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Model of dark matter and dark energy based on gravitational polarization

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A model of dark matter and dark energy based on the concept of gravitational polarization is investigated. We propose an action in standard general relativity for describing, at some effective or phenomenological level, the dynamics of a dipolar medium, i.e. one endowed with a dipole moment vector, and polarizable in a gravitational field. Using first-order cosmological perturbations, we show that the dipolar fluid is undistinguishable from standard dark energy (a cosmological constant Λ) plus standard dark matter (a pressureless perfect fluid), and therefore benefits from the successes of the Λ–cold-dark-matter scenario at cosmological scales. Invoking an argument of ''weak clusterization'' of the mass distribution of dipole moments, we find that the dipolar dark matter reproduces the phenomenology of the modified Newtonian dynamics at galactic scales. The dipolar medium action naturally contains a cosmological constant, and we show that if the model is to come from some fundamental underlying physics, the cosmological constant Λ should be of the order of a_0^2/c^4, where a_0 denotes the modified Newtonian dynamics constant acceleration scale, in good agreement with observations.

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I. INTRODUCTION

In the current concordance model of cosmology (the Λ–cold-dark-matter (CDM) scenario; see e.g. [1]) based on Einstein's general relativity (GR), the mass-energy content of the Universe is made of roughly 4% baryons, 23% CDM, and 73% dark energy in the form of a cosmological constant Λ. The dark matter accounts for the well-known discrepancy between the mass of a typical cluster of galaxies as deduced from its luminosity, and the Newtonian dynamical mass [2]. The model has so far been very successful in reproducing the observed cosmic microwave background (CMB) spectrum [3] and explaining the distribution of baryonic matter from galaxy cluster scales up to cosmological scales by the nonlinear growth of initial perturbations [4]. Although the exact nature of the hypothetical dark matter particle remains unknown, supersymmetric extensions of the standard model of particle physics predict well-motivated candidates (see [5] for a review). Simulations suggest some universal dark matter density profile around galaxies [6]. However, in that respect, the CDM hypothesis has some difficulties [7,8] at explaining in a natural way the distribution and properties of dark matter at galactic scales.

The modified Newtonian dynamics (MOND) was proposed by Milgrom [9–11] to account for the basic phenomenology of dark matter in galactic halos, as evidenced by the flat rotation curves of galaxies, and the Tully-Fisher relation [12] between the observed luminosity and the asymptotic rotation velocity of spiral galaxies. However, if MOND serves very well for these purposes (and some others also [8]), we know that MOND does not fully account for the inferred dark matter at the intermediate scale of clusters of galaxies [13–15]. In addition, MOND cannot be considered as a viable physical model, but only as an *ad-hoc*—though extremely useful—phenomenological ''recipe.'' In the usual interpretation, MOND is viewed (see [16] for a review) as a modification of the fundamental law of gravity or the fundamental law of dynamics, without the need for dark matter. The relativistic extensions of MOND, of which the tensor-vector-scalar (TeVeS) theory [14,17,18] is the prime example, share this view of modifying the gravity sector, by postulating some supplementary fields associated with the gravitational force, in addition to the metric tensor field of GR (see [19] for a review). Recently, such modified gravity theories have evolved toward Einstein-æther-like theories [20–24].

Each of these alternatives has proved to be very successful in complementary domains of validity: the cosmological scale (and cluster scale) for the CDM paradigm and the galactic scale for MOND. It is frustrating that two successful models seem to be fundamentally incompatible. In the present paper we shall propose a third approach, which has the potential of bringing together the main aspects of both Λ-CDM and MOND in a single relativistic model. Namely, we keep the standard law of gravity, i.e. GR and its Newtonian limit, but we add to the distribution of ordinary matter some specific nonstandard form of dark matter (described by a relativistic action in usual GR) in such a way as to naturally explain the phenomenology of MOND at galactic scales. Furthermore, we prove that this form of dark matter leads to the same predictions as for the Λ-CDM cosmological scenario at large scales. In particular, we find that the relativistic action for this matter model naturally contains the dark energy in the form of a cosmo-

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logical constant Λ. Thus, our model will benefit from both the successes of the Λ-CDM scenario, and the MOND phenomenology.

The model will be based on the observation [25,26] that the phenomenology of MOND can be naturally interpreted by an effect of "gravitational polarization" of some dipolar medium constituting the dark matter. The effect can be essentially viewed (in a Newtonian-like interpretation [25]) as the gravitational analogue of the electric polarization of a dielectric material, whose atoms can be modeled by electric dipoles, in an applied electric field [27]. In the quasi-Newtonian model of [25] the gravitational polarization follows from a microscopic description of the dipole moments in analogy with electrostatics. It was shown that the gravitational dipole moments require the existence of some internal nongravitational force to stabilize them in a gravitational field. Thanks to this internal force, an equilibrium state for the dipolar particle is possible, in which the dipole moment is aligned with the gravitational field and the medium is polarized. The MOND equation follows from that equilibrium configuration. However the model [25] cannot be considered as viable because it is nonrelativistic (NR), and involves negative gravitational-type masses (or gravitational charges) and, consecutively, a violation of the equivalence principle at a fundamental level.

In a second model [26] we showed that it is possible to describe dipolar particles consistently with the equivalence principle by an action principle in standard GR. The action depends on the particle's position in space-time (as for an ordinary particle action) and also on a four-vector dipole moment carried by the particle. The particle's position and the dipole moment are considered to be two dynamical variables to be varied independently in the action. Furthermore, a force internal to the dipolar particle was introduced in the form of a scalar potential function (say V) in this action. The potential V depends on some adequately defined norm of the dipole moment vector. Because of that force, the particle is not a "test" particle and its motion in space-time is nongeodesic. The nonrelativistic limit of the relativistic model [26] was found to be different from the quasi-Newtonian model [25] (hence the two models are distinct), but it was possible under some hypothesis to recover the same equilibrium state yielding the MOND equation as in [25]. However, the relativistic model [26], if considered as a model for dark matter, has some drawbacks—notably, the mechanism of alignment of the dipole moment with the gravitational field is unclear (so the precise link with MOND is questionable), and the dynamics of the dipolar particles in the special case of spherical symmetry does not seem to be very physical.

In the present paper, we shall propose a third model which will be based on an action similar to that of the relativistic model [26] but with some crucial modifications. First we shall add, with respect to [26], an ordinary mass term in the action to represent the (inertial or passive gravitational) mass of the dipolar particles. Second, the main improvement we shall make is to assume that the internal force derives from a potential function in the action (call it $\mathcal{W}$) which depends not on the dipole moment itself as in [26] but on the local *density* of dipole moments, i.e. the *polarization* field. In this new approach we are thus assuming that the motion of the dipolar particles is influenced by the density of the surrounding medium. This is analogous to the description of a plasma in electromagnetism in which the internal force, responsible for the plasma oscillations, depends on the density of the plasma (cf. the expression of the plasma frequency [27]).[1] Because the action [given by (2.2) with (2.7) below] will now depend on the density of the medium, it becomes more advantageous to write it as a *fluid* action rather than as a *particle* action.

This simple modification of the model, in which the potential $\mathcal{W}$ depends on the polarization field, will have important consequences. First of all, the relation with the phenomenology of MOND will become clear and straightforward. Second, we shall find that the motion of dipolar particles in the central field of a spherical mass (in the nonrelativistic limit) now makes sense physically. The drawbacks of the previous model [26] are thus cured. Last but not least, we shall find that the model naturally involves a cosmological constant. Then, with the equations of motion and evolution (and stress-energy tensor) derived from the action, we show the following:

(1) The dipolar fluid is undistinguishable from standard dark energy (a cosmological constant) plus standard CDM (say a pressureless perfect fluid) at cosmological scales, i.e. at the level of *first-order* cosmological perturbations.[2] The model is thus consistent with the observations of the CMB fluctuations. However, the model should differ from Λ-CDM at the level of second-order cosmological perturbations.

(2) The MOND phenomenology of the flat rotation curves of galaxies and the Tully-Fisher relation is recovered at galactic scales (for a galaxy at low redshift) from the effect of gravitational polarization. There is a one-to-one correspondence between the MOND function (say $\mu = 1 + \chi$) and the potential function $\mathcal{W}$ introduced in the action.

(3) The minimum of the potential function $\mathcal{W}$ is a cosmological constant Λ. We find that if $\mathcal{W}$ is to

[1] In the quasi-Newtonian model [25] the dipolar medium was formulated as the gravitational analogue of a plasma, oscillating at its natural plasma frequency.

[2] Note, however, that while in the standard scenario the CDM particle is, say, a well-motivated supersymmetric particle (perhaps to be discovered at the LHC in CERN), in our case the fundamental nature of the "dipolar particle" will remain unknown.

be considered as "fundamental," i.e. coming from some fundamental underlying theory (presumably a quantum field theory), the cosmological constant should be numerically of the order of a_0^2/c^4, where a_0 denotes the MOND constant acceleration scale.

A relation of the type $\Lambda \sim a_0^2/c^4$ between a cosmological observable Λ and a parameter a_0 measured from observations at galactic scales is quite remarkable and is in good agreement with observations. More precisely, if we define the natural acceleration scale associated with the cosmological constant,

$$a_\Lambda = \frac{c^2}{2\pi}\sqrt{\frac{\Lambda}{3}}, \tag{1.1}$$

then the current astrophysical measurements yield $a_0 \simeq 1.3 a_\Lambda$. The related numerical coincidence $a_0 \sim cH_0$ was pointed out very early on by Milgrom [9–11]. The near agreement between a_0 and a_Λ has a natural explanation within our model, although the exact numerical coefficient between the two acceleration scales cannot be determined presently.

Since the present model will not be connected to any (quantum) fundamental theory, it should be regarded merely as an "effective" or even "phenomenological" model. We shall even argue (though this remains open) that it may apply only at large scales, from the galactic scale up to cosmological scales, and not at smaller scales like in the Solar System. However, this model offers a nice unification between the dark energy in the form of Λ and the dark matter in the form of MOND (both effects of dark energy and dark matter occurring when gravity is weak). Furthermore, it reconciles in some sense the observations of dark matter on cosmological scales, where the evidence is for the standard CDM, and on galactic scales, which is the realm of MOND. It would be interesting to study the intermediate scale of clusters of galaxies and to see if the model is consistent with observations. Such a study should probably be performed using numerical methods.

The plan of this paper is as follows. In Sec. II we present the action principle for the dipolar medium, and we vary the action to obtain the equation of motion, the equation of evolution, and the stress-energy tensor. In Sec. III we apply first-order cosmological perturbations (on a homogeneous and isotropic background) to prove that the dipolar fluid reproduces all the features of the standard dark matter paradigm at cosmological scales. We investigate the nonrelativistic limit of the model in Sec. IV, and show that, under some hypothesis, the polarization of the dipolar dark matter in the gravitational field of a galaxy results in an apparent modification of the law of gravity in agreement with the MOND paradigm. Section V summarizes and concludes the paper. The dynamics of the dipolar dark matter in the central gravitational field of a spherically symmetric mass distribution is investigated in the Appendix.

II. DIPOLAR FLUID IN GENERAL RELATIVITY

A. Action principle

Our model will be based on a specific action functional for the dipolar fluid in standard GR. This fluid is described by the four-vector current density $J^\mu = \sigma u^\mu$, where u^μ is the four-velocity of the fluid, normalized to $g_{\mu\nu}u^\mu u^\nu = -1$, and where $\sigma = \sqrt{-g_{\mu\nu}J^\mu J^\nu}$ represents its rest-mass density.[3] In this paper we shall conveniently rescale most of the variables used in [26] by a factor of $2m$, where m is the mass parameter introduced in the action of [26]. Hence we have $\sigma = 2mn$, where n is the number density of dipole moments in the notation of [26]. The above current vector is conserved in the sense that

$$\nabla_\mu J^\mu = 0, \tag{2.1}$$

where ∇_μ denotes the covariant derivative associated with the metric $g_{\mu\nu}$. Our fundamental assumption is that the dipolar fluid is endowed with a dipole moment vector field ξ^μ which will be considered as a dynamical variable. We have $\xi^\mu = \pi^\mu/2m$ where π^μ is the dipole moment variable used in [26] (hence ξ^μ has the dimension of a length).

Adopting a fluid description of the dipolar matter rather than a particle formulation as in [26],[4] we postulate that the dynamics of the dipolar fluid in a prescribed gravitational field $g_{\mu\nu}$ is derived from an action of the type

$$S = \int \mathrm{d}^4x\sqrt{-g}L[J^\mu, \xi^\mu, \dot{\xi}^\mu, g_{\mu\nu}], \tag{2.2}$$

where $g = \det(g_{\mu\nu})$, the integration being performed over the entire 4-dimensional manifold. This action is to be added to the Einstein-Hilbert action for gravity, and to the actions of all the other matter fields. The Lagrangian L depends on the current density J^μ, the dipole moment vector ξ^μ, and its covariant derivative $\dot{\xi}^\mu$ with respect to the proper time τ (such that $\mathrm{d}\tau = \sqrt{-g_{\mu\nu}\mathrm{d}x^\mu \mathrm{d}x^\nu}$), which is defined using a fluid formulation by

$$\dot{\xi}^\mu \equiv \frac{\mathrm{D}\xi^\mu}{\mathrm{d}\tau} \equiv u^\nu \nabla_\nu \xi^\mu, \tag{2.3}$$

and where $\mathrm{D}/\mathrm{d}\tau$ is denoted by an overdot. In addition, the Lagrangian depends explicitly on the metric $g_{\mu\nu}$ which serves to lower and raise indices, so that, for instance, $\dot{\xi}_\mu = g_{\mu\nu}\dot{\xi}^\nu$.

[3] Greek indices take the space-time values $\mu, \nu, \ldots = 0, 1, 2, 3$ and Latin ones range on spatial values $i, j, \ldots = 1, 2, 3$. The metric signature is $(-, +, +, +)$. The convention for the Riemann curvature tensor $R^\mu{}_{\nu\rho\sigma}$ is the same as in [28]. Symmetrization of indices is $(\mu\nu) \equiv \frac{1}{2}(\mu\nu + \nu\mu)$ and $(ij) \equiv \frac{1}{2}(ij + ji)$. In Secs. II and III we make use of geometrical units $G = c = 1$.

[4] The fluid action is obtained from the particle one by the formal prescription $\sum \int \mathrm{d}\tau \to \int \mathrm{d}^4x\sqrt{-g}n$, where the sum runs over all the particles, and n is the number density of the fluid.

We shall consider an action for the dipolar medium similar to the one proposed in [26], with, however, a crucial generalization in that the potential function therein, which is supposed to describe a nongravitational force internal to the dipole moment, will be allowed to depend not only on the dipole moment variable ξ^μ, but also on the rest-mass density of the dipolar fluid σ. More precisely, we shall assume that the potential function $\mathcal{W}$ in the action depends on the dipole moment ξ^μ only through the *polarization*, namely, the number density of dipole moments, that is defined by

$$\Pi^\mu = \sigma\xi^\mu, \tag{2.4}$$

or equivalently $\Pi^\mu = n\pi^\mu$ in the notation of [26]. The dynamics of dipolar particles will therefore be influenced by the local density of the medium, in analogy with the physics of a plasma in which the force responsible for the plasma oscillations depends on the density of the plasma [27]. Our assumption is that $\mathcal{W}$ is a function solely of the norm $\Pi_\perp$ of the projection of the polarization field (2.4) perpendicular to the velocity, namely,

$$\Pi_\perp = \sqrt{g_{\mu\nu}\Pi^\mu_\perp\Pi^\nu_\perp} = \sqrt{\perp_{\mu\nu}\,\Pi^\mu\Pi^\nu}. \tag{2.5}$$

Here, the orthogonal projection of the polarization vector reads $\Pi^\mu_\perp = \perp^\mu_\nu\,\Pi^\nu$, with the associated projector defined by $\perp_{\mu\nu} \equiv g_{\mu\nu} + u_\mu u_\nu$. Similarly, we can define $\xi^\mu_\perp = \perp^\mu_\nu\,\xi^\nu$ and its norm $\xi_\perp$ so that the (scalar) polarization field reads

$$\Pi_\perp = \sigma\xi_\perp. \tag{2.6}$$

The chosen dependence of the internal potential on $\Pi_\perp$ will result in important differences and improvements with respect to the model of [26].

Our proposal for the Lagrangian of the dipolar fluid is

$$L = \sigma\left[-1 - \sqrt{(u_\mu - \dot\xi_\mu)(u^\mu - \dot\xi^\mu)} + \frac{1}{2}\dot\xi_\mu\dot\xi^\mu\right] - \mathcal{W}(\Pi_\perp), \tag{2.7}$$

where the two dynamical fields are the conserved current vector $J^\mu = \sigma u^\mu$ and the dipole moment vector ξ^μ. The fourth term is our fundamental potential which should, in principle, result from a more fundamental theory valid at some microscopic level. The third term in (2.7) is the same as in the previous model [26] and clearly represents a kineticlike term for the evolution of the dipole moment vector. This term will tell how this evolution should differ from parallel transport along the fluid lines. The second term in (2.7) (also the same as in [26]) is made of the norm of a spacelike vector and is inspired by the known action for the dynamics of particles with spin moving in a background gravitational field [29]. The motivation for postulating this term is that a dipole moment can be seen as the "lever arm" of the spin considered as a classical angular momentum (see a discussion in [26]).

Finally, we comment on the first term in (2.7) which is a mass term in an ordinary sense. The dipolar fluid we are considering will not be purely dipolar (or mostly dipolar) as in the previous model [26] but will involve a monopolar contribution as well. Here we shall thus have some dark matter in the ordinary sense. The mass term in (2.7) has been included for cosmological considerations, so that we recover the ordinary dark matter component at large scales (see Sec. III). However, one can argue that the presence of such a mass term σ is not fine-tuned. Indeed, this term corresponds to the simplest and most natural assumption that the relative contributions of this mass density and the second and third terms in (2.7) are comparable. In addition, we notice that $\sigma = 2mn$ corresponds to the inertial mass density of the dipole particles in the quasi-Newtonian model [25], so it is natural by analogy with this model to include that mass contribution in the action. Notice however that, even if the dipolar fluid is endowed with a mass density in an ordinary sense, its dynamics is well defined only when the dipole moment is nonzero. Indeed, we observe that the Lagrangian (2.7) becomes ill defined when $\xi^\mu = 0$ since the second term in (2.7) is imaginary.

B. Equations of motion and evolution

In order to obtain the equations governing the dynamics of the dipolar fluid, we vary the action (2.2) [with the explicit choice of the Lagrangian (2.7)] with respect to the dynamical variables ξ^μ and J^μ. The calculation is very similar to the one performed in [26], but because of the different notation adopted here for rescaled variables (e.g. $\xi^\mu = \pi^\mu/2m$), and especially because of the more general form of the potential function, we present all details of the derivation. Varying first with respect to the dipole moment variable ξ^μ, the resulting Euler-Lagrange equation reads, in general terms,[5]

$$\frac{\mathrm{D}}{\mathrm{d}\tau}\left(\frac{\partial L}{\partial\dot\xi^\mu}\right) + \nabla_\nu u^\nu\frac{\partial L}{\partial\dot\xi^\mu} = \frac{\partial L}{\partial\xi^\mu}, \tag{2.8}$$

in which the partial derivatives of the Lagrangian in (2.2) are applied considering the four variables ξ^μ, $\dot\xi^\mu$, J^μ, and $g_{\mu\nu}$ as independent. For the specific case of the Lagrangian (2.7), we get what shall be interpreted as the equation of *motion* of the dipolar fluid in the form

$$\dot K^\mu = -\mathcal{F}^\mu, \tag{2.9}$$

[5]We write the Euler-Lagrange equation in this particle-looking form to emphasize the fact that the action (2.7) is a *particle* (or fluid) action. Of course, this equation is equivalent to the usual *field* equation

$$\nabla_\nu\left(\frac{\partial L}{\partial\nabla_\nu\xi^\mu}\right) = \frac{\partial L}{\partial\xi^\mu}.$$

in which the left-hand side (LHS) is the proper time derivative of the linear momentum[6]

$$K^\mu = \dot{\xi}^\mu + k^\mu. \tag{2.10}$$

Here, we introduced, like in [26], a special notation for a four-vector k^μ which is *spacelike*, whose norm is normalized to $k^\mu k_\mu = 1$, and which reads

$$k^\mu = \frac{u^\mu - \dot{\xi}^\mu}{\Xi} \quad \text{with} \quad \Xi = \sqrt{-1 - 2u_\nu\dot{\xi}^\nu + \dot{\xi}_\nu\dot{\xi}^\nu}. \tag{2.11}$$

The spacelike four-vector k^μ will not represent the linear momentum (per unit mass) of the particle—that role will be taken by K^μ which, as we shall see, will normally be timelike; see (2.20a) below. The quantity Ξ has an important status in the present formalism because it represents the second term in the Lagrangian (2.7), and we shall be able to set it to 1 in Sec. II C as a particular way of selecting some physically interesting solution. On the right-hand side (RHS) of (2.9), the force per unit mass acting on a dipolar fluid element is given by

$$\mathcal{F}^\mu = \hat{\Pi}^\mu_\perp \mathcal{W}_{\Pi_\perp}, \tag{2.12}$$

in which we denote the unit direction of the polarization vector by $\hat{\Pi}^\mu_\perp \equiv \Pi^\mu_\perp/\Pi_\perp = \xi^\mu_\perp/\xi_\perp$ and the ordinary derivative of the potential $\mathcal{W}$ by $\mathcal{W}_{\Pi_\perp} \equiv \mathrm{d}\mathcal{W}/\mathrm{d}\Pi_\perp$. The "*internal*" force (2.12) being proportional to the spacelike four-vector $\xi^\mu_\perp = \perp^\mu_\nu \xi^\nu$, we immediately get the constraint

$$u_\mu \mathcal{F}^\mu = 0. \tag{2.13}$$

We now turn to the variation of the action with respect to the conserved current $J^\mu = \sigma u^\mu$ (hence we deduce $\sigma = \sqrt{-J_\nu J^\nu}$ and $u^\mu = J^\mu/\sigma$). The general form of the Lagrange equation for the conserved current density reads (see e.g. [30])[7]

$$\frac{\mathrm{D}}{\mathrm{d}\tau}\left(\frac{\partial L}{\partial J^\mu}\right) = u^\nu \nabla_\mu \left(\frac{\partial L}{\partial J^\nu}\right). \tag{2.14}$$

[6]The present notation is related to the one used in [26] by $K^\mu = P^\mu/2m$, $k^\mu = p^\mu/2m$, $\mathcal{F}^\mu = F^\mu/m$ (and $\xi^\mu = \pi^\mu/2m$). The quantity called Λ in [26] is now denoted Ξ in order to avoid confusion with the cosmological constant.

[7]This can alternatively be written with ordinary partial derivatives as

$$u^\nu\left[\partial_\nu\left(\frac{\partial L}{\partial J^\mu}\right) - \partial_\mu\left(\frac{\partial L}{\partial J^\nu}\right)\right] = 0.$$

For the case of the Lagrangian (2.7) at hand, we get the following equation, later to be interpreted as the *evolution* equation for the dipole moment,

$$\dot{\Omega}^\mu = \frac{1}{\sigma}\nabla^\mu(\mathcal{W} - \Pi_\perp \mathcal{W}_{\Pi_\perp}) - R^\mu{}_{\rho\nu\lambda} u^\rho \xi^\nu K^\lambda. \tag{2.15}$$

A new type of linear momentum Ω^μ—having the same meaning as in [26]—has been introduced and defined by

$$\begin{aligned} \Omega^\mu &= \omega^\mu - k^\mu \quad \text{with} \\ \omega^\mu &= u^\mu(1 + \tfrac{1}{2}\dot{\xi}_\nu\dot{\xi}^\nu + \xi_\perp \mathcal{W}_{\Pi_\perp}) - u_\nu \xi^\nu \mathcal{F}^\mu. \end{aligned} \tag{2.16}$$

The Riemann curvature term in the RHS of (2.15) represents the analogue of the coupling to curvature in the Papapetrou equations of motion of particles with spin in an arbitrary background [31]. The complete dynamics and evolution of the dipolar fluid is now encoded into the equations (2.9) and (2.15). Such equations constitute the appropriate generalization for the case of a density-dependent potential $\mathcal{W}$, and in the fluid formulation, of similar results in [26].

Notice that by contracting (2.15) with J_μ, the second term in the RHS of (2.15) cancels because of the symmetries of the Riemann tensor, and we get

$$J_\mu \dot{\Omega}^\mu = \frac{\mathrm{D}}{\mathrm{d}\tau}(\mathcal{W} - \Pi_\perp \mathcal{W}_{\Pi_\perp}). \tag{2.17}$$

One can readily check that this constraint (2.17) can alternatively be derived from the other equation (2.9) together with the definition of Ω^μ in (2.16). On the other hand, contracting (2.9) with u_μ yields $u_\mu \dot{K}^\mu = 0$, which according to the definition of K^μ, leads to the other constraint

$$u_\mu \frac{\mathrm{D}}{\mathrm{d}\tau}[(\Xi - 1)k^\mu] = 0. \tag{2.18}$$

This constraint can be viewed as a differential equation for the variable Ξ.

C. Particular solution of the equations

Following [26], we shall solve the constraint (2.18) with the most obvious and natural choice of solution,

$$\Xi = 1. \tag{2.19}$$

We shall see that this choice greatly simplifies the other equations we have. In particular, we are going to prove that the equations of motion (2.9) and equations of evolution (2.15), when reduced by the condition $\Xi = 1$, finally depend only on the *spacelike* component of the dipole moment that is orthogonal to the velocity, namely, $\xi^\mu_\perp$, so that the timelike component along the velocity, i.e. $u_\nu \xi^\nu$, will have no physically observable consequences (actually, in that case this unphysical component turns out to be complex [26]).

The structure of the subsequent equations and the physical properties of the model will heavily rely on the condition $\Xi = 1$. Note that we could regard this condition not as a choice of solution but rather as a choice of *theory*. Indeed, we are going to pick up the simplest theory out of a whole set of theories in which Ξ could have some nontrivial proper time evolution obeying (2.18). Actually, we can view the choice $\Xi = 1$ as an elegant way to impose into the Lagrangian formalism the condition that *in fine* the only physical component of the dipole moment should be $\xi_\perp^\mu$, namely, the one perpendicular to the four-velocity field. We can imagine that it would be possible to impose the same physical condition in a different way, for instance by using Lagrange multipliers in the initial action. For example, in TeVeS [14,17,18] or in Einstein-æther gravity [20–24], a dynamical timelike vector field whose norm is unity is introduced by this means. However, the present situation is different because our final physical vector $\xi_\perp^\mu$ is spacelike.

When the condition (2.19) holds, the two linear momenta (2.10) and (2.16) simplify appreciably and we obtain

$$K^\mu = u^\mu, \tag{2.20a}$$

$$\Omega^\mu = u^\mu(1 + \xi_\perp \mathcal{W}_{\Pi_\perp}) + \perp^\mu_\nu \dot{\xi}^\nu_\perp. \tag{2.20b}$$

We see that the linear momentum K^μ is finally timelike. These expressions depend only on the orthogonal component $\xi^\nu_\perp$, and we denote $\dot{\xi}^\nu_\perp \equiv \mathrm{D}\xi^\nu_\perp/\mathrm{d}\tau$. The equations of motion and evolution now take the simple forms

$$\dot{u}^\mu = -\mathcal{F}^\mu = -\hat{\Pi}^\mu_\perp \mathcal{W}_{\Pi_\perp}, \tag{2.21a}$$

$$\dot{\Omega}^\mu = \frac{1}{\sigma}\nabla^\mu(\mathcal{W} - \Pi_\perp \mathcal{W}_{\Pi_\perp}) - \xi^\nu_\perp R^\mu{}_{\rho\nu\lambda}u^\rho u^\lambda. \tag{2.21b}$$

Finally, the whole dynamics of the dipolar fluid only depends on the spacelike perpendicular projection $\xi_\perp^\mu$ of the dipole moment.

D. Expression of the stress-energy tensor

We vary the action (2.2) with respect to the metric $g_{\mu\nu}$ to obtain the stress-energy tensor. We must first consider the general case where Ξ is unconstrained, and then, only on the result, make the restriction that $\Xi = 1$. We properly take into account the metric contributions coming from the Christoffel symbols in the covariant time derivative $\dot{\xi}^\mu$ by using the Palatini formula [32]. We are also careful that, while the dipole moment ξ^μ should be kept fixed during the variation, the conserved current J^μ will vary because of the change in the volume element $\sqrt{-g}\mathrm{d}^4x$. Instead of J^μ, the relevant metric-independent variable that has to be fixed is the "coordinate" current density defined by $J^\mu_* = \sqrt{-g}J^\mu$. Straightforward calculations yield the expression of the stress-energy tensor for an action of the general type (2.2). We find

$$T^{\mu\nu} = 2\frac{\partial L}{\partial g_{\mu\nu}} + g^{\mu\nu}\left(L - J^\rho\frac{\partial L}{\partial J^\rho}\right) + u^\mu u^\nu \dot{\xi}^\rho \frac{\partial L}{\partial \dot{\xi}^\rho} + \nabla_\rho\left(u^\mu u^\nu \frac{\partial L}{\partial \dot{\xi}_\rho} - u^\rho \xi^{(\mu}\frac{\partial L}{\partial \dot{\xi}_{\nu)}} - \xi^\rho u^{(\mu}\frac{\partial L}{\partial \dot{\xi}_{\nu)}}\right), \tag{2.22}$$

in which we denote $\partial L/\partial \dot{\xi}_\rho \equiv g^{\rho\lambda}\partial L/\partial \dot{\xi}^\lambda$. The partial derivatives of the Lagrangian are performed assuming that its "natural" arguments J^μ, ξ^μ, $\dot{\xi}^\mu$, and $g_{\mu\nu}$ are independent. The application to the particular case of the Lagrangian (2.7) gives, for the moment for a general value of Ξ,

$$T^{\mu\nu} = -g^{\mu\nu}(\mathcal{W} - \Pi_\perp \mathcal{W}_{\Pi_\perp}) + \Omega^{(\mu}J^{\nu)} - \nabla_\rho([\xi^\rho K^{(\mu} - K^\rho \xi^{(\mu}]J^{\nu)}). \tag{2.23}$$

In the second term of (2.23) we see that the linear momentum Ω^μ is related to the monopolar contribution to the stress-energy tensor, while the other linear momentum K^μ parametrizes the dipolar contribution in the third term. Comparing with Eq. (2.14) of [26], we observe that a new term, proportional to the metric $g^{\mu\nu}$, has been introduced. This term will clearly be associated with a cosmological constant, and we shall discuss it in detail below. One can readily verify that the conservation law $\nabla_\nu T^{\mu\nu} = 0$ holds as a consequence of the equation of conservation of matter (2.1), and the equations of motion and evolution (2.9) and (2.15), for general Ξ.

In the next step we reduce the expression (2.23) by means of the condition $\Xi = 1$ and get

$$T^{\mu\nu} = -\mathcal{W}g^{\mu\nu} + \sigma(u^\mu u^\nu + \xi_\perp \mathcal{W}_{\Pi_\perp} \perp^{\mu\nu} + u^{(\mu} \perp^{\nu)}_\rho \dot{\xi}^\rho_\perp) - \nabla_\rho([\xi^\rho_\perp u^{(\mu} - u^\rho \xi^{(\mu}_\perp]J^{\nu)}). \tag{2.24}$$

Again we notice that this expression depends only on the perpendicular projection $\xi_\perp^\mu$ of the dipole moment.

It will be useful in the following to decompose the stress-energy tensor (2.24) according to the general canonical form

$$T^{\mu\nu} = r u^\mu u^\nu + \mathcal{P} \perp^{\mu\nu} + 2Q^{(\mu}u^{\nu)} + \Sigma^{\mu\nu}, \tag{2.25}$$

where r and $\mathcal{P}$ represent the energy density and pressure, where the "heat flow" Q^μ is orthogonal to the four-velocity, i.e. $u_\mu Q^\mu = 0$, and the symmetric anisotropic stress tensor $\Sigma^{\mu\nu}$ is orthogonal to the four-velocity and traceless, i.e. $u_\nu \Sigma^{\mu\nu} = 0$ and $\Sigma^\nu_\nu = 0$. We get

$$r = u_\rho u_\sigma T^{\rho\sigma}, \tag{2.26a}$$

$$\mathcal{P} = \tfrac{1}{3}\perp_{\rho\sigma} T^{\rho\sigma}, \tag{2.26b}$$

$$Q^\mu = -\perp^\mu_\rho u_\sigma T^{\rho\sigma}, \tag{2.26c}$$

while the anisotropic stress tensor is obtained by subtraction. In the case $\Xi = 1$ where the dipolar fluid is described by the stress-energy tensor (2.24), we find that the energy density, pressure, heat flow, and anisotropic stress tensor read, respectively,

$$r = \mathcal{W} - \Pi_\perp \mathcal{W}_{\Pi_\perp} + \rho, \tag{2.27a}$$

$$\mathcal{P} = -\mathcal{W} + \tfrac{2}{3}\Pi_\perp \mathcal{W}_{\Pi_\perp}, \tag{2.27b}$$

$$Q^\mu = \sigma \dot{\xi}^\mu_\perp + \Pi_\perp \mathcal{W}_{\Pi_\perp} u^\mu - \Pi^\lambda_\perp \nabla_\lambda u^\mu, \tag{2.27c}$$

$$\Sigma^{\mu\nu} = (\tfrac{1}{3}\perp^{\mu\nu} - \hat{\xi}^\mu_\perp \hat{\xi}^\nu_\perp)\Pi_\perp \mathcal{W}_{\Pi_\perp}, \tag{2.27d}$$

where we denote $\hat{\xi}^\mu_\perp \equiv \xi^\mu_\perp/\xi_\perp$, and where we introduced for future use the convenient notation

$$\rho = \sigma - \nabla_\lambda \Pi^\lambda_\perp. \tag{2.28}$$

By contrast to ordinary perfect fluids, the characteristic feature of the dipolar fluid is the existence of nonvanishing heat flow Q^μ and anisotropic stresses $\Sigma^{\mu\nu}$. Furthermore, we notice that the energy density r involves (*via* ρ) a dipolar contribution given by $-\nabla_\lambda \Pi^\lambda_\perp$. That contribution will play the crucial role, as we will see in Sec. IV, when recovering the phenomenology of MOND.

III. COSMOLOGICAL PERTURBATIONS AT LARGE SCALES

We are going to show in this section that the model of dipolar dark matter [i.e. based on the action (2.2) and (2.7), with equations of motion reduced by the condition $\Xi = 1$] contains the essential features of standard dark matter at cosmological scales. We shall indeed prove that, at *first order* in cosmological perturbations, it behaves like a pressureless perfect fluid. Furthermore, we shall see that the dipolar fluid naturally contains a cosmological constant (the interpretation of which will be discussed below), and is thus supported by the observations of dark energy. The model is therefore consistent with cosmological observations of the CMB fluctuations.

A. Perturbation of the gravitational sector

We apply the theory of first-order cosmological perturbations around a Friedman-Lemaître-Robertson-Walker (FLRW) background. For every generic scalar field or component of a tensor field, say F, we shall write $F = \bar{F} + \delta F$, where the background part $\bar{F}$ is the value of F in a FLRW metric, while δF is a first-order perturbation of this background value.

The FLRW metric is written in the usual way in terms of the conformal time η, such that $\mathrm{d}t = a\mathrm{d}\eta$ where $a(\eta)$ is the scale factor and t the cosmic time, as

$$\mathrm{d}\bar{s}^2 = \bar{g}_{\mu\nu}\mathrm{d}x^\mu \mathrm{d}x^\nu = a^2[-\mathrm{d}\eta^2 + \gamma_{ij}\mathrm{d}x^i\mathrm{d}x^j]. \tag{3.1}$$

Here γ_{ij} is the metric of maximally symmetric spatial hypersurfaces of constant curvature $K = 0$ or $K = \pm 1$. The perturbed FLRW metric $\mathrm{d}s^2 = g_{\mu\nu}\mathrm{d}x^\mu\mathrm{d}x^\nu$ will be of the general form [33]

$$\mathrm{d}s^2 = a^2[-(1+2A)\mathrm{d}\eta^2 + 2h_i \mathrm{d}\eta \mathrm{d}x^i + (\gamma_{ij} + h_{ij})\mathrm{d}x^i\mathrm{d}x^j]. \tag{3.2}$$

Making use of the standard scalar-vector-tensor (SVT) decomposition [34,35], the metric perturbations h_i and h_{ij} are decomposed according to

$$h_i = D_i B + B_i, \tag{3.3a}$$

$$h_{ij} = 2C\gamma_{ij} + 2D_iD_jE + 2D_{(i}E_{j)} + 2E_{ij}, \tag{3.3b}$$

where D_i denotes the covariant derivative with respect to the spatial background metric γ_{ij}. The vectors B^i and E^i are divergenceless, and the tensor E^{ij} is at once divergenceless and trace-free, i.e.

$$D_iB^i = D_iE^i = 0, \tag{3.4a}$$

$$D_iE^{ij} = E^i_i = 0. \tag{3.4b}$$

Spatial indices are lowered and raised with γ_{ij} and its inverse γ^{jk}. From these definitions, one can construct the gauge-invariant perturbation variables

$$\Phi \equiv A + (B' + \mathcal{H}B) - (E'' + \mathcal{H}E'), \tag{3.5a}$$

$$\Psi \equiv -C - \mathcal{H}(B - E'), \tag{3.5b}$$

$$\Phi_i \equiv E'_i - B_i, \tag{3.5c}$$

together with E_{ij} which is already gauge invariant. The prime stands for a derivative with respect to the conformal time η, and $\mathcal{H} \equiv a'/a$ denotes the conformal Hubble parameter. We shall also use the alternative definition for a gauge-invariant gravitational potential

$$X \equiv A - C - \left(\frac{C}{\mathcal{H}}\right)' = \Psi + \Phi + \left(\frac{\Psi}{\mathcal{H}}\right)'. \tag{3.6}$$

B. Kinematics of the dipolar fluid

The four-velocity of the dipolar fluid is decomposed into a background part and a perturbation, $u^\mu = \bar{u}^\mu + \delta u^\mu$. We have both $\bar{g}_{\mu\nu}\bar{u}^\mu\bar{u}^\nu = -1$ and $g_{\mu\nu}u^\mu u^\nu = -1$. The background part is supposed to be comoving, that is, $\bar{u}^i = 0$. This defines a zeroth order in the perturbation. In a FLRW background this means that it will satisfy the background geodesic equation $\dot{\bar{u}}^\mu = 0$. With standard notations, we have

$$\bar{u}^\mu = \frac{1}{a}(1, 0), \tag{3.7a}$$

$$\delta u^\mu = \frac{1}{a}(-A, \beta^i), \tag{3.7b}$$

while the covariant four-velocity will be written as $u_\mu = \bar{u}_\mu + \delta u_\mu$, with

$$\bar{u}_\mu = a(-1, 0), \tag{3.8a}$$

$$\delta u_\mu = a(-A, \beta_i + h_i). \tag{3.8b}$$

The velocities of all the other fluids (baryons, photons, neutrinos, ...) are decomposed in a similar way. The perturbation of the three-velocity β^i is split into scalar and vector parts,

$$\beta^i = D^i v + v^i \quad \text{with} \quad D_i v^i = 0, \tag{3.9}$$

and we introduce the gauge-invariant variables describing the perturbed motion,

$$V \equiv v + E', \tag{3.10a}$$
$$V_i \equiv v_i + B_i. \tag{3.10b}$$

The dipolar dark matter fluid differs from standard dark matter by the presence of the dipole moment $\xi_\perp^\mu$ (satisfying $u_\mu \xi_\perp^\mu = 0$) carried along the fluid trajectories. For the dipole moment we also write a decomposition into a background part plus a perturbation, namely, $\xi_\perp^\mu = \bar{\xi}_\perp^\mu + \delta\xi_\perp^\mu$. However, because a nonvanishing background dipole moment would break the isotropy of space, and would therefore be incompatible with a FLRW metric, we must make the assumption that the dipole moment is *zero* in the background, so that it is purely perturbative. Hence, we pose

$$\bar{\xi}_\perp^\mu = 0, \tag{3.11a}$$
$$\delta\xi_\perp^\mu = (0, \lambda^i), \tag{3.11b}$$

where λ^i represents the first-order perturbation of the dipole moment. Beware of our notation for which λ^i is a vector living in the background spatial metric γ_{ij}. Thus the covariant components of the dipole moment perturbation are $\delta\xi_{\perp\mu} = (0, a^2\lambda_i)$ where $\lambda_i \equiv \gamma_{ij}\lambda^j$. Note that there is no time component in the dipole moment perturbation because of the constraint $u_\mu \xi_\perp^\mu = 0$ which reduces to $\bar{u}_\mu \delta\xi_\perp^\mu = 0$ at linear order. Like for the three-velocity field β^i in (3.9), we split λ^i into a scalar and a vector part, namely,

$$\lambda^i = D^i y + y^i \quad \text{with} \quad D_i y^i = 0. \tag{3.12}$$

However, unlike for v and v^i, we notice that y and y^i are gauge-invariant perturbation variables. This is because the background quantity is zero, $\bar{\xi}_\perp^\mu = 0$; hence the perturbation $\delta\xi_\perp^\mu$ is gauge invariant according to the Stewart-Walker lemma [36,37].

C. Cosmological expansion of the fundamental potential

The next step is to make more specific our fundamental potential function $\mathcal{W}(\Pi_\perp)$ entering the Lagrangian (2.7). Such a function should be a "universal" function of the polarization of the dipolar medium, described by the polarization scalar field

$$\Pi_\perp = \sigma\xi_\perp. \tag{3.13}$$

Now, we have seen that in cosmology there is no background (FLRW) value for the dipole moment; hence the background value of the polarization field is zero: $\bar{\Pi}_\perp = 0$. In linear perturbations, the polarization is expected to stay around the background value. Therefore, it seems physically well motivated that the value $\bar{\Pi}_\perp = 0$ corresponds to a minimum of the potential function $\mathcal{W}$, so that $\Pi_\perp$ does not depart too much from this background value, at least in the linear perturbation regime. We therefore assume that $\mathcal{W}(\Pi_\perp)$ is given locally[8] by a harmonic potential of the form

$$\mathcal{W}(\Pi_\perp) = \mathcal{W}_0 + \tfrac{1}{2}\mathcal{W}_2\Pi_\perp^2 + \mathcal{O}(\Pi_\perp^3), \tag{3.14}$$

where $\mathcal{W}_0$ and $\mathcal{W}_2$ are two constant parameters, and we pose $\mathcal{W}_1 = 0$. For linear perturbations, because $\Pi_\perp = \delta\Pi_\perp$ is already perturbative, we shall be able to neglect the higher order terms $\mathcal{O}(\Pi_\perp^3)$ in (3.14) because these will contribute to second order at least in the internal force (2.12). Inserting the *ansatz* (3.14) into (2.12) we obtain

$$\mathcal{F}^\mu = \mathcal{W}_2\Pi_\perp^\mu + \mathcal{O}(\Pi_\perp^2). \tag{3.15}$$

We asserted in the previous section that the background motion of the dipolar fluid is geodesic, i.e. $\dot{\bar{u}}^\mu = 0$. This is now justified by the fact that the force (3.15) drives the nongeodesic motion *via* the equation of motion (2.21a); hence since this force vanishes in the background, the deviation from geodesic motion starts only at perturbation order.

In the present model the coefficients $\mathcal{W}_0$, $\mathcal{W}_2, \ldots$ of the expansion of our fundamental potential $\mathcal{W}(\Pi_\perp)$ are free parameters, and therefore will have to be measured by cosmological or astronomical observations. First of all, it is clear from inspection of the action (2.7), or from the general decomposition of the stress-energy tensor [see (2.27a) and (2.27b)], that $\mathcal{W}_0$ is nothing but a cosmological constant, and we find

$$\mathcal{W}_0 = \frac{\Lambda}{8\pi}. \tag{3.16}$$

The coefficient $\mathcal{W}_0$ is thereby determined by cosmological measurements of "dark energy." As we shall show in Sec. IV, the next two coefficients $\mathcal{W}_2$ and $\mathcal{W}_3$ will be fixed by requiring that our model reproduces the phenomenology of MOND at galactic scales [8], and we shall find that $\mathcal{W}_2 = 4\pi$ and $\mathcal{W}_3 = 32\pi^2/a_0$ where a_0 is the constant MOND acceleration scale.

Hence, in this model the cosmological constant Λ appears as the minimum value of the potential function $\mathcal{W}$, reached when the polarization field is exactly zero, that is, on an exact FLRW background (see Fig. 3). Thus, it is tempting to interpret Λ as a "vacuum polarization," i.e. the residual polarization which remains when the "classical" part of the polarization $\Pi_\perp \to 0$. Of course, our model is

[8]The domain of validity of this expansion will be made more precise in Sec. IV B.

only classical; hence there is no notion of vacuum polarization which would be due to quantum fluctuations. However, we can imagine that the present model is an effective one, describing at some macroscopic level a more fundamental underlying quantum field theory (QFT) in which there is a nonvanishing vacuum expectation value (VEV) of a quantum polarization field giving rise to the observed cosmological constant [38]. Then, the constant $\mathcal{W}_0$ would play the role of the VEV of this hypothetical quantum polarization field in such a more fundamental QFT.

D. Perturbation of the dipolar fluid equations

As for the four-velocity $u^\mu = \bar{u}^\mu + \delta u^\mu$, we consider a linear perturbation of the rest-mass energy density of the dipolar fluid according to $\sigma = \bar{\sigma} + \delta\sigma$. The conservation law $\nabla_\mu(\sigma u^\mu) = 0$ reduces in the case of the background to

$$\bar{\sigma}' + 3\mathcal{H}\bar{\sigma} = 0; \tag{3.17}$$

hence $\bar{\sigma}$ evolves like a^{-3}. Concerning the perturbation, we define $\sigma \equiv \bar{\sigma}(1+\varepsilon)$ so that the rest-mass density contrast reads

$$\varepsilon = \frac{\delta\sigma}{\bar{\sigma}}. \tag{3.18}$$

This quantity is not gauge invariant, and one can associate with it in the usual way a gauge-invariant variable by posing

$$\varepsilon_{\mathrm{F}} \equiv \varepsilon - \frac{\bar{\sigma}'}{\bar{\sigma}}\frac{C}{\mathcal{H}} = \varepsilon + 3C, \tag{3.19}$$

with the index F standing for "flat slicing." Alternatively, it is possible to introduce other gauge-invariant variables, like

$$\varepsilon_{\mathrm{N}} \equiv \varepsilon - 3\mathcal{H}(B - E') = \varepsilon_{\mathrm{F}} + 3\Psi, \tag{3.20}$$

where the index N stands for "Newtonian." For the linear perturbation, the conservation law $\nabla_\mu(\sigma u^\mu) = 0$ gives the gauge-invariant equations

$$\varepsilon'_{\mathrm{F}} + \Delta V = 0, \tag{3.21a}$$

$$\varepsilon'_{\mathrm{N}} + \Delta V = 3\Psi', \tag{3.21b}$$

where $\Delta = \gamma_{ij}D^i D^j$ denotes the usual Laplacian operator. In the following we shall choose to work only with the flat-slicing variable ε_{F}.

According to (2.21a), the motion of the dipolar fluid obeys the equation $\dot{u}^\mu = -\mathcal{F}^\mu$. A straightforward calculation yields the gauge-invariant expression for the four-acceleration,

$$\dot{u}^\mu = \frac{1}{a^2}(0, D^i(\Phi + V' + \mathcal{H}V) + V^{i\prime} + \mathcal{H}V^i). \tag{3.22}$$

On the other hand, the force is given by (3.15) at first order in the perturbation, in which we can use $\Pi^\mu_\perp = (0, \bar{\sigma}\lambda^i)$ to this order, with $\lambda^i = D^i y + y^i$. Hence, in terms of gauge-invariant quantities, the scalar and vector parts of the equation of motion read

$$V' + \mathcal{H}V + \Phi = -4\pi\bar{\sigma}a^2 y, \tag{3.23a}$$

$$V'_i + \mathcal{H}V_i = -4\pi\bar{\sigma}a^2 y_i. \tag{3.23b}$$

Here we are anticipating the results of Sec. IV and have replaced the constant $\mathcal{W}_2$ in the expression of the force (3.15) by its value 4π determined from the comparison with MOND predictions.

If there was no dipole moment (i.e. $y = y^i = 0$), we would recover the standard geodesic equations for a perturbed pressureless perfect fluid (see e.g. [33]), and according to (3.23b), the vector modes would satisfy $(aV_i)' = 0$, and therefore vanish like a^{-1}. In contrast with the standard perfect fluid case, the dipolar fluid may have nonvanishing vector modes because of the driving term proportional to y_i. Equation (3.23a) clearly shows that the scalar modes are also affected by a nonzero dipole moment.

The equation of evolution of the dipole moment was given by (2.21b). Now, Ω^μ reduces to $\dot{\xi}^\mu_\perp + u^\mu$ at first perturbation order; hence the evolution equation gives at that order

$$\ddot{\xi}^\mu_\perp + \dot{u}^\mu = -\xi^\nu_\perp \bar{R}^\mu{}_{\rho\nu\sigma}\bar{u}^\rho\bar{u}^\sigma, \tag{3.24}$$

where $\bar{R}^\mu{}_{\rho\nu\sigma}$ is the Riemann tensor of the FLRW background. By easy calculations we find for the derivatives of the dipole moment variable

$$\dot{\xi}^\mu_\perp = \frac{1}{a}(0, \lambda^{i\prime} + \mathcal{H}\lambda^i), \tag{3.25a}$$

$$\ddot{\xi}^\mu_\perp = \frac{1}{a^2}(0, \lambda^{i\prime\prime} + \mathcal{H}\lambda^{i\prime} + \mathcal{H}'\lambda^i). \tag{3.25b}$$

The scalar and vector parts of the equation of evolution are thus given by

$$y'' + \mathcal{H}y' = -(V' + \mathcal{H}V + \Phi), \tag{3.26a}$$

$$y''_i + \mathcal{H}y'_i = -(V'_i + \mathcal{H}V_i). \tag{3.26b}$$

Notice that the equation for the vector modes can be integrated, giving the simple relation

$$y'_i + V_i = \frac{s_i}{a}, \tag{3.27}$$

where s_i is an integration constant three-vector.

A comment is in order at this stage. Recall that we have included in the original Lagrangian (2.7) a mass term in the ordinary sense, with the most natural value of the mass density simply given by σ. This choice was made having in mind the physical analogy with the quasi-Newtonian model [25] where $\sigma = 2mn$ represented the inertial mass of the dipolar particles. Now we can see on a more technical level that such a mass term is, in fact, essential for the model to work properly. If this mass term was set to zero in the action, then the RHS of both Eqs. (3.26a) and (3.26b) would be zero. We would then find that y' and y'_i vanish like a^{-1}, so that the dipole moment would, in fact, rapidly disappear or at least become nondynamical, and the whole model would turn out to be meaningless.

Combining the equations of motion (3.23) and the evolution equations (3.26), we obtain some differential equations for the scalar and vector contributions y and y^i of the dipole moment $\lambda^i = D^i y + y^i$, which turn out to be decoupled from the equations giving V and V^i, and to be exactly the same, *viz*

$$y'' + \mathcal{H} y' - 4\pi\bar{\sigma}a^2 y = 0, \tag{3.28a}$$

$$y_i'' + \mathcal{H} y_i' - 4\pi\bar{\sigma}a^2 y_i = 0. \tag{3.28b}$$

We find it remarkable that the dipole moment decouples from the other perturbation variables so that its evolution depends *in fine* only on background quantities, namely, $\bar{\sigma}$ and the scale factor a. Since the equations for the scalar and vector modes are the same, we have also the same equation for the dipole moment itself,

$$\lambda_i'' + \mathcal{H}\lambda_i' - 4\pi\bar{\sigma}a^2\lambda_i = 0. \tag{3.29}$$

Clearly, the solutions of (3.29) behave typically as increasing and decreasing exponentials moderated by a cosmological damping term $\mathcal{H}\lambda_i'$. We can also write this equation in terms of the cosmic time $t = \int a d\eta$, namely,[9]

$$\ddot{\lambda}_i + 2H\dot{\lambda}_i - 4\pi\bar{\sigma}\lambda_i = 0, \tag{3.30}$$

where $H \equiv \dot{a}/a = a'$ is the usual Hubble parameter. We find that Eq. (3.29) or (3.30) is the same as the equation governing the growth of the density contrast of a perfect fluid with vanishing pressure for the sub-Hubble modes (say $k \gg H$) and when we neglect the contribution of other fluids; see (3.50) below. In particular, this means that, like for the case of the density of a perfect fluid, there is no problem of divergence (i.e. blowing up) of the components of the dipole moment λ_i between, say, the end of the inflationary era and the recombination. We can thus apply the theory of first-order cosmological perturbations even for the components of the dipole moment itself, which should stay perturbative.

Notice that the value of the coefficient $\mathcal{W}_2 = 4\pi$ used in (3.29) or (3.30), which makes such equations identical with the equation of growth of cosmological structures in the standard CDM scenario, will only be determined in Sec. IV from a comparison with MOND predictions. There is thus an interesting interplay between the cosmology at large scales and the gravitational physics of smaller scales.[10]

[9]In this equation, the dot stands for a derivative with respect to the coordinate time t, and not the proper time τ as everywhere else.

[10]Actually, the coefficient 4π in (3.29) could be changed if we had assumed a mass term in the action (2.7) different from σ (say 2σ or $\sigma/2$). The simplest choice we have made (for different reasons), that σ is the correct mass term in the action, corresponds also to the usual-looking evolution equation (3.29).

E. The perturbed stress-energy tensor

Consider next the stress-energy tensor of the dipolar fluid, which we decomposed as (2.25) with the expressions (2.27) and (2.28). At first perturbation order, these expressions reduce to

$$r = \mathcal{W}_0 + \rho, \tag{3.31a}$$

$$\mathcal{P} = -\mathcal{W}_0, \tag{3.31b}$$

$$Q^\mu = \frac{1}{a}(0, \bar{\sigma}\lambda^{i\prime}), \tag{3.31c}$$

$$\Sigma^{\mu\nu} = 0, \tag{3.31d}$$

together with

$$\rho = \bar{\sigma}(1 + \varepsilon - D_i\lambda^i). \tag{3.32}$$

We first note that part of the dipolar medium is actually made of a fluid of "dark energy" satisfying $\rho_{\rm de} = -P_{\rm de} = \mathcal{W}_0 = \Lambda/8\pi$ where Λ is the cosmological constant. Accordingly, we shall write the decomposition

$$T^{\mu\nu} = T^{\mu\nu}_{\rm de} + T^{\mu\nu}_{\rm dm}, \tag{3.33}$$

where the stress-energy tensor associated with the cosmological constant is denoted by $T^{\mu\nu}_{\rm de}$, and where the other part represents specifically a fluid of "dark matter" whose stress-energy tensor is $T^{\mu\nu}_{\rm dm}$. Their explicit expressions read

$$T^{\mu\nu}_{\rm de} = -\mathcal{W}_0 g^{\mu\nu}, \tag{3.34a}$$

$$T^{\mu\nu}_{\rm dm} = \rho u^\mu u^\nu + 2Q^{(\mu}u^{\nu)}. \tag{3.34b}$$

Note that the dark matter part of the dipolar fluid, which may be called dipolar dark matter, has no pressure P and no anisotropic stresses $\Sigma^{\mu\nu}$, but a heat flow Q^μ given by (3.31c) and an energy density ρ given by (3.32), or alternatively

$$\rho = \bar{\sigma}(1 + \varepsilon - \Delta y). \tag{3.35}$$

The background energy density is simply given by the background rest-mass energy density, $\bar{\rho} = \bar{\sigma}$, and the corresponding energy density contrast is

$$\delta \equiv \frac{\delta\rho}{\bar{\rho}} = \varepsilon - \Delta y. \tag{3.36}$$

It differs from the rest-mass energy density contrast ε because of the internal dipolar energy. Like for ε, one can construct several gauge-invariant perturbations associated with δ. We shall limit ourselves to the flat-slicing (F) one defined by (recall that y is gauge invariant)

$$\delta_{\rm F} \equiv \delta + 3C = \varepsilon_{\rm F} - \Delta y, \tag{3.37}$$

and whose evolution equation is

$$\delta_{\rm F}' + \Delta V + \Delta y' = 0. \tag{3.38}$$

Similar, gauge-invariant, density contrast variables are also defined for the other fluids. Next, we split the dark matter stress-energy tensor (3.34b) into a background part plus a linear perturbation, namely, $T^{\mu\nu}_{\rm dm} = \bar{T}^{\mu\nu}_{\rm dm} + \delta T^{\mu\nu}_{\rm dm}$, and find

$$\bar{T}^{\mu\nu}_{\rm dm} = \bar{\rho}\bar{u}^\mu\bar{u}^\nu, \tag{3.39a}$$
$$\delta T^{\mu\nu}_{\rm dm} = \delta\rho\bar{u}^\mu\bar{u}^\nu + 2\bar{\rho}\delta u^{(\mu}\bar{u}^{\nu)} + 2Q^{(\mu}\bar{u}^{\nu)}. \tag{3.39b}$$

We made use of the fact that the heat flow Q^μ is already perturbative to replace the four-velocity in the last term by its background value.

We are now going to show that the dipolar dark matter stress-energy tensor is undistinguishable, at linear perturbation order, from that of a perfect fluid with vanishing pressure. To this end, we introduce the effective perturbed four-velocity

$$\delta\tilde{u}^\mu \equiv \delta u^\mu + \frac{Q^\mu}{\bar{\rho}}. \tag{3.40}$$

Notice that $\tilde{u}^\mu = \bar{u}^\mu + \delta\tilde{u}^\mu$ is still an admissible velocity field because $\delta\tilde{u}^0 = -A/a$ by virtue of the transversality property $\bar{u}_\mu Q^\mu = 0$. The perturbed part of the dark matter stress-energy tensor (3.39b) can then be written in the form

$$\delta T^{\mu\nu}_{\rm dm} = \delta\rho\bar{u}^\mu\bar{u}^\nu + 2\bar{\rho}\delta\tilde{u}^{(\mu}\bar{u}^{\nu)}, \tag{3.41}$$

which, together with (3.39a), is precisely the stress-energy tensor of a perfect fluid with vanishing pressure P, vanishing anisotropic stresses $\Sigma^{\mu\nu}$, and a four-velocity field $\tilde{u}^\mu = \bar{u}^\mu + \delta\tilde{u}^\mu$. Using the definition (3.40) of the perturbed four-velocity $\delta\tilde{u}^\mu$, with the explicit expression of the heat flow (3.31c), one can check that this perfect fluid consistently follows a geodesic motion, i.e. $\delta\dot{\tilde{u}}^\mu = 0$.

More explicitly, we can write the latter effective perturbation of the four-velocity in the standard form $\delta\tilde{u}^\mu = a^{-1}(-A, \tilde{\beta}^i)$, and find that the effective ordinary velocity reads

$$\tilde{\beta}^{\,i} = \beta^i + \lambda^{i\prime}, \tag{3.42}$$

which can be viewed as a modification of the spacelike component of the dipolar dark matter four-velocity. This allows one to build a new four-velocity which would be tangent to the worldline of the effective perfect fluid (cf. Fig. 1). In terms of scalar and vector parts, if we write $\tilde{\beta}_i = D_i\tilde{v} + \tilde{v}_i$, then

$$\tilde{v} = v + y', \tag{3.43a}$$
$$\tilde{v}_i = v_i + y_i'. \tag{3.43b}$$

Like for the perturbed four-velocity δu^μ, we can introduce the gauge-invariant variables

$$\tilde{V} \equiv \tilde{v} + E' = V + y', \tag{3.44a}$$
$$\tilde{V}_i \equiv \tilde{v}_i + B_i = V_i + y_i'. \tag{3.44b}$$

In terms of the gauge-invariant variables $\tilde{V}$, $\tilde{V}_i$, and $\delta_{\rm F}$, the dipolar dark matter fluid equations (3.23) and (3.38) finally read

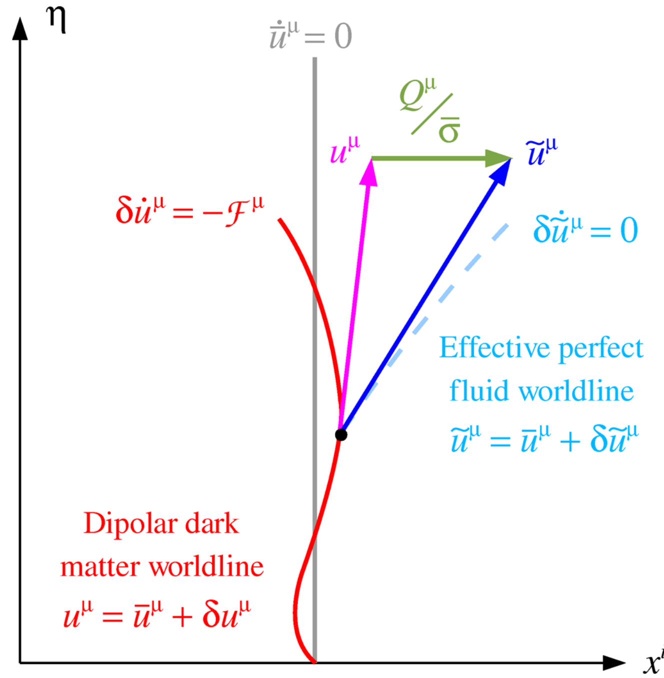

FIG. 1 (color online). Sketch of the equivalence at *first order* of cosmological perturbations between dipolar dark matter and an effective perfect fluid. The dipolar dark matter has a four-velocity $u^\mu = \bar{u}^\mu + \delta u^\mu$, and follows a nongeodesic motion driven by the internal force $\mathcal{F}^\mu$, namely, $\dot{u}^\mu = -\mathcal{F}^\mu$. One can construct from u^μ and the heat flux Q^μ an effective four-velocity $\tilde{u}^\mu = \bar{u}^\mu + \delta\tilde{u}^\mu$ satisfying a geodesic motion, i.e. $\dot{\tilde{u}}^\mu = 0$. This is the four-velocity field of the effective perfect fluid associated with dipolar dark matter.

$$\tilde{V}' + \mathcal{H}\tilde{V} + \Phi = 0, \tag{3.45a}$$
$$\tilde{V}_i' + \mathcal{H}\tilde{V}_i = 0, \tag{3.45b}$$
$$\delta_{\rm F}' + \Delta\tilde{V} = 0. \tag{3.45c}$$

These are precisely the standard evolution equations of a perfect fluid with no pressure and no anisotropic stresses (see e.g. [33]).

To summarize, we have proved that at *first order* of perturbation theory—and only at that order—the dipolar fluid behaves exactly as ordinary dark energy (i.e. a cosmological constant) plus ordinary dark matter (i.e. a perfect fluid). If we specify the background rest-mass energy density $\bar{\sigma}$ so that $\Omega_{\rm dm} \equiv 8\pi\bar{\sigma}_0/3H_0^2 \simeq 0.23$ today as evidenced in cosmological observations, we can assert that the first-order cosmological perturbation theory with the dipolar fluid described by the stress-energy tensor (3.33) and (3.34) will lead to the same predictions as the standard Λ-CDM scenario—and is therefore consistent with cosmological observations at large scales. However, at *second order* of cosmological perturbations, the dipole moment entering the stress-energy tensor cannot be absorbed in an effective perturbed velocity field, which means that the dipolar dark matter fluid could, in principle, be distinguished from a standard perturbed dark matter fluid.

Working out the theory of second-order cosmological perturbations could thus yield distinctive features of the present model and reveal a signature of the dipolar nature of dark matter. We particularly have in mind effects linked with the non-Gaussianity of the CMB fluctuations that are associated with second-order perturbations.

F. Perturbation of the Einstein equations

The Einstein equations at first perturbation order around the FLRW background read

$$\delta G^{\mu\nu} = 8\pi\left(\delta T^{\mu\nu} + \sum_{\mathrm{f}} \delta T_{\mathrm{f}}^{\mu\nu}\right), \qquad (3.46)$$

where $G^{\mu\nu} \equiv R^{\mu\nu} - \frac{1}{2} g^{\mu\nu} R$ is the Einstein tensor and where $\delta T^{\mu\nu} = \delta T_{\mathrm{de}}^{\mu\nu} + \delta T_{\mathrm{dm}}^{\mu\nu}$ is the perturbative part of the stress-energy tensor of the dipolar fluid given by (3.34). The summation runs over all the other cosmological fluids present (baryons, photons, neutrinos, …) which are described by stress-energy tensors $T_{\mathrm{f}}^{\mu\nu}$. Separating out the dark matter from the dark energy (using the link $\mathcal{W}_0 = \Lambda/8\pi$), we get

$$\delta G^{\mu\nu} + \Lambda \delta g^{\mu\nu} = 8\pi\left(\delta T_{\mathrm{dm}}^{\mu\nu} + \sum_{\mathrm{f}} \delta T_{\mathrm{f}}^{\mu\nu}\right). \qquad (3.47)$$

As we have seen in the previous section, the dark matter fluid is entirely described at linear perturbation order by the gauge-invariant variables $\tilde{V}$, $\tilde{V}_i$, and δ_{F} (and the background density $\bar{\rho}$) obeying the evolution equations (3.45) like for an ordinary pressureless fluid. We can thus immediately write the gauge-invariant perturbation equations in the standard SVT formalism (see e.g. [33]). Though these are well known, we reproduce them here for completeness. For the scalar modes, we have

$$\Delta\Psi - 3\mathcal{H}^2 X = 4\pi a^2\left(\bar{\rho}\,\delta_{\mathrm{F}} + \sum_{\mathrm{f}} \bar{\rho}_{\mathrm{f}} \delta_{\mathrm{f}}^{\mathrm{F}}\right), \qquad (3.48a)$$

$$\Psi - \Phi = 8\pi a^2 \sum_{\mathrm{f}} \bar{\rho}_{\mathrm{f}} w_{\mathrm{f}} \sigma_{\mathrm{f}}, \qquad (3.48b)$$

$$\Psi' + \mathcal{H}\Phi = -4\pi a^2\left(\bar{\rho}\,\tilde{V} + \sum_{\mathrm{f}} \bar{\rho}_{\mathrm{f}}(1 + w_{\mathrm{f}}) V_{\mathrm{f}}\right), \qquad (3.48c)$$

$$\mathcal{H} X' + (\mathcal{H}^2 + 2\mathcal{H}')X = 4\pi a^2 \sum_{\mathrm{f}} \bar{\rho}_{\mathrm{f}}\left(w_{\mathrm{f}}\Gamma_{\mathrm{f}} + c_{\mathrm{f}}^2 \delta_{\mathrm{f}}^{\mathrm{F}} + \frac{2}{3} w_{\mathrm{f}} \Delta\sigma_{\mathrm{f}}\right), \qquad (3.48d)$$

where we have singled out the contribution of the dipolar dark matter (cf. the variables $\tilde{V}$, δ_{F}, and $\bar{\rho}$) from the other fluid contributions described by their background density $\bar{\rho}_{\mathrm{f}}$, equation of state w_{f}, adiabatic sound velocity c_{f}, and gauge-invariant entropy perturbation Γ_{f}. We also introduced the SVT components of the perturbative part of the anisotropic stress tensor, defined by $\delta\Sigma_{\mathrm{f}}^{ij} = a^2 \bar{\rho}_{\mathrm{f}} w_{\mathrm{f}} [\Delta^{ij}\sigma_{\mathrm{f}} + D^{(i}\sigma_{\mathrm{f}}^{j)} + \sigma_{\mathrm{f}}^{ij}]$ with $\Delta^{ij} \equiv D^i D^j - \gamma^{ij}\Delta/3$. The variables σ_{f}, σ_{f}^i, and σ_{f}^{ij} are gauge invariant because the background part of Σ_{f}^{ij} vanishes in the case of a perfect fluid. The equations for the vector and tensor modes are

$$(\Delta + 2K)\Phi^i = -16\pi a^2\left(\bar{\rho}\tilde{V}^i + \sum_{\mathrm{f}} \bar{\rho}_{\mathrm{f}} \times (1 + w_{\mathrm{f}}) V_{\mathrm{f}}^i\right), \qquad (3.49a)$$

$$\Phi^{i\prime} + 2\mathcal{H}\Phi^i = 8\pi a^2 \sum_{\mathrm{f}} \bar{\rho}_{\mathrm{f}} w_{\mathrm{f}} \sigma_{\mathrm{f}}^i, \qquad (3.49b)$$

$$E^{ij\prime\prime} + 2\mathcal{H}E^{ij\prime} + (2K - \Delta)E^{ij} = 8\pi a^2 \sum_{\mathrm{f}} \bar{\rho}_{\mathrm{f}} w_{\mathrm{f}} \sigma_{\mathrm{f}}^{ij}. \qquad (3.49c)$$

We highlight once more the fact that at first perturbation order, the dipolar dark matter is like ordinary dark matter, as can be seen from the fluid equations (3.45) and the Einstein equations (3.48) and (3.49). Indeed, these sets of equations can be evolved without any reference to the dipole moment λ^i.

Combining the dipolar dark matter equations (3.45a) and (3.45c) with the Einstein equations (3.48a) and (3.48b), we get the equation governing the growth of the dipolar dark matter density contrast as

$$\delta_{\mathrm{F}}'' + \mathcal{H}\delta_{\mathrm{F}}' - 4\pi\bar{\rho}a^2\delta_{\mathrm{F}} = 3\mathcal{H}^2 X + 4\pi a^2 \times \sum_{\mathrm{f}} \bar{\rho}_{\mathrm{f}}(\delta_{\mathrm{f}}^{\mathrm{F}} - 2w_{\mathrm{f}}\Delta\sigma_{\mathrm{f}}). \qquad (3.50)$$

Again, we find that the growth of structures driven by Eq. (3.45c) or equivalently (3.50) for the dipolar dark matter of the present model is identical with that in the standard CDM model at linear perturbation order. For sub-Hubble modes one can neglect the first term in the RHS, and we expect that the contribution of the dark matter dominates that of the other fluids, so we can neglect also the second term in the RHS of (3.50).

Interestingly, we have found in (3.29) that each of the components of the dipole moment obeys the same equation as (3.50) but with a RHS of exactly zero. Recall that the dipolar dark matter density contrast is defined by (3.37) as

$$\delta_{\mathrm{F}} = \varepsilon_{\mathrm{F}} - D^i\lambda_i. \qquad (3.51)$$

From (3.29) we see that the internal energy due to the dipole moment satisfies the "homogeneous" equation that is associated with (3.50), *viz* (recalling $\bar{\rho} = \bar{\sigma}$)

$$D^i\lambda_i'' + \mathcal{H} D^i\lambda_i' - 4\pi\bar{\rho}a^2 D^i\lambda_i = 0. \tag{3.52}$$

This result indicates that, in the *nonlinear* regime, the internal energy related to the dipole moment may contribute significantly to the growth of perturbations (see Sec. IV B for more comments). Finally, it is clear that the rest-mass density contrast obeys the same "inhomogeneous" equation, i.e.

$$\varepsilon_{\rm F}'' + \mathcal{H}\varepsilon_{\rm F}' - 4\pi\bar{\rho}a^2\varepsilon_{\rm F} = 3\mathcal{H}^2 X + 4\pi a^2 \times \sum_{\rm f}\bar{\rho}_{\rm f}(\delta_{\rm f}^{\rm F} - 2w_{\rm f}\Delta\sigma_{\rm f}). \tag{3.53}$$

IV. DIPOLAR DARK MATTER AT GALACTIC SCALES

In this section, we shall show that, under some well-motivated assumptions, the dipolar dark matter naturally recovers the phenomenology of MOND for a typical galaxy at low redshift. Such a link between a form of dipolar dark matter and MOND was the primary motivation of previous works [25,26]. We shall see that with the present improvement of the model with respect to [26], thanks to the fact that the fundamental potential in the action now depends on the polarization field $\Pi_\perp = \sigma\xi_\perp$ (instead of $\xi_\perp$ in the previous model [26]), the relation with MOND is straightforward and physically appealing.

A. Nonrelativistic limit of the model

We investigate the NR limit of the dipolar fluid dynamics described by Eqs. (2.21a) and (2.21b), and by the stress-energy tensor (2.24). To do so, we consider the formal limit $c \to +\infty$,[11] which is equivalent to the condition $v \ll c$, where v is the typical value of the coordinate three-velocity of the dipolar fluid. To consistently keep track of the order of relativistic corrections, we systematically write as $\mathcal{O}(c^{-n})$ a typical neglected remainder.

We are interested in the dynamics of dipolar dark matter and ordinary baryonic matter in a typical galaxy at low redshift. Let us introduce a *local* Cartesian coordinate system $\{ct, z^i\}$, centered on this galaxy around some cosmological epoch, and which is *inertial* in the sense that it is without any rotation, or acceleration with respect to some averaged cosmological matter distribution at large distances from the galaxy. Such a local coordinate system can be derived from the cosmological coordinate system $\{\eta, x^i\}$ used in Sec. III by posing

[11] From now on, we reintroduce for convenience all factors of c and G.

$$ct = a(\eta_0)(\eta - \eta_0), \tag{4.1a}$$

$$z^i = a(\eta_0)(x^i - x_0^i), \tag{4.1b}$$

near an event occurring at cosmological time η_0 and at the galaxy's center x_0^i. In the local coordinate system, the metric developed at the lowest NR order reads

$$g_{00} = -1 + \frac{2U}{c^2} + \mathcal{O}(c^{-4}), \tag{4.2a}$$

$$g_{0i} = \mathcal{O}(c^{-3}), \tag{4.2b}$$

$$g_{ij} = \left(1 + \frac{2U}{c^2}\right)\delta_{ij} + \mathcal{O}(c^{-4}), \tag{4.2c}$$

where $U \ll c^2$ is a Newtonian-like potential. For the motion of massive (nonrelativistic) particles we need only to include the contribution of U in the 00 metric coefficient. Thanks to the standard general relativistic coupling to gravity in the ij metric coefficient, the motion of photons agrees with the general relativistic prediction with the Newtonian-like potential U.

In the NR limit, the equation of motion (2.21a) is readily seen to reduce to

$$\frac{{\rm d}v^i}{{\rm d}t} - g^i = -\hat{\Pi}_\perp^i \mathcal{W}_{\Pi_\perp} + \mathcal{O}(c^{-2}), \tag{4.3}$$

where $a^i \equiv {\rm d}v^i/{\rm d}t = (\partial_t + v^j\partial_j)v^i$ is the standard Newtonian acceleration of a fluid in the Eulerian picture, v^i being the coordinate three-velocity, and $g^i = \partial_i U$ the nonrelativistic local gravitational field. Note that g^i is generated by both the ordinary baryonic matter and the dipolar dark matter. Similarly, the equation of evolution (2.21b) for the dipole moment reads in the NR limit [using also (4.3)]

$$\frac{{\rm d}^2\xi_\perp^i}{{\rm d}t^2} - \hat{\Pi}_\perp^i \mathcal{W}_{\Pi_\perp} = \frac{1}{\sigma}\partial_i(\mathcal{W} - \Pi_\perp \mathcal{W}_{\Pi_\perp}) + \xi_\perp^j\partial_j g^i + \mathcal{O}(c^{-2}), \tag{4.4}$$

where we explicitly have ${\rm d}^2\xi_\perp^i/{\rm d}t^2 = (\partial_t^2 + a^j\partial_j + 2v^j\partial_{jt}^2 + v^jv^k\partial_{jk}^2)\xi_\perp^i$. Notice the second term in the RHS which is a tidal term coming from the Riemann curvature coupling in (2.21b). Finally, Eq. (2.1) reduces to the classical continuity equation

$$\partial_t\sigma + \partial_i(\sigma v^i) = \mathcal{O}(c^{-2}). \tag{4.5}$$

Next, we need to be cautious about the relativistic order of magnitude of the potential function $\mathcal{W}$ appearing in the Lagrangian (2.7). It is clear that $\mathcal{W}$ has the dimension either of a mass density or an energy density, depending on where we would reinstall the factors c in (2.7). We shall, from now on, assume that $\mathcal{W}$ is an *energy* density, and has a finite nonzero limit when $c \to +\infty$. This will be justified when we show in (4.22) below that the coefficients $\mathcal{W}_2$, $\mathcal{W}_3$, ... in the expansion of $\mathcal{W}$ considered as an energy density can be expressed solely in terms of G and the MOND acceleration a_0 (without any c's). Therefore, our assumption means that we are viewing a_0 as a new fundamental acceleration scale *a priori* independent from c. With such a hypothesis, if we reintroduce the factors of c in the expression of the density r considered as a *mass* density and given by (2.27a), we get $r = \rho + (\mathcal{W} - \Pi_\perp \mathcal{W}_{\Pi_\perp})/c^2$, where ρ is given by (2.28). Thus, the term $(\mathcal{W} - \Pi_\perp \mathcal{W}_{\Pi_\perp})/c^2$ becomes negligible in the formal limit $c \to +\infty$, and we have $r = \rho + \mathcal{O}(c^{-2})$. In particular, we observe that the term $\mathcal{W}_0$, which is linked to the cosmological constant by (restoring the c's and G)

$$\mathcal{W}_0 = \frac{\Lambda c^4}{8\pi G}, \tag{4.6}$$

does not enter the expression of the dipolar fluid density r, and therefore has no influence on the local dynamics of the dipolar dark matter in the NR limit. Our assumption that $\mathcal{W}$ has a finite nonzero limit when $c \to +\infty$ means that the cosmological constant Λ should scale with c^{-4}, which will be justified later when we show that $\Lambda \sim a_0^2/c^4$.

Thus, in the NR limit we need to consider only the mass density of the dipolar dark matter given by ρ. Now, from (2.28) we have $\rho = \sigma - \nabla_\lambda \Pi^\lambda_\perp$ which becomes, when $c \to +\infty$,

$$\rho = \sigma - \partial_i \Pi^i_\perp + \mathcal{O}(c^{-2}). \tag{4.7}$$

At that order the dipolar term involves only an ordinary partial space derivative. Finally, we get the Poisson equation in the standard way as the NR limit of the 00 and ii components of the Einstein equations, and find

$$\Delta U = -4\pi G(\rho_b + \sigma - \partial_i \Pi^i_\perp) + \mathcal{O}(c^{-2}), \tag{4.8}$$

where ρ_b is the Newtonian mass density of baryonic matter. This equation can be written in the alternative form

$$\partial_i(g^i - 4\pi G \Pi^i_\perp) = -4\pi G(\rho_b + \sigma) + \mathcal{O}(c^{-2}). \tag{4.9}$$

To summarize, the equations governing the dynamics of the dipolar dark matter and the gravitational field in the NR limit are as follows: the equation of motion (4.3), the evolution equation (4.4), the continuity equation (4.5), and the Poisson equation (4.9). On the other hand, baryons and photons obey the geodesic equation, which means $\mathrm{d}v_b^i/\mathrm{d}t = \partial_i U + \mathcal{O}(c^{-2})$ for baryons, and the standard GR formula for light deflection in a potential U for photons, where U is generated by (4.8).

B. The weak clustering hypothesis

We have shown in Sec. III that at linear perturbation order, in a cosmological context, the dynamics of dipolar dark matter cannot be distinguished from that of baryonic matter or standard dark matter. We now argue that, with the motion of dipolar dark matter being nongeodesic, its *nonlinear* dynamics should be different. Our main motivation for the argument is the existence of an *exact* solution of the equations governing the dynamics of the dipolar dark matter in the NR limit. Indeed, we show in the Appendix that, in the simple case where the baryonic matter is modeled by a spherically symmetric mass distribution, there is a solution to the equations for which the dipole moments are *in equilibrium* ($\xi_\perp = \text{const}$), and *at rest* ($v^i = 0$), with the internal force $\mathcal{F}^i$ exactly balancing the gravitational field g^i. In such a solution, the dipolar medium is uniformly distributed or, more generally, spherically symmetrically distributed, and the polarization $\Pi^i_\perp$ is aligned with the gravitational field g^i; the dipolar fluid is thus polarized. Furthermore, we show in the Appendix that the latter solution is stable against dynamical perturbations.

From that solution, we expect that the dipolar medium will not cluster much during the cosmological evolution because the internal force may balance part of the local gravitational field generated by an overdensity (see Fig. 2 for a pictorial view of the argument). From this we infer that the dark matter density contrast in a typical galaxy at low redshift should be small, at least smaller than in the standard Λ-CDM scenario. Such a galaxy would therefore be essentially baryonic, with a typical mass density of the dipolar dark matter σ rather small compared to the baryonic one, and perhaps around its mean cosmological value $\bar{\sigma}$. Thus, the crucial hypothesis we are making (based on the solution in the Appendix) is that

$$\sigma \ll \rho_b, \tag{4.10}$$

or perhaps that σ stays essentially at a value of the order of its mean cosmological value,

$$\sigma \sim \bar{\sigma} \ll \rho_b. \tag{4.11}$$

Note that for standard CDM (or baryonic matter), the density contrast between the value of $\rho_{\rm cdm}$ (or ρ_b) in a galaxy and the mean cosmological one $\bar{\rho}_{\rm cdm}$ (or $\bar{\rho}_b$) is typically of order 10^5. This means that even if dipolar dark matter clusters enough so that, for instance, $\sigma \sim 10^3 \bar{\sigma}$ in a galaxy at low redshift, it would still satisfy the condition (4.10).

Note also that with this hypothesis, the nonlinear growth of structures in our model will not be triggered by the rest mass σ of dipolar dark matter (since it does not cluster much), but by the *internal* energy $\rho_{\rm int}$ of the dipolar medium, which is such that $\rho = \sigma + \rho_{\rm int}$ and is explicitly given by $\rho_{\rm int} = -\nabla_\lambda \Pi^\lambda_\perp$ [recall (2.28)]. We have seen that, at first cosmological perturbation order, the density contrast associated with $\rho_{\rm int}$ reduces to $-D^i\lambda_i$, and obeys the standard evolution equation (3.52). We expect that at nonlinear order it will take over the dominant role as compared to the rest-mass density contrast ε in the formation of structures. On the other hand, in the NR limit $\rho_{\rm int}$ reduces to $-\partial_i \Pi^i_\perp$ [see (4.7)] and, as we shall see in the following section, will be at the origin of the MOND effect.

We shall refer to the condition (4.10) [or even to the stronger condition (4.11)] as the hypothesis of *weak clus-*

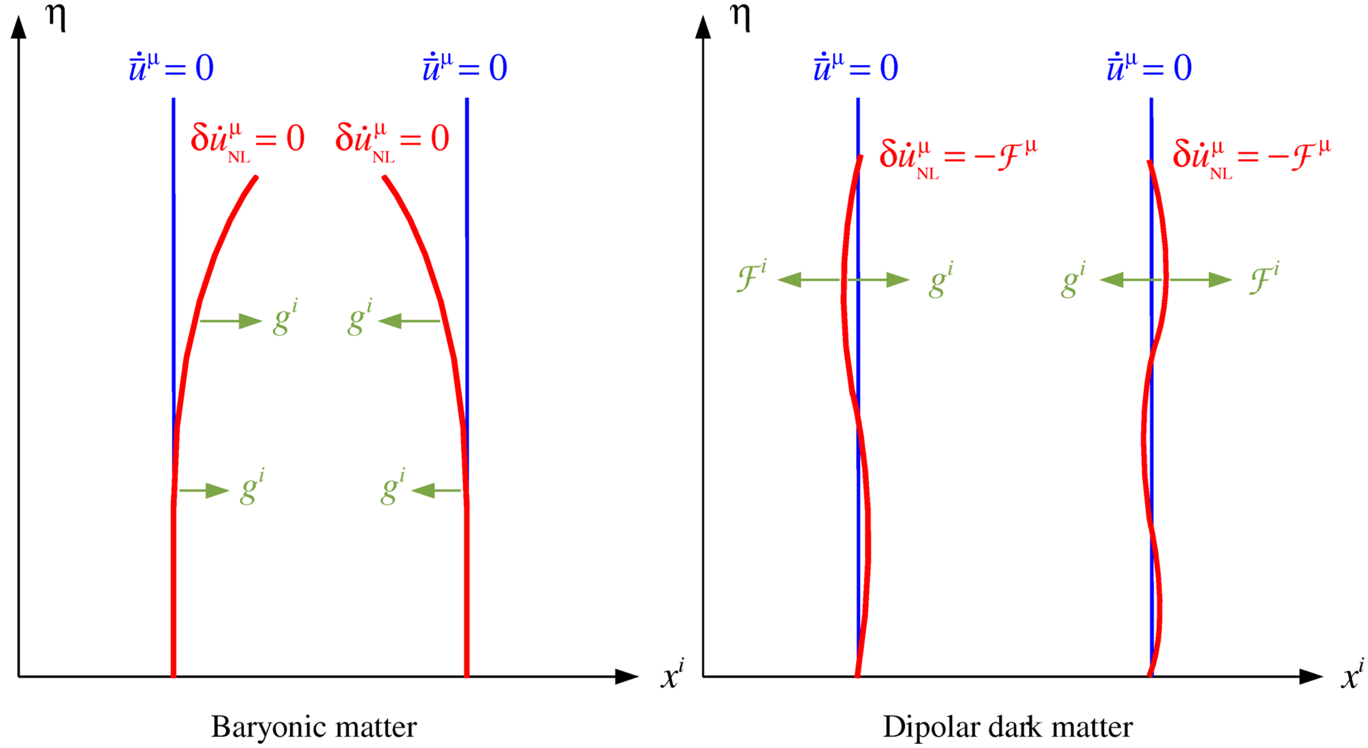

FIG. 2 (color online). Schematic view of two worldlines of baryonic matter and dipolar dark matter. The baryonic matter follows a geodesic motion, $\dot{u}^\mu = 0$, and therefore collapses in the regions of overdensity. Obeying the nongeodesic equation of motion $\dot{u}^\mu = -\mathcal{F}^\mu$, the dipolar dark matter is expected to have a different behavior in the *nonlinear* (NL) regime. Namely, the internal force $\mathcal{F}^i$ can balance the gravitational field g^i created by an overdensity, in order to keep the rest-mass density of dipolar dark matter close to its mean cosmological value, $\sigma \sim \bar{\sigma}$, or at least far smaller than the baryonic one.

tering of the dipolar dark matter fluid. Obviously, the validity of this hypothesis cannot be addressed with the formalism of first-order cosmological perturbations in Sec. III, because it is a consequence of the nonlinear cosmological evolution. The hypothesis of weak clustering of dipolar dark matter should be validated through numerical N-body simulations.

Let us thus assume that the dipolar dark matter has not clustered very much, and even that σ might stay more or less at the cosmological mean value $\bar{\sigma}$ (such that $\Omega_{\rm dm} \simeq 0.23$). Because of its size and typical time scale of evolution, a galaxy is almost unaffected by the cosmological expansion of the Universe. Therefore, the cosmological mass density $\bar{\sigma}$ of the dipolar dark matter is not only homogeneous, but also almost constant in this galaxy. Thus, the continuity equation (4.5) reduces to $\partial_i(\bar{\sigma} v^i) \simeq 0$. The most simple solution obviously corresponds to a static fluid verifying

$$v^i \simeq 0. \tag{4.12}$$

It is therefore natural to consider that the dipolar dark matter is almost at rest with respect to some averaged cosmological matter distribution. This is supported by the exact solution found in the Appendix, which indicates that the dipolar dark matter in the presence of an ordinary mass does indeed behave essentially like a static medium. Because of the internal force, the motion is not geodesic, and the force acts like a "rocket" to compensate the gravitational field and to keep the dipolar particle at rest with respect to ordinary matter (see Fig. 2).

C. Link with the phenomenology of MOND

Let us now show that under the weak clustering hypothesis, Eqs. (4.3), (4.4), (4.5), and (4.9) naturally reproduce the phenomenology of MOND. First of all, if (4.12) holds, Eq. (4.3) tells us that the polarization $\Pi^i_\perp$ should be aligned with the local gravitational field g^i, namely,[12]

$$g^i = \hat{\Pi}^i_\perp \mathcal{W}_{\Pi_\perp}. \tag{4.13}$$

This proportionality relation will be the crucial ingredient for recovering MOND.

We must now further specify the fundamental potential $\mathcal{W}$ entering the original action (2.7). In Sec. III, we considered the dipolar fluid at early cosmological times, where the polarization field was perturbative. We shall now consider it at late cosmological times (around the value η_0)

[12]From now on, we no longer indicate the neglected remainder terms $\mathcal{O}(c^{-2})$. Furthermore, we assume for the discussion that (4.12) is exactly verified, i.e. $v^i = 0$.

but still in a regime where the polarization field is weak. This will correspond to the outer zone of a galaxy at low redshift, where the local gravitational field generated by the galaxy is weak. We therefore assume that the potential $\mathcal{W}$ can still be expanded in powers of $\Pi_\perp$, and we keep only a few terms in the expansion. Next, we introduce a fundamental acceleration scale a_0 to later be identified with the MOND constant acceleration whose commonly accepted value is $a_0 \simeq 1.2 \times 10^{-10}$ m/s^2 [8]. Associated with a_0 we can define a fundamental surface density scale

$$\Sigma \equiv \frac{a_0}{2\pi G}, \tag{4.14}$$

whose numerical value is $\Sigma \simeq 0.3\ \mathrm{kg/m^2} \simeq 130 M_\odot/\mathrm{pc}^2$. The numerical value of Σ is close to the observed upper limit of the surface brightness of spiral galaxies—the so-called Freeman's law which is seen as empirical evidence for MOND [8]. We now assert that the expansion of $\mathcal{W}$ when $\Pi_\perp \to 0$ is physically valid when the condition $\Pi_\perp \ll \Sigma$ is satisfied. As will become obvious, this condition can equivalently be written $g \ll a_0$, where $g = |g^i|$ is the norm of the local gravitational field of the galaxy, and this will correspond to the deep MOND regime (see Fig. 3). With respect to the expansion (3.14) already used in cosmology, we shall be able to add an extra term. We now write this expansion, for $\Pi_\perp \ll \Sigma$, as

$$\mathcal{W}(\Pi_\perp) = \mathcal{W}_0 + \frac{1}{2}\mathcal{W}_2\Pi_\perp^2 + \frac{1}{6}\mathcal{W}_3\Pi_\perp^3 + \mathcal{O}\left[\left(\frac{\Pi_\perp}{\Sigma}\right)^4\right]. \tag{4.15}$$

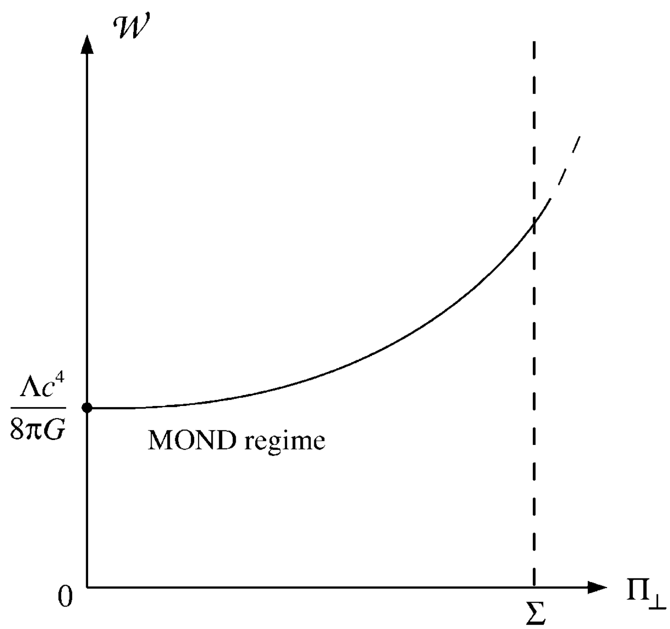

FIG. 3. The minimum of the potential function $\mathcal{W}(\Pi_\perp)$, reached when $\Pi_\perp = 0$, is a cosmological constant Λ. Small deviations around the minimum, corresponding to $\Pi_\perp \ll \Sigma = a_0/2\pi G$, describe the MOND regime $g \ll a_0$.

The first term $\mathcal{W}_0$ is related to the cosmological constant Λ through (4.6). We now show that the next two coefficients $\mathcal{W}_2$ and $\mathcal{W}_3$ are uniquely determined if we want to recover the phenomenology of MOND. Indeed, by inserting (4.15) into the relation (4.13) we obtain

$$g^i = \Pi_\perp^i\left\{\mathcal{W}_2 + \frac{1}{2}\mathcal{W}_3\Pi_\perp + \mathcal{O}\left[\left(\frac{\Pi_\perp}{\Sigma}\right)^2\right]\right\}, \tag{4.16}$$

which can be inverted to yield the polarization as an expansion in powers of (the norm of) the gravitational field. Anticipating that $\mathcal{W}_2\Sigma \sim a_0$, this expansion will be valid whenever $g \ll a_0$. We obtain

$$\Pi_\perp^i = \frac{g^i}{\mathcal{W}_2}\left\{1 - \frac{1}{2}\frac{\mathcal{W}_3}{\mathcal{W}_2^2}g + \mathcal{O}\left[\left(\frac{g}{a_0}\right)^2\right]\right\}. \tag{4.17}$$

Next, following the conventions of [25,26], we introduce the coefficient of "*gravitational susceptibility*" χ of the dipolar medium through

$$\Pi_\perp^i = -\frac{\chi}{4\pi G}g^i. \tag{4.18}$$

Inserting that definition[13] into the LHS of the Poisson equation (4.9), we find

$$\partial_i[(1+\chi)g^i] = -4\pi G(\rho_\mathrm{b} + \sigma). \tag{4.19}$$

Finally, invoking our hypothesis of weak clustering (4.10), or (4.11) in the more extreme variant, we can neglect the mass density σ of the dipole moments with respect to the baryonic one, so we obtain the MOND equation which is generated solely by the distribution of baryonic matter as [39]

$$\partial_i(\mu g^i) = -4\pi G\rho_\mathrm{b}. \tag{4.20}$$

The MOND function μ is related to the susceptibility coefficient by $\mu = 1 + \chi$ and can actually be interpreted as the "digravitational" coefficient of the dipolar medium [25]. Again, let us stress that in this model we do have some distribution of dark matter σ in an ordinary sense, but we expect its contribution to become negligible in galactic halos at low redshifts (after cosmological evolution), so that the MOND fit of rotation curves of galaxies is unaffected by this "monopolar" dark matter.[14] The MOND effect is due to the dipolar part of the dark matter given by the internal energy $\rho_\mathrm{int} = -\partial_i\Pi_\perp^i$.

[13]Note that this definition is valid in both MOND and Newtonian regimes whenever the polarization is aligned with the gravitational field.

[14]However, at the larger scale of clusters of galaxies the monopolar part of the dipolar medium σ may play a role to explain the missing dark matter in MOND estimates of the dynamical mass [8,40]. Note that in the present model, the motion of photons, needed to interpret weak-lensing experiments, is given by the standard general relativistic prediction; see (4.2) with the potential U solution of the MOND equation (4.20).

Now, from astronomical observations we know that the gravitational susceptibility χ in the deep MOND regime $g \ll a_0$ should behave like

$$\chi = -1 + \frac{g}{a_0} + \mathcal{O}\left(\left[\frac{g}{a_0}\right]^2\right). \tag{4.21}$$

The fact that χ should be *negative* was interpreted in the quasi-Newtonian model [25] as evidence for gravitational polarization—the gravitational analogue of the electric polarization in dielectric media. By inserting (4.21) into (4.18), and comparing with the prediction of our model as given by (4.17), we uniquely fix the unknown coefficients therein as

$$\mathcal{W}_2 = 4\pi G, \tag{4.22a}$$

$$\mathcal{W}_3 = 32\pi^2\frac{G^2}{a_0}. \tag{4.22b}$$

This, together with $\mathcal{W}_0$ fixed by (4.6), determines the potential function $\mathcal{W}$ up to third order from astronomical observations. As we see, the MOND acceleration a_0 enters at third order in the expansion, and therefore does not show up in the linear cosmological perturbations of Sec. III. At third order, the potential $\mathcal{W}$ deviates from a purely harmonic potential, and a_0 can be seen as a measure of its anharmonicity.

To express $\mathcal{W}$ in the best way, we prefer using the surface density scale $\Sigma = a_0/2\pi G$ rather than the acceleration scale a_0. To do so, we must introduce a purely numerical dimensionless coefficient α to express the cosmological constant Λ (which is positive and has the dimension of an inverse length squared) in units of a_0^2/c^4, and we pose

$$\Lambda = 3\alpha^2\left(\frac{2\pi a_0}{c^2}\right)^2. \tag{4.23}$$

The definition of α is such that $a_\Lambda = \alpha a_0$ represents the natural acceleration scale associated with the cosmological constant, and is already given by (1.1) as $a_\Lambda = \sqrt{\Lambda/3}c^2/2\pi$. Then, the cosmological term (4.6) becomes $\mathcal{W}_0 = 6\pi^3 G\Sigma^2\alpha^2$, and we obtain

$$\mathcal{W} = 6\pi G\Sigma^2\left\{\alpha^2\pi^2 + \frac{1}{3}\left(\frac{\Pi_\perp}{\Sigma}\right)^2 + \frac{4}{9}\left(\frac{\Pi_\perp}{\Sigma}\right)^3 + \mathcal{O}\left[\left(\frac{\Pi_\perp}{\Sigma}\right)^4\right]\right\}. \tag{4.24}$$

In the present model there is nothing which can give the relation between Λ and a_0; hence α is not determined. However, if the dipolar fluid action (2.7) is intended to describe at some macroscopic level a more fundamental theory (presumably a QFT), we expect that the potential $\mathcal{W}$ should depend only on certain more or less fundamental constants, and some dimensionless variables built from "fundamental fields." Introducing the dimensionless quantity $x \equiv \Pi_\perp/\Sigma$, we can rewrite (4.24) as $\mathcal{W} = 6\pi G\Sigma^2 w(x)$, where

$$w(x) = \alpha^2\pi^2 + \tfrac{1}{3}x^2 + \tfrac{4}{9}x^3 + \mathcal{O}(x^4) \tag{4.25}$$

represents some universal function coming from some fundamental albeit unknown physics. Therefore, we expect that the numerical coefficients in the expansion of $w(x)$ should be of the order of 1 or, say, 10. In particular, it is natural to expect that α should be of the order of 1 (to within a factor 10 say), and we deduce from (4.23) that the magnitude of Λ should scale approximately with the square of the MOND acceleration, namely, $\Lambda \sim a_0^2/c^4$.

The numerical coincidence between the measured values of Λ and a_0 is well known [16]. The observed value of the cosmological constant is around $\Lambda \simeq 0.12\ \mathrm{Gpc}^{-2}$ [33] which, together with $a_0 \simeq 1.2 \times 10^{-10}\ \mathrm{m/s^2}$, corresponds to a value for α which is very close to 1: $\alpha \simeq 0.8$. Thus a_0 is very close to the scale a_Λ associated with the cosmological constant, which is related to the Gibbons-Hawking temperature $T_{\rm GH} = \hbar a_\Lambda/kc$ derived from semiclassical theory on de Sitter space-time [41]. From the previous discussion, we see that the "cosmic" coincidence between Λ and a_0 has a natural explanation if dark matter is made of a medium of dipole moments.

D. The Newtonian regime

For the moment, we look at the explicit expression of the potential function $\mathcal{W}$ in the MOND regime $g \ll a_0$. We would also like to get some information about this function in the Newtonian regime $g \gg a_0$. To do so, we first derive the general expression of the gravitational susceptibility coefficient χ. Here we assume that the MOND function $\mu = 1 + \chi$ is always less than 1. This implies $\chi < 0$, and thus using (4.13) and (4.18) we must have $\mathcal{W}_{\Pi_\perp} > 0$ (where we recall that $\mathcal{W}_{\Pi_\perp} \equiv d\mathcal{W}/d\Pi_\perp$). Taking the norm of (4.13) we get $g = \mathcal{W}_{\Pi_\perp}(\Pi_\perp)$. Next, we introduce the function $\Theta(g)$ which is the inverse of $\mathcal{W}_{\Pi_\perp}(\Pi_\perp)$; i.e. it satisfies

$$g = \mathcal{W}_{\Pi_\perp}(\Pi_\perp) \iff \Pi_\perp = \Theta(g). \tag{4.26}$$

According to (4.18), the susceptibility χ is then given as the following function of the gravitational field g,

$$\chi(g) = -4\pi G\frac{\Theta(g)}{g}. \tag{4.27}$$

This is the general relation linking χ (or equivalently the MOND function $\mu = 1 + \chi$) to the potential function $\mathcal{W}$ in the dipolar action (2.7). Of course, in the present model $\mathcal{W}$ is to be considered as more fundamental than χ which is a derived quantity.

In the Newtonian regime $g \gg a_0$, the MOND function μ should tend to 1, so that χ vanishes in this regime. To discuss more concretely this condition, we assume that in the formal limit $g \to +\infty$, the gravitational susceptibility behaves as $\chi \sim g^{-\gamma}$, with γ a strictly positive real number.

More precisely, it should behave like $\chi \sim -\epsilon(g/a_0)^{-\gamma}$, where ϵ is a strictly positive real number. Beware that even if this power-law behavior is a simple assumption, nothing guaranties that it is verified. Then, when $g \to +\infty$, we get from (4.26) and (4.27) that

$$\Pi_\perp \sim A g^{1-\gamma}, \tag{4.28a}$$

$$\mathcal{W} \sim \frac{1-\gamma}{2-\gamma} A g^{2-\gamma} + \kappa, \tag{4.28b}$$

where $A = \epsilon a_0^\gamma / 4\pi G > 0$ and κ is an integration constant. We have to distinguish several cases, depending on the value of the exponent γ:

(i) If $0 < \gamma < 1$, then both the polarization $\Pi_\perp$ and the potential $\mathcal{W}$ diverge. This would correspond to the curve (a) of Fig. 4.

(ii) If $\gamma = 1$, the polarization $\Pi_\perp$ tends to a maximum "saturation" value $\Pi_{\max} = A$, and the potential $\mathcal{W}$ equals the constant κ. See curve (b) in Fig. 4.

(iii) If $1 < \gamma < 2$, the polarization goes to zero while the potential diverges to $-\infty$ like a power law. This implies that $\mathcal{W}$ cannot be a univalued function of $\Pi_\perp$. Therefore, there must exist two branches corresponding to the Newtonian and MOND regimes.

(iv) If $\gamma = 2$, according to (4.28b) the potential diverges to $-\infty$ logarithmically, i.e. $\mathcal{W} \sim -A \ln g$, while the polarization still vanishes. The same conclusions as in case (iii) apply.

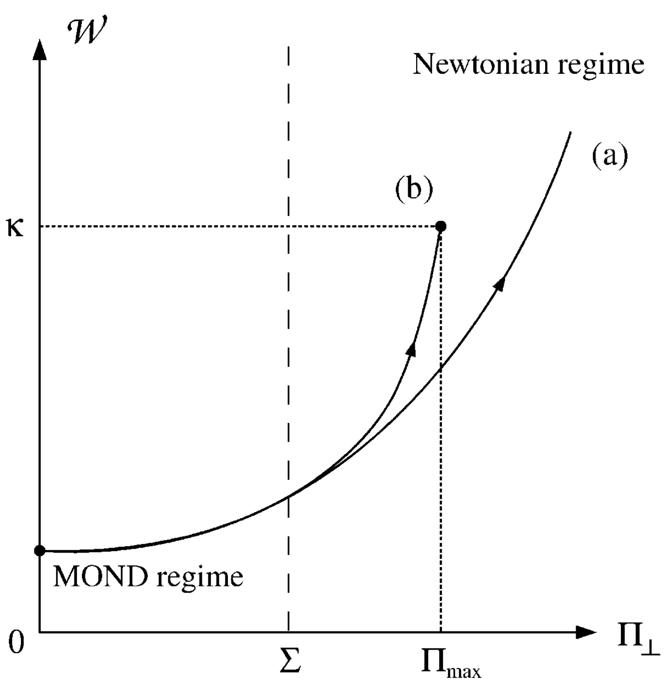

FIG. 4. The potential $\mathcal{W}$ as a function of the polarization $\Pi_\perp$ for different asymptotic behaviors of the gravitational susceptibility χ in the Newtonian regime $g \gg a_0$. The arrows indicate the direction of increasing gravitational field g.

(v) Finally, if $\gamma > 2$, the polarization goes to zero while the potential tends to κ. The same conclusions as in (iii) apply.

If we believe that the potential $\mathcal{W}$ represents a fundamental function in the action, and that our model should, strictly speaking, be valid in a Newtonian regime (and not merely valid in the MOND regime), we should *a priori* expect that $\mathcal{W}$ is a univalued function of $\Pi_\perp$. Then, the susceptibility coefficient should be like $\chi \sim g^{-\gamma}$ with $0 < \gamma \leq 1$ in the Newtonian regime. This would mean that the MOND function μ behaves like

$$\mu \sim 1 - \epsilon\left(\frac{a_0}{g}\right)^\gamma, \tag{4.29}$$

with $0 < \gamma \leq 1$. Such rather slow transition of μ toward the Newtonian regime is consistent with the recent results of [42] which fitted the rotation curves of the Milky Way and galaxy NGC 3198, and of [43] which fitted 17 early-type disc galaxies and concluded that the Newtonian regime is rather slowly reached. For instance, the authors of [42–44] agreed that $\gamma = 1$ yields a better fit to the data than $\gamma = 2$.

The case $\gamma = 1$ [curve (b) in Fig. 4] corresponds to an interesting physical situation in which the dipolar medium *saturates* when $g \to +\infty$, at the maximum value $\Pi_{\max} = A$, or

$$\Pi_{\max} = \frac{\epsilon}{2}\Sigma, \tag{4.30}$$

where Σ is the surface density scale (4.14). In this saturation case, the gravitational susceptibility coefficient behaves as

$$\chi \sim -\epsilon \frac{a_0}{g}. \tag{4.31}$$

However, let us remind the reader that such a slow transition from MOND toward the Newtonian regime is *a priori* ruled out by Solar System observations. Indeed, according to the MOND equation, a planet orbiting the Sun feels a gravitational field g obeying $(1+\chi)g = g_N$, where g_N is the Newtonian gravitational field. Hence, if χ scales like g^{-1} when $g \gg a_0$ like in (4.31), the gravitational field experienced by planets will involve a constant supplementary acceleration directed toward the Sun (i.e. a "Pioneer-type" anomaly) given by

$$g \sim g_N + \epsilon a_0. \tag{4.32}$$

Of course, it is striking that the order of magnitude of the Pioneer anomaly is the same as the MOND acceleration a_0. Unfortunately, the presence of a constant acceleration such as in (4.32) should be detected in the motion of planets, and this is incompatible with current measurements (see e.g. [45,46] for a discussion).

Despite the fact that a slow transition to the Newtonian regime (like, for example, the case $\gamma = 1$) seems to be favored by observations at the galactic scale [42–44], it does not seem to be viable when extrapolated up to the scale of the Solar System. In our model, we found that such a behavior is the result of our belief that the fundamental function $\mathcal{W}$ be univalued. In this respect, the validity of the model should be limited to large scales, from the galactic scale up to cosmological scales, i.e. in a regime of weak gravity. At smaller scales the description in terms of a single univalued function $\mathcal{W}$ should break down. But with our model being an effective one, or even a phenomenological one, the question of whether the potential $\mathcal{W}$ is univalued or not remains an open issue.

V. SUMMARY AND CONCLUSION

In this paper, we proposed a model of dark matter and dark energy based on the concept of gravitational polarization of a medium of dipole moments. The dynamics of the dipolar fluid is governed by the Lagrangian (2.7) in standard general relativity, and constitutes a generalization of the previous model [26]. Namely, this Lagrangian involves a potential function $\mathcal{W}$, describing at some effective level a nongravitational internal force influencing the dynamics of the dipolar fluid, and which depends on the polarization field or density of dipole moments $\Pi_\perp = \sigma\xi_\perp$ instead of merely the dipole moment itself $\xi_\perp$ in the model [26]. This new form of the potential permits recovering, in a most elegant way, the phenomenology of MOND in a typical galaxy at low redshift. In addition, we show that the model naturally contains a cosmological constant Λ.

We proved in Sec. III that within the framework of the theory of first-order cosmological perturbations, the dipolar fluid behaves exactly as standard dark energy (i.e. a cosmological constant) plus standard dark matter (i.e. a pressureless perfect fluid). Thus, our model is consistent with the cosmological observations at large scales. In particular, it leads to the same predictions as the standard Λ-CDM model for the CMB fluctuations. However, at second order in the cosmological perturbations, we expect that the dipolar dark matter should differ from a perfect fluid because of the influence of the internal force resulting in a nongeodesic motion. The model could thus be checked by working out the second-order cosmological perturbations and comparing with CMB fluctuations (notably the effects linked with the non-Gaussianity).

The dynamics of the dipolar dark matter being different from that of standard dark matter (at the level of nonlinear perturbations), we expect the monopolar part of the dipolar dark matter not to cluster much during the cosmological evolution. We call this expectation the hypothesis of "weak clustering." It is supported by an exact solution worked out in the Appendix for the dynamics of dipolar dark matter in the nonrelativistic limit and in spherical symmetry. In this solution, the internal force balances the local gravitational field produced by a spherical mass, so that the dark matter remains at rest with respect to the central mass. The weak clustering hypothesis should be checked *via* N-body numerical simulations. Under that hypothesis, we show that the Poisson equation for the gravitational field generated by the baryonic and dipolar dark matter reduces to the MOND equation in the regime of weak gravitational fields $g \ll a_0$. Our model of dipolar dark matter therefore naturally explains all the successes of the MOND phenomenology.

To achieve this result (in Sec. IV) we have to adjust the fundamental potential $\mathcal{W}$ in the action. We find that it should be given by an anharmonic potential, the minimum of which, reached when $\Pi_\perp = 0$, is directly related to the cosmological constant Λ. It is tempting to interpret Λ as a "vacuum polarization" of some hypothetical quantum field, when the classical part of the polarization $\Pi_\perp \to 0$. The expansion around that minimum is fine-tuned in order to recover MOND. In particular, we show that the MOND acceleration a_0 parametrizes the coefficient of the third-order deviation of $\mathcal{W}$ from the minimum. Although fine-tuned to fit with observations, this potential function $\mathcal{W}$ offers a nice unification between the dark energy in the form of Λ and the dark matter in the form of MOND (see Fig. 3). A consequence of such unification is that the cosmological constant should scale with the MOND acceleration according to $\Lambda \sim a_0^2/c^4$. This scaling relation is in good agreement with observations and has a very natural explanation in our model.

To conclude, we proposed to modify the matter sector rather than the gravity sector as in modified gravity theories [14,18,23,24]. Namely, we investigated a model of dark matter, but of such an exotic form that it naturally explains the phenomenology of MOND at galactic scales. Furthermore, that form of dark matter has a simple physical interpretation in terms of the well-known mechanism of polarization by an exterior field. More work is necessary to test the model, either by studying second-order perturbations in cosmology, or by computing numerically the nonlinear growth of perturbations and comparing with large-scale structures.

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APPENDIX: DARK MATTER IN A CENTRAL GRAVITATIONAL FIELD

We investigate the dynamics of the dipolar dark matter fluid in the presence of a spherically symmetric mass distribution of ordinary baryonic matter in the NR limit $c \to +\infty$. The equations to solve are the equation of motion (4.3), the equation of evolution (4.4), the continuity equation (4.5), and the Poisson equation for the gravitational field (4.9). Let us rewrite those equations here for convenience[15]:

$$\frac{\mathrm{d}\boldsymbol{v}}{\mathrm{d}t} = \boldsymbol{g} - \boldsymbol{\mathcal{F}}, \tag{A1a}$$

$$\partial_t \sigma = -\boldsymbol{\nabla}\cdot(\sigma\boldsymbol{v}), \tag{A1b}$$

$$\boldsymbol{\nabla}\cdot\boldsymbol{g} = -4\pi G(\sigma + \rho_{\mathrm{b}} - \boldsymbol{\nabla}\cdot\boldsymbol{\Pi}), \tag{A1c}$$

$$\frac{\mathrm{d}^2\boldsymbol{\xi}}{\mathrm{d}t^2} = \boldsymbol{\mathcal{F}} + \frac{1}{\sigma}\boldsymbol{\nabla}(\mathcal{W} - \Pi\mathcal{W}') + (\boldsymbol{\xi}\cdot\boldsymbol{\nabla})\boldsymbol{g}, \tag{A1d}$$

where the internal force reads $\boldsymbol{\mathcal{F}} = \hat{\boldsymbol{\Pi}}\mathcal{W}'$, with $\hat{\boldsymbol{\Pi}} \equiv \boldsymbol{\Pi}/\Pi$.

Our aim is to solve Eqs. (A1) in the special case where the baryonic matter is modeled by a time-independent, spherically symmetric distribution of mass $\rho_{\mathrm{b}}(r)$, say with compact support. Let us show that there is a simple solution to such a set of equations, in the case where

$$\boldsymbol{v}_0 = 0, \tag{A2a}$$

$$\sigma_0 = \sigma_0(r), \tag{A2b}$$

which corresponds to a static fluid whose mass distribution is time independent and spherically symmetric. We denote such a particular solution with a lower index 0. From (A2) we observe that the continuity equation (A1b) is immediately satisfied. In such a solution, according to (A1a) the internal force balances exactly the gravitational field, i.e. $\boldsymbol{\mathcal{F}}_0 = \boldsymbol{g}_0$ (this is somewhat similar to the case of a nonrotating star in hydrostatic equilibrium, where the pressure gradient plays the role of the internal force). We deduce that the polarization field $\boldsymbol{\Pi}_0 = \sigma_0\boldsymbol{\xi}_0$ is aligned with the gravitational field $\boldsymbol{g}_0$. Hence, from Eq. (A1c) both $\boldsymbol{\Pi}_0$ and $\boldsymbol{g}_0$ are radial. We shall pose $\boldsymbol{g}_0 = -g_0(r,t)\boldsymbol{e}_r$ and $\boldsymbol{\Pi}_0 = -\Pi_0(r,t)\boldsymbol{e}_r$, where in our notation $g_0 > 0$ and $\Pi_0 > 0$.

Furthermore, let us show that in addition the polarization field is practically in "equilibrium," i.e. Π_0 is not dependent on time t, and neither is g_0. We replace $\boldsymbol{g}_0$ by the explicit expression of the internal force $\boldsymbol{\mathcal{F}}_0 = \hat{\boldsymbol{\Pi}}_0\mathcal{W}_0'$ into the evolution equation (A1d), use (A2a), and get

$$\partial_t^2\boldsymbol{\Pi}_0 - \sigma_0\mathcal{W}_0'\hat{\boldsymbol{\Pi}}_0 = \boldsymbol{\nabla}(\mathcal{W}_0 - \Pi_0\mathcal{W}_0') + (\boldsymbol{\Pi}_0\cdot\boldsymbol{\nabla})(\hat{\boldsymbol{\Pi}}_0\mathcal{W}_0'). \tag{A3}$$

Here $\hat{\boldsymbol{\Pi}}_0 = \boldsymbol{\Pi}_0/\Pi_0 = -\boldsymbol{e}_r$, and we introduced the shorthand notation $\mathcal{W}_0' \equiv \mathcal{W}'(\Pi_0)$. Now, it turns out that the RHS of this equation vanishes in the special case where the polarization field is radial; hence we get

$$\partial_t^2\Pi_0 = \sigma_0\mathcal{W}_0'. \tag{A4}$$

In order to determine the time evolution of Π_0, an explicit expression for the potential $\mathcal{W}$ is, in principle, required. However, we saw in Sec. IV C that the potential $\mathcal{W}$ only depends on the polarization Π and the constants a_0 and G. The only time scale one can build with a_0, G, and σ_0 is the dipolar dark matter self-gravitating time scale $\tau_{\mathrm{g}} = (\pi/G\sigma_0)^{1/2}$, or equivalently, in terms of frequency, $\omega_{\mathrm{g}}^2 = 4\pi G\sigma_0$. Therefore, the polarization Π_0 can only evolve on this time scale. For instance, in the MOND regime $g \ll a_0$, we have at leading order $\mathcal{W}_0' = 4\pi G\Pi_0$; hence (A4) reduces to

$$\partial_t^2\Pi_0 = \omega_{\mathrm{g}}^2\Pi_0. \tag{A5}$$

The most general solution of this equation is a linear combination of hyperbolic $\cosh\omega_{\mathrm{g}}t$ and $\sinh\omega_{\mathrm{g}}t$. For a monopolar dark matter mass density σ_0 of, say, the mean cosmological value $\bar{\sigma} \simeq 10^{-26}\ \mathrm{kg/m^3}$ [in agreement with our weak clustering hypothesis (4.11)], the typical time scale of evolution of Π_0 will be larger than 6×10^{10} years. This is large enough to neglect any time variation of Π_0 with respect to a typical orbital time scale in a galaxy. Our solution is therefore given by

$$\boldsymbol{\Pi}_0 = -\Pi_0(r)\boldsymbol{e}_r, \tag{A6}$$

together with (A2). The dipole moments are at rest and in equilibrium. The explicit function $\Pi_0(r)$ is determined from the radial gravitational field $g_0(r)$ as[16]

$$\Pi_0(r) = \Theta(g_0(r)), \tag{A7}$$

where $\Theta(g_0)$ denotes the inverse function of $\mathcal{W}'(\Pi_0)$ following the notation (4.26). The gravitational field $g_0(r)$ is determined by the Poisson equation (A1c) as

$$g_0 - 4\pi G\Pi_0 = \frac{GM_0(r)}{r^2}, \tag{A8}$$

where $M_0(r) = 4\pi\int_0^r \mathrm{d}s s^2[\rho_{\mathrm{b}}(s) + \sigma_0(s)]$ is the mass enclosed within radius r.

The existence of this physically simple solution represents notable progress compared to the more complicated solution found in the previous model [26] (see Sec. IV there). Such a solution is quite interesting for the present model because it indicates that during the cosmological evolution (at nonlinear perturbation order) the dipolar dark matter may not cluster very much toward regions of overdensity. Most of the effect will be in the dipole moment vectors which acquire a spatial distribution. This is our motivation for the "weak clustering" assumption (4.10) and (4.11) stating that $\sigma \ll \rho_{\mathrm{b}}$, which was used in Sec. IV C to obtain MOND. In the present case, neglecting σ_0 with respect to ρ_{b} in the RHS of (A8), we recover the

[15]In this appendix, we adopt 3-dimensional notations with boldface vectors, e.g. $\boldsymbol{\mathcal{F}} = (\mathcal{F}^i)$. We also remove the subscript $\perp$ from the variables $\boldsymbol{\xi}_\perp$ and $\boldsymbol{\Pi}_\perp$ for notational simplicity. The derivatives of the potential $\mathcal{W}$ with respect to its argument Π will be denoted with a prime, e.g. $\mathcal{W}' \equiv \mathcal{W}_\Pi \equiv \mathrm{d}\mathcal{W}/\mathrm{d}\Pi$.

[16]Note that if in this solution the polarization field $\Pi_0(r) = \sigma_0(r)\xi_0(r)$ is determined, the density $\sigma_0(r)$ and dipole moment $\xi_0(r)$ are not specified separately. For instance, the density could be at the uniform cosmological value $\bar{\sigma}$ so that $\xi_0(r) = \Pi_0(r)/\bar{\sigma}$. This degeneracy of $\sigma_0(r)$ is an artifact of our assumptions of spherical symmetry and staticity.

usual MOND equation generated by the baryonic density only. This being said, such an appealing solution may be physically irrelevant if the spherically symmetric configuration appears to be unstable with respect to linear perturbations. This motivates the following study of the stability of the previous solution.

Consider a general perturbation of the background solution, namely,

$$\sigma = \sigma_0 + \delta\sigma, \tag{A9a}$$

$$\boldsymbol{v} = \delta\boldsymbol{v}, \tag{A9b}$$

$$\boldsymbol{\Pi} = \boldsymbol{\Pi}_0 + \delta\boldsymbol{\Pi}. \tag{A9c}$$

We have also $\boldsymbol{g} = \boldsymbol{g}_0 + \delta\boldsymbol{g}$ and $\boldsymbol{\mathcal{F}} = \boldsymbol{\mathcal{F}}_0 + \delta\boldsymbol{\mathcal{F}}$, where the expression of the perturbed force in terms of the perturbed polarization explicitly reads

$$\delta\boldsymbol{\mathcal{F}} = \mathcal{W}_0''(\hat{\boldsymbol{\Pi}}_0\cdot\delta\boldsymbol{\Pi})\hat{\boldsymbol{\Pi}}_0 + \mathcal{W}_0'\left[\frac{\delta\boldsymbol{\Pi}}{\Pi_0} - \left(\hat{\boldsymbol{\Pi}}_0\cdot\frac{\delta\boldsymbol{\Pi}}{\Pi_0}\right)\hat{\boldsymbol{\Pi}}_0\right]. \tag{A10}$$

Assuming a Fourier decomposition for any perturbative quantity δX, we write for a given mode of frequency ω and wave number $\boldsymbol{k}$,

$$\delta X(\boldsymbol{x}, t) = \delta X(\boldsymbol{k}, \omega)e^{\mathrm{i}(\boldsymbol{k}\cdot\boldsymbol{x}-\omega t)}. \tag{A11}$$

We want to find the relation between $\boldsymbol{k}\cdot\boldsymbol{e}_r$ and ω, the so-called dispersion relation, which contains all the physical information about the behavior of the generic perturbation (A11). Introducing this *ansatz* into (A1), and simplifying the resulting equations by making use of the background solution, we find

$$\delta\boldsymbol{v} = \frac{\mathrm{i}}{\omega}(\delta\boldsymbol{g} - \delta\boldsymbol{\mathcal{F}}), \tag{A12a}$$

$$\delta\sigma = \frac{1}{\omega}(\sigma_0\boldsymbol{k}\cdot\delta\boldsymbol{v} - \mathrm{i}\delta\boldsymbol{v}\cdot\boldsymbol{\nabla}\sigma_0), \tag{A12b}$$

$$\delta\boldsymbol{g} = 4\pi G\frac{\mathrm{i}\boldsymbol{k}}{k^2}(\delta\sigma - \mathrm{i}\boldsymbol{k}\cdot\delta\boldsymbol{\Pi}). \tag{A12c}$$

These algebraic expressions can be combined to express $\delta\sigma$, $\delta\boldsymbol{g}$, and $\delta\boldsymbol{v}$ in terms of $\delta\boldsymbol{\Pi}$ only. After some algebra, we get from the evolution equation (A1d) a relation expressing the perturbed polarization field $\delta\boldsymbol{\Pi} = \sigma_0\delta\boldsymbol{\xi} + \delta\sigma\boldsymbol{\xi}_0$ as

$$\begin{aligned}\omega^2\delta\boldsymbol{\Pi} = {}& \omega^2\frac{\delta\sigma}{\sigma_0}\boldsymbol{\Pi}_0 + \frac{\mathrm{i}\omega}{\sigma_0}(\delta\boldsymbol{v}\cdot\boldsymbol{\nabla}\sigma_0)\boldsymbol{\Pi}_0 - \mathrm{i}\omega(\delta\boldsymbol{v}\cdot\boldsymbol{\nabla})\boldsymbol{\Pi}_0 \\ & + (\hat{\boldsymbol{\Pi}}_0\cdot\delta\boldsymbol{\Pi})\boldsymbol{\nabla}(\Pi_0\mathcal{W}_0'') + \Pi_0\mathcal{W}_0''(\hat{\boldsymbol{\Pi}}_0\cdot\delta\boldsymbol{\Pi})\mathrm{i}\boldsymbol{k} \\ & - (\mathrm{i}\boldsymbol{k}\cdot\boldsymbol{\Pi}_0)\delta\boldsymbol{g} - (\delta\boldsymbol{\Pi}\cdot\boldsymbol{\nabla})\boldsymbol{g}_0 - \sigma_0\delta\boldsymbol{\mathcal{F}}.\end{aligned} \tag{A13}$$

When replacing $\delta\sigma$, $\delta\boldsymbol{g}$, $\delta\boldsymbol{v}$, and $\delta\boldsymbol{\mathcal{F}}$ into (A13), we obtain a master equation for the perturbed polarization $\delta\boldsymbol{\Pi}$ which is quite complicated. Given the complexity of the problem, we restrict our analysis to the simplest modes in a spherically symmetric background, namely, those propagating radially. We shall thus write $\boldsymbol{k} = k\boldsymbol{e}_r$, and study successively the transverse and longitudinal perturbations.

First, let us consider a transverse perturbation $\delta\boldsymbol{\Pi}$, i.e. one which satisfies $\delta\boldsymbol{\Pi}\cdot\boldsymbol{e}_r = 0$. Projecting the master equation (A13) in the direction of $\delta\boldsymbol{\Pi}$, we get that

$$\left[\omega^2 + \mathcal{W}_0'\left(\frac{1}{\xi_0} - \frac{2}{r}\right)\right]\delta\Pi = 0, \tag{A14}$$

which simply states that no transverse perturbations propagating radially are allowed, i.e. $\delta\Pi = 0$. Consider now the case of a longitudinal perturbation $\delta\boldsymbol{\Pi} = -\delta\Pi(r, t)\boldsymbol{e}_r$, where $\delta\Pi$ can be positive or negative (with our convention, the norm of $\boldsymbol{\Pi}$ reads $\Pi = \Pi_0 + \delta\Pi$), and represents the arbitrary amplitude of the applied linear perturbation. After some lengthy calculations, we get the dispersion relation

$$k = \mathrm{i}\frac{\partial_r\sigma_0}{\sigma_0}\left(1 + \frac{\omega^2}{\omega_\mathrm{g}^2}\left[1 + \frac{(4\pi G - \mathcal{W}_0'')\partial_r\Pi_0}{\omega^2 + \sigma_0\mathcal{W}_0'' + \Pi_0\partial_r\mathcal{W}_0''}\right]\right)^{-1}. \tag{A15}$$

Notice first that, as the wave number k is purely imaginary, such a perturbation cannot propagate. Second, the stability of the background solution with respect to this perturbation is related to the sign of k/i, so an explicit expression for the potential $\mathcal{W}$ is required to conclude. Such an expression is available in the MOND regime $g_0 \ll a_0$ using the expansion (4.24). Assuming the MOND equation with a (baryonic) point mass M for simplicity, i.e. Eq. (A8) with $\rho_\mathrm{b} = M\delta(\boldsymbol{x})$ and negligible σ_0, we find that the dispersion relation can be recast at the leading order in the form

$$k = \mathrm{i}\frac{\partial_r\sigma_0}{\sigma_0}\frac{\omega_\mathrm{g}^2(\omega^2 + \omega_\mathrm{g}^2 - 2\omega_\mathrm{K}^2)}{\omega^4 + 2\omega_\mathrm{g}^2\omega^2 + \omega_\mathrm{g}^2(\omega_\mathrm{g}^2 - 2\omega_\mathrm{K}^2)}, \tag{A16}$$

where $\omega_\mathrm{K}^2 = GM/r^3$ denotes the Keplerian frequency. We now turn to the analysis of the two factors in (A16), namely, the ω-dependent and σ_0-dependent ones.

At a given distance r from the center of the galaxy, the ω-dependent factor becomes very large in the vicinity of the resonant frequency

$$\omega_\mathrm{R}^2 = \omega_\mathrm{g}(\sqrt{2}\omega_\mathrm{K} - \omega_\mathrm{g}). \tag{A17}$$

But we are restricting our attention to perturbations in the MOND regime where $g_0 \ll a_0$, which means at distances r from the galactic center that are far larger than the MOND radius $r_\mathrm{M} \equiv \sqrt{GM/a_0}$, or equivalently at Keplerian frequencies $\omega_\mathrm{K} \ll \omega_\mathrm{M}$ with $\omega_\mathrm{M}^2 = GM/r_\mathrm{M}^3$. For a typical galaxy of mass $M \sim 10^{11}\mathrm{M}_\odot$, and a monopolar dark matter mass density around the mean cosmological value, i.e. $\sigma_0 \sim \bar{\sigma} \simeq 10^{-26}\ \mathrm{kg/m^3}$, we find by reporting the constraint $\omega_\mathrm{K} \ll \omega_\mathrm{M}$ into (A17) the upper bound $\omega_\mathrm{R}^2 \ll \sqrt{2}\omega_\mathrm{g}\omega_\mathrm{M}$, which gives numerically $\omega_\mathrm{R} \ll 10^{-17}\mathrm{s}^{-1}$. Because perturbations with a typical time scale $2\pi/\omega \gg 2\times 10^{10}$ years are out of the present scope, the ω-dependent part of (A16) reduces to a numerically small factor.

Finally, we consider the σ_0-dependent part of (A16). Consistent with the "weak clustering hypothesis" (4.10) and (4.11), we are expecting the background density profile σ_0 to be almost homogeneous. Thus, the factor $\partial_r\sigma_0/\sigma_0$ will be of the order of the inverse of the characteristic length scale L of variation of σ_0 assumed to be far larger than the typical size ℓ of the galaxy, which implies $|\boldsymbol{k}\cdot\boldsymbol{x}| \lesssim \ell/L \simeq 0$ in (A11). A longitudinal perturbation would therefore keep oscillating at the frequency ω without propagating, and we conclude that it would be stable.

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