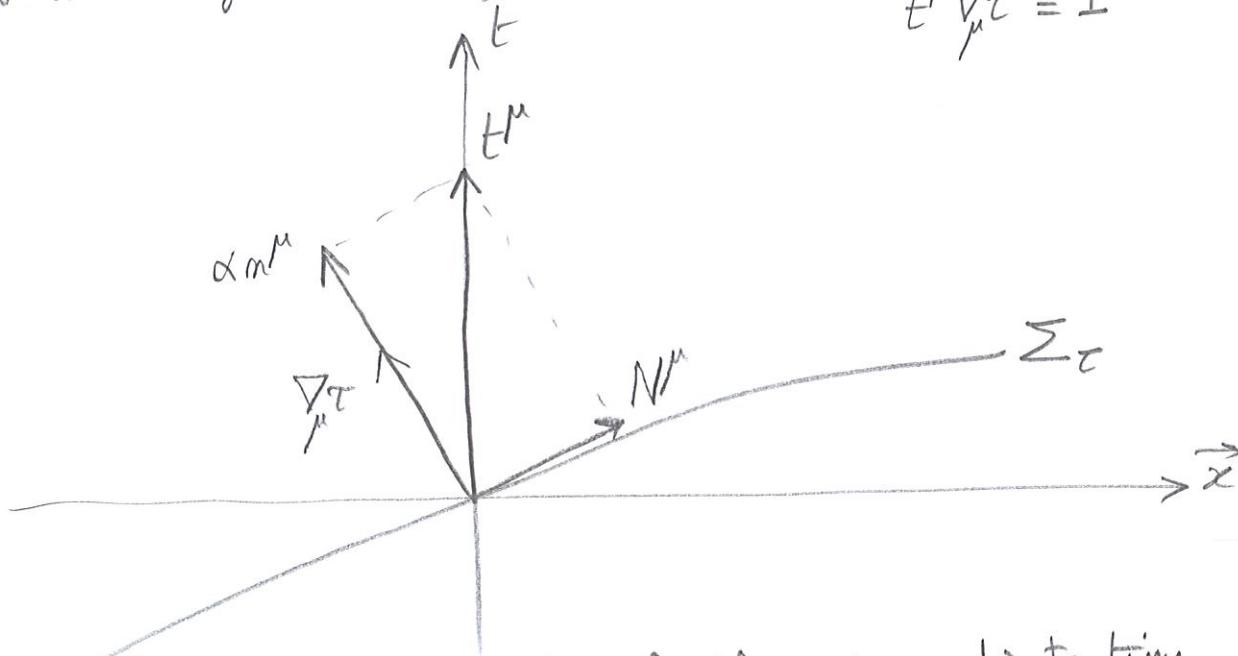
The unit vector m^μ is orthogonal to Σ_τ and thus proportional to $\nabla^\mu \tau$. We pose

$$\boxed{m^\mu = -\alpha \nabla^\mu \tau}$$

where α is called the "lapse". Minus sign chosen so that m^μ is future directed.

To further describe the foliation $\Sigma_\tau (\tau \in \mathbb{R})$ we introduce a vector t^μ which will be in the time direction t of some coordinate system $\{t, \vec{x}\}$. We can always normalize it by

$$t^\mu \nabla_\mu \tau = 1$$



The vector t^μ describes the flow of coordinate time through the surface.

Then we decompose

$$t^\mu = N^\mu + \alpha m^\mu$$

where N^μ belongs to Σ i.e. $m_\mu N^\mu = 0$ and is called the "shift".

Thus we have

$$\begin{cases} \alpha = -m_\mu t^\mu \\ N_\mu = \gamma_{\mu\nu} t^\nu \end{cases}$$

The unit vector m^μ ($m^2 = -1$) is hypersurface-orthogonal since $m_\mu = -\alpha \nabla_\mu \tau$. One can check directly that

$$m_{[\mu} \nabla_\nu m_{\rho]} = 0 \quad (\text{Frobenius conditions})$$

Reciprocally any vector satisfying these conditions can be shown to be hypersurface-orthogonal (Frobenius theorem).

"Acceleration" of the congruence of time-like curves m^μ

$$a^\mu = m^\nu \nabla_\nu m^\mu$$

a^μ belongs to Σ
 $m_\mu a^\mu = 0$

$$\begin{aligned} a^\mu &= m^\nu \nabla_\nu (-\alpha \nabla^\mu \tau) = m^\nu \left(+ \nabla_\nu \frac{m^\mu}{\alpha} - \alpha \nabla^\mu \left(-\frac{1}{\alpha} m_\nu \right) \right) \\ &= m^\mu m^\nu \nabla_\nu \ln \alpha - \alpha \nabla^\mu \left(\frac{1}{\alpha} \right) \end{aligned}$$

$$a^\mu = \gamma^{\mu\nu} \nabla_\nu \ln \alpha$$

Frobenius $0 = m^\rho \left(m_{[\mu} \nabla_\nu m_{\rho]} \right) = m^\rho \frac{1}{3} \left(m_\mu \nabla_\nu m_{\rho]} + m_\nu \nabla_\rho m_{\mu]} + m_\rho \nabla_\mu m_{\nu]} \right)$

$$0 = \nabla_\mu m_{\nu]} + m_{[\mu} a_{\nu]}$$

Define the extrinsic curvature (or 2^a "fundamental form") 39

$$K_{\mu\nu} = \nabla_\mu m_\nu + m_\mu a_\nu = \gamma_\mu^\rho \nabla_\rho m_\nu$$

which is indeed symmetric.

For any tensor [0] $\mathcal{L}_m \gamma_{\mu\nu} = m^\rho \nabla_\rho \gamma_{\mu\nu} + 2 \gamma_{\rho(\mu} \nabla_{\nu)} m^\rho$
 $= m^\rho \nabla_\rho (m_\mu m_\nu) + 2 \nabla_{[\mu} m_{\nu]}$

$$K_{\mu\nu} = \frac{1}{2} \mathcal{L}_m \gamma_{\mu\nu}$$

represents the "time" derivative of $\gamma_{\mu\nu}$ in direction m^μ .

Covariant derivative induced in the surface Σ

For any scalar S define $D_\mu S = \gamma^\nu \nabla_\nu S$

hence in particular $a_\mu = D_\mu l^m \alpha$

For any vector $A^\mu \in \Sigma$ i.e. $m_\mu A^\mu = 0$

$$D_\mu A^\nu = \gamma_\mu^\rho \gamma_\nu^\sigma \nabla_\rho A^\sigma \quad \text{and so on}$$

This is the covariant derivative associated with the induced metric $\gamma_{\mu\nu}$ on the surface. Indeed $\gamma_{\mu\nu}$ is compatible with D_μ

$$D_\mu \gamma_{\nu\rho} = 0$$

Riemann curvature induced on the surface.

Defined from the non-commutation of covariant derivative D_μ for any tensor in the surface

$$m^\nu A_\mu = 0$$

$$(D_\mu D_\nu - D_\nu D_\mu) A_\rho = R_{\mu\nu\rho}^{\sigma} A_\sigma$$

Note that

$$\boxed{R_{\mu\nu\rho\sigma} \equiv R_{\mu\nu\rho\sigma}^{(3)}}$$

$$\begin{cases} K_{\mu\nu} = \text{extrinsic curvature of } \Sigma \text{ as viewed from the exterior of surface} \\ R_{\mu\nu\rho\sigma} = \text{intrinsic curvature of } \Sigma \text{ as measured by experiments confined in the surface } \Sigma \end{cases}$$

and of course $R_{\mu\nu\rho\sigma} = \text{intrinsic curvature of 4-dimensional space time } \Sigma \times \mathbb{R}$.

$$\begin{aligned} D_\mu D_\nu A_\rho &= D_\mu (\gamma^\lambda \gamma^\epsilon \nabla_\lambda A_\epsilon) \\ &= \gamma^\pi \gamma^\tau \gamma^\varphi \nabla_\pi (\gamma^\lambda \gamma^\epsilon \nabla_\lambda A_\epsilon) \\ &= \gamma^\pi \gamma^\lambda \gamma^\epsilon \nabla_\pi \nabla_\lambda A_\epsilon - \gamma^\pi \nabla_\pi m^\lambda + \gamma^\pi m_\tau \nabla_\pi^\lambda \rightarrow K_{\mu\pi}{}^\lambda \\ &\quad + \underbrace{\gamma^\pi \nabla_\pi (m_\tau m^\lambda)}_{\gamma^\pi \nabla_\pi m^\lambda} \gamma^\tau \gamma^\epsilon \nabla_\lambda A_\epsilon \\ &\quad + \underbrace{\gamma^\pi \nabla_\pi (m_\rho m^\epsilon)}_{\gamma^\pi \nabla_\pi m^\epsilon} \gamma^\lambda \gamma^\varphi \nabla_\lambda A_\epsilon - \gamma^\pi m_\rho \nabla_\pi^\epsilon \rightarrow K_{\mu\rho}{}^\epsilon \\ &= \gamma^\pi \gamma^\lambda \gamma^\epsilon \nabla_\pi \nabla_\lambda A_\epsilon + K_{\mu\nu}{}^\lambda \gamma^\epsilon \nabla_\lambda A_\epsilon \\ &\quad + K_{\mu\rho}{}^\epsilon \gamma^\lambda \nabla_\lambda A_\epsilon \\ &\quad - \gamma^\lambda A_\epsilon \nabla_\lambda^\epsilon \end{aligned}$$

Anti-symmetrize

$$D_{\mu} D_{\nu} A_{\rho} = \gamma^{\pi}_{\mu} \gamma^{\lambda}_{\nu} \gamma^{\epsilon}_{\rho} \nabla_{[\pi} \nabla_{\lambda]} A_{\epsilon} + o - K_{\rho[\mu} K_{\nu]\epsilon}^{\epsilon} A_{\epsilon}$$

$$R_{\rho[\mu\nu]}^{\epsilon} A_{\epsilon} = \gamma^{\pi}_{\mu} \gamma^{\lambda}_{\nu} \gamma^{\sigma}_{\rho} R_{\sigma\pi\lambda}^{\epsilon} A_{\epsilon} - \frac{1}{2} K_{\rho[\mu} K_{\nu]\epsilon}^{\epsilon} A_{\epsilon}$$

which is true for any A_{ϵ} in the surface ($m^{\epsilon} A_{\epsilon} = 0$)

First Gauss-Codazzi equation

$$\boxed{R_{\mu\nu\rho\sigma} = \gamma^{\lambda}_{\mu} \gamma^{\tau}_{\nu} \gamma^{\epsilon}_{\rho} \gamma^{\pi}_{\sigma} R_{\lambda\tau\epsilon\pi} - K_{\mu\rho} K_{\nu\sigma} + K_{\mu\sigma} K_{\nu\rho}}$$

$$\begin{aligned} D_{\nu} K_{\mu}^{\nu} &= \gamma^{\rho}_{\nu} \gamma^{\sigma}_{\sigma} \gamma^{\epsilon}_{\mu} \nabla_{\rho} K_{\epsilon}^{\sigma} \\ &= \gamma^{\rho}_{\nu} \gamma^{\sigma}_{\sigma} \gamma^{\epsilon}_{\mu} \nabla_{\rho} (\gamma^{\tau}_{\epsilon} \nabla_{\tau} m^{\sigma}) \\ &= \gamma^{\rho}_{\nu} \gamma^{\sigma}_{\sigma} \gamma^{\tau}_{\mu} \nabla_{\rho} \nabla_{\tau} m^{\sigma} + \gamma^{\rho}_{\nu} \gamma^{\sigma}_{\sigma} \gamma^{\epsilon}_{\mu} \nabla_{\rho} (m_{\epsilon}^{\tau}) \nabla_{\tau} m^{\sigma} \\ &= \gamma^{\rho}_{\sigma} \gamma^{\tau}_{\mu} \nabla_{\rho} \nabla_{\tau} m^{\sigma} + \gamma^{\rho}_{\sigma} \gamma^{\epsilon}_{\mu} \nabla_{\rho} (m_{\epsilon}^{\tau}) \nabla_{\tau} m^{\sigma} \\ &= \gamma^{\tau}_{\mu} \nabla_{\sigma} \nabla_{\tau} m^{\sigma} + \gamma^{\tau}_{\mu} m_{\sigma}^{\rho} \nabla_{\rho} \nabla_{\tau} m^{\sigma} \\ &\quad + \gamma^{\epsilon}_{\mu} K_{\sigma\epsilon}^{\sigma} a^{\sigma} \\ &= \gamma^{\tau}_{\mu} \nabla_{\sigma} \nabla_{\tau} m^{\sigma} - \gamma^{\tau}_{\mu} m^{\rho} \nabla_{\rho} m_{\sigma} \nabla_{\tau} m^{\sigma} + K_{\mu\sigma} a^{\sigma} \\ &= \gamma^{\tau}_{\mu} \nabla_{\sigma} \nabla_{\tau} m^{\sigma} \end{aligned}$$

$$\begin{aligned}
 D_\mu K &= \gamma^\nu \nabla_\nu K \quad (\text{where } K = K_\nu = \nabla_\nu m^\nu) \\
 &= \gamma^\nu \nabla_\nu (\gamma_\rho \nabla^\rho m^\mu) \\
 &= \gamma^\nu \gamma_\rho \nabla_\nu \nabla^\rho m^\mu + \gamma^\nu \nabla_\nu (m_\rho m^\sigma) \nabla^\sigma m^\mu \\
 &= \gamma^\nu \nabla_\nu \nabla_\rho m^\mu + \gamma^\nu m_\rho m^\sigma \nabla_\nu \nabla^\sigma m^\mu + K_{\rho\mu} a^\rho \\
 &= \gamma^\nu \nabla_\nu \nabla_\rho m^\mu - \gamma^\nu \nabla_\nu m_\rho m^\sigma \nabla^\sigma m^\mu + K_{\rho\mu} a^\rho \\
 &= \gamma^\nu \nabla_\nu \nabla_\rho m^\mu
 \end{aligned}$$

$$\begin{aligned}
 D_\nu K_\mu^\nu - D_\mu K &= \gamma^\nu (\nabla_\rho \nabla_\nu - \nabla_\nu \nabla_\rho) m^\rho \\
 &= \gamma^\nu R_{\nu\rho\epsilon\sigma}^{\rho} m^\epsilon
 \end{aligned}$$

Second Gauss-Codazzi equation

$$D_\nu K_\mu^\nu - D_\mu K = \gamma^\nu G_{\nu\rho} m^\rho$$

$$\text{where } G_{\nu\rho} = R_{\nu\rho} - \frac{1}{2} g_{\nu\rho} R$$

We compute from 1st GC equation the 3-dimensional scalar curvature

$$R = \gamma^\mu \gamma^\nu R_{\mu\nu\rho\sigma} = \gamma^{\lambda\epsilon} \gamma^{\tau\pi} R_{\lambda\tau\epsilon\pi} - K^2 + K_{\mu\nu} K^{\mu\nu}$$

$$R + K^2 - K_{\mu\nu} K^{\mu\nu} = 2 m^\mu m^\nu G_{\mu\nu}$$

The initial value problem in GR can be formulated as given in some initial (Cauchy) surface Σ

- the induced metric $\gamma_{\mu\nu}$
- the extrinsic curvature $K_{\mu\nu} = \frac{1}{2}\mathcal{L}_m \gamma_{\mu\nu}$

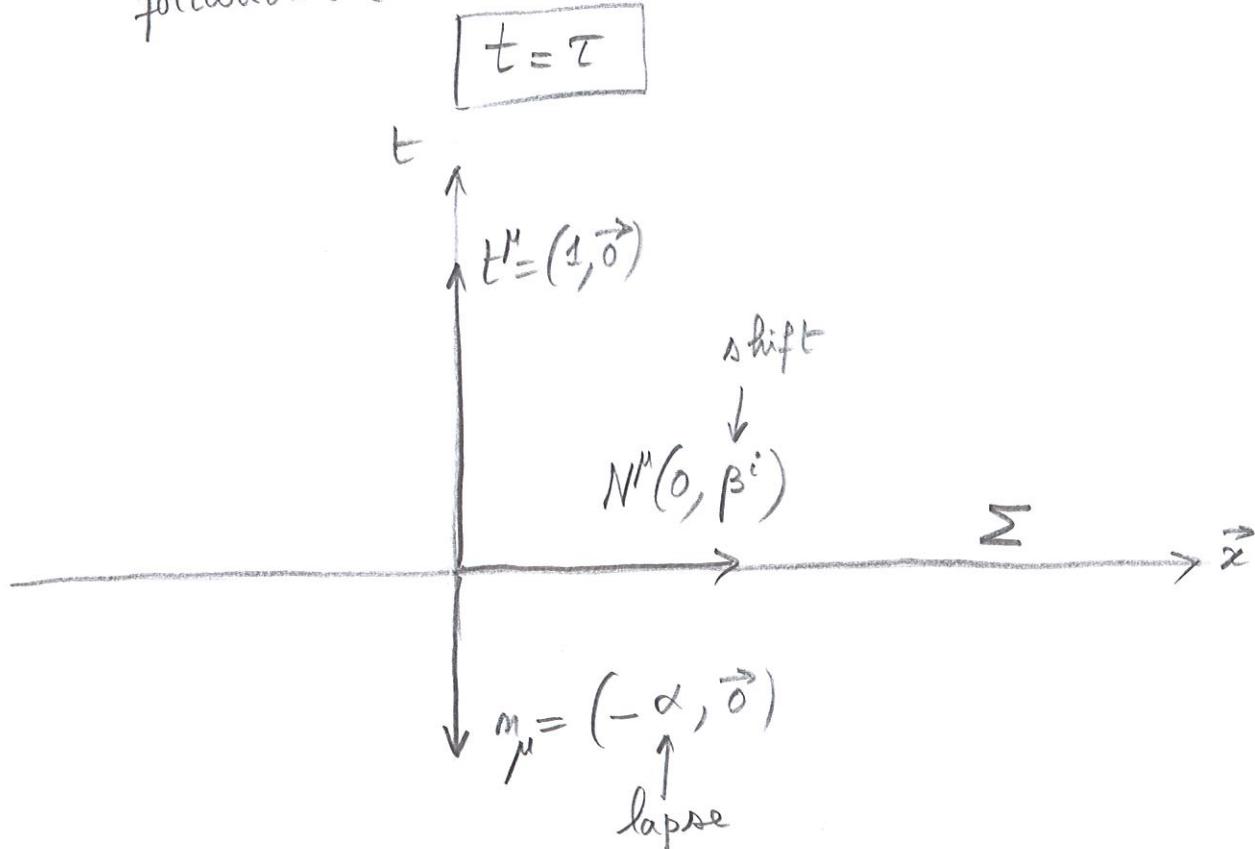
satisfying the two constraint equations (in vacuum case $G_{\mu\nu} = 0$)

$$\boxed{\begin{array}{l} R + K^2 - K_{\mu\nu} K^{\mu\nu} = 0 \quad \leftarrow \text{Hamiltonian constraint equation} \\ D_\nu K^\nu_\mu - D_\mu K = 0 \quad \leftarrow \text{Momentum constraint equation} \end{array}}$$

One can then show that this initial value problem is well-posed in the sense of Hadamard (Choquet-Bruhat).

ADM (or "3+1") decomposition (Arnowitt-Deser-Misner)

Choose a coordinate system $\{t, \vec{x}\}$ which is adapted to the foliation i.e.



Then (exercise)

$$\begin{cases} g_{00} = -\alpha^2 + \beta_i \beta^i & g_{0i} = \beta_i & g_{ij} = \gamma_{ij} \\ g^{00} = -\frac{1}{\alpha^2} & g^{0i} = \frac{\beta^i}{\alpha^2} & g^{ij} = \gamma^{ij} - \frac{1}{\alpha^2} \beta^i \beta^j \end{cases}$$

where γ_{ij} is the spatial metric, with γ^{ik} its inverse ($\gamma_{ij} \gamma^{jk} = \delta_i^k$)
and where $\beta_i = \gamma_{ij} \beta^j$

The 10 components of the metric $g_{\mu\nu}$ are then decomposed into

$$\begin{cases} \alpha & \text{lapse function} \\ \beta_i & \text{shift} \\ \gamma_{ij} & \text{spatial metric} \end{cases}$$

Note also

$$\boxed{\sqrt{-g} = \alpha \sqrt{\gamma}}$$

\uparrow
determinant
of γ_{ij}

$$\begin{aligned} ds^2 &= g_{00} dt^2 + 2 g_{0i} dt dx^i + g_{ij} dx^i dx^j \\ &= (-\alpha^2 + \beta_i \beta^i) dt^2 + 2 \beta_i dt dx^i + \gamma_{ij} dx^i dx^j \end{aligned}$$

$$\boxed{ds^2 = -\alpha^2 dt^2 + \gamma_{ij} (dx^i + \beta^i dt)(dx^j + \beta^j dt)}$$

ADM (or 3+1) decomposition of the metric

Acceleration of the congruence

$$\boxed{a_\mu^\nu = \gamma_\mu^\nu \partial_\nu \ln \alpha = (0, D^i \ln \alpha)}$$

where D_i is the covariant derivative
associated with γ_{ij} ($D_i \gamma_{jk} = 0$)

Extrinsic curvature (exercise)

$$K_{\mu\nu} = \gamma^\rho_\mu \nabla_\rho m_\nu$$

$$K_{ij} = \underbrace{\gamma_i^0 \nabla_0 m_j}_{=0} + \underbrace{\gamma_i^k \nabla_k m_j}_{\delta_i^k}$$

$$= \nabla_i m_j = - \Gamma_{ij}^0 m_0 \quad \text{since } m_j = 0 \text{ in adapted coord.}$$

$$= \alpha \Gamma_{ij}^0$$

$$= \frac{\alpha}{2} \left(-\frac{1}{\alpha^2} \right) (\partial_i \beta_j + \partial_j \beta_i - \partial_k \gamma_{ij})$$

$$+ \frac{\alpha}{2} \left(\frac{\beta^k}{\alpha^2} \right) (\partial_i \gamma_{jk} + \partial_j \gamma_{ik} - \partial_k \gamma_{ij})$$

$\frac{1}{\alpha} \Pi_{ij}^k \beta_k$ where Π_{ij}^k is the Christoffel symbol associated with γ_{ij}

$$\boxed{K_{ij} = \frac{1}{2\alpha} (\partial_i \gamma_{jj} - D_i \beta_j - D_j \beta_i)}$$

where D_i is the covariant derivative associated with γ_{ij}
 (for instance $D_i V^j = \partial_i V^j + \Pi_{ik}^j V^k$).

(other components $K_{oi} = \beta^j K_{ij}$ and $K_{oo} = \beta^i \beta^j K_{ij}$).

Hamiltonian formalism of GR

In GR, described by the Einstein-Hilbert Lagrangian, all quantities are spacetime covariant. By contrast, in a Hamiltonian formulation, one has to specify the dynamical variable and to consider its time evolution. So to derive the Hamiltonian for GR one has to break the manifest general covariance. We have just reviewed the appropriate formalism for doing this, which is by a space-like foliation Σ_t (or Σ_t in adapted coordinates), with induced metric γ_{ij} on Σ_t . So we shall look for an Hamiltonian which will depend (in particular) on γ_{ij} and its conjugate momentum (we shall denote Π^{ij})

We shall first express the Einstein-Hilbert Lagrangian

$$\mathcal{L}_g = \frac{\sqrt{-g}}{16\pi} R \quad (G=c=1)$$

by means of the 3+1 variables, that are lapse α , shift β^i and spatial metric.

We first use a general foliation not necessarily in adapted coordinates, with unit vector m^μ ($\eta_\mu m^\mu = -1$).

$$R = 2(G_{\mu\nu} - R_{\mu\nu})^{m^l m^v}$$

and we know from Gauss-Codazzi relation

$$2G_{\mu\nu}^{m^l m^v} = R + K^2 - K_{\mu\nu} K^{\mu\nu}$$

Also

$$R_{\mu\nu}^{m^l m^v} = -m^l (\nabla_\mu \nabla_\nu - \nabla_\nu \nabla_\mu)^{m^l m^v}$$

Since we work at the level of the Lagrangian we can discard a total (covariant) derivative

$$R_{\mu\nu}^{m^l m^v} = \nabla_\mu^{m^l} \nabla_\nu^{m^v} - \nabla_\nu^{m^l} \nabla_\mu^{m^v} + \text{div}$$

$$\text{which implies } = K^2 - K_{\mu\nu} K^{\mu\nu} + \text{div}$$

Hence

$$R = R + K_{\mu\nu} K^{\mu\nu} - K^2 + \text{div}$$

and the EH Lagrangian reads (ignoring the surface term)

$$\mathcal{L}_g = \frac{\sqrt{-g}}{16\pi} [R + K_{\mu\nu} K^{\mu\nu} - K^2]$$

In adapted coordinates (we then have $\sqrt{-g} = \alpha \sqrt{\gamma}$)

$$\boxed{\mathcal{L}_g = \frac{\alpha \sqrt{\gamma}}{16\pi} [R + K_{ij} K^{ij} - K^2]}$$

$$\text{where we recall } K_{ij} = \frac{1}{2\alpha} (\dot{\gamma}_{ij} - 2D_i \beta_j)$$

Hence we have an ordinary Lagrangian (density)

$$\mathcal{L}_g[\gamma_{ij}, \dot{\gamma}_{ij}, \alpha, \beta_i] \quad \begin{array}{l} \text{(and spatial gradients)} \\ \text{of } \gamma_{ij} \text{ and } \beta_i \end{array}$$

and we shall apply an ordinary Legendre transformation.

We define the conjugate momentum

$$\Pi^{ij} = \frac{\partial \mathcal{L}_g}{\partial \dot{\gamma}_{ij}}$$

The conjugate momenta
of α and β_i are zero.

$$\Pi^{ij} = \frac{\alpha \sqrt{g}}{16\pi} \left(2K^{ij}\frac{1}{2\alpha} - 2K\gamma^{ij}\frac{1}{2\alpha} \right)$$

$$\boxed{\Pi^{ij} = \frac{\sqrt{g}}{16\pi} (K^{ij} - \gamma^{ij} K)}$$

and the Hamiltonian is

$$\mathcal{H}_g = \Pi^{ij} \dot{\gamma}_{ij} - \mathcal{L}_g$$

which must be expressed in term of the conj. momentum

$$\mathcal{H}_g = \underbrace{\Pi^{ij} (2\alpha K_{ij} + 2D_i \beta_j)}_{\frac{32\pi\alpha}{\sqrt{g}} (\Pi_{ij} \Pi^{ij} - \frac{1}{2} \Pi^2)} - \frac{\alpha \sqrt{g}}{16\pi} \left[R + \underbrace{K_{ij} K^{ij} - K^2}_{\left(\frac{16\pi}{\sqrt{g}}\right)^2 (\Pi_{ij} \Pi^{ij} - \frac{1}{2} \Pi^2)} \right]$$

Also integrate by part the term with shift

$$2\Pi^{ij} D_i \beta_j = 2\sqrt{g} \frac{\Pi^{ij}}{\sqrt{g}} D_i \beta_j = -2\sqrt{g} \beta_i D_j \left(\frac{\Pi^{ij}}{\sqrt{g}} \right)$$

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Final Hamiltonian

$$\mathcal{H}_g = \sqrt{8} \alpha \left[-\frac{R}{16\pi} + \frac{16\pi}{8} \left(\Pi^{ij} \Pi_{ij} - \frac{1}{2} \Pi^2 \right) \right] - 2\sqrt{8} \beta_i D_j \left(\frac{\Pi^{ij}}{\sqrt{8}} \right)$$

The canonical variables are

$$\mathcal{H}_g [\gamma_{ij}, \Pi^{ij}, \alpha, \beta_i] \quad \begin{matrix} \text{(and spatial gradients)} \\ \text{of } \gamma_{ij} \end{matrix}$$

Since the conjugate momenta of α and β_i vanish we have

$$\frac{\partial \mathcal{H}_g}{\partial \alpha} = 0 \quad \text{hence} \quad R - \frac{(16\pi)^2}{8} \left(\Pi^{ij} \Pi_{ij} - \frac{1}{2} \Pi^2 \right) = 0$$

"Hamiltonian" constraint equation

$$\frac{\partial \mathcal{H}_g}{\partial \beta_i} = 0 \quad D_j \left(\frac{\Pi^{ij}}{\sqrt{8}} \right) = 0 \quad \text{Momentum constraint equation}$$

Thus the non-dynamical Hamilton equations for α and β_i nicely reduce the constraint equations to be imposed on initial conditions γ_{ij}, K_{ij} on an initial Cauchy surface.

Then we have dynamical equations for δ_{ij}

$$\boxed{\frac{\partial \mathcal{H}_g}{\partial \dot{\pi}^{ij}} = \ddot{\delta}_{ij}}$$

gives the definition of extrinsic curvature as
 $K_{ij} = \frac{1}{2\alpha} (\ddot{\delta}_{ij} - 2 D_i \beta_j)$

$$\boxed{\frac{\partial \mathcal{H}_g}{\partial \delta_{ij}} = - \overset{\circ}{\Pi}{}^{ij}}$$

gives

Einstein tensor of spatial metric δ_{ij}

$$\boxed{\begin{aligned} \overset{\circ}{\Pi}{}^{ij} &= - \frac{\alpha \sqrt{\delta}}{16\pi} \delta^{ij} \\ &\quad - \frac{32\pi\alpha}{\sqrt{\delta}} \left(\overset{\circ}{\Pi}{}^{ik} \overset{\circ}{\Pi}{}^j_k - \frac{1}{2} \overset{\circ}{\Pi}{}^{ll} \overset{\circ}{\Pi}{}^{ij} \right) \\ &\quad + \dots \end{aligned}}$$

evolution equation for $\overset{\circ}{\Pi}{}^{ij}$

The fact that there are "constraints" in our Hamiltonian formalism indicates that we have not yet identified the true dynamical degrees of freedom, and as a result the phase space is "too large".

The Hamiltonian "on-shell" is zero because of the two constraint equations. Does it mean that the total energy of our space-time is zero? No because we did not include the boundary term (we neglected all total divergences) and this is that boundary surface term which gives the total mass-energy called the ADM mass.

The computation is similar to the one of the

calculation of the surface term in the action. When we vary γ_{ij} and T^i_j in the total Hamiltonian

$$H_g = \int d^3x \mathcal{H}_g$$

the surface term will come from the variation of $\sqrt{\alpha} R_{ij}$. We have a formula exactly similar to the 4-dimensional case

$$\delta(\sqrt{\alpha} R_{ij}) = \sqrt{\alpha} (\delta\gamma^{ij} (g_{ij} + D_i v^j))$$

$$\text{where } v_i = D^j (\delta\gamma_{ij} - \gamma_{ij} \gamma^{kl} \delta\gamma_{kl})$$

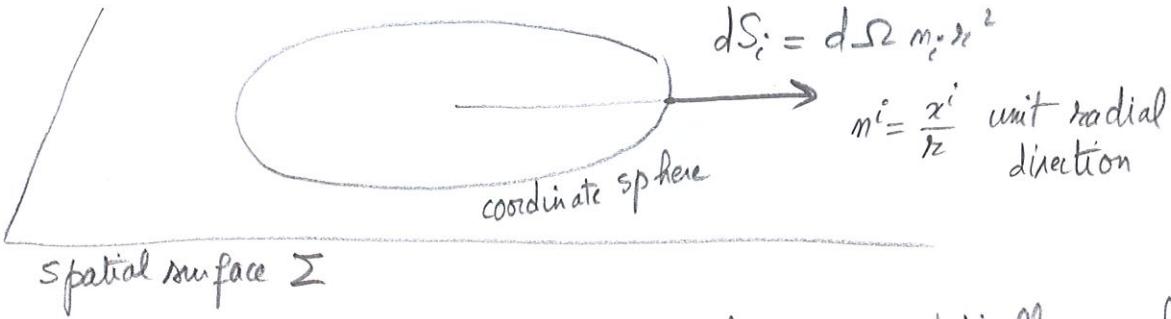
Suppose that s.t. is asymptotically flat. We shall vary γ_{ij} in such a way that s.t. remains asymptotically flat at infinity.

Thus we have $\alpha = 1 + O(\frac{1}{r})$ when $r \rightarrow \infty$, $\beta^i = O(\frac{1}{r^2})$, $\gamma_{ij} = \delta_{ij} + O(\frac{1}{r})$. We thus also assume that $\delta\gamma_{ij} = O(\frac{1}{r})$.

$$\delta H_g = \int d^3x \left[\frac{\delta H_g}{\delta \gamma_{ij}} \delta \gamma^{ij} + \frac{\delta H_g}{\delta T^i_j} \delta T^i_j \right] - \frac{1}{16\pi} \int d^3x \sqrt{\alpha} D_i v^i$$

where $\frac{\delta H_g}{\delta \gamma_{ij}}$ and $\frac{\delta H_g}{\delta T^i_j}$ have been computed before.

We evaluate the surface term say on a coordinate sphere located at infinity $r \rightarrow \infty$ (with $t = \text{const}$).



$$\delta H_g = \dots - \frac{1}{16\pi} \int d^3x \partial_i v^i$$

where asymptotically we have replaced the covariant derivative by ordinary derivative since $\partial_k \gamma_{ij} = O(\frac{1}{r^2})$ when $r \rightarrow \infty$

$$= \dots - \frac{1}{16\pi} \int d\Omega r^2 n^i (\partial_j \delta \gamma_{ij} - \partial_i \delta \gamma_{jj})$$

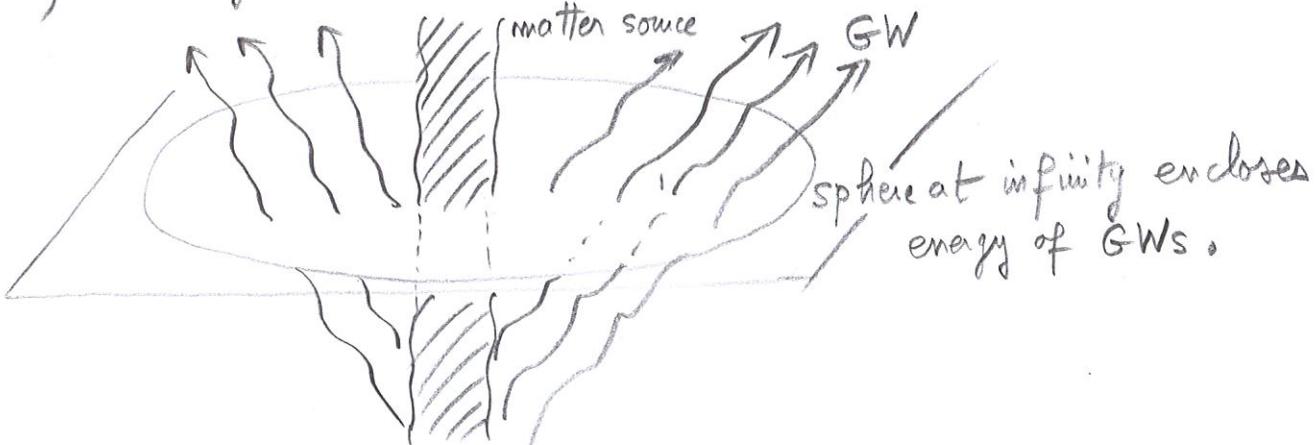
Again this calculation is done in asymptotically Minkowskian coordinate system where only the leading terms when $r \rightarrow \infty$ contribute.

$$H'_g = H_g + M_{ADM} c^2$$

where

$$M_{ADM} = \frac{1}{16\pi} \int_{\text{sphere at infinity}} d\Omega r^2 n^i (\partial_j \gamma_{ij} - \partial_i \gamma_{jj})$$

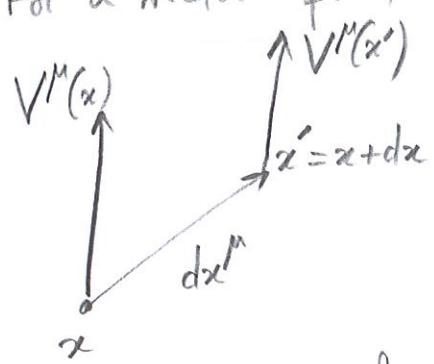
For s.t. generated by mass M $\gamma_{ij} \approx (1 + \frac{2M}{r}) \delta_{ij}$ hence we recover $M_{ADM} = M$. The ADM mass is the total constant mass of space-time, including all matter sources and energy in gravitational waves.



Einstein-Cartan theory

It is based on a Riemann-Cartan space-time manifold.
Basically we keep the notion of parallel transport

For a vector field $V^{\mu}(x)$ parallelly transported from x to x'



\exists affine connection $\Gamma_{\cdot \nu p}^{\mu}$ such that

$$\boxed{dV^{\mu} = - \Gamma_{\cdot \nu p}^{\mu} dx^{\nu} V^{\rho}}$$

$$\text{where } dV^{\mu} = V^{\mu}(x+dx) - V^{\mu}(x) = dx^{\nu} \partial_{\nu} V^{\mu}$$

In a coord. transformation $\{x\} \rightarrow \{x'\}$ this connection transforms like the Christoffel symbol

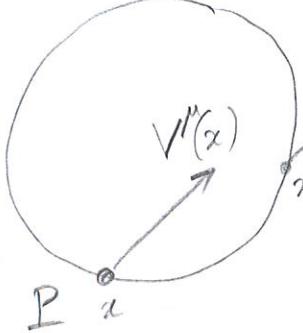
$$\tilde{\Gamma}_{\cdot \nu p}^{\mu}(x') = \frac{\partial x'^{\mu}}{\partial x^{\varepsilon}} \frac{\partial x^{\lambda} \partial x^{\sigma}}{\partial x'^{\nu} \partial x'^{p}} \Gamma_{\cdot \lambda \sigma}^{\varepsilon} + \frac{\partial x'^{\mu}}{\partial x^{\lambda}} \frac{\partial^2 x^{\lambda}}{\partial x'^{\nu} \partial x'^{p}}$$

But $\tilde{\Gamma}_{\cdot \nu p}^{\mu}$ is not necessarily symmetric in νp . However its anti-symmetric part is a tensor, the Cartan tensor or torsion

$$\boxed{C_{\cdot \nu p}^{\mu} = \tilde{\Gamma}_{\cdot \nu p}^{\mu} - \tilde{\Gamma}_{\cdot p \nu}^{\mu}}$$

The curvature is defined by the parallel transport of a vector along an infinitesimal closed contour (exactly like in GR)

closed contour
(infinitesimal around P)



$$V''(x') = V''(x)$$

Parallel transport equation gives
after a round trip

$$\Delta V^{\mu} = - \oint \Gamma_{\nu\rho}^{\mu} dx^{\nu} V^{\rho}$$

Near P $\Gamma_{\nu\rho}^{\mu}(x) = \Gamma_{\nu\rho}^{\mu}(x_2) + (x^{\sigma} - x_2^{\sigma}) \partial_{\sigma} \Gamma_{\nu\rho}^{\mu}(x_2) + O(|x - x_2|^2)$
(Taylor expansion to first order)

and the sol. of the parallel transport equation to first order is

$$V^{\mu}(x) = V^{\mu}(x_2) - (x^{\sigma} - x_2^{\sigma}) \Gamma_{\sigma\lambda}^{\mu}(x_2) V^{\lambda}(x_2) + O(|x - x_2|^2)$$

$$\Delta V^{\mu} = - \oint \left[\Gamma_{\nu\rho}^{\mu}(x_2) V^{\rho}(x_2) - (x^{\sigma} - x_2^{\sigma}) \Gamma_{\sigma\lambda}^{\mu} \Gamma_{\nu\rho}^{\lambda} V^{\rho} \right. \\ \left. + (x^{\sigma} - x_2^{\sigma}) \partial_{\sigma} \Gamma_{\nu\rho}^{\mu} V^{\rho} + O(2) \right] dx^{\nu}$$

But $\oint dx^{\nu} = 0$ around a closed contour

$$\Delta V^{\mu} = - \left[\partial_{\sigma} \Gamma_{\nu\rho}^{\mu} V^{\rho} - \Gamma_{\sigma\lambda}^{\mu} \Gamma_{\nu\rho}^{\lambda} V^{\rho} \right] (x_2) \underbrace{\oint x^{\sigma} dx^{\nu}}_{= 0}$$

From Stoke's theorem

$$\oint x^{\sigma} dx^{\nu} = \iint_{\text{surface limited by the contour}} dx^{\sigma} \wedge dx^{\nu}$$

NB: Stoke's theorem is best expressed in terms of differential forms

If $f = f_{\mu} dx^{\mu}$ then $df = \partial_{\mu} f_{\nu} dx^{\mu} \wedge dx^{\nu}$ and we have

$\oint f$	$= \iint df$
contour	surface

We define (like in GR)

$$\Delta V^{\mu} = -\frac{1}{2} R_{\nu\rho\sigma}^{\mu} \sqrt{-g} \int_{\text{surface}} dx^{\rho} \wedge dx^{\sigma}$$

(for an infinitesimal contour)

$$R_{\nu\rho\sigma}^{\mu} = \partial_{\rho} \Gamma_{\nu\sigma}^{\mu} - \partial_{\sigma} \Gamma_{\nu\rho}^{\mu} + \Gamma_{\nu\lambda}^{\mu} \Gamma_{\rho\sigma}^{\lambda} - \Gamma_{\nu\lambda}^{\mu} \Gamma_{\rho\sigma}^{\lambda}$$

(antisymmetric w.r.t. 2^d pair of indices $\rho\sigma$)

On the Riemann-Cartan space-time we have a covariant derivative defined by

$$dx^{\nu} \nabla_{\nu} V^{\mu} = 0 \quad \text{for parallelly transported vectors}$$

Hence

$$\nabla_{\nu} V^{\mu} = \partial_{\nu} V^{\mu} + \Gamma_{\nu\rho}^{\mu} V^{\rho}$$

↑
index of derivation comes first

The s.t. is metric, and the metric $g_{\mu\nu}$ (symmetric in $\mu\nu$) is defined to be compatible with the covariant derivative

$$\nabla_{\lambda} g_{\mu\nu} = 0$$

$$\partial_{\lambda} g_{\mu\nu} - \cancel{\Gamma_{\lambda\mu}^{\rho} g_{\rho\nu}} - \cancel{\Gamma_{\lambda\nu}^{\rho} g_{\mu\rho}} = 0$$

$$\partial_{\nu} g_{\mu\lambda} - \cancel{\Gamma_{\nu\mu}^{\rho} g_{\rho\lambda}} - \cancel{\Gamma_{\nu\lambda}^{\rho} g_{\mu\rho}} = 0$$

$$-\partial_{\mu} g_{\nu\lambda} + \cancel{\Gamma_{\mu\nu}^{\rho} g_{\rho\lambda}} + \cancel{\Gamma_{\nu\lambda}^{\rho} g_{\mu\rho}} = 0$$

$$0 = \partial_\lambda g_{\mu\nu} + \partial_\nu g_{\mu\lambda} - \partial_\mu g_{\nu\lambda} - 2 C_{\cdot\lambda\mu}^{\rho} g_{\nu\rho} - 2 C_{\cdot\nu\mu}^{\rho} g_{\lambda\rho} - 2 \Gamma_{\cdot(\lambda\nu)}^{\rho} g_{\mu\rho}$$

Define the usual Christoffel symbol as

$$\left\{ \begin{matrix} \mu \\ \nu\lambda \end{matrix} \right\} = \frac{1}{2} g^{\mu\rho} \left(\partial_\nu g_{\rho\lambda} + \partial_\rho g_{\nu\lambda} - \partial_\lambda g_{\nu\rho} \right)$$

$$0 = \left\{ \begin{matrix} \mu \\ \nu\lambda \end{matrix} \right\} - C_{\nu\lambda}^{\mu} - C_{\lambda\nu}^{\mu} - \Gamma_{\cdot(\nu\lambda)}^{\mu}$$

$$\Gamma_{\cdot\nu\lambda}^{\mu} = \Gamma_{\cdot(\nu\lambda)}^{\mu} + \Gamma_{\cdot[\nu\lambda]}^{\mu}$$

$$= \left\{ \begin{matrix} \mu \\ \nu\lambda \end{matrix} \right\} + C_{\nu\lambda}^{\mu} + C_{\lambda\nu}^{\mu} + \Gamma_{\cdot[\nu\lambda]}^{\mu}$$

$$\boxed{\Gamma_{\cdot\nu\lambda}^{\mu} = \left\{ \begin{matrix} \mu \\ \nu\lambda \end{matrix} \right\} + K_{\cdot\nu\lambda}^{\mu}}$$

where the tensor $K_{\cdot\nu\lambda}^{\mu}$ is called the contorsion

$$\boxed{K_{\cdot\nu\lambda}^{\mu} = C_{\nu\lambda}^{\mu} + C_{\lambda\nu}^{\mu} + C_{\cdot\nu\lambda}^{\mu}}$$

It is anti-symmetric in the 1st and 3^d indices

$$\boxed{K_{\mu\nu\lambda} = - K_{\lambda\nu\mu}}$$

On a Riemann-Cartan space-time the Einstein equivalence principle (EEP) is not valid. One cannot define a locally inertial frame $\{X^\alpha\}$ such that $g_{\alpha\beta}(X)$ equals $h_{\alpha\beta}$ at a point P and differs around P by small corrections of second order in the distance.

If $g_{\alpha\beta} = h_{\alpha\beta} + O(2)$ then $\partial g_{\alpha\beta} = O(1)$ hence $\{\overset{\circ}{\alpha\beta}\} = O(1)$

But by def. of the covariant derivative $\nabla g_{\alpha\beta} = 0 = \Gamma^\lambda_{\alpha\beta} g_{\lambda\beta}$ hence we should have $\Gamma = K$ which cannot be zero. There is a non-trivial parallel transport.

One can thus erect loc. inertial coord. (and thus recover local Lorentz covariance LLI) only if the contorsion and torsion are zero.

The Ricci tensor $R_{\mu\nu} = R^\lambda_{\cdot\mu\lambda\nu}$ is no longer symmetric

$$\boxed{R_{[\mu\nu]} = (\nabla_\rho + T_\rho) T^\rho_{\cdot\mu\nu}}$$

where $T^\rho_{\cdot\mu\nu} = C^\rho_{\cdot\mu\nu} + 2 \delta^\rho_{[\nu} C^\lambda_{\cdot\mu]\lambda}$ is the modified torsion and $T_\rho = T_{\cdot\rho\rho}$. The Riemann tensor $R_{\mu\nu\rho\sigma}$ is still anti-symmetric wrt $\mu\nu$ and $\rho\sigma$ but the cyclic symmetry is modified. Bianchi identities are modified too, as well as Ricci identity

$$\boxed{(\nabla_\mu \nabla_\nu - \nabla_\nu \nabla_\mu) V^\rho = R^\rho_{\cdot\sigma\mu\nu} V^\sigma - 2 C^\sigma_{\cdot\mu\nu} \nabla_\sigma V^\rho}$$

Einstein-Cartan theory is defined by the same action as GR

$$S = \frac{1}{16\pi} \int d^4x \sqrt{-g} R + S_m[\psi, g, C]$$

where R is the scalar curvature (depending on metric and torsion) and the matter action depend also on the torsion.

To get a non-trivial theory we need that the matter tensor depends on $C_{\mu\nu\rho}$. This dependence will be through a covariant derivative in the matter action, so we need particles with spins. (Indeed, matter with spins implies derivatives

$$\rho(\vec{x}) \sim m \delta^{(2)}_{(3)}(\vec{x}) + \frac{1}{c} S^i \partial_i \delta^{(2)}_{(3)}(\vec{x}) + \dots$$

and the minimal coupling to gravity $\partial_i \rightarrow \nabla_\mu$ will imply covariant derivatives and hence a dependence on torsion. See the description of particles with spins.

The field equations are (exercise)

$$G_{\mu\nu} + (\nabla_\rho + T_\rho) U_{\mu\nu\rho}^P = 8\pi T_{\mu\nu}$$

$$U_{\rho}^{\mu\nu} = 8\pi \sum_p \delta_{\rho}^{\mu\nu}$$

where $U_{\rho}^{\mu\nu} = T_{\rho 00}^{\mu\nu} + T_{\rho 0\sigma}^{\mu\nu} + T_{\rho\sigma 0}^{\mu\nu}$ and

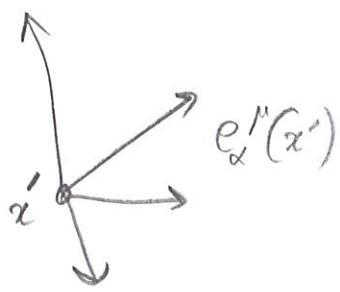
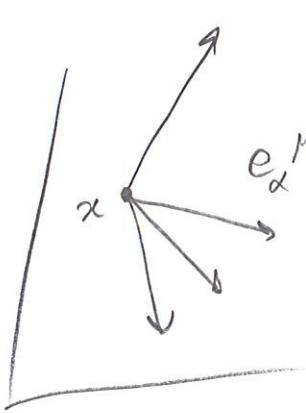
$$T_{\mu\nu} = -\frac{2}{\sqrt{-g}} \frac{\delta S_m}{\delta g^{\mu\nu}}$$

$$\sum_p \delta_{\rho}^{\mu\nu} = -\frac{1}{\sqrt{-g}} \frac{\delta S_m}{\delta C_{\rho\mu\nu}^P}$$

If S_m does not depend on $C_{\rho\mu\nu}$ Einstein-Cartan simply reduces to Einstein.

Connection coefficients and form

Introduce at each point in space-time a tetrad field $e_\alpha^\mu(x)$



These are four vectors $\begin{bmatrix} 1 \\ 0 \end{bmatrix}$

labeled by $\alpha = 0, 1, 2, 3$

Thus in a change of
coordinates $\{x\} \rightarrow \{x'\}$
they transform as

$$e'_\alpha^\mu(x') = \frac{\partial x'^\mu}{\partial x^\nu} e_\alpha^\nu(x)$$

These vectors form an orthonormal basis w.r.t. the metric, i.e.

$$g_{\mu\nu} e_\alpha^\mu e_\beta^\nu = \eta_{\alpha\beta} \quad \eta_{\alpha\beta} = \text{diag}(-1, 1, 1, 1)$$

μ, ν, \dots are space-time indices while α, β, \dots are Lorentz indices. Space-time indices are raised and lowered with $g_{\mu\nu}$, Lorentz indices are raised/lowered with $\eta^{\alpha\beta}$

Thus $e_\mu^\alpha = \eta^{\alpha\beta} g_{\mu\nu} e_\beta^\nu$ and the orthonormality condition means that

e_μ^α is the inverse of e_β^ν

$$e_\alpha^\mu e_\nu^\alpha = \delta_\nu^\mu \quad e_\alpha^\mu e_\beta^\nu = \delta_\alpha^\beta$$

Since a matrix always commutes with its inverse we have these two relations.

At each space-time point one can perform an arbitrary
change of tetrad

$$\tilde{e}_\alpha^\mu = \Lambda_\alpha^\beta e_\beta^\mu$$

where $\Lambda_\alpha^\beta(x)$ is a Lorentz matrix, possibly depending on the s.t. event.
The new tetrad is orthonormal thanks to the condition of Lorentz matrices.

$$g_{\mu\nu} \tilde{e}_\alpha^\mu \tilde{e}_\beta^\nu = g_{\mu\nu} \Lambda_\alpha^\gamma e_\gamma^\mu \Lambda_\beta^\delta e_\delta^\nu = \eta_{\alpha\beta} \Lambda_\alpha^\gamma \Lambda_\beta^\delta = \eta_{\alpha\beta}$$

V^μ a vector $\begin{bmatrix} 1 \\ 0 \end{bmatrix}$. We define its tetradic components as

$$\boxed{V^\alpha = e_\mu^\alpha V^\mu \quad (\text{4 scalars } \begin{bmatrix} 0 \\ 0 \end{bmatrix})}$$

and its tetradic derivative as $\partial_\beta V^\alpha = e_\beta^\nu \partial_\nu V^\alpha$ (still a scalar)

We want now to define its corariant tetradic derivative such that

- $\nabla_\beta V^\alpha$ is a scalar $\begin{bmatrix} 0 \\ 0 \end{bmatrix}$ under $\{x\} \rightarrow \{x'\}$

- $\nabla_\beta V^\alpha$ transforms as $\Lambda_\beta^\gamma(x) \Lambda_\gamma^\alpha(x) \nabla V^\alpha(x)$

under arbitrary changes of tetrad, depending on the event x.

Clearly

$$\boxed{\nabla_\beta V^\alpha = e_\beta^\nu e_\mu^\alpha \nabla_\nu V^\mu}$$

will do the job.

$$\nabla_\beta V^\alpha = e_\beta^\nu e_\mu^\alpha (\partial_\nu V^\mu + \Gamma_{\nu\rho}^\mu V^\rho) \quad \left(\begin{array}{l} \text{We include some possible} \\ \text{torsion so that } \Gamma_{\nu\rho}^\mu \\ \text{is not necessarily symmetric} \end{array} \right)$$

$$= e_\beta^\nu \partial_\nu V^\alpha - e_\beta^\nu \partial_\nu e_\mu^\alpha V^\mu + e_\beta^\nu e_\mu^\alpha \Gamma_{\nu\rho}^\mu V^\rho$$

$$= \partial_\beta V^\alpha - e_\beta^\nu (\partial_\nu e_\mu^\alpha - \Gamma_{\nu\rho}^\mu e_\mu^\rho) V^\mu$$

$$= \partial_\beta V^\alpha - e_\beta^\nu \nabla_\nu e_\mu^\alpha V^\mu$$

$$\boxed{\nabla_\beta V^\alpha = \partial_\beta V^\alpha + \omega_{\beta\rho}^\alpha V^\rho}$$

where

$$\boxed{\omega_{\beta\rho}^\alpha = - e_\beta^\nu \nabla_\nu e_\mu^\alpha e_\rho^\mu}$$

is the connection

The connection is antisymmetric in 1st and 2^d indices

$$\omega_{\alpha\beta\gamma} = -\omega_{\gamma\beta\alpha}$$

Various forms (including mixed forms)

$$\omega_{\alpha\beta\gamma}^{\alpha} = e_{\mu}^{\alpha} e_{\beta}^{\nu} \nabla_{\nu} e_{\gamma}^{\mu}$$

$$\omega_{\alpha\mu\nu}^{\alpha} = -\nabla_{\mu} e_{\nu}^{\alpha}, \quad \omega_{\alpha\gamma\alpha}^{\mu} = \nabla_{\gamma} e_{\alpha}^{\mu}$$

We can also extend the tetradic (ordinary) derivative to any tensor as $\partial_{\alpha} = e_{\alpha}^{\mu} \nabla_{\mu}$. Then we have also

$$\omega_{\alpha\beta\mu}^{\alpha} = -\partial_{\beta} e_{\mu}^{\alpha} \quad \omega_{\alpha\mu\beta}^{\mu} = \partial_{\alpha} e_{\beta}^{\mu}$$

Minkowski metric is compatible with covariant derivative and tetrad is constant

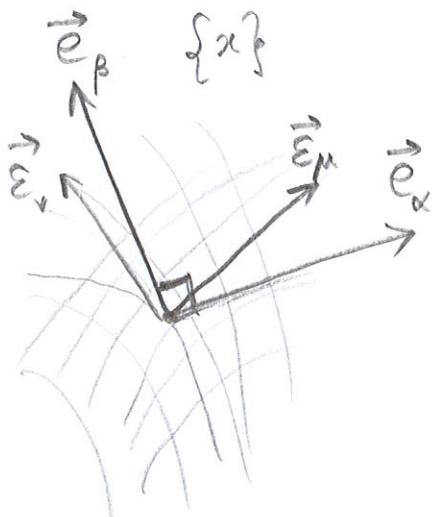
$$\nabla_{\alpha} h_{\beta\gamma} = 0 \quad \nabla_{\alpha} e_{\beta}^{\mu} = 0$$

Structure constants

The tetrad e_{α}^{μ} is considered as a set of 4 vectors

say $\vec{e}_{\alpha} = e_{\alpha}^{\mu} \partial_{\mu}$

(in a global coord. system $\{x^{\mu}\}$)



We can denote $\vec{E}_{\mu} = \partial_{\mu}$ the coordinate basis (which is a priori non orthonormal) so that

$$\vec{e}_{\alpha} = e_{\alpha}^{\mu} \vec{E}_{\mu}$$

The vector is a partial derivative acting, say, on scalars ϕ .

$$[\vec{e}_\alpha, \vec{e}_\beta] = f_{\alpha\beta}^\gamma \vec{e}_\gamma$$

↑
structure constants

On a scalar $[\vec{e}_\alpha, \vec{e}_\beta]\phi = e_\alpha^\nu \partial_\nu (e_\beta^\mu \partial_\mu \phi) - e_\beta^\nu \partial_\nu (e_\alpha^\mu \partial_\mu \phi)$

$$= (e_\alpha^\nu \partial_\nu e_\beta^\mu - e_\beta^\nu \partial_\nu e_\alpha^\mu) \partial_\mu \phi$$

$$= f_{\alpha\beta}^\gamma e_\gamma^\mu \partial_\mu \phi$$

$$f_{\alpha\beta}^\gamma = e_\alpha^\nu \partial_\nu e_\beta^\mu - e_\beta^\nu \partial_\nu e_\alpha^\mu$$

For a coordinate basis like $\vec{e}_\mu = \partial_\mu$ the commutator vanishes and the structure constants are zero.

In a coord. sys. associated with tetrad (i.e. which would be orthonormal) $e_\alpha^\mu = \delta_\alpha^\mu$ — this would be a loc. inertial coordinate system in absence of torsion — the f 's vanish.

Another form is

$$f_{\mu\nu}^\alpha = - \partial_\mu e_\nu^\alpha + \partial_\nu e_\mu^\alpha$$

and we can relate this to the connection. Indeed

$$f_{\mu\nu}^\alpha = - \nabla_\mu e_\nu^\alpha - \Gamma_{\mu\nu}^\rho e_\rho^\alpha + (\mu \leftrightarrow \nu)$$

$$= \omega_{\mu\nu}^\alpha - \omega_{\nu\mu}^\alpha - 2 \Gamma_{[\mu\nu]}^\rho e_\rho^\alpha$$

$$f_{\mu\nu}^\alpha = \omega_{\cdot\mu\nu}^\alpha - \omega_{\cdot\nu\mu}^\alpha - 2 C_{\cdot\mu\nu}^\alpha$$

This can be inverted to give the connection in terms of the structure constants and the contorsion

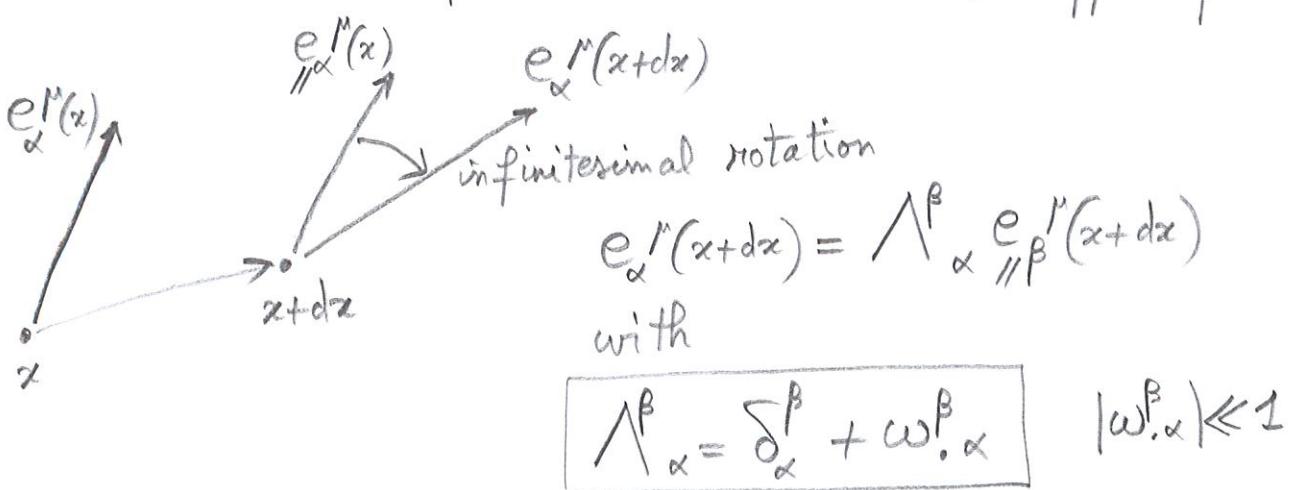
$$\omega_{\cdot\mu\nu}^\alpha = \frac{1}{2} (f_{\mu\nu}^\alpha + f_{\nu\mu}^\alpha + f_{\mu\nu}^\alpha) + K_{\cdot\mu\nu}^\alpha$$

Again we see that torsion (or contorsion) is an obstacle for loc. inertial frames. If we have a "loc. inertial" frame $\{\tilde{x}\}$ this means that $e_\alpha^I = \delta_\alpha^I$ thus $f_{\cdot\alpha\beta}^I = 0$. Furthermore

$$\omega_{\cdot\mu\nu}^\alpha = - \nabla_\mu e_\nu^\alpha = - \partial_\mu e_\nu^\alpha + \Gamma_{\mu\nu}^\rho e_\rho^\alpha \text{ hence } \omega_{\cdot\mu\nu}^\alpha = \Gamma_{\mu\nu}^\alpha.$$

Hence $\Gamma_{\mu\nu}^\alpha = K_{\cdot\mu\nu}^\alpha$ in that coord. system and there is a non trivial parallel transport in that frame.

The connection $\omega_{\alpha\beta}^\gamma$ connects together different tetrad basis at different points



with $\omega_{\alpha\beta} = -\omega_{\beta\alpha}$ to satisfy the law of Lorentz matrices.

By def. of parallel transport (in Riemann-Cartan space-time) 64

$$e_{\alpha}^{\mu}(x+dx) = e_{\alpha}^{\mu}(x) - \Gamma_{\nu\rho}^{\mu} dx^{\nu} e_{\alpha}^{\rho}$$

Hence $e_{\alpha}^{\mu}(x+dx) = e_{\alpha}^{\mu}(x) - \Gamma_{\nu\rho}^{\mu} dx^{\nu} e_{\alpha}^{\rho} + \omega_{\nu\rho}^{\mu} e_{\alpha}^{\rho}$

$$dx^{\nu} \partial_{\nu} e_{\alpha}^{\mu} = - \Gamma_{\nu\rho}^{\mu} dx^{\nu} e_{\alpha}^{\rho} + \omega_{\nu\rho}^{\mu}$$

$$\omega_{\nu\rho}^{\mu} = \nabla_{\nu} e_{\alpha}^{\mu} dx^{\alpha} = \omega_{\nu\rho}^{\mu} dx^{\alpha}$$

This is a one-form called the connection 1-form

$$\boxed{\omega_{\alpha\beta} = \omega_{\alpha\mu\beta} dx^{\mu}}$$

We introduce also the basis 1-form

$$\boxed{\theta^{\alpha} = e_{\mu}^{\alpha} dx^{\mu}} \quad (\text{so } \omega_{\alpha\beta} = \omega_{\alpha\mu\beta} \theta^{\mu})$$

Cartan's structure equations

We shall define now a 2-form associated with curvature. Start with Ricci identity applied to a tetrad vector

$$(\nabla_{\mu} \nabla_{\nu} - \nabla_{\nu} \nabla_{\mu}) e_{\beta}^{\rho} = R_{\mu\nu}^{\rho} e_{\beta}^{\lambda} - \underbrace{2 C_{\mu\nu}^{\lambda} \nabla_{\lambda} e_{\beta}^{\rho}}_{\text{torsion}}$$

$$\underbrace{e_{\alpha\rho} (\nabla_{\mu} \nabla_{\nu} - \nabla_{\nu} \nabla_{\mu}) e_{\beta}^{\rho}}_{\text{operate by parts}} = R_{\alpha\beta\mu\nu}^{\rho} - 2 C_{\mu\nu}^{\lambda} \omega_{\alpha\lambda\beta}^{\rho}$$

operate by parts

$$\nabla_\mu (e_\alpha^\beta \omega_{\nu\beta}^\gamma) - \omega_{\mu\nu}^\lambda \omega_{\nu\beta}^\gamma - (\mu \leftrightarrow \nu) \\ = R_{\alpha\beta\mu\nu} - 2 C_{\cdot\mu\nu}^\lambda \omega_{\lambda\beta}^\gamma$$

$$\partial_\mu \omega_{\nu\beta} - \cancel{\Gamma_{\mu\nu}^\lambda} \omega_{\lambda\beta} - \omega_{\mu\nu}^\lambda \omega_{\nu\beta}^\gamma - (\mu \leftrightarrow \nu) \\ = R_{\alpha\beta\mu\nu} - 2 \cancel{C_{\cdot\mu\nu}^\lambda} \omega_{\lambda\beta}^\gamma$$

Explicit torsion dependence cancels out. Best form to express the curvature is the mixed form

$$R_{\alpha\beta\mu\nu} = \partial_\mu \omega_{\nu\beta} - \partial_\nu \omega_{\alpha\beta} - \omega_{\mu\nu}^\lambda \omega_{\nu\beta}^\gamma + \omega_{\nu\alpha}^\lambda \omega_{\nu\beta}^\gamma$$

is explicitly antisymmetric in two pairs of indices

Curvature 2-form

$$R_{\alpha\beta} = R_{\alpha\beta\mu\nu} dx^\mu \wedge dx^\nu$$

↑
exterior product (antisymmetric)

Cartan's structure equations are differential identities which specify the link between connection and torsion, and between curvature and connection. We define also the torsion 2-form

$$C^\alpha = C_{\cdot\mu\nu}^\alpha dx^\mu \wedge dx^\nu$$

We use language of exterior calculus, with the exterior differentiation d .

$$\begin{aligned}
 d\theta^\alpha &= d(e^\alpha_v dx^v) \\
 &= \partial_\mu e^\alpha_v dx^\mu \wedge dx^v \\
 &= (\nabla_\mu e^\alpha_v + \Gamma_{\mu\nu}^\rho e^\alpha_\rho) dx^\mu \wedge dx^v \\
 &= (\omega_{v\mu}^\alpha + \underbrace{C_{\mu\nu}^\alpha}_{\text{torsion}}) dx^\mu \wedge dx^v \\
 &= -\omega_{\nu\mu}^\alpha dx^\mu \wedge \theta^\beta + C^\alpha
 \end{aligned}$$

Hence the first Cartan structure equation

$$\boxed{d\theta^\alpha + \omega_{\nu\mu}^\alpha \wedge \theta^\mu = C^\alpha}$$

$$\begin{aligned}
 d\omega_{\alpha\beta} &= d(\omega_{\alpha\gamma\beta} dx^\gamma) = \partial_\mu \omega_{\alpha\gamma\beta} dx^\mu \wedge dx^\gamma \\
 &= \frac{1}{2} (\partial_\mu \omega_{\alpha\gamma\beta} - \partial_\gamma \omega_{\alpha\mu\beta}) dx^\mu \wedge dx^\gamma \\
 &= \frac{1}{2} (R_{\alpha\beta\mu\nu} + \omega_{\alpha\mu\gamma} \omega_{\gamma\beta}^\nu - \omega_{\alpha\beta\gamma} \omega_{\gamma\mu}^\nu) dx^\mu \wedge dx^\nu
 \end{aligned}$$

hence the second Cartan structure equation

$$\boxed{d\omega_{\alpha\beta} + \omega_{\alpha\gamma} \wedge \omega_{\gamma\beta}^\nu = \frac{1}{2} R_{\alpha\beta}}$$

Spinorial connection

When coupling matter fields to gravity we have used for the moment the rule of minimal coupling to the metric

$$\eta_{\alpha\beta} \rightarrow g_{\mu\nu}$$

$$\partial_\alpha \rightarrow \nabla_\mu$$

which is a direct consequence of the equivalence principle. Actually the rule works for tensorial fields, describing integer spins, but not for spinorial fields, corresponding to half-integer spins.

To couple spinors to gravity we need to introduce a tetrad (which is the "square root" of the metric).

We need thus to generalize the notion of (tetradic) covariant derivative to any field transforming in a change of tetrad $e_\alpha^\mu \rightarrow \tilde{e}_\alpha^\mu = \Lambda_\alpha^\beta e_\beta^\mu$ (where the Lorentz matrix depends on the point in question) as some representation of the Lorentz group.

The field Ψ is composed of a set of scalars $\Psi_m(x)$

$$\Psi = \{\Psi_m(x)\}_{m \in \mathbb{N}}$$

which transform under a Lorentz transformation Λ as

$$\boxed{\tilde{\Psi}_m = \sum_m [D(\Lambda)]_{mm} \Psi_m}$$

where the matrices $[D(\Lambda)]_{mm}$ form a representation

of the Lorentz group i.e. satisfy (for any Λ_1 and Λ_2) 68

$$\sum_p [D(\Lambda_1)]_{\mu p} [D(\Lambda_2)]_{\nu m} = [D(\Lambda_1 \Lambda_2)]_{\mu m}$$

or, in matrix notation

$$[D(\Lambda_1) D(\Lambda_2)] = [D(\Lambda_1 \Lambda_2)]$$

(group multiplication rule)

For instance, if ψ is a contravariant vector V^α we have $V^\alpha = e^\alpha_\mu V^\mu$ and $\tilde{V}^\alpha = \tilde{e}^\alpha_\mu V^\mu$ with $\tilde{e}^\alpha_\mu = \Lambda^\alpha_\beta e^\beta_\mu$ hence $\tilde{V}^\alpha = \Lambda^\alpha_\beta V^\beta$

$$[D(\Lambda)]^\alpha_\beta = \Lambda^\alpha_\beta \text{ in this case}$$

If ψ is a covariant vector U_α

$$[D(\Lambda)]_\alpha^\beta = \Lambda_\alpha^\beta$$

The most general representations of the Lorentz group (satisfying $D(\Lambda_1) D(\Lambda_2) = D(\Lambda_1 \Lambda_2)$) are the tensorial representations.

However when we consider infinitesimal Lorentz transformations, there are additional representations called the spinor representations.

Consider an infinitesimal Lorentz transformation

$$\Lambda^\alpha_\beta = \delta^\alpha_\beta + \omega^\alpha_\beta \quad (\text{where } |\omega^\alpha_\beta| \ll 1)$$

with $\omega_{\alpha\beta} = -\omega_{\beta\alpha}$ to satisfy at first order $\eta_{\alpha\beta} \Lambda^\alpha_\gamma \Lambda^\beta_\delta = \eta_{\gamma\delta}$

For such infinitesimal transformation the matrix representation $D(1)$ will take the form

$$D(1+\omega) = \mathbb{1} + \frac{1}{2}\omega^{\alpha\beta} \tau_{\alpha\beta}$$

where $\mathbb{1} = (\mathbb{1}_{mm})$ and $\tau_{\alpha\beta} = ([\tau_{\alpha\beta}]_{mm})$ are matrices.

We can always choose $\tau_{\alpha\beta} = -\tau_{\beta\alpha}$.

For the vector V^α we have

$$(1 + \frac{1}{2}\omega^{\gamma\delta} \tau_{\gamma\delta})^\alpha_\beta = \Lambda^\alpha_\beta = \delta^\alpha_\beta + \omega^\alpha_\beta$$

hence $(\tau_{\gamma\delta})^\alpha_\beta = \delta^\alpha_\gamma \eta_{\beta\delta} - \delta^\alpha_\delta \eta_{\beta\gamma}$

The matrices $\tau_{\alpha\beta}$ must satisfy an algebra in order that $D(1)$ satisfies the group multiplication rule. To see that we apply it to the product $\hat{\Lambda} = \Lambda(1+\omega)\bar{\Lambda}^{-1}$ where ω is infinitesimal.

$$\begin{aligned} \hat{\Lambda}^\alpha_\beta &= \Lambda^\alpha_\gamma (\delta^\gamma_\varepsilon + \omega^\gamma_\varepsilon) (\bar{\Lambda}^{-1})^\varepsilon_\beta \\ &= \Lambda^\alpha_\gamma (\delta^\gamma_\varepsilon + \omega^\gamma_\varepsilon) \Lambda^\varepsilon_\beta \\ &= \delta^\alpha_\beta + \underbrace{\Lambda^\alpha_\gamma \omega^\gamma_\varepsilon \Lambda^\varepsilon_\beta}_{{\omega^\alpha}_\beta} \end{aligned}$$

$$\begin{aligned}
 D(\hat{\Lambda}) &= \mathbb{I} + \frac{1}{2} \hat{\omega}^{\alpha\beta} \tau_{\alpha\beta} = \mathbb{I} + \frac{1}{2} \omega^{\alpha\beta} \Lambda^\delta_\alpha \Lambda^\delta_\beta \tau_{\delta\delta} \\
 &= D(\Lambda) D(1+\omega) D(\Lambda^{-1}) \\
 &= D(\Lambda) \left(\mathbb{I} + \frac{1}{2} \omega^{\alpha\beta} \tau_{\alpha\beta} \right) D(\Lambda^{-1}) \\
 &= \mathbb{I} + \frac{1}{2} \omega^{\alpha\beta} D(\Lambda) \tau_{\alpha\beta} D(\Lambda^{-1})
 \end{aligned}$$

hence $D(\Lambda) \tau_{\alpha\beta} D(\Lambda^{-1}) = \Lambda^\delta_\alpha \Lambda^\delta_\beta \tau_{\delta\delta}$

If we now choose $\Lambda = 1 + \omega$ (not necessarily the same ω)
 $\Lambda = 1 - \omega$

$$\left(\mathbb{I} + \frac{1}{2} \omega^{\delta\delta} \tau_{\delta\delta} \right) \tau_{\alpha\beta} \left(\mathbb{I} - \frac{1}{2} \omega^{\pi\epsilon} \tau_{\pi\epsilon} \right) = (\delta^\delta_\alpha + \omega^\delta_\alpha)(\delta^\delta_\beta + \omega^\delta_\beta) \tau_{\delta\delta}$$

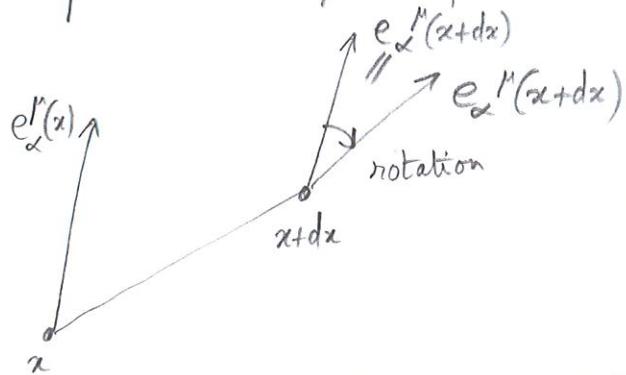
from which we deduce

$$[\tau_{\alpha\beta}, \tau_{\gamma\delta}] = h_{\alpha\beta} \tau_{\gamma\delta} - h_{\beta\gamma} \tau_{\alpha\delta} - h_{\alpha\delta} \tau_{\beta\gamma} + h_{\beta\delta} \tau_{\alpha\gamma}$$

Finding all representations of the infinitesimal Lorentz group is reduced to the problem of finding all matrices $\tau_{\alpha\beta}$ satisfying this algebra.

This algebra contains (besides tensorial representations) the spinorial representations associated to transformations of half-integer spin fields.

We define the spinorial derivative $\nabla_\alpha \psi$ using
the parallel transport from x to $x+dx$



$$e_\alpha^\mu(x+dx) = \Lambda_\alpha^\beta e_\beta^\mu(x+dx)$$

$$\text{where } \Lambda_\alpha^\beta = \delta_\alpha^\beta + \omega_\alpha^\beta$$

$$\text{hence } e_\alpha^\mu(x+dx) = e_{\parallel\alpha}^\mu(x+dx) + \omega_{\alpha\beta}^\mu e_\beta^\mu$$

We shall say that the components of $\psi(x+dx)$
in the basis $e_\alpha^\mu(x+dx)$ differ from the components of the //transported
 $\psi_{\parallel}(x+dx)$ in the same basis $e_{\parallel\alpha}^\mu(x+dx)$ by

$$\boxed{\psi(x+dx) - \psi_{\parallel}(x+dx) = \theta^\alpha \nabla_\alpha \psi}$$

By definition of the // transported $\psi_{\parallel}(x+dx)$ its components in
the basis $e_{\parallel\alpha}^\mu(x+dx)$ are the same as $\psi(x)$ in the basis
 $e_\alpha^\mu(x)$ thus

$$\left. \psi_{\parallel}(x+dx) \right|_{\text{basis } e_{\parallel\alpha}^\mu(x+dx)} = \psi(x)$$

$\psi_{\parallel}(x+dx)$ can be deduced from $\left. \psi_{\parallel}(x+dx) \right|_{\text{basis } e_{\parallel\alpha}^\mu}$
by using the rotation $\Lambda_\alpha^\beta = \delta_\alpha^\beta + \omega_\alpha^\beta$

$$D(\lambda) \Psi_{\parallel}(x+dx) = \left. \Psi_{\parallel}(x+dx) \right|_{\text{basis } e_{\alpha}} = \Psi(x)$$

$$\left(\mathbb{I} + \frac{1}{2} \omega^{\alpha\beta} \nabla_{\alpha\beta} \right) \Psi_{\parallel}(x+dx) = \Psi(x)$$

$$\Psi_{\parallel}(x+dx) = \left(\mathbb{I} - \frac{1}{2} \omega^{\alpha\beta} \nabla_{\alpha\beta} \right) \Psi(x)$$

Plugging in the defining relation of the cov. derivative

$$\Psi(x+dx) = \Psi(x) - \frac{1}{2} \omega^{\alpha\beta} \nabla_{\alpha\beta} \Psi + \theta^{\alpha} \nabla_{\alpha} \Psi$$

$$\theta^{\alpha} \nabla_{\alpha} \Psi + \frac{1}{2} \omega^{\beta\gamma} \nabla_{\beta\gamma} \Psi = \theta^{\alpha} \nabla_{\alpha} \Psi$$

But we have $\omega^{\beta\gamma} = \omega^{\beta}_{\alpha\gamma} \theta^{\alpha}$ hence

$$\boxed{\nabla_{\alpha} \Psi = \partial_{\alpha} \Psi + \frac{1}{2} \omega^{\beta}_{\alpha\gamma} \nabla_{\beta} \Psi}$$

For the vector we have seen $(\nabla_{\beta})^{\pi}_{\varepsilon} = \delta_{\beta}^{\pi} \delta_{\varepsilon}^{\gamma} - h_{\beta\varepsilon} h^{\gamma\pi}$

$$\nabla_{\alpha} V^{\pi} = \partial_{\alpha} V^{\pi} + \frac{1}{2} \omega^{\beta}_{\alpha\gamma} (\delta_{\beta}^{\pi} \delta_{\varepsilon}^{\gamma} - h_{\beta\varepsilon} h^{\gamma\pi}) V^{\varepsilon}$$

$$= \partial_{\alpha} V^{\pi} + \omega^{\pi}_{\alpha\varepsilon} V^{\varepsilon}$$

and we recover the previous cov. derivative acting on (tetrad components of) tensorial fields.

To couple any field (classical or quantum) to gravity we write the action of this field in special relativity and replace all derivatives

$$\frac{\partial}{\partial x^\alpha} \rightarrow \nabla_\alpha \quad (\text{cov. derivative, tetradiic or spinorial})$$

For instance to couple a Dirac spinor Ψ we shall write the Dirac equation as

$$\boxed{\gamma^\alpha \nabla_\alpha \Psi + \frac{mc}{\hbar} \Psi = 0}$$

where $\nabla_\alpha \Psi = \partial_\alpha \Psi + \frac{1}{2} \omega_{\alpha\beta\gamma} \gamma^\beta \cdot \boldsymbol{\sigma} \gamma^\gamma \Psi$ and $\boldsymbol{\sigma}_{\alpha\beta} = \frac{1}{4} [\gamma_\alpha, \gamma_\beta]$

for the representation of Dirac spinors.

(Here the gamma-matrices satisfy $\gamma_\alpha \gamma_\beta + \gamma_\beta \gamma_\alpha = 2 \delta_{\alpha\beta}$)

Gauge-fixed Einstein-Hilbert action

We come back to the EH action without torsion.

$$S = \frac{1}{16\pi} \int d^4x \sqrt{-g} R + S_m \quad (G=c=1)$$

The Ricci scalar R is a function of the metric g , and space-time derivative ∂g and $\partial^2 g$. But the second-order derivatives appear in the form of a surface-term (already studied). We derive the Landau-Lifchitz form of the action, which depends only on g and ∂g (ignoring the surface term).

$$\sqrt{-g} R = \sqrt{-g} g^{\mu\nu} \left(\partial_\lambda \Gamma^\lambda_{\mu\nu} - \partial_\nu \Gamma^\lambda_{\mu\lambda} + \Gamma^\epsilon_{\mu\nu} \Gamma^\lambda_{\epsilon\lambda} - \Gamma^\epsilon_{\mu\lambda} \Gamma^\lambda_{\nu\epsilon} \right)$$

\uparrow
operate by parts and throw the pure divergence

$$= - \underbrace{\partial_\lambda (\sqrt{-g} g^{\mu\nu})}_{\text{re-expres } \partial g^{\mu\nu}} \Gamma^\lambda_{\mu\nu} + \underbrace{\partial_\nu (\sqrt{-g} g^{\mu\nu})}_{\text{in terms of Christoffel symbols}} \Gamma^\lambda_{\mu\lambda} + \sqrt{-g} \left(\Gamma^\epsilon_{\mu\nu} \Gamma^\lambda_{\epsilon\lambda} - \Gamma^\epsilon_{\mu\lambda} \Gamma^\lambda_{\nu\epsilon} \right)$$

Hence the Landau-Lifchitz form (which is not manifestly covariant, but in fact it's modulo a surface term)

$$S = \frac{1}{16\pi} \int d^4x \sqrt{-g} g^{\mu\nu} \left(\Gamma^\rho_{\mu\lambda} \Gamma^\lambda_{\nu\rho} - \Gamma^\rho_{\mu\nu} \Gamma^\lambda_{\rho\lambda} \right) + S_m$$

Posing $\tilde{g}^{\mu\nu} = \sqrt{-g} g^{\mu\nu}$ "gothic" metric
 (and its inverse $\tilde{g}_{\mu\nu} = \frac{1}{\sqrt{-g}} g_{\mu\nu}$) we have the completely explicit form

$$S = -\frac{1}{64\pi} \int d^4x \left[\left(\tilde{g}_{\mu\rho} \tilde{g}_{\nu\sigma} - \frac{1}{2} \tilde{g}_{\mu\nu} \tilde{g}_{\rho\sigma} \right) \tilde{g}^{\lambda\tau} \partial_\lambda \tilde{g}^{\mu\nu} \partial_\tau \tilde{g}^{\rho\sigma} - 2 \tilde{g}_{\mu\nu} \partial_\rho \tilde{g}^{\mu\sigma} \partial_\sigma \tilde{g}^{\nu\rho} \right] + S_m$$

This action is generally covariant ("general diffeomorphism" invariance). In particular, in an infinitesimal coord. transformation

$$x'^\mu = x^\mu + \xi^\mu(x)$$

it is invariant when $\tilde{g}_{\mu\nu} \rightarrow \tilde{g}_{\mu\nu} + \mathcal{L}_\xi \tilde{g}_{\mu\nu}$ with $\mathcal{L}_\xi \tilde{g}_{\mu\nu} = -2 \nabla_{\mu} \xi^\nu$

or equivalently $\tilde{g}^{\mu\nu} \rightarrow \tilde{g}^{\mu\nu} + \mathcal{L}_\xi \tilde{g}^{\mu\nu}$ with (exercise)

$$\mathcal{L}_\xi \tilde{g}^{\mu\nu} = \sqrt{-g} \left(2 \nabla^\mu \xi^\nu - g^{\mu\nu} \nabla^\rho \xi^\rho \right)$$

If we consider an expansion around a given background (for instance flat)

$$\tilde{g}^{\mu\nu}(x) = \tilde{g}_B^{\mu\nu}(x) + h^{\mu\nu}(x)$$

$$\tilde{g}^\mu(x) = \tilde{g}_B^\mu(x) + h^\mu(x)$$

the diffeo invariance implies a gauge invariance for the potentials $h \rightarrow h'$ with

$$h'^{\mu\nu}(x) = h^{\mu\nu}(x) + \sqrt{-g_B} \left(2 \nabla_B^\mu \xi^\nu - g_B^{\mu\nu} \nabla_B^\rho \xi^\rho \right)$$

↑ cov. derivative w.r.t.
the background

We now write the gauge-fixed action corresponding to harmonic (or de Donder) coordinates defined by

$$\boxed{\partial_\nu y^\mu = 0}$$

or equivalently $\Gamma^\mu \equiv g^{\rho\sigma} \Gamma_{\rho\sigma}^\mu = 0$ since $\boxed{\sqrt{-g} \Gamma^\mu = -\partial_\nu y^\mu}$

We simply have to add a term $\frac{1}{2} \sqrt{-g} g_{\mu\nu} \Gamma^\mu \Gamma^\nu = -\frac{1}{2} g_{\mu\nu} \partial_\rho y^\mu \partial^\rho y^\nu$ to the LL action

$$S_{\text{gauge fixed}} = \frac{1}{16\pi} \int d^4x \sqrt{-g} \left[g^{\mu\nu} \left(\Gamma_{\mu\lambda}^\rho \Gamma_{\nu\rho}^\lambda - \Gamma_{\mu\nu}^\rho \Gamma_{\rho\lambda}^\lambda \right) - \frac{1}{2} g_{\mu\nu} \Gamma^\mu \Gamma^\nu \right] + S_m$$

$$S_{\text{g.f.}} = -\frac{1}{64\pi} \int d^4x \left[\left(\frac{y_{\mu\rho} y_{\nu\sigma} - \frac{1}{2} y_{\mu\nu} y_{\rho\sigma}}{g_{\mu\nu} g_{\rho\sigma}} \right) y^{\lambda\rho} \partial_\lambda y^\nu \partial_\rho y^\sigma - 2 y_{\mu\nu} \left(\partial_\rho y^\mu \partial_\sigma y^\nu - \partial_\sigma y^\mu \partial_\rho y^\nu \right) \right] + S_m$$

pure divergence at
"quadratic order"
in an expansion $y^\mu = h^\mu + \tilde{h}^\mu$

General form of the harmonic-gauge Einstein field eqs

$$\boxed{y^{\mu\rho} \partial_\rho^2 y^\nu = 16\pi |g| T^{\mu\nu} + \underbrace{\sum_{\lambda}^{\mu\nu} [y, \partial_\lambda y]}_{(\text{gravitational source})}}$$

and the gauge condition $\partial_\nu y^\mu = 0$ which is in fact implied by the latter equations when the EOM of matter are satisfied, $\nabla_\nu T^{\mu\nu} = 0$.

The field eqs in such form appear as wave-like equations, with an hyperbolic wave operator $\gamma^{\mu\nu} \partial_\mu \partial_\nu$, with only first order derivatives $\partial_\mu f$ in the right-hand-side, and form a well defined hyperbolic system of eqs, with a well-posed Cauchy problem.

If we further assume that space-time is a small deviation from a flat Minkowski background

$$\gamma^{\mu\nu} = g^{\mu\nu} + h^{\mu\nu} \quad (|h^{\mu\nu}| \ll 1)$$

we have (posing $\square = \square_g = \gamma^{\mu\nu} \partial_\mu \partial_\nu$ the flat D'Alembertian operator)

$$\boxed{\square h^{\mu\nu} = 16\pi |g| T^{\mu\nu} + \Lambda^{\mu\nu}[h, \partial h, \partial^2 h]}$$

where $\Lambda^{\mu\nu} = \sum \Gamma^{\mu\nu} - h^{\rho\sigma} \partial_\rho \partial_\sigma h^{\mu\nu}$. The gravitational source term is at least quadratic in h and is amenable to a perturbative expansion

$$\Lambda \sim \underbrace{h \partial^2 h}_{\text{quadratic}} + \underbrace{\partial h \partial h}_{\text{cubic}} + \underbrace{h \partial h \partial h}_{\text{quartic}} + \dots$$

$$\text{Posing } \boxed{T^{\mu\nu} = |g| T^{\mu\nu} + \frac{1}{16\pi} \Lambda^{\mu\nu}}$$

this is the pseudo-stress energy tensor of the matter fields and gravitational fields (in harmonic coordinates). The coord. condition is equivalent to the EOM of matter fields

$$\boxed{\partial_\nu h^{\mu\nu} = 0 \Leftrightarrow \partial_\nu T^{\mu\nu} = 0 \Leftrightarrow \nabla^\mu T^{\mu\nu} = 0}$$

The Fokker action

We consider the problem of motion of particles under their mutual gravitation. Thus this is a self-gravitating problem, where the dynamics is purely gravitational.

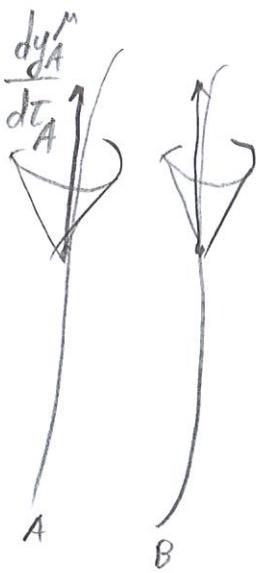
Thus we are not considering the motion of particles in a given background metric (as we did in a previous course) but the motion of particles in the metric generated by the particles.

The matter action will be

$$S_m = \sum_A -m_A \int_{-\infty}^{+\infty} d\tau_A$$

proper time

$$d\tau_A = \sqrt{-g_{\mu\nu} \frac{dy^{\mu}}{d\tau_A} \frac{dy^{\nu}}{d\tau_A}}$$



with stress-energy tensor

$$T^{\mu\nu} = \sum_A m_A \int_{-\infty}^{+\infty} d\tau_A u_A^{\mu} u_A^{\nu} \frac{\delta q_4(x-y_A)}{Fg}$$

$$T^{\mu\nu} = \sum_A m_A \frac{v_A^{\mu} v_A^{\nu}}{\sqrt{-g_{\mu\nu}}} \frac{\delta_3(\vec{x}-\vec{y}_A)}{\sqrt{Fg}}$$

or
(in 3-dimensional form)

where $y_A^{\mu} = (ct, \vec{y}_A(t))$ $v_A^{\mu} = (\epsilon, \vec{v}_A(t))$ $\vec{v}_A = \frac{d\vec{y}_A}{dt}$ coordinate velocity

This $T^{\mu\nu}$ is on the RHS of the Einstein field eqs and, solving them, we get a solution of the field eqs as an explicit functional of the positions, velocities, ... of particles.

$$\boxed{\bar{h}^{\mu\nu} = \bar{h}^{\mu\nu}(\vec{x}; \vec{y}_A(t), \vec{v}_A(t), \vec{a}_A(t), \dots)}$$

dependence on particles
with in general presence of accelerations \vec{a}_A ,
derivative of accelerations \vec{D}_A, \dots

For instance the metric evaluated at position $\vec{y}_A(t)$ will be

$$(\bar{g}_{\mu\nu})_A = \bar{g}_{\mu\nu}(\vec{y}_A(t); \vec{y}_B(t), \vec{v}_B(t), \dots)$$

Such solution $\bar{h}^{\mu\nu}$ (or $\bar{g}_{\mu\nu}$) can be constructed order by order as a post-Newtonian (PN) expansion, see below.

The Fokker action is obtained when we insert this explicit solution back into the (gauge-fixed) EH action thus

$$\boxed{S_{\text{Fokker}}[\vec{y}_A, \vec{v}_A, \vec{a}_A, \dots] = \int d^4x \mathcal{L}_g [\bar{h}^{\mu\nu}(\vec{x}; \vec{y}_A(t), \vec{v}_A(t), \dots)] - \sum_B m_B \int dt \sqrt{-\bar{g}_{\mu\nu}(\vec{y}_B; \vec{y}_A, \vec{v}_A)} v_B^\mu v_B^\nu}$$

The E.O.M. of particles follow from varying the Fokker action wrt the particles.

$$\frac{\delta S_F}{\delta y_A^\mu} = \underbrace{\frac{\delta S}{\delta g_{\mu\nu}}}_{\delta g_{\mu\nu}} \frac{\delta \bar{g}_{\mu\nu}}{\delta y_A^\mu} + \underbrace{\frac{\delta S_m}{\delta y_A^\mu}}_{\delta y_A^\mu}$$

functional variation
of the full EH action
w.r.t. metric

variation of
matter action

But $\frac{\delta S}{\delta g_{\mu\nu}} [\bar{g}(\vec{x}; \vec{y}_A, \dots)] = 0$ because \bar{g} is a solution of the field eqs

$$\boxed{\frac{\delta S_F[\bar{g}]}{\delta y_A^i} = \frac{\delta S_m[\bar{g}]}{\delta y_A^i} = 0}$$

which are thus the correct EOM of particles in the metric $\bar{g}_{\mu\nu}(\vec{x}; \vec{y}_A, \dots)$ generated by the particles themselves.

Equations of motion in the PN approximation

Matter source (say N particles)

- isolated ($T^{\mu\nu}$ has a compact support)

- PN, i.e. "slowly moving", existence of a small parameter

$$\epsilon \sim \frac{v}{c} \ll 1$$

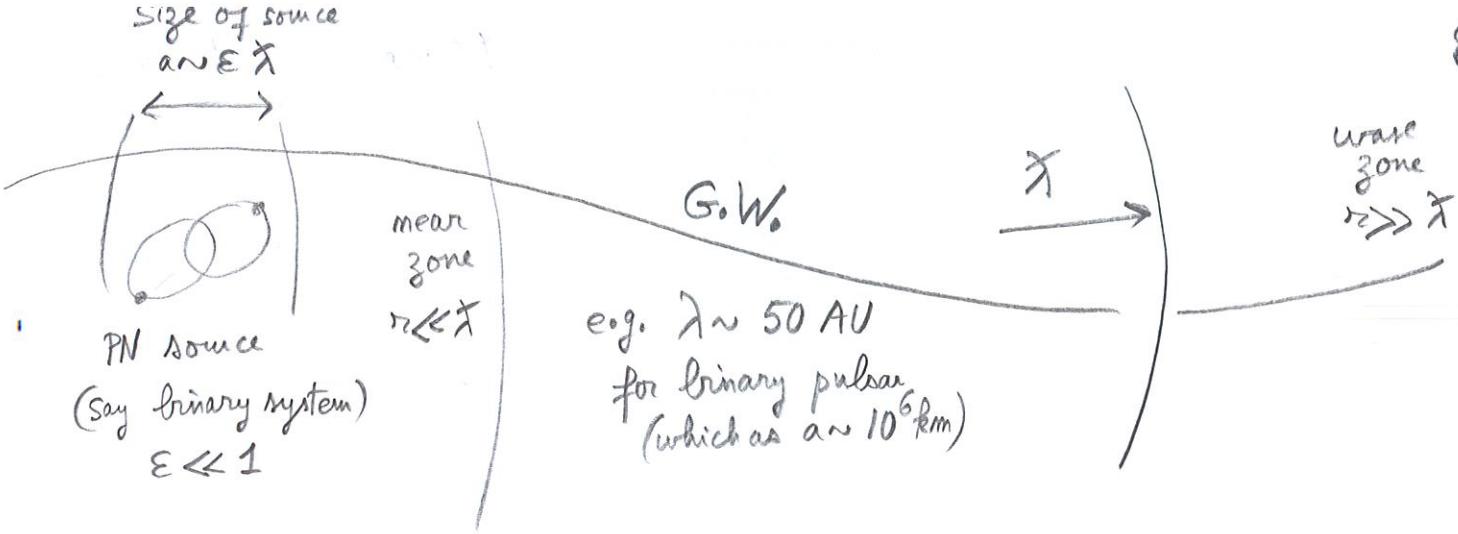
- self-gravitating, internal motion is due to gravitational forces

$$\gamma \sim \frac{v^2}{a} \sim \frac{GM}{a^2} \quad \begin{aligned} a &= \text{size of source} \\ M &= \text{its mass} \end{aligned}$$

$$\text{Period of motion } P \sim \frac{2\pi a}{v}$$

$$\text{Gravitational wave length } \lambda = cP \quad \tilde{\lambda} = \frac{\lambda}{2\pi}$$

$$\boxed{\frac{a}{\tilde{\lambda}} \sim \frac{v}{c} \sim \epsilon}$$



The near zone $r \ll \lambda$ covers entirely the PN source.

In the mean zone the PN expansion $\epsilon \rightarrow 0$ is valid

$$\square h^{mn} = \underbrace{\frac{16\pi G}{c^4} |g| T^{mn}}_{\text{neglect to leading Newtonian order}} + \underbrace{N^{mn}}_{}$$

$$T^{00} \sim \rho c^2 = O(c^2)$$

$$T^{0i} \sim \rho c v^i = O(c)$$

$$T^{ij} \sim \rho v^i v^j = O(1)$$

$$\Delta h^{00} = \frac{16\pi G}{c^2} \rho + O(\frac{1}{c^4})$$

U usual Newtonian potential of source

$$\Delta h^{0i} = O(\frac{1}{c^3})$$

$$\Delta U = -4\pi G \rho$$

$$\Delta h^{ij} = O(\frac{1}{c^4})$$

$$h^{00} \approx -\frac{4U}{c^2}$$

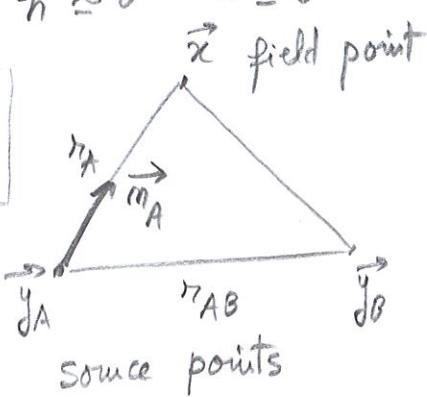
$$h^{0i} \approx 0 \quad h^{ij} \approx 0$$

For N point particles

$$U = \sum_{A=1}^N \frac{G m_A}{r_A}$$

$$r_A^i = \vec{x}^i - \vec{y}_A^i(t)$$

$$r_A = |\vec{r}_A| \quad m_A^i = \frac{r_A^i}{r_A}$$



$$\mathcal{L}_g \approx -\frac{c^4}{64\pi G} \left[\partial_i h^{00} \partial_i h^{00} - \frac{1}{2} \partial_i h \partial_i h \right] + O(\frac{1}{c^2})$$

$h = -h^{00} + h^{ii} \approx -h^{00}$

$$= -\frac{1}{8\pi G} \partial_i U \partial_i U + O(\frac{1}{c^2})$$

$$g_{00} \approx -1 + \frac{2U}{c^2}$$

$$L_m = -\sum_A m_A c \sqrt{-g_{\mu\nu}}_A v_A^\mu v_A^\nu$$

$$g_{ij} \approx \delta_{ij}$$

$$= \sum_A m_A \left(-c^2 + \frac{\vec{v}_A^2}{2} + U_A \right) + O(\frac{1}{c^2})$$

Folklore Lagrangian at Newtonian order

$$L^F = -\frac{1}{8\pi G} \int d^3x \partial_i U \partial_i U + \sum_A m_A \left(-c^2 + \frac{\vec{v}_A^2}{2} + U_A \right) + O(\frac{1}{c^2})$$

$$\int d^3x \partial_i U \partial_i U = - \int d^3x U \Delta U = 4\pi G \int d^3x U \sum_A m_A \delta^{(3)}(\vec{x} - \vec{r}_A)$$

$$= 4\pi G \sum_A m_A U_A$$

$$L^F = \sum_A m_A \left(-c^2 + \frac{\vec{v}_A^2}{2} + \frac{U_A}{2} \right) + O(\frac{1}{c^2})$$

We need a self-field regularization to remove the infinite self-field of particles.

$$U_A = \left(\sum_B \frac{G m_B}{r_B} \right)_A = \sum_{B \neq A} \frac{G m_B}{r_{AB}}$$

$$\begin{aligned}\frac{1}{2} \sum_A m_A U_A &= \frac{1}{2} \sum_{A=1}^N m_A \sum_{B \neq A} \frac{G m_B}{r_{AB}} \\ &= \sum_{A=1}^N \sum_{B=A+1}^N \frac{G m_A m_B}{r_{AB}} \\ &= \sum_{A < B} \frac{G m_A m_B}{r_{AB}}\end{aligned}$$

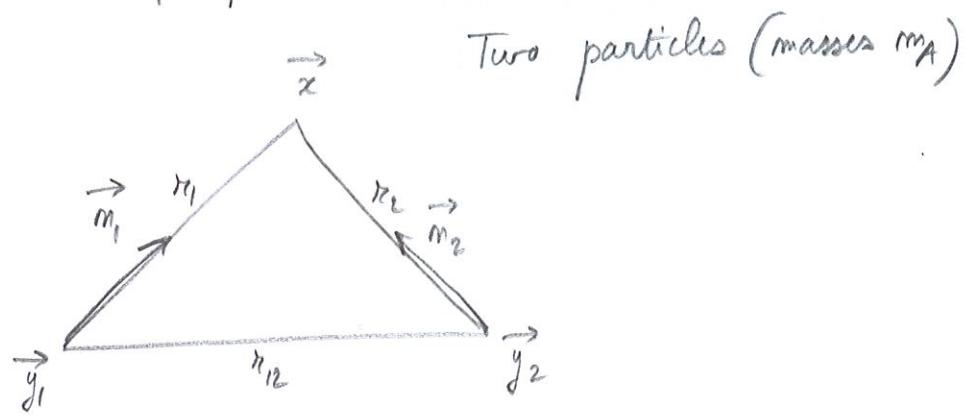
$$\boxed{L^F = \sum_A m_A \left(-c^2 + \frac{\vec{v}_A^2}{2} \right) + \sum_{A < B} \frac{G m_A m_B}{r_{AB}} + O\left(\frac{1}{c^2}\right)}$$

and we recover the Lagrangian of N point masses in Newtonian dynamics. The Fokker Lagrangian of 2 point-masses is known at 4PN order, $\left(\frac{v}{c}\right)^8$ beyond Newtonian term.

For two particles the EOM, which derive from the Lagrangian for the conservative part, and on which we add some radiation-reaction terms (studied later) read

$$\begin{aligned}\frac{dv_1^i}{dt} &= - \underbrace{\frac{G m_2 m_1^i}{r_{12}^2}}_{\text{Lorentz-Droste-Einstein-Infeld-Hoffmann 1PN term}} \\ &\quad + \frac{1}{c^2} \left\{ \left(\frac{5G^2 m_1 m_2}{r_{12}^3} + \frac{4G^2 m_1^2}{r_{12}^3} + \frac{G m_2}{r_{12}} \left(\frac{3}{2} (m_2 v_2)^2 + \dots \right) \right) m_1^i \right. \\ &\quad \left. + \frac{G m_2}{r_{12}^2} (4(m_2 v_1) - 3(m_2 v_2)) v_{12}^i \right\} \\ &\quad + \frac{1}{c^4} \{ \text{2PN term} \} + \frac{1}{c^5} \{ \text{2.5PN radiation reaction term} \} \\ &\quad + \underbrace{\frac{1}{c^6} + \frac{1}{c^7} + \frac{1}{c^8} + O\left(\frac{1}{c^9}\right)}_{\text{also known}}\end{aligned}$$

Problem of point-particles



Newtonian potential generated by the masses

$$\Delta U = -4\pi G \rho = -4\pi G [m_1 \delta(\vec{x} - \vec{r}_1) + m_2 \delta(\vec{x} - \vec{r}_2)]$$

Using $\Delta \frac{1}{r} = -4\pi \delta(\vec{x})$ $U(\vec{x}) = \frac{Gm_1}{r_1} + \frac{Gm_2}{r_2}$

acceleration $\frac{d^2 r_i^i}{dt^2} = (\partial_i U)(\vec{r}_i) = \left(-\underbrace{\frac{Gm_1}{r_1^2} m_1^i - \frac{Gm_2}{r_2^2} m_2^i}_{\text{infinite self-force}} \right) (\vec{r}_i)$
of the point particle

If $F(\vec{x})$ is singular at \vec{r}_1 and \vec{r}_2 (with power-like singular expansion around \vec{r}_1, \vec{r}_2) what are the meanings of

$$F(\vec{r}_1) ?$$

$$F(\vec{x}) \delta(\vec{x} - \vec{r}_1) ?$$

$$\partial_i F ? \quad (\text{for instance } \partial_i \partial_j \frac{1}{r_i} = \frac{3m_1^i m_1^j \delta^{ij}}{r_1^3} - \underbrace{\frac{4\pi}{3} \delta^{ij} \delta(\vec{x} - \vec{r}_1)}_{\text{distributional term}})$$

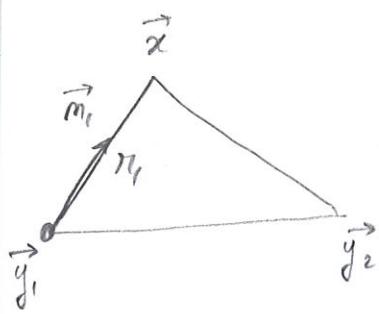
$$\int d^3x F(\vec{x}) ?$$

Answer to all these problems needed to answer the problems of motion and radiation from compact objects modelled by point-particles in PN approximations.

Hadamard self-field regularization

$F(\vec{x})$ smooth except at \vec{y}_1 and \vec{y}_2 .

When $\eta_i \rightarrow 0$

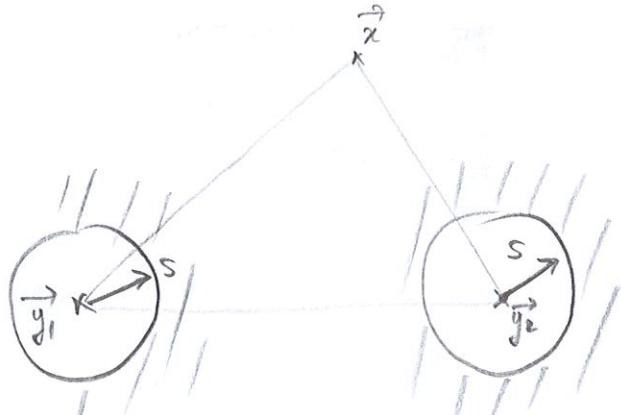


$$F(\vec{x}) = \sum_{a_0 \leq a \leq N} \eta_i^a f_a(\vec{m}_i) + o(\eta_i^N)$$

$$a_0, a \in \mathbb{Z}$$

Hadamard's "partie finie" (Pf)

$$\boxed{(F)_i = \int \frac{d\Omega}{4\pi} f_0(\vec{m}_i)}$$



Two balls (radius s) excised

$$\boxed{\begin{aligned} \text{Pf}_{s_1, s_2} \int d^3x F(\vec{x}) &= \lim_{s \rightarrow 0} \left[\int d^3x F(\vec{x}) \right. \\ &\quad \left. + \sum_{a+3 < 0} \frac{s^{a+3}}{a+3} \int d\Omega_i f_a(\vec{m}_i) \right. \\ &\quad \left. + \lim_{\frac{s}{s_1} \rightarrow 0} \left(\frac{s}{s_1} \right)^{\alpha} \int d\Omega_i f_{-3}(\vec{m}_i) + i \epsilon^2 \right] \end{aligned}}$$

s_1, s_2 two arbitrary UV cut-off scales.

$$\boxed{\text{Pf}_{s_1, s_2} \int d^3x F = \underset{\alpha \rightarrow 0}{\text{FP}} \underset{\beta \rightarrow 0}{\text{FP}} \int d^3x \left(\frac{s_1}{s_2} \right)^\alpha \left(\frac{s_2}{s_1} \right)^\beta F}$$

using analytic continuation in α and $\beta \in \mathbb{C}$.

Hadamard's regularization is very convenient in practical calculations but in higher PN orders is plagued with ambiguities. Basic reason is that

$$(FG) \neq (F)(G), \text{ in general.}$$

Hence basic symmetries of GR such as diffeomorphism invariance are broken in high PN approximations (starting in fact at 3PN).

Dimensional self-field regularization (t'Hooft, Veltman)

We work in a space with d dimensions (so space-time has $D=d+1$ dimensions).

In D dimensions EFE take the same form

$$\boxed{R^{\mu\nu} - \frac{1}{2} g^{\mu\nu} R = \frac{16\pi G}{c^4} T^{\mu\nu}}$$

↑
coeff $\frac{1}{2}$ in any dim

but

$$\boxed{R^{\mu\nu} = \frac{8\pi G}{c^4} \left(T^{\mu\nu} - \frac{1}{d-1} g^{\mu\nu} T \right)}$$

↑
coeff depends on d

Here

$$\boxed{G = G_N l_0^{d-3}}$$

where G_N is the usual Newton's constant and l_0 is the characteristic length of dim-reg.

The dimension $d \in \mathbb{C}$ and we apply complex analytic regularization in d .

Volume element $\boxed{d^d x = r^{d-1} dr d\Omega_{d-1}}$

Volume of $(d-1)$ -dimensional sphere $\Omega_{d-1} = \int d\Omega_{d-1}$

To compute it we use the Gaussian integral

$$\begin{aligned} \int_{R^d}^d e^{-r^2} &= \left(\int_{-\infty}^{+\infty} dx e^{-x^2} \right)^d = \pi^{d/2} \\ &= \int r^{d-1} dr d\Omega_{d-1} e^{-r^2} = \Omega_{d-1} \int_0^{+\infty} dr r^{d-1} e^{-r^2} = \frac{\Omega_{d-1}}{2} \Gamma\left(\frac{d}{2}\right) \\ \boxed{\Omega_{d-1} = \frac{2\pi}{\Gamma\left(\frac{d}{2}\right)}} \end{aligned}$$

$\Omega_1 = 4\pi$ $\Omega_2 = 2\pi$ and $\Omega_0 = 2$ (since sphere with 0 dimension is made of 2 points!)

Tools used in PN calculations

Green's function of the Laplace operator

$$\boxed{\Delta u = -4\pi \delta^{(d)}(\vec{x})}$$

$$u = k r^{2-d} \quad k = \frac{\Gamma\left(\frac{d-2}{2}\right)}{\pi^{\frac{d-1}{2}}}$$

Riesz Euclidean kernels (generalize $\delta^{(d)}$ and u)

$$\boxed{\delta_\alpha^{(d)}(\vec{x}) = K_\alpha r^{\alpha-d}}$$

$$\boxed{\Delta \delta_{\alpha+2}^{(d)} = -\delta_\alpha^{(d)}}$$

$$K_\alpha = \frac{\Gamma\left(\frac{d-\alpha}{2}\right)}{2^{\alpha} \pi^{d/2} \Gamma(d/2)}$$

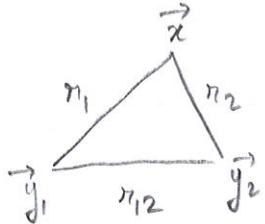
(so that $\delta^{(d)} = \delta_0^{(d)}$ and $u = 4\pi \delta_0^{(d)}$)

Permits to integrate term by term the PN series

Convolution relation $\left| \delta_{\alpha}^{(d)} * \delta_{\beta}^{(d)} = \delta_{\alpha+\beta}^{(d)} \right|$

which is an elegant formulation of Riesz's formula

$$\boxed{\int d^d x \, \gamma_1^\alpha \gamma_2^\beta = \pi^{d/2} \frac{\Gamma(\frac{\alpha+d}{2}) \Gamma(\frac{\beta+d}{2}) \Gamma(-\frac{\alpha+\beta+d}{2})}{\Gamma(-\frac{\alpha}{2}) \Gamma(-\frac{\beta}{2}) \Gamma(\frac{\alpha+\beta+2d}{2})} \gamma_{12}^{\alpha+\beta+d}}$$



which is extensively used in PN calculations.

Difference between Had. reg. and Dim. reg.

iterating in PN form we have to solve Poisson equations

$$\Delta P = F$$

where F is a non-compact support function singular on \vec{y}_1, \vec{y}_2 .

Had. reg. is
given by "partie finie"

$$\boxed{P(\vec{x}') = -\frac{1}{4\pi} \int_{S_1 S_2} \frac{d^3 \vec{x}}{|\vec{x} - \vec{x}'|} F(\vec{x}')}}$$

Dim. reg.

$$\boxed{P^{(d)}(\vec{x}') = -\frac{k}{4\pi} \int \frac{d^d \vec{x}}{|\vec{x} - \vec{x}'|^{d-2}} F^{(d)}(\vec{x}')}}$$

We have to compute the values of these solutions at the location of singular point \vec{y}_1 .

$(P)_1$ (in the sense
of "partie finie")

and

$$P^{(d)}(\vec{y}_1) = -\frac{k}{4\pi} \int \frac{d^d \vec{x}}{|\vec{x}|^{d-2}} F^{(d)}(\vec{x})$$

In d dim we have when $\gamma_i \rightarrow 0$

$$F(\vec{x}) = \sum_{p,q} \underbrace{\gamma_i^{p+q\varepsilon}}_{\text{complex powers of } \gamma_i \text{ appear}} f_{p,q}^{(\varepsilon)}(\vec{m}_i) + o(\gamma_i^N) \quad \text{where } \varepsilon = d-3$$

$$\mathcal{D}P(1) = P^{(d)}(\vec{y}_i) - (P), \quad \text{difference}$$

$$\begin{aligned} \mathcal{D}P(1) = & -\frac{1}{\varepsilon(1+\varepsilon)} \sum_q \left(\frac{1}{q} + \varepsilon [\ln s_1 - 1] \right) \langle f_{-2,q}^{(\varepsilon)}(\vec{m}_1) \rangle \\ & - \frac{1}{\varepsilon(1+\varepsilon)} \sum_q \left(\frac{1}{q+1} + \varepsilon \ln s_2 \right) \sum_{l=0}^{\infty} \frac{(-)^l}{l!} \partial_l \left(\frac{1}{\gamma_1^{1+\varepsilon}} \right) \langle m_2^L f_{-l+3,q}^{(\varepsilon)}(\vec{m}_2) \rangle \\ & + O(\varepsilon) \end{aligned}$$

Apparition of poles $\propto \frac{1}{\varepsilon}$

when logarithmic divergences in Hdd. reg.

(can be computed locally
when $\gamma_1 \rightarrow 0, \gamma_2 \rightarrow 0$)

It can be shown that the poles can be renormalized

by a redefinition of the trajectories of the particles

$$\left(\dot{y}_A^i(t) \right)^{\text{bare}} \rightarrow \left(\dot{y}_A^i(t) \right)^{\text{dressed}} = \left(\dot{y}_A^i \right)^{\text{bare}} + O\left(\frac{1}{\varepsilon}\right) + O(\varepsilon^\circ) + \dots$$

so that the final result is UV finite.

Gravitational waves from isolated systems

$$h^{\mu\nu} = \sqrt{-g} g^{\mu\nu} - h^{\mu\nu}$$

small perturbation around Minkowski metric

$$\square h^{\mu\nu} = \frac{16\pi G}{c^4} T^{\mu\nu}$$

ordinary flat

$$\text{d'Alembertian } \square = \eta^{\rho\sigma} \partial_\rho \partial_\sigma$$

pseudo stress-energy tensor (actually Lorentz tensor)
of matter and gravitational field in harmonic coordinates

To solve this equation we need to impose boundary conditions at infinity, saying that the source of GW is isolated from other sources in the Universe.

Boundary conditions are imposed at past null infinity

In GR one defines the spatio-temporal infinities

$$I^+ = \text{future temporal infinity} \quad (t \rightarrow +\infty, r = \text{const})$$

$$J^+ = \text{future null infinity} \quad (r \rightarrow +\infty, t - r/c = \text{const})$$

$$I^0 = \text{spatial infinity} \quad (r \rightarrow \infty, t = \text{const})$$

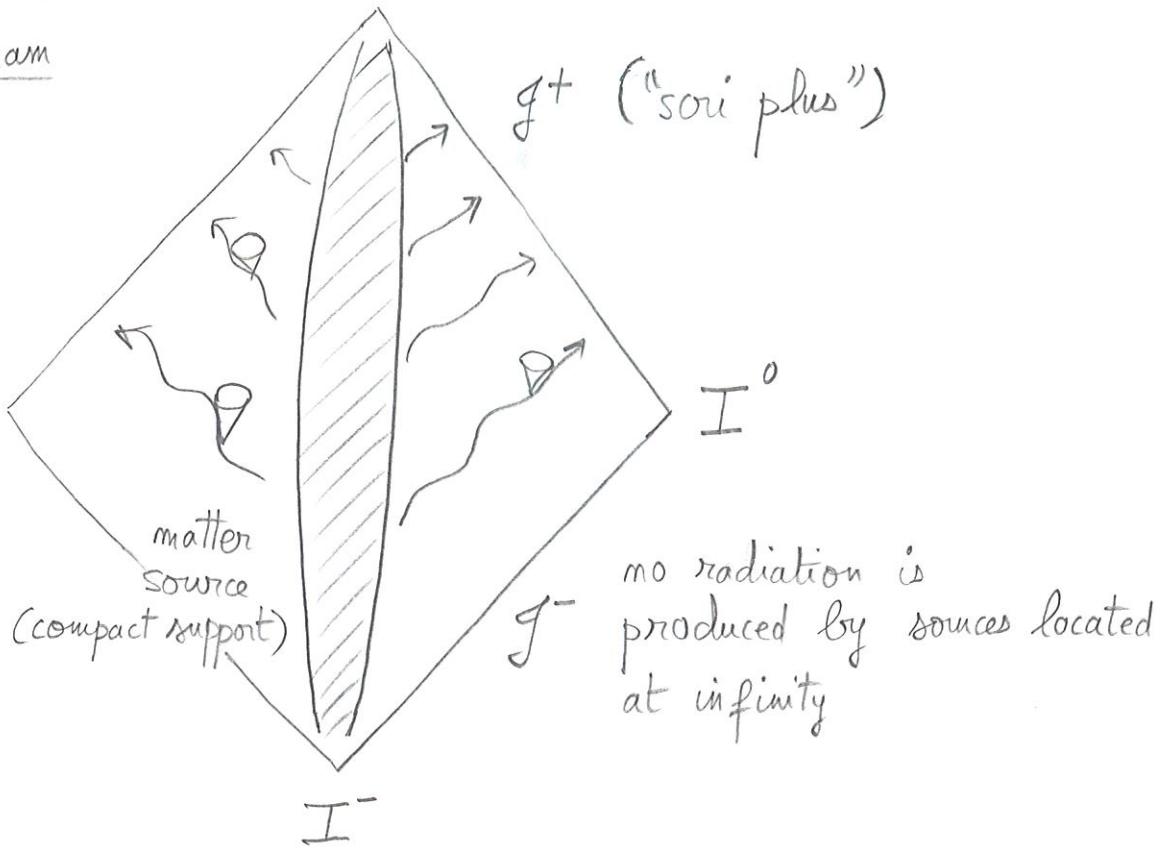
$$J^- = \text{past null infinity} \quad (r \rightarrow +\infty, t + r/c = \text{const})$$

$$I^- = \text{past temporal infinity} \quad (t \rightarrow -\infty, r = \text{const})$$

Carter-Penrose

91

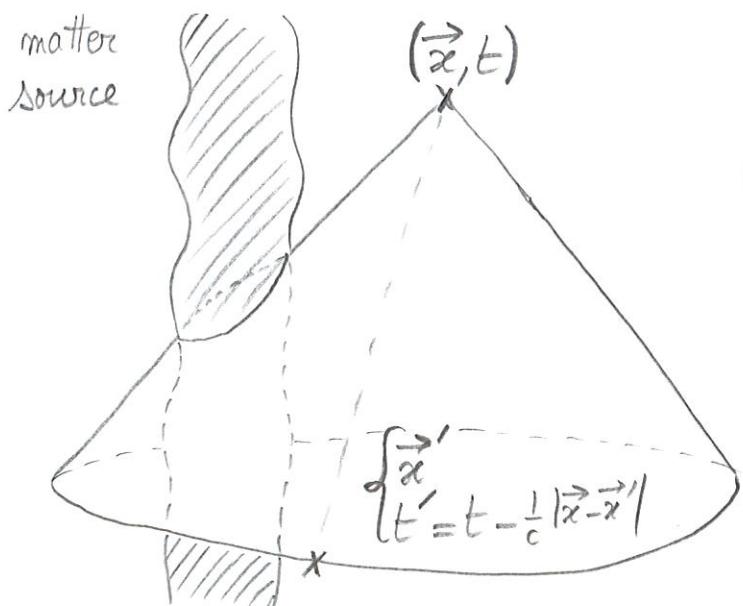
diagram



Kirchhoff's formula for the homogeneous sol. of

$$\square h_{\text{Hom}} = 0$$

$$h_{\text{Hom}}(\vec{x}, t) = \lim_{|\vec{x}'| \rightarrow \infty} \iint \frac{d\Omega'}{4\pi} \left(\frac{\partial}{\partial n} + \frac{1}{c} \frac{\partial}{\partial t} \right) (n h_{\text{Hom}})(\vec{x}', t - \frac{|\vec{x}-\vec{x}'|}{c})$$



(\vec{x}, t) = field point

(\vec{x}', t') = source point

No-incoming rad. cond. is

$$\lim_{r \rightarrow -} \left(\frac{\partial}{\partial r} + \frac{1}{c} \frac{\partial}{\partial t} \right) (r h^{\mu\nu}) = 0$$

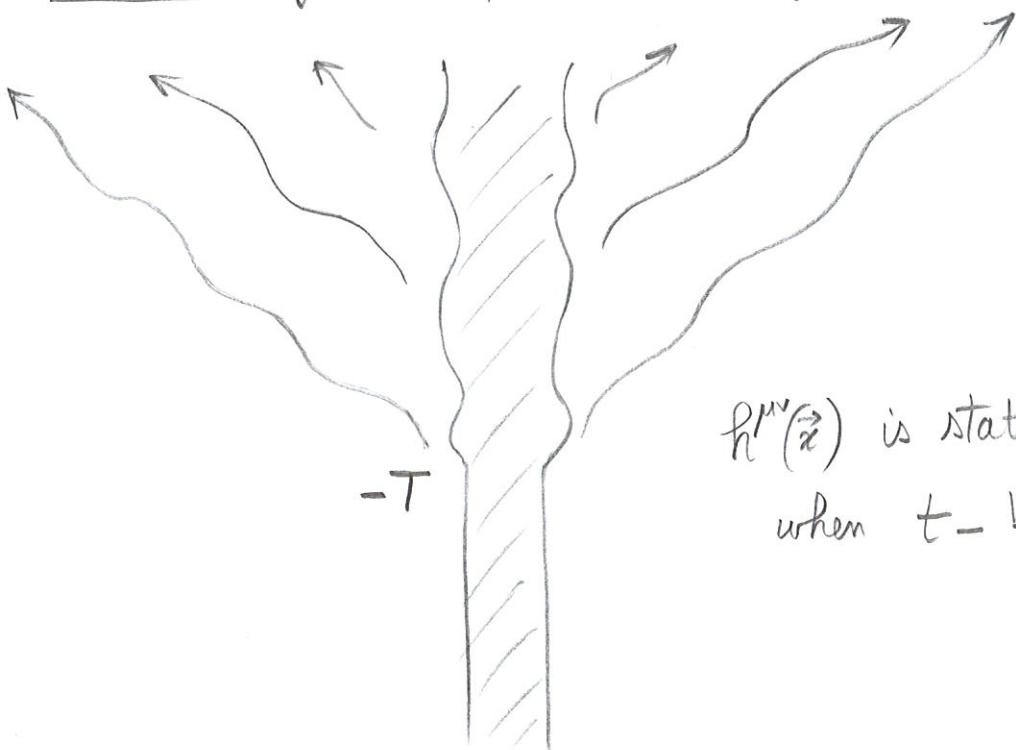
This excludes advanced waves $r h_{\text{adv}} \sim f(t+r/c)$ at \mathcal{T}^-

Einstein field eqs. can be solved (in an iterative way) by means of standard retarded integral in 3+1 dimensions

$$h^{\mu\nu}(\vec{x}, t) = -\frac{4G}{c^4} \iiint \frac{d^3 \vec{x}'}{|\vec{x} - \vec{x}'|} T^{\mu\nu}(\vec{x}', t - \frac{1}{c} |\vec{x} - \vec{x}'|)$$

note this is in fact an integro-differential equation because $T^{\mu\nu}$ depends on $h, \partial h, \partial^2 h$

Stationarity in the past (simple way to implement the no-incoming rad. condition)



$h^{\mu\nu}(\vec{x})$ is stationary (ind. of t)
when $t - \frac{|\vec{x}|}{c} \leq -T$

Linearized GWs in vacuum

$$\begin{cases} \square h^{\mu\nu} = 0 \\ \partial_\nu h^{\mu\nu} = 0 \end{cases} \quad (\text{we neglect } O(h^2))$$

Gauge transformation preserving the harmonic cond. $\partial h = 0$

$$h'^{\mu\nu} = h^{\mu\nu} + \partial^\mu \xi^\nu + \partial^\nu \xi^\mu - g^{\mu\nu} \partial_\rho \xi^\rho$$

$$\text{where } \square \xi^\mu = 0$$

Fourier decomposition

$$h^{\mu\nu}(x) = \int d^4k H^{\mu\nu}(k) e^{ik_\lambda x^\lambda}$$

↑ Fourier amplitude of
monochromatic wave $k_\lambda = \begin{pmatrix} \text{wave} \\ \text{vector} \end{pmatrix}$

$$k^2 = g_{\mu\nu} k^\mu k^\nu = 0$$

$$k_\nu H^{\mu\nu} = 0$$

Can perform a gauge transf.

$$\text{with any } \xi^\mu(x) = \int d^4k \varepsilon^\mu(k) e^{ik_\lambda x^\lambda}$$

TT coordinates u^μ four-vector constant (independent of x)
and not orthogonal to k_μ (i.e. $u_\mu k^\mu \neq 0$) for instance

u^μ = four velocity of an observer (time-like)

There exists a gauge such that (at once)

$$\boxed{u_\nu H^{\mu\nu} = 0}$$

\leftarrow transverse (T) condition

$$\boxed{H \equiv h_{\mu\nu} H^{\mu\nu} = 0}$$

\leftarrow traceless (T) condition

Proof: perform a gauge transf. in Fourier domain

$$H^\mu = H_0^{\mu\nu} + i k^\mu \varepsilon^\nu + i k^\nu \varepsilon^\mu - i h^{\mu\nu} k_\nu \varepsilon^\mu$$

Then TT conditions are satisfied with gauge vector

$$\varepsilon^\mu = \frac{i}{(u \cdot k)} \left[u_\nu \bar{H}_0^{\mu\nu} - \frac{k^\mu}{2(u \cdot k)} u_\rho u_\sigma \bar{H}_0^{\rho\sigma} \right]$$

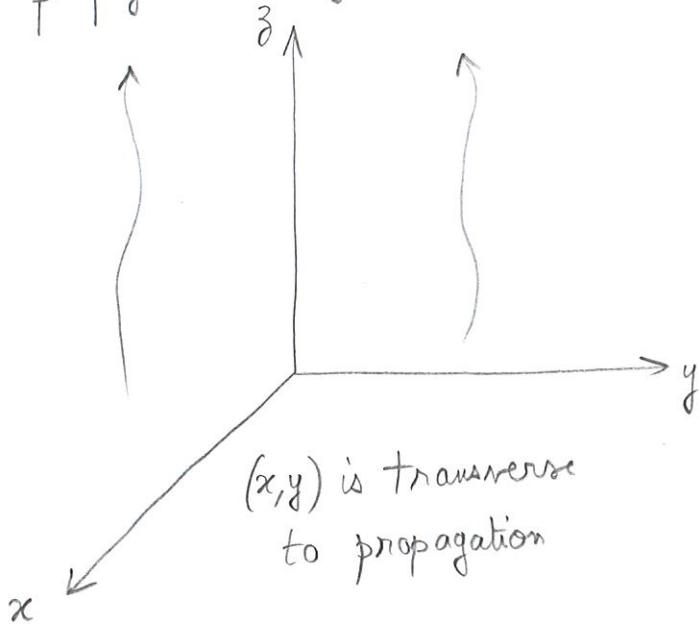
$$\text{where } \bar{H}_0^{\mu\nu} = H_0^{\mu\nu} - \frac{1}{2} h^{\mu\nu} H_0$$

$$\boxed{10 - 4 - (4-1) - 1 = 2 \text{ independent components of } H^{\mu\nu}}$$

2 polarization states

$$u^\mu = (1, \vec{0}) \text{ in rest frame of observer}$$

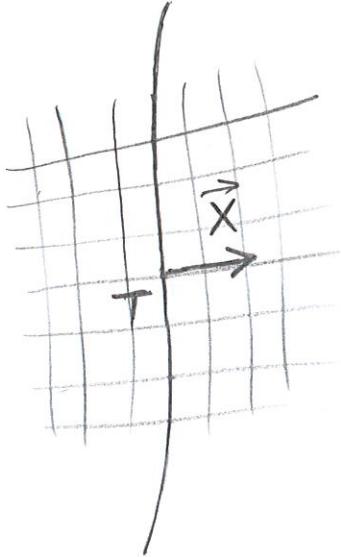
propagation in z-direction



$$h_{\mu\nu}^{TT} = \begin{pmatrix} t & x & y & z \\ 0 & 0 & 0 & 0 \\ 0 & h_+(t-z/c) & h_x(t-z/c) & 0 \\ 0 & h_x(t-z/c) & -h_+(t-z/c) & 0 \\ 0 & 0 & 0 & 0 \end{pmatrix}$$

Action of GWs on matter

central geodesics ($X^i = 0$)



Fermi coordinates (X^i, T) in the neighborhood of central geodesics

T = proper time along central geodesics

$$g_{\mu\nu}(\vec{X}, T) = \eta_{\mu\nu} + \underbrace{F_{\mu\nu ij}(T)}_{\text{function of time } T} X^i X^j + \mathcal{O}(|\vec{X}|^3)$$

Geodesic equ. in vicinity of central geodesic ($|\vec{X}| \ll \lambda^{GW}$)

$$\frac{d^2 X^i}{dT^2} = -c^2 \frac{\partial \Gamma_{00}^i}{\partial X^j}(T, \vec{o}) X^j = -c^2 R_{.0j0}^i(T, \vec{o}) X^j$$

(to first order in X^i)

Riemann in Fermi coord.
($-c^2 R_{.0j0}^i$ is a relativistic version
of the tidal tensor $\partial_i \partial_j U$)

$$R_{.0j0}^i = \frac{\partial X^i}{\partial x^\lambda} \frac{\partial x^\mu}{\partial X^0} \dots \quad R_{,\mu\nu\rho}^{\lambda} \stackrel{TT}{\approx} R_{.0j0}^i \stackrel{TT}{\approx} -\frac{1}{2c^2} \frac{\partial^2 h_{ij}}{\partial t^2}$$

Riemann in TT coordinates

$$\frac{d^2 X^i}{dT^2} = \frac{1}{2} \frac{\partial^2 h_{ij}}{\partial t^2} \stackrel{TT}{(T, \vec{o})} X^j$$

acceleration in
Fermi coord.

wave form in TT
coord. evaluated on central geodesic

$$X^i(T) = X^i(0) + \frac{1}{2} \stackrel{TT}{h_{ij}}(T, \vec{o}) X^j(0)$$

position before passage of GW

(to first
order in h)

Quadrupole moment formalism

Matter source is

- isolated ($T^{\mu\nu}$ has a compact support)

- post-Newtonian

$$\epsilon \approx \frac{v}{c} \ll 1$$

- self-gravitating: internal motion is due to gravitational forces

$$\gamma \sim \frac{v^2}{a} \sim \frac{GM}{a^2} \quad \begin{aligned} a &= \text{size of source} \\ M &= \text{its mass} \end{aligned}$$

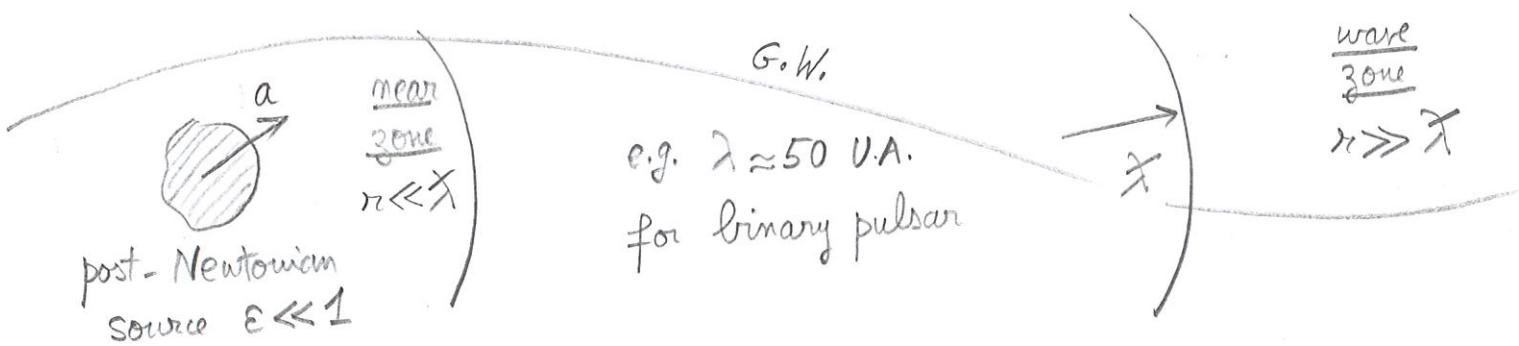
Period of motion $P \sim \frac{2\pi a}{v}$

Gravitational wave length

$$\lambda = cP$$

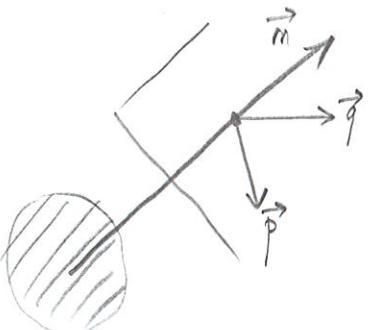
$$\chi = \frac{\lambda}{2\pi}$$

$$\frac{a}{\chi} \sim \frac{v}{c} \approx \epsilon$$



The mean zone ($r \ll \chi$) covers entirely the post-Newtonian source

$$Q_{ij}(t) = \int_{\text{source}} d^3x \rho(\vec{x}, t) (x_i x_j - \frac{1}{3} \delta_{ij} \vec{x}^2)$$



$$h_{ij}^{TT} = \frac{2G}{c^4 n} P_{ijkl} (\vec{n}) \left\{ \ddot{Q}_{kl} \left(t - \frac{r}{c} \right) + O(\epsilon) \right\} + O\left(\frac{1}{n^2}\right)$$

TT projection operator

$$P_{ijkl} = P_{ik} P_{jl} - \frac{1}{2} P_{ij} P_{kl} \quad \text{where } P_{ij} = \delta_{ij} - m_i m_j$$

Polarization

states

u.r.t. \vec{P}, \vec{q}

$$h_+ = \frac{\vec{P}_i \vec{P}_j - \vec{q}_i \vec{q}_j}{2} h_{ij}^{TT}$$

\vec{P}, \vec{q} polarization vectors

$$h_x = \frac{\vec{P}_i \vec{q}_j + \vec{P}_j \vec{q}_i}{2} h_{ij}^{TT}$$

Einstein
quadrupole
formula

$$\mathcal{F}^{GW} = \left(\frac{dE}{dt} \right)^{GW} = \frac{G}{5c^5} \left\{ \ddot{Q}_{ij} \ddot{Q}_{ij} + O(\epsilon^2) \right\}$$

↑
order of magnitude of radiation reaction
 $O(\epsilon^5)$ called also 2.5 PN

Typically $Q \sim Ma^2$ $\ddot{Q} \sim Ma^2 \omega^3$ $\omega = \frac{2\pi}{P}$
Self-gravitating source $\omega^2 \sim \frac{GM}{a^3}$

$$\mathcal{F}^{GW} \sim \left(\frac{c^5}{G} \right) \left(\frac{GM\omega}{c^3} \right)^{10/3}$$

Ultra-relativistic source $\omega \sim c$ or $\frac{GM\omega}{c^3} \sim 1$

$$\mathcal{F}^{GW} \underset{\text{ultra relativistic}}{\sim} \frac{c^5}{G} = 3.63 \cdot 10^{52} W$$

value independent of source

GW has typically the frequency $\omega \sim \frac{c^3}{GM}$

$$M \sim 1 M_\odot$$

$$\omega \sim 10^3 \text{ Hz}$$

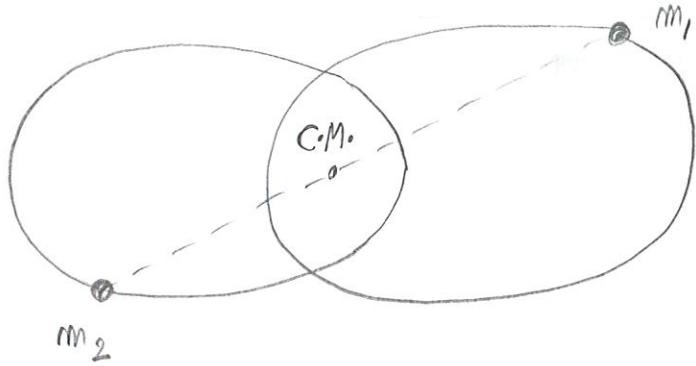
bandwidth
of LIGO/VIRGO

$$M \sim 10^6 M_\odot$$

$$\omega \sim 10^{-3} \text{ Hz}$$

bandwidth
of LISA

Secular decrease of orbital period of binary pulsar



Two compact objects (without spin)
on a Keplerian ellipse

a = semi-major axis

e = eccentricity

$$M = m_1 + m_2$$

$$\mu = \frac{m_1 m_2}{M}$$

$$\nu = \frac{\mu}{M} \text{ such that } 0 < \nu \leq \frac{1}{4}$$

test-mass
limit

equal masses

$$\langle \mathcal{F}^{GW} \rangle = \frac{1}{P} \int_0^P dt \mathcal{F}^{GW}(t) = \frac{32}{5} \frac{c^5}{G} \nu^2 \left(\frac{GM}{ac^2} \right)^5 \frac{1 + \frac{73}{24} e^2 + \frac{37}{96} e^4}{(1-e^2)^{7/2}}$$

eccentricity dependent
"enhancement" factor $f(e)$
(Peters & Mathews 1964)

Energy balance argument

$$\frac{dE}{dt} = -\langle \mathcal{F}^{GW} \rangle$$

with

$$E = -\frac{GMv^2}{2a}$$

$$GM = \omega^2 a^3$$

$$\dot{P} = -\frac{192\pi}{5c^5} \left(\frac{2\pi GM}{P} \right)^{5/3} \nu f(e) = -2.4 \cdot 10^{-12} \text{ s/s}$$

Binary pulsar
PSR 1913+16

in agreement with observations (Taylor et al.).

Actually the masses m_1 and m_2 (or equivalently M and γ) are measured by the relativistic effects themselves.

One measures

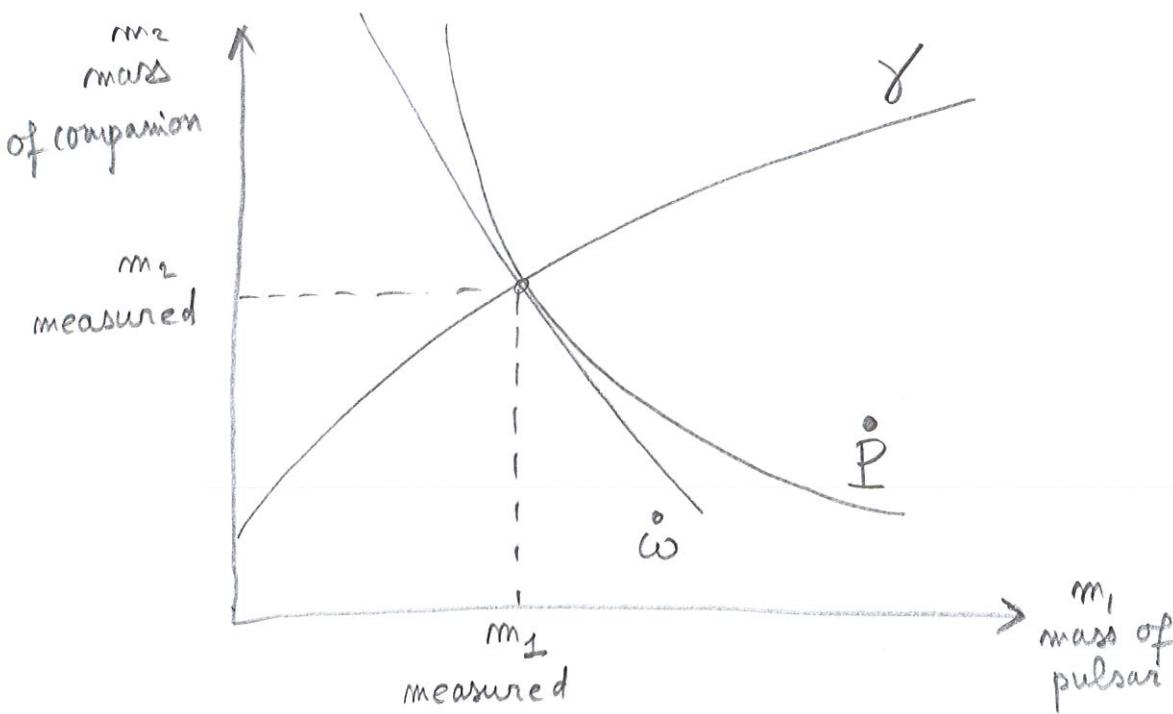
- $\dot{\omega}$ relativistic advance of periastron (like for Mercury, but here the effect is of order of 4° per year)
- γ combination between the gravitational redshift due to the companion and the second-order Doppler effect
- \dot{P} due to gravitational radiation

In GR

$$\dot{\omega} = \frac{3G^{2/3}}{c^2} \left(\frac{P}{2\pi} \right)^{-5/3} \frac{M^{2/3}}{1-e}$$

$$\gamma = \frac{G^{2/3}}{c^2} e \left(\frac{P}{2\pi} \right)^{1/3} m_2 (m_1 + 2m_2) M^{-4/3}$$

and we draw the mass plane



Inspiring compact binaries

Evolution of eccentricity $e(t)$

Orbit's energy and angular momentum

$$\boxed{\frac{E}{\gamma} = -\frac{GM^2}{2a}}$$

$$\boxed{\frac{J}{\gamma} = \sqrt{GM^3 a(1-e^2)}}$$

$$\gamma = \frac{\mu}{M}$$

Apply quadrupole formulas for both E and J

$$\dot{E} = - \left\langle \frac{G}{5c^5} \ddot{Q}_{ij} \ddot{Q}_{ij} \right\rangle$$

$$\dot{J}^i = - \left\langle \frac{2G}{5c^5} \varepsilon_{ijk} \ddot{Q}_{jl} \ddot{Q}_{kl} \right\rangle$$

$$\boxed{\frac{e^2}{(1-e^2)^{19/6}} \left(1 + \frac{121}{304} e^2\right)^{145/121} = \left(\frac{\omega}{\omega_0}\right)^{-\frac{19}{9}}}$$

gives $e(t)$ as a function of $\omega(t)$ during the inspiral
 $(\omega_0$ is determined from initial conditions) ($e^2 \sim P^{19/9}$ for small e)

For the binary pulsar $e_{\text{now}} = 0.617$

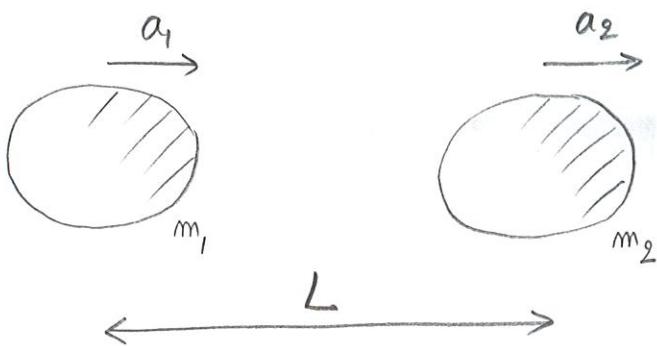
$$\omega_{\text{now}} = 2.24 \times 10^{-4} \text{ Hz}$$

hence GWs are visible by VIRGO/LIGO when

$$\boxed{\omega \sim 30 \text{ Hz} \Rightarrow e \sim 5 \times 10^{-6}}$$

eccentricity is negligible in general.

Finite size effects



Look for influence of quadrupole moments Q_1 and Q_2 induced by tidal interactions between

non-spinning compact objects

$$\boxed{Q_1 = k_1 m_2 \frac{a_1^5}{L^3} \quad Q_2 = k_2 m_1 \frac{a_2^5}{L^3}} \quad \begin{matrix} k_{1,2} = \text{Love} \\ \text{numbers} \\ (\text{depend on internal} \\ \text{structure}) \end{matrix}$$

$Q_{1,2}$ scale like L^{-3} because of tidal field $\partial_{ij}U \sim \frac{1}{L^3}$

Introduce the compacity parameters

$$K_1 = \frac{2Gm_1}{a_1 c^2} \quad K_2 = \frac{2Gm_2}{a_2 c^2}$$

The quadrupoles modify the energy and GW flux and the orbital frequency ω and phase $\phi = \int \omega dt$

$$\dot{E} = -\mathcal{F}^{GW} \Rightarrow \phi = - \int \frac{\omega dE}{\mathcal{F}^{GW}}$$

Effect of quadrupoles is

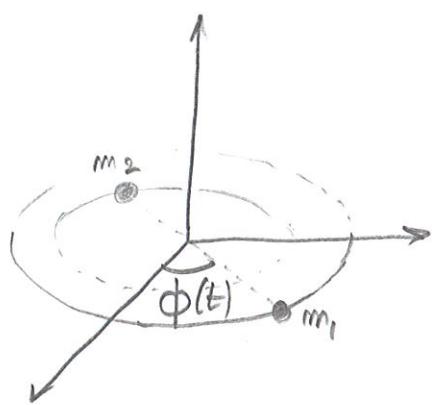
$$\boxed{\phi^{\text{finite-size}} = \phi_0 - \frac{1}{8x^{5/2}} \left\{ 1 + (\text{const}) \left(\frac{x}{K} \right)^5 \right\}}$$

point-mass result

$x \equiv \left(\frac{GM\omega}{c^3} \right)^{2/3}$ Since $K \sim 1$ for compact objects the formal order of magnitude of the finite-size effect is 5PN (namely $x^5 \sim \frac{1}{c^{10}}$)

Orbital phase evolution $\phi(t)$

(same as for binary pulsar, i.e. based on



$$\frac{dE}{dt} = -\mathcal{F}^{GW}$$

$$\text{where } \frac{E}{M} = -\frac{c^2}{2} \nu x$$

$$\mathcal{F}^{GW} = \frac{32}{5} \frac{c^5}{G} \nu^2 x^5$$

$$x = \left(\frac{GM\nu}{c^3} \right)^{2/3} = \text{PN parameter } \mathcal{O}(\varepsilon^2)$$

$$\dot{E} = -\mathcal{F}^{GW} \Rightarrow \dot{x} = \frac{64}{5} \frac{c^3}{G} \frac{\nu}{M} x^5 \Rightarrow x(t) = \left[\frac{256}{5} \frac{c^3}{G} \frac{\nu}{M} (t_c - t) \right]^{-1/4}$$

t_c = instant of coalescence

$$\phi(t) = \int \omega dt = \frac{5}{64\nu} \int x^{-7/2} dx \Rightarrow \boxed{\phi(t) = \phi_c - \frac{x(t)}{32\nu}^{-5/2}}$$

Number of orbital cycles left till coalescence from time t

$$N = \frac{\phi_0 - \phi(t)}{\pi} = \frac{1}{32\pi\nu} \underbrace{\left(\frac{GM\nu}{c^3} \right)^{-5/3}}_{\mathcal{O}(\varepsilon^{-5})}$$

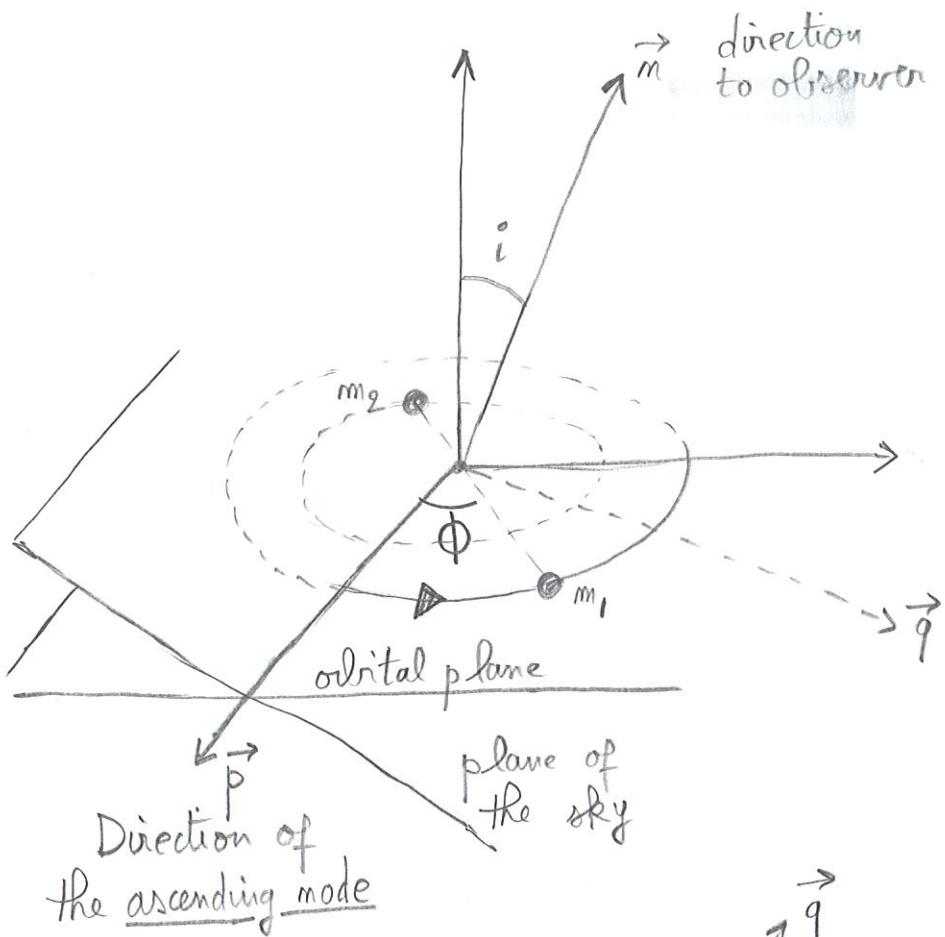
inverse of order of
radiation reaction $\varepsilon^{-5} \sim \left(\frac{c}{r} \right)^5$

But N should be monitored in LIGO/VIRGO with precision

$$\delta N \sim 1$$

so it is evident that PN corrections in the phase will play a crucial role up to at least the 2.5PN order. Detailed analysis show that good templates for inspiralling compact binaries should have 3PN accuracy. Current theoretical prediction is 3.5PN.

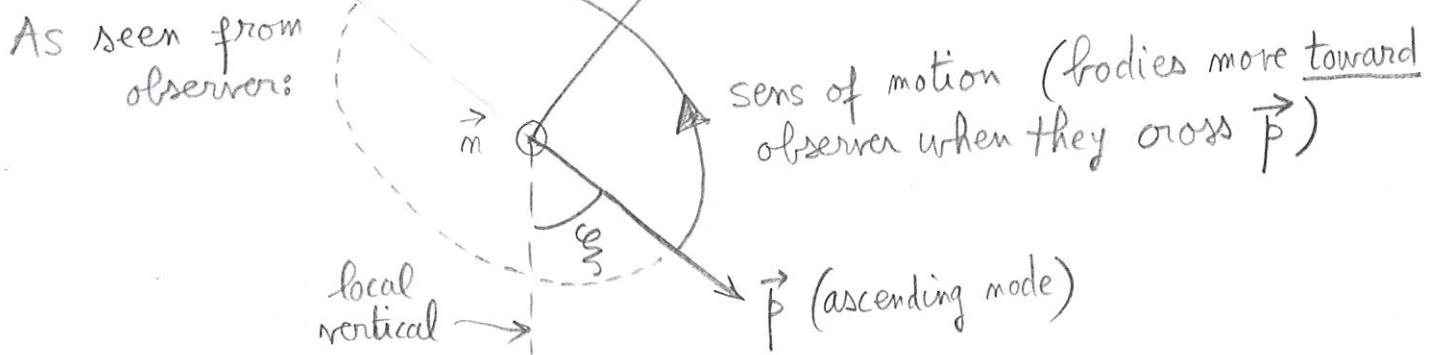
Wave form of inspiralling compact binaries (ICBs)



\vec{p}, \vec{q} = polarization vectors
(in the plane of sky)

i = inclination angle

$\phi(t)$ = orbital phase



ξ = polarization angle (between \vec{p} and local vertical of observer)

Response of detector

$$h \equiv \frac{2\delta L}{L} = F_+ h_+ + F_X h_X$$

$F_{+,X}$ = detector's pattern functions
depend on $-\vec{m}$ (direction of source) and ξ

In quadrupole approximation

$$h_+ = \frac{2G\mu}{c^2 D} \left(\frac{GM\omega}{c^3} \right)^{2/3} (1 + \cos^2 i) \cos(2\phi)$$

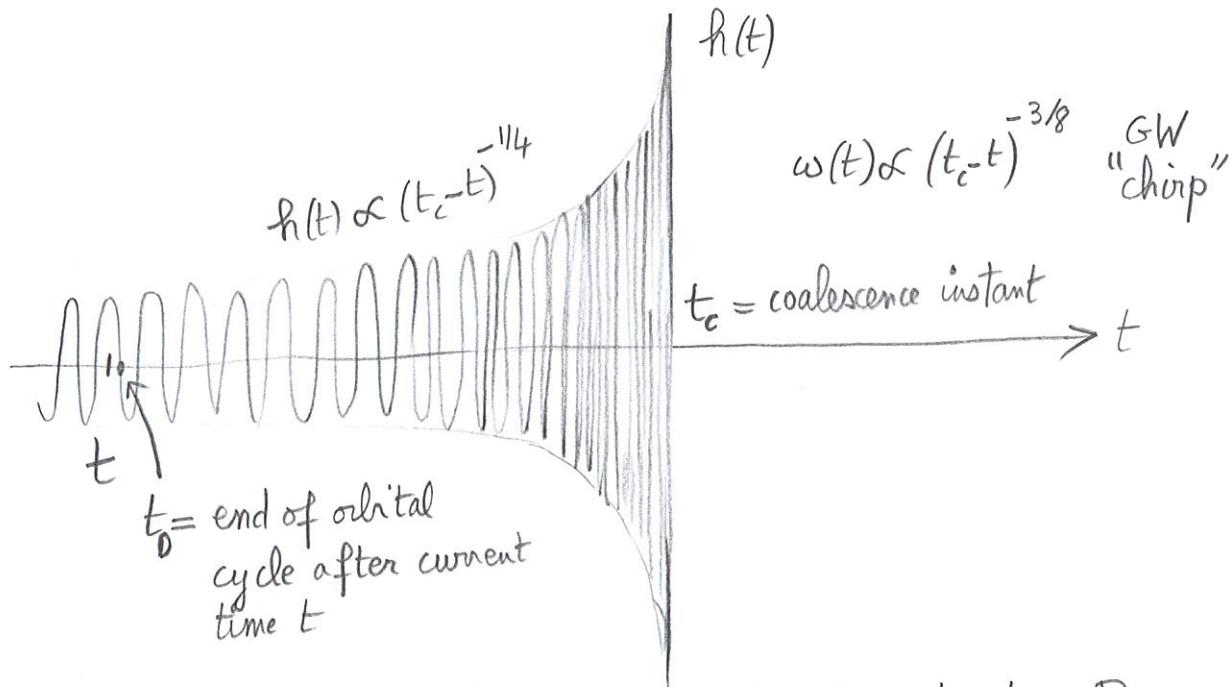
$$h_x = - (2 \cos i) \sin(2\phi)$$

D = distance of source
 $=$ luminosity distance in cosmology

where

$$\phi(t) = \phi_c - \frac{1}{\nu} \left(\frac{\gamma c^3}{5GM} (t_c - t) \right)^{5/8}$$

$$\omega(t) = \frac{c^3}{8GM} \left(\frac{\gamma c^3}{5GM} (t_c - t) \right)^{-3/8}$$



Suppose current time t is such that $t_c - t \gg P$
 (non-relativistic limit, two bodies are well-separated)

$$t_c - t = (t_c - t_0) \left[1 + \frac{t_0 - t}{t_c - t_0} \right] \quad \text{with} \quad \frac{t_0 - t}{t_c - t_0} \ll 1$$

$$\begin{aligned} \phi(t) &\approx \phi_c - \frac{1}{\nu} \left(\frac{\gamma c^3}{5GM} (t_c - t_0) \right)^{5/8} \left[1 + \frac{5}{8} \frac{t_0 - t}{t_c - t_0} + \dots \right] \\ &\approx \phi_0 + \frac{5}{8\nu} \left(\frac{\gamma c^3}{5GM} \right)^{5/8} (t_c - t_0)^{-3/8} t + \dots \end{aligned}$$

thus

$$\boxed{\phi(t) \approx \phi_0 + \omega_0 t + \dots}$$

constant orbital motion
 in the non-relativistic limit

Orders of magnitude

$$h \sim \frac{GM\nu}{c^2 D} \left(\frac{GM\omega}{c^3} \right)^{2/3}$$

Number of cycles around frequency ω

$$n = \frac{\omega^2}{\dot{\omega}} \sim \frac{1}{\nu} \left(\frac{GM\omega}{c^3} \right)^{-5/3} = O(\varepsilon^{-5})$$

inverse of
rad. reaction
order

Effective amplitude after matched filtering

$$\boxed{h_{\text{eff}} = h \sqrt{n} \sim \frac{GM\sqrt{\nu}}{c^2 D} \left(\frac{GM\omega}{c^3} \right)^{-11/6}}$$

Example: coalescence of two supermassive BHs in LISA

Characteristic frequency $\omega_c \sim \omega_{\text{I.C.O.}}$

innermost circular orbit (defined by
the minimum of the energy function)

$$\frac{GM\omega_c}{c^3} \sim 0.1 \quad \Rightarrow \quad f_c \sim 10^4 \text{ Hz} \left(\frac{M_\odot}{M} \right)$$

(from 3PN theory) For LISA $f_c \in [10^{-4} \text{ Hz}, 10^1 \text{ Hz}]$

Hence LISA should observe

$$\boxed{10^5 M_\odot \lesssim M \lesssim 10^8 M_\odot}$$

$$\boxed{h_{\text{eff}} \sim 10^{-14} \left(\frac{1 \text{ Gpc}}{D} \right) \left(\frac{\gamma}{0.25} \right)^{1/2} \left(\frac{M}{10^7 M_\odot} \right)^{-5/6} \left(\frac{f}{10^{-4} \text{ Hz}} \right)^{-1/6}}$$

Separation of BHs ($M \sim 10^7 M_\odot$) at entry frequency of LISA

$$r = \left(\frac{GM}{\omega^2} \right)^{1/3} \sim 1 \text{ A.U.}$$

Time left till coalescence

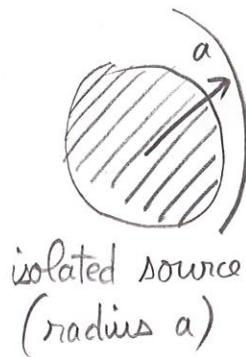
$$T = \frac{5GM}{\gamma c^3} \left(\frac{8GM\omega}{c^3} \right)^{-8/3} \sim 10 \text{ days}$$

The signal-to-noise of the supermassive BH coalescence
in LISA is enormous

$$\boxed{\frac{S}{N} = \left(\int_{-\infty}^{+\infty} d\omega \frac{|\tilde{h}(\omega)|^2}{S_m(\omega)} \right)^{1/2} \sim \frac{h_{\text{eff}}}{\sqrt{\omega S_m(\omega)}} \sim 10^4}$$

$$S_m(\omega) \sim 10^{-34} \text{ Hz}^{-1} \text{ for LISA}$$

Non-linearity (or post-Minkowskian) expansion



In exterior region ($r > a$)

$$\left\{ \begin{array}{l} \square h_{\text{ext}}^{\mu\nu} = \Lambda^{\mu\nu}(h_{\text{ext}}) \\ \partial_\nu h_{\text{ext}}^{\mu\nu} = 0 \end{array} \right. \quad \begin{array}{l} \text{of order} \\ O(h_{\text{ext}}^2) \end{array}$$

harmonic coordinate condition

We solve these equations by means of post-Minkowskian (PM) or non-linearity expansion

$$h_{\text{ext}}^{\mu\nu} = \sum_{m=1}^{+\infty} G^m h_{(m)}^{\mu\nu}$$

G = Newton's constant

(viewed here as a "bookkeeping" parameter to label the successive PM orders)

Insert PM expansion into vacuum Einstein field eqs.

$$\square \left(G h_{(1)}^{\mu\nu} + G^2 h_{(2)}^{\mu\nu} + \dots \right) = G^2 \Lambda_{(2)}^{\mu\nu}(h_{(1)}) + G^3 \Lambda_{(3)}^{\mu\nu}(h_{(1)}, h_{(2)}) + \dots$$

$$\partial_\nu \left(\right) = 0$$

where $\Lambda_{(2)} \sim h_{(1)} \partial^2 h_{(1)} + \partial h_{(1)} \partial h_{(1)}$

$$\Lambda_{(3)} \sim h_{(1)} \partial h_{(1)} \partial h_{(1)} + h_{(1)} \partial^2 h_{(2)} + h_{(2)} \partial^2 h_{(1)} + \partial h_{(1)} \partial h_{(2)}$$

...

$\forall n \geq 1$

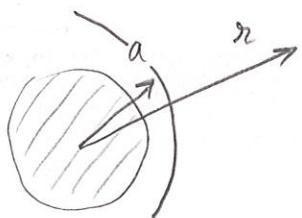
$$\square h_{(n)}^{\mu\nu} = \Lambda_{(n)}^{\mu\nu}(h_{(1)}, h_{(2)}, \dots, h_{(n-1)})$$

$$\partial_r h_{(n)}^{\mu\nu} = 0$$

The source term $\Lambda_{(n)}$ is known from previous iterations

Linearized solution

Solve $\square h_{(1)} = 0$ by means of multipole expansion (valid in exterior $r > a$)



"Monopolar" general solution

$$h_{(1)}^{\text{Mono.}}(\vec{x}, t) = \frac{R(t - r/c) + A(t + r/c)}{r}$$

Impose no incoming rad. cond.

$$0 = \lim_{\substack{t \rightarrow -\infty \\ t + \frac{r}{c} = \text{const}}} \left[\partial_r(r h_{(1)}) + \partial_t(r h_{(1)}) \right] = 2A'(t + \frac{r}{c}) \quad \text{hence } A(u) \text{ is constant and can be included into definition of } R(t - \frac{r}{c}).$$

$$h_{(i)}^{\text{Mono.}} = \frac{R(t - r/c)}{r} \quad (i=1,2,3)$$

"Dipolar" solution is obtained by applying $\partial_i \equiv \frac{\partial}{\partial x^i}$

hence $h_{(1)}^{\text{Dip.}} = \partial_i \left(\frac{R_i(t-\tau_c)}{r} \right)$. General multipolar solution is obtained by applying l spatial derivatives

$$h_{(1)}^{(\mu\nu)}(\vec{x}, t) = \sum_{L=0}^{+\infty} \partial_L \left(\frac{R_L^{(\mu\nu)}(u)}{r} \right) \quad \left| \begin{array}{l} u = t - \frac{r}{c} \\ L = i_1 i_2 \dots i_l \text{ a multi-index with } l \text{ spatial indices} \end{array} \right.$$

$L = i_1 i_2 \dots i_l$ a multi-index with l spatial indices

$$\partial_L = \partial_{i_1 i_2 \dots i_l} = \frac{\partial}{\partial x^{i_1}} \dots \frac{\partial}{\partial x^{i_l}}$$

Without loss of generality we can assume that R_L is symmetric and trace-free (STF)

$$R_L = \hat{R}_L + \sum_{j \leq l-1} \epsilon \underbrace{\delta \delta \dots \delta}_{\substack{1 \text{ to } [\frac{l}{2}] \\ \text{Kronecker symbol}}} \hat{U}_j$$

STF tensors

ϵ Levi-Civita symbol

where the \hat{U}_j 's are linear in the $\epsilon \delta \dots \delta R_K$'s.

For example:

$$\begin{cases} \hat{R}_{ij} = \hat{R}_{ij} + \epsilon_{ijk} \hat{U}_k + \delta_{ij} \hat{U} \\ \hat{U}_k = \frac{1}{2} \epsilon_{kab} R_{ab} \\ \hat{U} = \frac{1}{3} R_{kk} \end{cases}$$

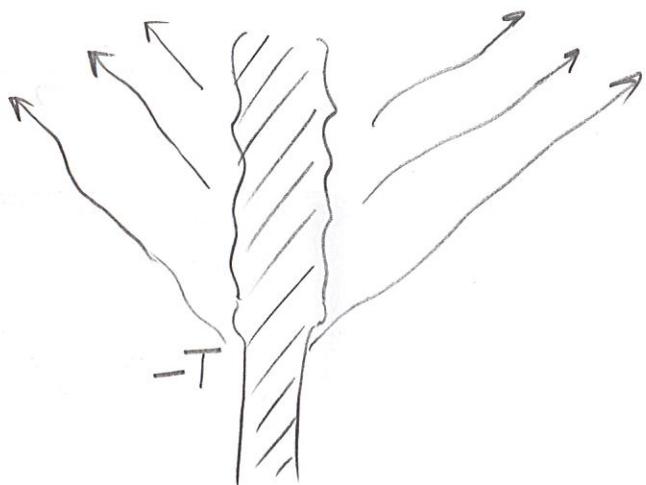
$\hat{R}_{ij} = \frac{R_{ij} + R_{ji}}{2} - \frac{1}{3} \delta_{ij} R_{kk}$ is the STF part of R_{ij} .

$$\partial_L \left(\frac{1}{r} R_L \right) = \partial_L \left(\frac{1}{r} \hat{R}_L \right) + \sum_{k \geq 1} \Delta^k \uparrow \partial_{L-2k} \left(\frac{1}{r} \hat{U}_{L-2k} \right)$$

because of k Kronecker δ s
(terms with one E cancelled by symmetry of ∂_L)

$$\Delta^k \partial \left(\frac{1}{r} \hat{U}(u) \right) = \partial \left(\frac{1}{r} \frac{d^{2k} \hat{U}}{c^{2k} du^{2k}}(u) \right) \text{ takes same structure}$$

For simplicity assume that source emits GWs only from some finite instant $-T$ in the past (stationarity in the past)



$h_{\text{ext}}^{\mu\nu}(x)$ is independent of time when $t \leq -T$

(and even when
 $t - \frac{r}{c} - \underbrace{\frac{2GM \ln(r/r_0)}{c^3} + \dots}_{\leq -T} \leq -T$)

"light cone" in coordinates (t, r)

There are 10 independent functions $R_L^{\mu\nu}(u)$ (for each multi-index L) at this stage.

We impose now the harmonicity condition $\partial_\nu h_{(1)}^{\mu\nu} = 0$ which gives 4 differential relations between the R_L 's.

Hence we end up with 6 independent functions (6 types of "source" multipole moments).

Most general solution of $\square h_{(1)}^{\mu\nu} = 0 = \partial h_{(1)}^{\mu\nu}$ is (Thorne 1980)

$$h_{(1)}^{\mu\nu} = R_{(1)}^{\mu\nu} + \underbrace{\partial^\mu \varphi_{(1)}^\nu + \partial^\nu \varphi_{(1)}^\mu - g^{\mu\nu} \partial_\rho \varphi_{(1)}^{\rho\nu}}_{\text{linearized gauge transformation}}$$

where $R_{(1)}^{\mu\nu}$ depends on two sets of STF multipole moments

$$\boxed{\begin{array}{ll} I_L(u) & \text{and} \\ \uparrow L & \\ \text{mass-moment of order } l & \end{array} \quad \boxed{\begin{array}{ll} J_L(u) \\ \uparrow \\ \text{current-moment of order } l \end{array}}}$$

and $\varphi_{(1)}^\mu$ depends on four sets of moments (for its four components)
 $\mu = 0, 1, 2, 3$

$$W_L(u) \quad X_L(u) \quad Y_L(u) \quad \text{and} \quad Z_L(u)$$

$$R_{(1)}^{00} = -\frac{4}{c^2} \sum_{l=0}^{+\infty} \frac{(-)^l}{l!} \partial \left(\frac{1}{r} I_L(u) \right)$$

$$R_{(1)}^{0i} = \frac{4}{c^3} \sum_{l=1}^{+\infty} \frac{(-)^l}{l!} \left\{ \partial \left(\frac{1}{r} \dot{I}_{iL-1}(u) \right) + \frac{l}{l+1} \epsilon_{iab} \partial_{aL-1} \left(\frac{1}{r} J_{bL-1}(u) \right) \right\}$$

$$R_{(1)}^{ij} = -\frac{4}{c^4} \sum_{l=2}^{+\infty} \frac{(-)^l}{l!} \left\{ \partial_{L-2} \left(\frac{1}{r} \ddot{I}_{ijL-2} \right) + \frac{2l}{l+1} \partial_{aL-2} \left(\frac{1}{r} \epsilon_{ab(i} \dot{J}_{j)L-2} \right) \right\}$$

Dots mean derivative w.r.t. time $u = t - r/c$

$I_L(u)$ and $J_L(u)$ are arbitrary functions of time u except for the conservation laws (directly issued from the harmonicity condition $\partial h_{(1)} = 0$)

$M \equiv I = \text{const}$	total mass
$X_i \equiv \frac{I_i}{I} = \text{const}$	center-of-mass position
$P_i \equiv \dot{I}_i = 0$	linear momentum
$S_i \equiv J_i = \text{const}$	angular momentum

These conservation laws are exact (by definition of the moments) and refer to the total quantities associated with the source and including the contributions of GWs emitted by the source.

They describe the initial state of the source before emission of GWs.

In particular $M=I$ is the total ADM mass of source

Finally $h_{(1)}$ (and hence $h_{\text{ext}} = \sum G^m h_{(m)}$) will be described by

$$\underbrace{I_L(u) \quad J_L(u)}_{\substack{\text{main moments} \\ (\text{source at linear order})}} \quad \underbrace{W_L(u) \dots Z_L(u)}_{\substack{\text{gauge moments} \\ (\text{will play a role} \\ \text{at non-linear order})}} = \text{six source multipole moments}$$

Non-linear vacuum solution

When $r \rightarrow 0$ $h_{(1)} \sim \partial \left(\frac{R(t-r)}{r} \right)$ diverges. This is because $h_{(1)}$ is valid only in the exterior $r > a_0$. Inserting $h_{(1)}$ into $\Lambda_{(2)}$ we get

$$\Lambda_{(2)} \sim \partial \left(\frac{R(t-r)}{r} \right) \partial \left(\frac{S(t-r)}{r} \right)$$

$$\sim \sum_{k \geq 2} \frac{\hat{m}_L^k}{r^k} F(t-r)$$

STF product of unit vectors m_i $\hat{m}_L = \langle m_{i_1} \dots m_{i_l} \rangle$
is equivalent to spherical harmonics $Y_{lm}(\theta, \varphi)$

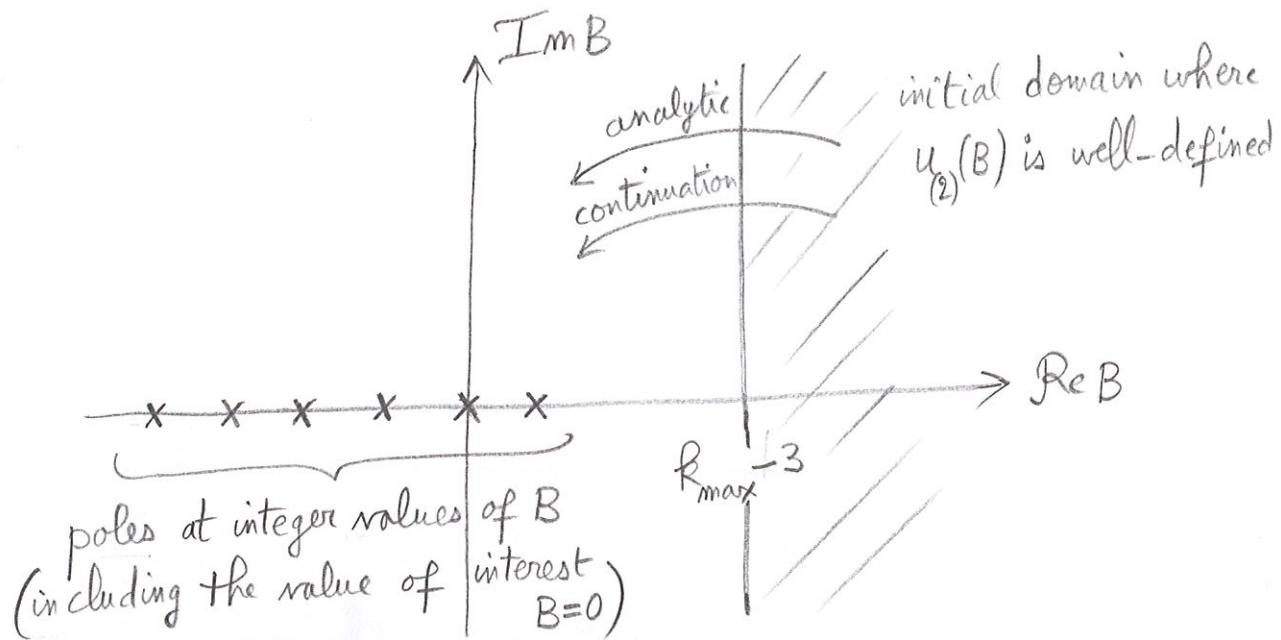
$$\begin{cases} \hat{m}_L(\theta, \varphi) = \sum_{m=-l}^l \alpha_L^m Y_{lm}(\theta, \varphi) \\ \alpha_L^m = \int d\Omega \hat{m}_L^* Y_{lm}^* \end{cases} \quad \text{← constant STF tensor}$$

Because of divergence when $r \rightarrow 0$ one cannot apply the standard retarded integral.

If we assume $h_{(1)}$ is made of a finite set of moments, say $l \leq l_{\max}$, there is a maximal order of divergencies in $\Lambda_{(2)}$, $k \leq k_{\max}$. We can regularize $\Lambda_{(2)}$ by multiplying by some factor r^B (where $B \in \mathbb{C}$). Next we define:

$$u_{(2)}^{\mu\nu}(B) \equiv \square^{-1} \underset{\text{Ret}}{\left[\left(\frac{r}{r_0}\right)^B \Lambda_{(2)}^{\mu\nu} \right]}$$

The retarded integral is convergent when $\operatorname{Re} B > R_{\max}^{-3}$



$$u_{(2)}(B) = \sum_{p=p_0}^{+\infty} \lambda_p B^p \quad \begin{array}{l} \text{Laurent expansion} \\ \text{when } B \rightarrow 0 \\ (p \in \mathbb{Z}) \end{array}$$

Applying \square we get $\left(\frac{r}{r_0}\right)^B \Lambda_{(2)} = \sum (\square \lambda_p) B^p$

$$\begin{aligned} p_0 \leq p \leq -1 &\Rightarrow \square \lambda_p = 0 \\ p \geq 0 &\Rightarrow \square \lambda_p = \frac{(\ln(r/r_0))^p}{p!} \Lambda_{(2)} \end{aligned}$$

In particular when $p=0$ we obtain a solution of the eq. we want ($\square u_{(2)} = \Lambda_{(2)}$). Pose $u_{(2)}^{\mu\nu} \equiv \lambda_0^{\mu\nu}$

$$\boxed{u_{(2)}^{\mu\nu} = \underset{B \rightarrow 0}{\text{Finite Part}} \square_{\text{Ret}}^{-1} \left[r^B \Lambda_{(2)}^{\mu\nu} \right]} \quad (r_0 = 1)$$

Thus $\square u_{(2)} = \Lambda_{(2)}$ is satisfied and $u_{(2)}$ has the same structure $\sim \sum \frac{m_L}{r^k} G(t-r)$ as $\Lambda_{(2)}$ but $\partial_\gamma u_{(2)}^{\mu\nu} \neq 0$ in general.

$$\boxed{w_{(2)}^\mu \equiv \partial_\gamma u_{(2)}^{\mu\nu} = \underset{B \rightarrow 0}{\text{FP}} \square_{\text{Ret}}^{-1} \left[B^{m_i} r^{B-1} \Lambda_{(2)}^{\mu i} \right]}$$

↑
computed from the fact
that $\partial_\gamma \Lambda_{(2)}^{\mu\nu} = 0$

Because of factor B (coming from $\partial_i^n B = B r^{B-1} m_i$) $w_{(2)}^\mu$ is non zero when the integral develops a pole $\propto \frac{1}{B}$. The structure of the pole is that of a source-free (retarded) solution of d'Alembert's eq.

$$\boxed{w_{(2)}^\mu = \sum_{l=0}^{\infty} \partial_L \left(\frac{S_L^\mu(u)}{r} \right)}$$

Indeed $\underset{B \rightarrow 0}{\square w_{(2)}} = \text{FP} \left(B m_i r^{B-1} \Lambda \right) = 0$. From that structure one can construct "algorithmically"

$$\boxed{w_{(2)}^{\mu\nu} = \mathcal{H}^{\mu\nu}(w_{(2)})}$$

\mathcal{H} is an algorithm which gives a unique $w_{(2)}^{\mu\nu}$ starting from any $w_{(2)}^\mu$ (source-free solution)

such that (at once) $\square v_{(2)} = 0$ and $\partial v_{(2)} = - u_{(2)}$

$$v_{(2)}^{\mu\nu} = \sum_{l=0}^{\infty} \partial \left(\frac{T_L^{\mu\nu}(u)}{r} \right)$$

where the $T_L^{\mu\nu}$'s are given in terms of the S_L^{μ} 's by the algorithm Ml. Solution is thus

$$h_{(2)}^{\mu\nu} = u_{(2)}^{\mu\nu} + v_{(2)}^{\mu\nu}$$

Same method applies by induction to any m
(Blanchet & Damour 1986)

$$\begin{aligned} u_{(m)}^{\mu\nu} &= \text{Finite Part } \underset{B \rightarrow 0}{\square}^{-1} \underset{\text{Ret}}{\left[\left(\frac{r}{r_0}\right)^B \bigwedge_{(m)} (h_{(1)} \dots h_{(m-1)}) \right]} \\ v_{(m)}^{\mu\nu} &= \mathcal{H}^{\mu\nu}(\partial u_{(m)}) \\ h_{(m)}^{\mu\nu} &= u_{(m)}^{\mu\nu} + v_{(m)}^{\mu\nu} \end{aligned}$$

To $h_{(m)}$ one can still add a homogeneous solution
(such that $\square h_{(m)}^{\text{Hom}} = 0 = \partial h_{(m)}^{\text{Hom}}$) but $h_{(m)}^{\text{Hom}}$ is necessarily of the form $h_{(0)}[\text{some momenta}]$. Hence

$$h_{(n)}^{\text{gen}} = h_{(n)}[I_L \dots Z_L] + \underbrace{h_{(1)}[\delta I_L \dots \delta Z_L]}_{\text{can be re-absorbed into } h_{(1)}[I_L \dots Z_L] \text{ by posing}}$$

$$\begin{cases} I_L^{\text{new}} = I_L + G^{m-1} \delta I_L \\ \vdots \\ Z_L^{\text{new}} = Z_L + G^{m-1} \delta Z_L \end{cases}$$

Hence the previous construction represents the most general solution of Einstein's field eqs. outside the source

Resulting metric

$$g_{\mu\nu}^{\text{ext}}(x; \underbrace{I_L J_L W_L X_L Y_L Z_L}_{6 \text{ source moments}}) \quad \overbrace{\qquad \qquad \qquad}^{\text{4 gauge moments}}$$

One can define by coord. transformation $x \rightarrow x'$ a "canonical" metric which depends only on 2 moments $M_L S_L$.

Thus

$$g_{\mu\nu}^{\text{can}}(x'; \underbrace{M_L S_L}_{2 \text{ canonical moments}})$$

is isometric to $g_{\mu\nu}^{\text{ext}}$ i.e. $g_{\mu\nu}^{\text{can}}(x') = \frac{\partial x^\rho}{\partial x'^\mu} \frac{\partial x^\sigma}{\partial x'^\nu} g_{\rho\sigma}^{\text{ext}}(x)$ where

$$x'^\mu = x^\mu + G \underbrace{q_{(1)}^\mu(x; W_L X_L Y_L Z_L)}_{\text{gauge vector in the general linear solution}} + O(G^2)$$

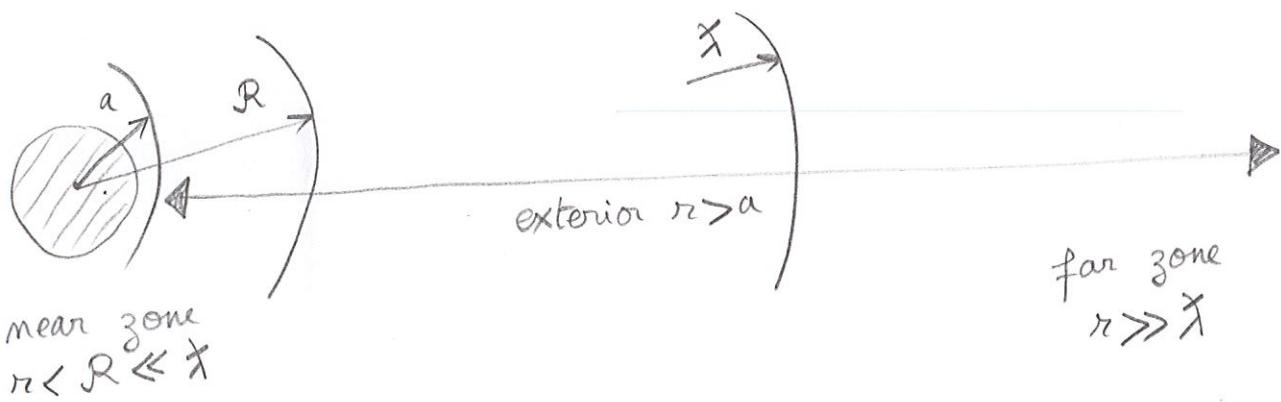
↑
crucial non-linear corrections

Hence any isolated system can be described by 2 sets of moments

$$\begin{array}{cc} M_L(u) & S_L(u) \\ \text{mass-type} & \text{current-type} \end{array}$$

$$\boxed{\begin{aligned} M_L &= I_L + O(G) \\ S_L &= J_L + O(G) \end{aligned}} \quad \begin{array}{l} \text{complicated non-linear} \\ \text{functionals of} \\ I_L J_L X_L \dots Z_L \end{array}$$

General structure of the solution SOLUTION



The solution $h_{\text{ext}} = \sum G^m h_{(m)}$ is physically valid in the exterior $r > a$ but is defined for any $r > 0$. When $r \rightarrow 0$

$$\boxed{h_{(m)} = \sum_{p \leq N} \hat{m}_L(\theta, \varphi) r^p (\ln r)^q F(t) + O(r^N)}$$

(proved by induction on m in the construction of $h_{(m)}$).

Note appearance of powers of $\ln r$ with $q \leq m-2$.

Since $r \rightarrow 0$ means $\frac{r}{c} \rightarrow 0$ or $c \rightarrow \infty$ we have the general structure of the post-Newtonian (PN) expansion

$$h_{(m)}(c) = \sum_{p \leq N} \frac{(\ln c)^p}{c^p} + \mathcal{O}\left(\frac{1}{c^N}\right)$$

When $r \rightarrow \infty$ (wave zone) we find also a "poly-logarithmic" structure

$$h_{(m)} = \sum_{k \leq N} \frac{(\ln r)^k}{r^k} G(u) + \mathcal{O}\left(\frac{1}{r^N}\right) \quad \text{where } u = t - r/c$$

(expansion at g^+)

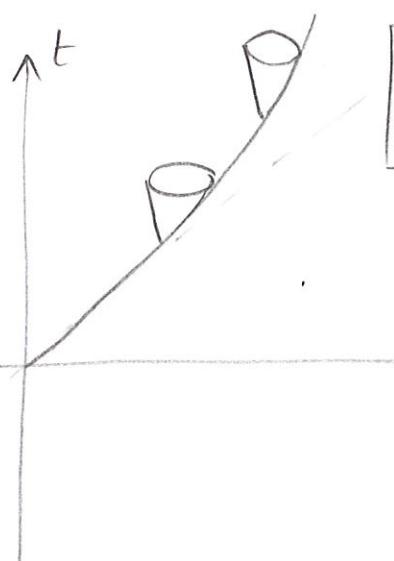
The logs here come from the well-known deviation of light rays in harmonic coordinates.

Schwarzschild
in harmonic coord.

$$ds^2 = -\left(\frac{r-M}{r+M}\right)dt^2 + \left(\frac{r+M}{r-M}\right)dr^2 + (r+M)^2 d\Omega^2$$

For an outgoing radial ($\theta = \text{const}$ $\varphi = \text{const}$) photon

$$dt = \frac{r+M}{r-M} dr \Rightarrow t = r + 2M \ln\left(\frac{r-M}{\text{const}}\right)$$



$$t = \frac{r}{c} + \frac{2GM}{c^3} \ln\left(\frac{r}{r_0}\right) + \mathcal{O}(G^2)$$

We shall see that all these logs (in the FZ) can be removed by a coord. transformation

The matching equation

We have constructed the exterior field (physically valid when $r > a$) of any isolated source

$$h_{\text{ext}} = \sum_{m=1}^{+\infty} G^m h_{(m)} \left[\underbrace{I_L J_L W_L \dots Z_L}_{\text{source moments (for the moment arbitrary)}} \right]$$

We suppose that h_{ext} comes from the multipole expansion of h defined everywhere inside and outside the source (for any r)

$$h_{\text{ext}} = M(h)$$

↑
operation of taking
the multipole expansion

Note that $M(h)$ is defined of any $r > 0$ but agrees with the "true" field h only when $r > a$

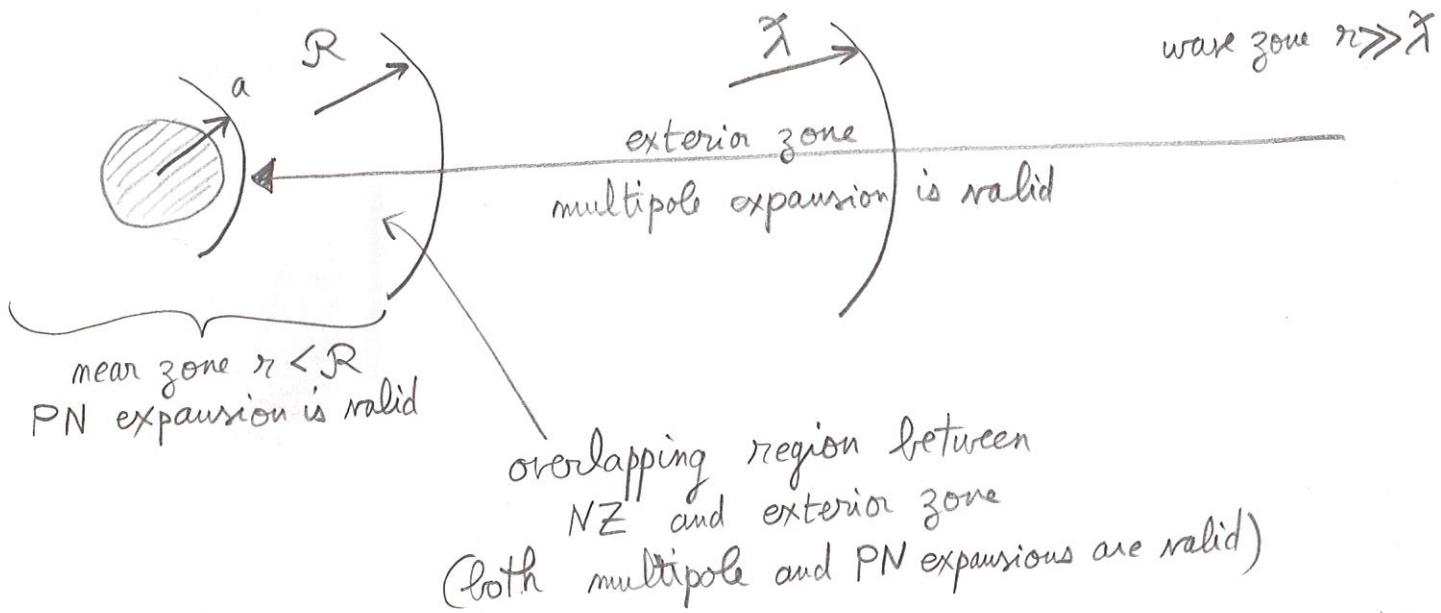
$$r > a \Rightarrow M(h) = h \quad (\text{numerically})$$

But when $r \rightarrow 0$ $M(h)$ diverges while h is a perfectly smooth solution of Einstein field eqs. inside the matter (of the extended source).

Suppose the source is post-Newtonian (existence of the PN parameter $\varepsilon = \frac{v}{c} \ll 1$). We know that the near zone $r < R$ where $R \ll \lambda$ encloses totally the PN source ($R > a$).

In the NZ the field h can be expanded as a PN expansion ($\bar{h} = \sum c^{-1} (lmc)^q$)

$$r < R \Rightarrow h = \bar{h} \quad (\text{numerically})$$



$$a < r < R \Rightarrow M(h) = \bar{h} \quad (\text{numerically})$$

The matching equation follows from transforming the latter numerical equality in a functional identity (valid $\forall (\vec{x}, t)$ in $\mathbb{R}_+^3 \times \mathbb{R}$) between two formal asymptotic series

Matching equation:

$$\overline{\mathcal{M}(\mathbf{r})} \equiv \mathcal{M}(\overline{\mathbf{r}})$$

NZ expansion ($\frac{r}{c} \rightarrow 0$)
of each multipolar coeff.
of $\mathcal{M}(\mathbf{r})$

multipole expansion of
each PN coefficient of $\overline{\mathbf{r}}$

We assume (as part of our fundamental assumptions) that the matching eq. is correct (in the sense of formal series)

$$\boxed{\text{NZ expansion } \frac{r}{c} \rightarrow 0 \quad \left(\begin{array}{l} \text{multipolar expansion} \\ \frac{a}{n} \rightarrow 0 \end{array} \right) \equiv \text{FZ expansion } r \rightarrow \infty \quad \left(\begin{array}{l} \text{PN series} \\ c \rightarrow \infty \end{array} \right)}$$

The NZ expansion $\frac{r}{c} \rightarrow 0$ is "equivalent" to the PN expansion $c \rightarrow \infty$ for fixed r

The multipole expansion $\frac{a}{n} \rightarrow 0$ is "equivalent" to the FZ expansion $r \rightarrow \infty$ for a given source (fixed a)

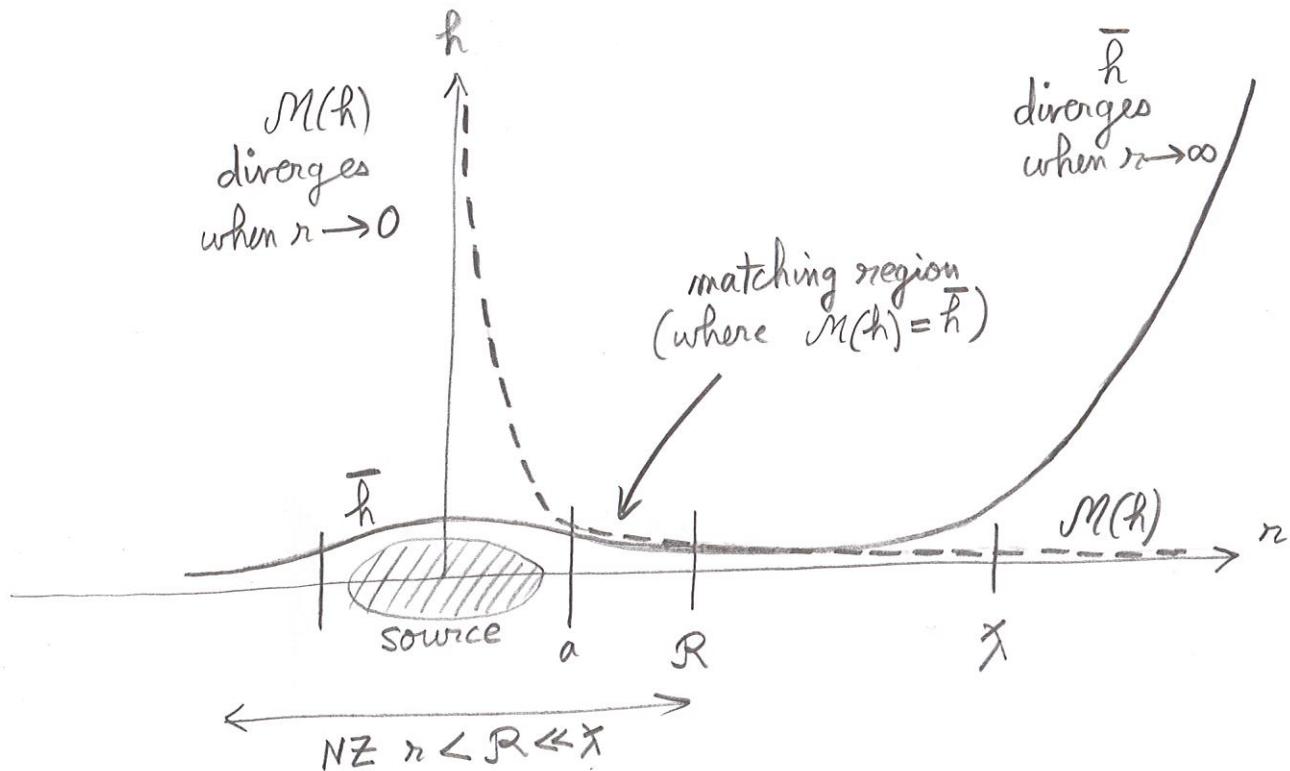
The matching equation says basically the NZ and multipole expansions can be commuted.

Thus there is a common structure for the formal NZ and FZ expansions

$$\overline{M(h)} = \sum_{n=1}^{\infty} n^p (ln n)^q F(t) = M(\bar{h})$$

can be interpreted either as

- NZ singular expansion when $n \rightarrow 0$
- FZ ————— $n \rightarrow \infty$



Expression of the multipole moments

h is the sol. of Einstein eqs (in harmonic coord. $\partial h = 0$)
valid everywhere inside and outside the source

$$h = \frac{16\pi G}{c^4} \square_{\text{Ret}}^{-1} T \quad (\text{suppress indices } \mu\nu)$$

where $T = 1g/ T + \underbrace{\frac{c^4}{16\pi G} \Delta}_{\substack{\text{gravitational source-term} \\ \text{(non-linearity in } h\text{)}}}$

Define

$$\boxed{\Delta = h - \text{FP} \square_{\text{Ret}}^{-1} M(\lambda)}$$

where $M(\lambda) = \Lambda[M(\lambda)] = \Lambda_{\text{ext}}$ and FP is the finite part
when $B \rightarrow 0$ (plays a crucial role because Λ_{ext} diverges when $r \rightarrow 0$)

$$\Delta = \underbrace{\frac{16\pi G}{c^4} \square_{\text{Ret}}^{-1} T}_{\text{no FP here}} - \text{FP} \square_{\text{Ret}}^{-1} M(\lambda)$$

since T is regular (C^∞)

However we can add FP on the first term (do not change the value because it converges). Using also $M(T) = 0$ since T has a compact support

$$\Delta = \frac{16\pi G}{c^4} \text{FP} \square_{\text{Ret}}^{-1} [T - M(T)]$$

Hence Δ appears as the retarded integral of a source with compact support. Indeed

$$T = M(T) \quad \text{when } r > a$$

$$\boxed{M(\Delta) = -\frac{4G}{c^4} \sum_{l=0}^{+\infty} \frac{(-)^l}{l!} \mathcal{J}_L \left(\frac{1}{r} \mathcal{H}_L(u) \right)}$$

This is standard expression of multipolar expansion outside a compact-support source. Here the moments are

$$\mathcal{H}_L = \text{FP} \int d^3x \chi_L [\tau - \bar{M}(\tau)]$$

since this has compact support
($r < a$, inside the NZ) we can
replace by the NZ or PN expansion

$$\mathcal{H}_L = \text{FP} \int d^3x \chi_L [\bar{\tau} - \bar{M}(\tau)]$$

But we know the structure $\bar{M}(\tau) = \sum \hat{n}_Q^P (l_{mn})^P F(t)$
which is sufficient to prove that the second term is zero
by analytic continuation

$$\text{FP} \int d^3x \chi_L \bar{M}(\tau) = \sum \text{FP} \int d^3x \chi_L \hat{n}_Q^P n^P (l_{mn})^P$$

$$= \sum \underset{B \rightarrow 0}{\text{Finite Part}} \int dr r^{B+s} (l_{mn})^P$$

$$= \sum \underset{B \rightarrow 0}{\text{FP}} \left(\frac{d}{dB} \right)^P \int_0^{+\infty} dr r^{B+s}$$

$$\int_0^{+\infty} dr r^{B+s} = \underbrace{\int_0^R dr r^{B+s}}_{\text{computed when } \Re B > -s-1} + \underbrace{\int_R^{+\infty} dr r^{B+s}}_{\text{computed when } \Re B < -s-1}$$

$$= \underbrace{\frac{R^{B+s+1}}{B+s+1}}_{\text{by analytic continuation}}$$

$$= - \underbrace{\frac{R^{B+s+1}}{B+s+1}}_{\text{by analytic continuation}}$$

Analytic Continuation $\int_0^{+\infty} dr r^{B+S} (lmn)^\dagger = 0 \quad \forall B \in \mathbb{C}$

The general multipole expansion outside the domain of a PN isolated source reads (Blanchet 1995, 1998)

$$\mathcal{M}(r) = \text{FP } \square_{\text{Ret}}^{-1} \mathcal{M}(\lambda) - \frac{4G}{c^4} \sum_{l=0}^{+\infty} \frac{(-)^l}{l!} \partial_L^l \left(\frac{1}{r} \mathcal{H}_L(u) \right)$$

where

$$\mathcal{H}_L(u) = \text{FP } \int d^3x \vec{x}_L \bar{T}(\vec{x}, u)$$

PN expansion crucial here
(this is where the formalism applies only to PN sources)

Same result but in STF guise

$$\mathcal{M}(r) = \text{FP } \square_{\text{Ret}}^{-1} \mathcal{M}(\lambda) - \frac{4G}{c^4} \sum_{l=0}^{+\infty} \frac{(-)^l}{l!} \partial_L^l \left(\frac{1}{r} \mathcal{F}_L(u) \right)$$

where

$$\mathcal{F}_L(u) = \text{FP } \int d^3x \vec{x}_L \int_{-1}^1 dz \delta_L(z) \bar{T}(\vec{x}, u + z|\vec{x}|/c)$$

$$\delta_L(z) = \frac{(2l+1)!!}{2^{l+1} l!} (1-z^2)^l \quad \text{such that}$$

$$\begin{aligned} \int_{-1}^1 dz \delta_L(z) &= 1 \\ \lim_{l \rightarrow +\infty} \delta_L(z) &= \delta(z) \end{aligned}$$

Practical way to implement the STF multipole expansion is to use the PN series

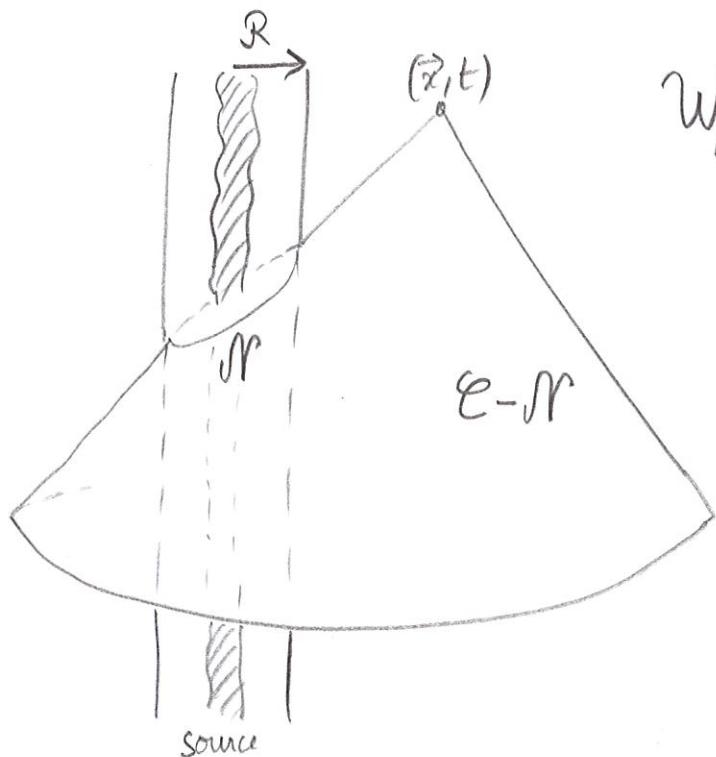
$$\int_{-1}^1 dz \delta_\ell(z) \bar{T}(\vec{x}, u+z|\vec{x}|/c) = \sum_{k=0}^{+\infty} \alpha_k^\ell \left(\frac{|\vec{x}|}{c} \frac{\partial}{\partial u} \right)^{2k} \bar{T}(\vec{x}, u)$$

$\frac{(2\ell+1)!!}{(2k)!! (2\ell+2k+1)!!}$

There is an alternative formalism for writing the general multipole expansion (Will & Wiseman 1996)

$$\mathcal{M}(h) = \boxed{\square_{\text{Ret}}^{-1} \mathcal{M}(\lambda)} - \frac{4G}{c^4} \sum_{l=0}^{+\infty} \frac{(-)^l}{l!} \mathcal{J} \left(\frac{1}{r} W_L(t-n) \right)$$

$\underbrace{\quad}_{\text{the retarded integral excludes the NZ of source}}$ where



$$W_L(u) = \int_{r < R} d^3x \chi_L \bar{T}(\vec{x}, u)$$

$\underbrace{\quad}_{\text{volume integral limited to the NZ of the source (N)}}$

The two formalisms are equivalent

Next we identify $\mathcal{L}_{\text{ext}} = \mathcal{M}(h)$ which means

$$\begin{aligned}
 G h_{(1)} [I_L J_L W_L \dots Z_L] + G^2 h_{(2)} + \dots + G^m h_{(m)} + \dots \\
 = - \frac{4G}{c^4} \sum_{l=0}^{+\infty} \frac{(-)^l}{l!} \underbrace{\partial_L \left(\frac{1}{r} \mathcal{F}_L(u) \right)}_{\text{has the form of the linear metric } G h_{(1)} \text{ where the } \mathcal{F}_L \text{'s}} + \underbrace{\text{FP } \square_{\text{Ret}}^{-1} \mathcal{M}(h)}_{\text{represents the non-linear corrections } G^2 h_{(2)} + \dots + G^m h_{(m)} + \dots}
 \end{aligned}$$

Note that for the identification to work the \mathcal{F}_L 's in the right-hand-side should be considered as of zero-th order in G

Then we obtain $I_L \dots Z_L$ in terms of the components of $\mathcal{F}_L^{\mu\nu}$ and hence of the source's pseudo-tensor $\bar{T}^{\mu\nu}$.

Decompose the $\mathcal{F}_L^{\mu\nu}$'s into ten irreducible STF tensors

$$R_L \quad T_{L+1}^{(+)} \dots U_{L-2}^{(-2)} \quad V_L$$

$$\left\{
 \begin{array}{l}
 \mathcal{F}_L^{00} = R_L \\
 \mathcal{F}_L^{oi} = T_{iL}^{(+)} + \epsilon_{ai<il} T_{L>a}^{(0)} + \delta_{i<il} T_{L>}^{(-)} \\
 \mathcal{F}_L^{ij} = U_{ijL}^{(-2)} + \underset{L}{\text{STF}} \underset{ij}{\text{STF}} \left[\epsilon_{aiil} U_{ojL-1}^{(+1)} + \delta_{iil} U_{jL-1}^{(0)} \right. \\
 \quad \left. + \delta_{iil} \epsilon_{ajil-1} U_{aL-2}^{(-1)} + \delta_{iil} \delta_{jil-1} U_{L-2}^{(-2)} \right] + \delta_{ij} V_L
 \end{array}
 \right.$$

The final result is

$$I_L = \text{FP} \int d\vec{x} \int_{-1}^1 dz \left\{ \delta_\ell(z) \hat{x}_L^\ell \sum - \frac{4(2\ell+1)}{c^2(\ell+1)(2\ell+3)} \delta_{\ell+1} \hat{x}_{iL}^\ell \overset{(1)}{\sum}_i + \frac{2(2\ell+1)}{c^4(\ell+1)(\ell+2)(2\ell+5)} \delta_{\ell+2} \hat{x}_{ijL}^\ell \overset{(2)}{\sum}_{ij} (\vec{x}, u+z/c) \right\}$$

$$J_L = \text{FP} \int d\vec{x} \int_{-1}^1 dz \epsilon_{abf<ip} \left\{ \delta_\ell \hat{x}_{L-1>a}^\ell \sum_b - \frac{2\ell+1}{c^2(\ell+2)(2\ell+3)} \delta_{\ell+1} \hat{x}_{L-1>ac}^\ell \overset{(1)}{\sum}_{bc} (\vec{x}, u+z/c) \right\}$$

where

$$\begin{cases} \sum = \frac{\bar{T}^{00} + \bar{T}^{ii}}{c^2} \\ \sum_i = \frac{\bar{T}^{0i}}{c} \\ \sum_{ij} = \bar{T}^{ij} \end{cases}$$

There are similar expressions for $\mathcal{W}_L \dots \mathcal{Z}_L$

These expressions give the source moments of any isolated PN source, up to any PN order (formally).

PN expansion in the near-zone

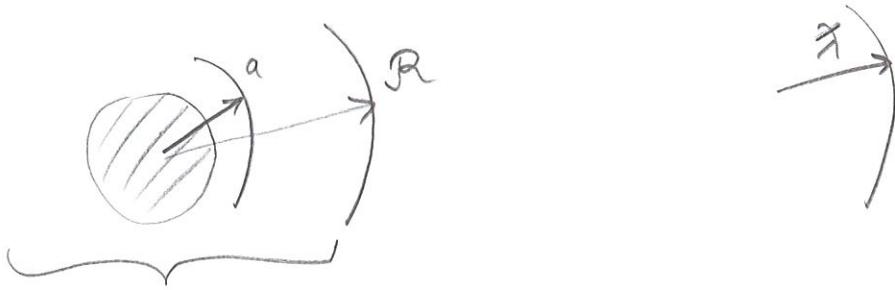
Consider the PN expansion of the field in the NZ ($r < R$)

$$\bar{h}(\vec{x}, t, c) = \sum_{p=2}^{+\infty} \frac{1}{c^p} \bar{h}_p(\vec{x}, t, lmc)$$

Note: \bar{h}_p denotes the PN coefficient of $\frac{1}{c^p}$
 while h_{lmn} denotes the PM coefficient of G^m

formal PN series
 (appearance of lmc 's
 at 4 PN order)

① Problem of NZ limitation



\bar{h} is valid only in NZ
(and diverges in the FZ, when $n \rightarrow \infty$)

How to incorporate into the PN series the information about boundary conditions at infinity (notably the no-incoming radiation condition which is imposed at \mathcal{T}^-)?

② Problem of divergencies

$$\Delta \bar{h}_P = \left(\begin{array}{l} \text{source term} \\ \text{with non-compact} \\ \text{support} \\ \text{which blows up when } n \rightarrow +\infty \end{array} \right)$$

Then the usual Poisson integral is divergent

$$\bar{h}_P = \int \frac{d^3 \vec{x}'}{|\vec{x} - \vec{x}'|} \text{ (source term)}$$

diverges at the bound $|\vec{x}'| = +\infty$
(for high P)

Problem ① will be solved by matching: $\mathcal{M}(\bar{h}) = \mathcal{M}(\bar{h})$

Problem ② will be solved by finding a suitable solution of the Poisson equation (different from the Poisson integral)

Insert $\bar{h} = \sum \frac{1}{c^p} \bar{h}_p$ into $\begin{cases} \square \bar{h} = \frac{16\pi G}{c^4} \bar{T} \\ \partial \bar{h} = 0 \end{cases}$

Hierarchy of PN equations ($\forall m \geq 2$)

$$\boxed{\begin{aligned} \Delta \bar{h}_p^{(m)} &= 16\pi G \bar{T}_{p-4}^{(m)} + \partial_t^2 \bar{h}_{p-2}^{(m)} \\ \partial_r \bar{h}_p^{(m)} &= 0 \end{aligned}}$$

At any given p the right-hand-side is known from previous iteration (using recursive treatment).

Construct first a particular solution of these equations using the generalized Poisson integral (Poujade & Blanchet 2002)

$$\text{FP } \Delta^{-1}[\bar{T}_p] \equiv \underset{B \rightarrow 0}{\text{Finite Part}} \underbrace{\frac{1}{4\pi} \int \frac{d^3 \vec{x}' |\vec{x}'|^B}{|\vec{x} - \vec{x}'|} \bar{T}_p(\vec{x}', t)}_{\text{defined by analytic continuation}}$$

Then we add the general homogeneous solution of Laplace's equation which is regular in the source ($r \rightarrow 0$)

$$\Delta \left[a \hat{x}_L + b \hat{\partial}_L \frac{1}{r} \right] = 0$$

\uparrow
solution
regular
when $r \rightarrow 0$

\uparrow
solution
regular
when $r \rightarrow \infty$

Most general solution is

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$$\bar{h}_p^{\mu\nu} = \text{FP} \Delta^{-1} \left\{ 16\pi G \bar{T}_{p-4}^{\mu\nu} + \partial_t^2 \bar{h}_{p-2}^{\mu\nu} \right\} + \sum_{l=0}^{+\infty} \frac{B_l^{\mu\nu}(t)}{pL} \tilde{x}_L$$

particular solution homogeneous solution
 (well-defined thanks to (unknown for the
 the Finite Part) moment)

To compute the homogeneous solution we require that it matches the external field in the sense

$$\mathcal{M} \left(\sum \frac{1}{c^p} \bar{h}_p^{\mu\nu} \right) = \overline{\mathcal{M}(h)} = \overline{\sum G^m h_m}$$

where $\mathcal{M}(h) = h_{\text{ext}} = \sum G^m h_m$. This fixes uniquely the homogeneous solution which is associated with radiation reaction forces inside the source, appropriate to an isolated system emitting GWs but not receiving GWs from \mathcal{I}^- .

Summing up $\bar{h} = \sum \frac{1}{c^p} \bar{h}_p^{\mu\nu}$ we get

$$\bar{h}^{\mu\nu} = \frac{16\pi G}{c^4} \left\{ \sum_{k=0}^{\infty} \left(\frac{2}{c \partial t} \right)^{2k} \text{FP} \Delta^{-k-1} \bar{T}^{\mu\nu} \right\} - \frac{4G}{c^4} \sum_{l=0}^{+\infty} \frac{(-1)^l}{l!} \partial_t^l \left\{ \frac{A_L^{\mu\nu}(t-n) - A_L^{\mu\nu}(t+n)}{2n} \right\}$$

particular solution homogeneous solution
 of d'Alembert eq. of d'Alembert eq.
 denoted $\text{FP} \mathcal{I}^{-1} \bar{T}^{\mu\nu}$

It's an anti-symmetric wave
(retarded)-(advanced)

Result of the matching is (Poujade & Blanchet 2002)

$$\mathcal{A}_L^{\mu\nu}(u) = \mathcal{F}_L^{\mu\nu}(u) + \mathcal{R}_L^{\mu\nu}(u)$$

where $\mathcal{F}_L^{\mu\nu}$ is the source's multipole moment (computed previously)

$$\mathcal{F}_L^{\mu\nu}(u) = \text{FP} \int d^3x \hat{x}_L \int_{-1}^1 dz \delta_\ell(z) \overline{T}^{\mu\nu}(\vec{x}, u+z/\vec{x}/c)$$

PN expansion of T

and where $\mathcal{R}_L^{\mu\nu}(u)$ is a new type of moment which turns out to parametrize non-linear radiation reaction effects in the source (Blanchet 1993)

$$\boxed{\mathcal{R}_L^{\mu\nu}(u) = \text{FP} \int d^3x \hat{x}_L \int_1^{+\infty} dz \gamma_\ell(z) \mathcal{M}(\overline{T}^{\mu\nu})(\vec{x}, u-z/\vec{x}/c)}$$

multipole expansion of T

where $\gamma_\ell(z) = -2\delta_\ell(z)$ satisfies (by analytic continuation in ℓ)

$$\int_1^{+\infty} dz \gamma_\ell(z) = 1 \quad \gamma_\ell(z) = (-)^{\ell+1} \frac{(2\ell+1)!!}{2^\ell \ell!} (z^2 - 1)^\ell$$

This comes from

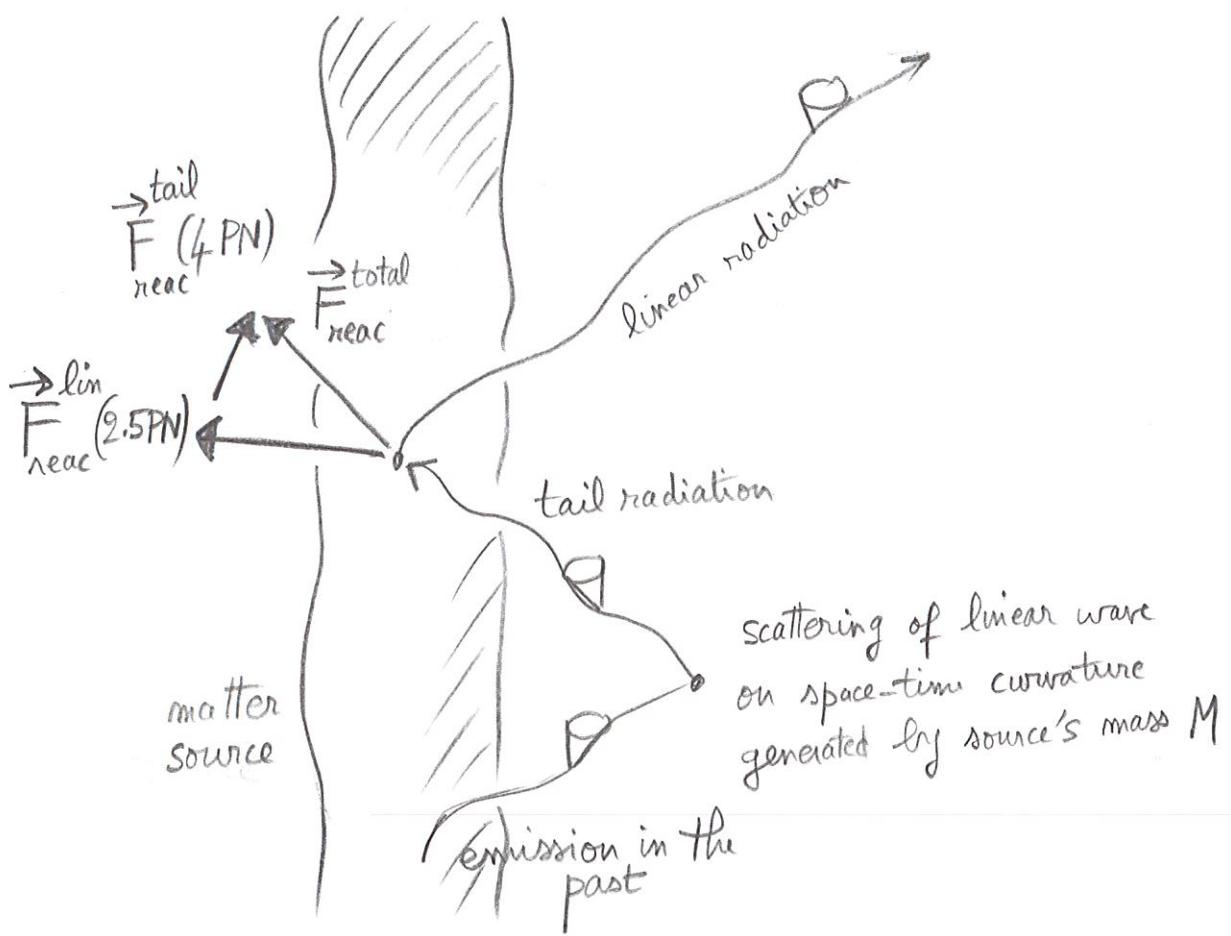
$$0 = \int_{-\infty}^{+\infty} dz \delta_\ell(z) = 2 \int_1^\infty dz \delta_\ell(z) + \int_{-1}^1 dz \delta_\ell(z) = - \int_1^\infty dz \gamma_\ell(z) + 1$$

by analytic continuation in $\ell \in \mathbb{C}$

Note that the PN expansion in the NZ ($r < R$) depends on the multipole exp. $M(\tau^{\mu\nu})$ and therefore on the properties of the field in the FZ ($r \gg \chi$).

Indeed the PN exp. includes the radiation reaction terms appropriate to an isolated system, satisfying the correct boundary conditions at infinity (notably \mathcal{J}^-).

$$\mathcal{F}_L^{\mu\nu} = \underbrace{\mathcal{F}_L^{\mu\nu}}_{\text{describes "linear" radiation reaction terms and starts at 2.5PN}} + \underbrace{\mathcal{R}_L^{\mu\nu}}_{\text{describes "non-linear" effects (tails) in the radiation reaction and starts at 4PN}}$$



The linear rad. reac. (parametrized by $\mathcal{F}_L^{\mu\nu}$) can be recombined with the particular solution

$$\text{FP } \mathcal{I}^{-1} \bar{T}^{\mu\nu} = \sum_{k=0}^{+\infty} \left(\frac{2}{c\partial t} \right)^{2k} \text{FP } \Delta^{-k-1} \bar{T}^{\mu\nu}$$

to give simply the retarded integral

$$\text{FP } \square_{\text{Ret}}^{-1} \bar{T}^{\mu\nu} = -\frac{1}{4\pi} \sum_{p=0}^{+\infty} \frac{\alpha_p}{p!} \left(\frac{2}{c\partial t} \right)^p \text{FP} \int d^3x' |x-x'|^{p-1} \bar{T}^{\mu\nu}(x', t)$$

formal expansion $c \rightarrow +\infty$
of the retardation $t - \frac{1}{c} |\vec{x} - \vec{x}'|$
(well-defined thanks to the FP)

The sol. $\text{FP } \mathcal{I}^{-1}$ corresponds to the even-parity part $p=2k$.
The odd-parity $p=2k+1$ is exactly given by the terms with $\mathcal{F}_L^{\mu\nu}$
Final result is thus (Blanchet, Faye & Nisanke 2005)

$$\bar{h}^{\mu\nu} = \underbrace{\frac{16\pi G}{c^4} \text{FP } \square_{\text{Ret}}^{-1} \bar{T}^{\mu\nu}}_{\text{corresponds to the old way of performing the PN expansion (Anderson & DeCanio 1975)}} - \underbrace{\frac{4G}{c^4} \sum_{l=0}^{+\infty} \partial_l \left[\frac{R_L^{\mu\nu}(t-r) - R_L^{\mu\nu}(t+r)}{2r} \right]}_{\text{starts at 4PN}}$$