Post-Newtonian approximation for isolated systems calculated by matched asymptotic expansions

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(Received 21 December 2001; published 12 June 2002)

Two long-standing problems with the post-Newtonian approximation for isolated slowly moving systems in general relativity are (i) the appearance at high post-Newtonian orders of divergent Poisson integrals, casting doubt on the soundness of the post-Newtonian series, and (ii) the domain of validity of the approximation which is limited to the near-zone of the source, and prevents one, *a priori*, from incorporating the condition of no-incoming radiation to be imposed at past null infinity. In this paper, we resolve problem (i) by iterating the post-Newtonian hierarchy of equations by means of a new (Poisson-type) integral operator that is free of divergencies, and problem (ii) by matching the post-Newtonian near-zone field to the exterior field of the source, known from previous work as a multipolar-post-Minkowskian expansion satisfying the relevant boundary conditions at infinity. As a result, we obtain an algorithm for iterating the post-Newtonian series up to any order, and we determine the terms, present in the post-Newtonian field, that are associated with the gravitational-radiation reaction onto an isolated slowly moving matter system.

DOI: 10.1103/PhysRevD.65.124020 PACS number(s): 04.25.Nx, 04.30.−w

I. INTRODUCTION

A. Problems with the post-Newtonian expansion

The post-Newtonian approximation, or expansion when the speed of light $c\rightarrow+\infty$, has been formalized in the early days of general relativity by Einstein [1], Droste [2], and de Sitter [3]. Since then, it has provided us with our best insight into the problems of motion and gravitational radiation, two of general relativity's most important issues. Concerning the problem of motion, we quote the dynamics of N separated bodies at the first post-Newtonian (1 PN, or $1/c^2$) order; works of Einstein, Infeld, and Hoffmann [4] and other authors [5–7], and the dynamics of extended fluid systems up to the 2.5 PN level of gravitational radiation reaction; works of Chandrasekhar and collaborators [8–10] and followers [11–19]. In the case of two compact objects, we know the 2.5 PN equations of motion of the binary pulsar [20–23], and the 3 PN equations of motion of inspiraling compact binaries [24–28]. The specific contribution of the gravitational-radiation reaction has been obtained up to the 1.5 PN relative order by the method of matched asymptotic expansions for extended fluids [29–33], and by means of balance equations for compact binary systems [34,35]. Concerning the problem of gravitational radiation, the work has focused on the expressions of the multipole moments of general fluid systems [36–40], and on the gravitational-wave flux emitted by inspiraling compact binaries, including the specific effects of wave tails, up to the 3.5 PN order [41–45].

The "standard" post-Newtonian approximation, at the basis of most of the body of work quoted previously, is known to be plagued with some apparently inherent difficulties, which crop up at some high post-Newtonian order like 3 PN. Up to the 2.5 PN order, the approximation can be worked out without problems, and at the 3 PN order the problems can be solved specifically for each case at hand (see, for instance, Ref. [27]). However, it must be admitted that these difficulties, even appearing at higher approximations, cast doubt on the actual soundness, from a theoretical point of view, of the post-Newtonian expansion. What may be worse is that they pose the practical question of the reliability of this approximation when comparing the theory's predictions with very precise experimental results. It is therefore highly desirable to assess the nature of these difficulties—are they purely technical or linked with some fundamental drawback of the approximation scheme—and eventually to resolve them. This is especially important in view of the fact that inspiraling compact binaries, when they are detected and analyzed by gravitational-wave experiments, will necessitate *a priori* theoretical knowledge of the gravitational-wave signal at some very high post-Newtonian order [41–45]. In this paper, let us distinguish (and resolve) the two basic problems faced by the post-Newtonian expansion.

The first problem is that in higher approximations some *divergent* Poisson-type integrals appear. Recall that the post-Newtonian expansion replaces the resolution of a hyperboliclike d'Alembertian equation by a perturbatively equivalent hierarchy of ellipticlike Poisson equations. Rapidly it is found during the post-Newtonian iteration that the right-hand side of the Poisson equations acquires a noncompact support (it is distributed over all space), and that the standard Poisson integral diverges because of the bound of the integral at spatial infinity, i.e., $r\equiv|\mathbf{x}|\rightarrow+\infty$, with $t=$const. For instance, some of the potentials occurring at the 2 PN order in Chandrasekhar's work [9] are divergent, so the corresponding metric is formally infinite.[1] In fact, Kerlick [14,15] showed that the post-Newtonian computation in the manner of Chandrasekhar [8–10], following the iteration scheme of Anderson and DeCanio [11], can be made well-defined up to the 2.5 PN order, by keeping some derivatives inside some crucial integrals to make them finite [12,13]. However, the latter remedy does not solve the problem at the next 3 PN order, which has been found to involve some inexorably divergent Poisson integrals [14,15].

[1]Nevertheless, these divergencies were not a problem when considering the equations of motion because the gradients of these potentials, which parametrize the equations, were finite.

These divergencies come from the fact that the post-Newtonian expansion is actually a singular perturbation, in the sense that the coefficients of the successive powers of $1/c$ are not uniformly valid in space, since they typically blow up at spatial infinity like some positive powers of r. For instance, Rendall [46] has shown that the post-Newtonian expansion cannot be ''asymptotically flat'' starting at the 2 PN or 3 PN level, depending on the adopted coordinate system. The result is that the Poisson integrals are in general badly behaving at infinity. Physically, this can be understood by the fact that the post-Newtonian approximation is valid only in the near zone of the source (see below) while the Poisson integral extends over the whole three-dimensional space, including the regions far from the source where the approximation breaks down. Therefore, trying to solve the post-Newtonian equations by means of the standard Poisson integral does not *a priori* make sense. This does not mean that there is no solution to the problem, but simply that the Poisson integral does not constitute the correct solution of the Poisson equation in the context of post-Newtonian expansions. So the difficulty is purely of a technical nature, and will be solved once we succeed in finding the appropriate solution to the Poisson equation.[2] A solution to the problem of divergencies has been proposed by Futamase and Schutz [47] and Futamase [48]. Their approach is alternative to the one we shall follow below. It is based on an initial-value formalism, which avoids the appearance of divergencies because of the finiteness of the integration region.

The second problem has to do with the near-zone limitation of the approximation. Indeed the post-Newtonian expansion assumes that all retardations r/c are small, so it can be viewed as a formal *near-zone* expansion when $r\rightarrow 0$, which is valid only in the region surrounding the source that is of small extent with respect to the typical wavelength of the emitted radiation: $r\ll\lambda$ (if we locate the origin of the coordinates $r=0$ inside the source). Therefore, the fact that the coefficients of the post-Newtonian expansion blow up at spatial infinity, when $r\rightarrow+\infty$, has nothing to do with the actual behavior of the field at infinity. The serious consequence is that it is not possible, *a priori*, to implement within the post-Newtonian iteration the physical information that the matter system is isolated from the rest of the universe. Most importantly, the no-incoming-radiation condition, imposed at past null infinity, cannot be taken into account, *a priori*, into the scheme. In a sense the post-Newtonian approximation is not ''self-supporting,'' because it necessitates some information taken from outside its own domain of validity.

To the lowest post-Newtonian orders one can circumvent this difficulty by considering *retarded* integrals that are formally expanded when $c\rightarrow+\infty$ as series of ''instantaneous'' Poisson-like integrals [11]. This procedure works well up to the 2.5 PN level and has been shown to correctly fix the dominant radiation reaction term at the 2.5 PN order [14,15]. Unfortunately, such a procedure assumes fundamentally that the gravitational field, after expansion of all retardations $r/c\rightarrow 0$, depends on the state of the source at a single time, in keeping with the instantaneous character of the Newtonian interaction. However, we know that from the 4 PN order the post-Newtonian field (as well as the source's dynamics) ceases to be given by a functional of the source parameters at a single time, because of the imprint of gravitational-wave tails in the near zone field, in the form of some modification, at the 1.5 PN relative order, of the radiation reaction force [31–33]. Therefore, the formal post-Newtonian expansion of retarded Green functions is no longer valid starting at the 4 PN order. We face here a true difficulty, which is fundamentally linked to the nature of the post-Newtonian approximation.

The aim of the present paper is to resolve the two latter problems. We shall prove that the post-Newtonian expansion can be *indefinitely* reiterated, while incorporating the correct boundary conditions satisfied by the wave field at infinity. In particular, we shall get new insights about the problem of gravitational-radiation reaction inside an isolated (post-Newtonian) system. To solve the problem of divergencies, we introduce, at any post-Newtonian order, a generalized solution of the Poisson equation with a noncompact support source, in the form of an appropriate *finite part* of the usual Poisson integral: namely, we regularize the bound at infinity of the Poisson integral by means of a process of analytic continuation, analogous to the one already used to regularize the retarded integrals in Refs. [36,39,40]. Our generalized solution constitutes a particular (well-defined) solution of the problem; the most general solution is the sum of that particular solution and the most general solution of the corresponding homogeneous equation, i.e., the source-free Laplace equation. The homogeneous solution should be regular all over the matter system (we are considering smooth matter distributions), and we find, after summing up the post-Newtonian series, that it can be thoroughly written with the help of some tensorial functions of time $A_L^{\mu\nu}(t)$, where $L=i_1\cdots i_l$ denotes a multi-index with l indices [49]. At this stage, considering the post-Newtonian iteration scheme alone, we cannot do more and therefore we leave the functions $A_L^{\mu\nu}(t)$ unspecified. We refer to them as some ''radiation-reaction'' functions.

The solution of the problem of the near-zone limitation of the post-Newtonian expansion resides in the matching of the near-zone field to the exterior field, a solution of the vacuum equations outside the source which has been developed in previous works [36,32] using some post-*Minkowskian* and multipolar expansions. In the case of post-Newtonian sources, the near zone, i.e., $r\ll\lambda$, covers entirely the source, because the source's radius itself is such that $a\ll\lambda$. Thus the near zone overlaps with the exterior zone where the multi-

[2]The problem is somewhat similar to what happens in Newtonian cosmology. Here we have to solve the Poisson equation $\Delta U=-4\pi G\rho$, where the density ρ of the cosmological fluid is constant all over space: $\rho=\rho(t)$. Clearly the Poisson integral of a constant density does not make sense, as it diverges at the bound at infinity like the integral $\int r\,dr$. This nonsensical result has occasionally been referred to as the ''paradox of Seeliger.'' However, the problem is solved once we realize that the Poisson integral does not constitute the appropriate solution of the Poisson equation in the context of Newtonian cosmology. A well-defined solution is simply given by $U=-\frac{2}{3}\pi G\rho r^2$.

pole expansion is valid. Matching together the post-Newtonian and multipolar–post-Minkowskian solutions in this overlapping region is an application of the method of matched asymptotic expansions, and has frequently been applied in the present context, both for radiation-reaction [29–33] and wave-generation [37–40] problems.

The exterior multipolar–post-Minkowskian field originally obtained in Ref. [36] depends on some "multipole-moment" functions, say $X_L^{\mu\nu}(t)$ [whose components are associated with some source multipole moments, e.g., $I_L(t)$, $J_L(t)$, ...], which must be left unspecified as long as we consider only the external vacuum solution. In the work [40], we have shown that the multipole moments $X_L^{\mu\nu}(t)$ are entirely determined, up to any post-Newtonian order, from the requirement of matching to a post-Newtonian solution. In the present paper, we shall further show that the radiation-reaction functions $A_L^{\mu\nu}(t)$, parametrizing the post-Newtonian solution, are also uniquely fixed, up to any post-Newtonian order, by the matching. In particular, we shall find that the latter functions include correctly the contribution of wave tails, arising at the 4 PN order, as determined in Refs. [31–33]. We shall also recover by a different method the result of Ref. [40] concerning the multipole moments $X_L^{\mu\nu}(t)$.

A comment is in order regarding the possibility of determining the near-zone field by matched asymptotic expansions up to *any* post-Newtonian order. Indeed, the method pre-supposes the existence of the exterior near zone for which $a<r\ll\lambda$. Now if a given physical system, whose dynamics is described by Newton's theory, emits gravitational radiation at some Newtonian fundamental wavelength $\lambda_{\rm N}$, we expect that when taking into account the post-Newtonian corrections up to the post-Newtonian order *n*, it will have a radiation spectrum composed of harmonics between $\sim 2\lambda_{\rm N}/n$ and $\sim 2\lambda_{\rm N}$. Indeed, this is the case of the radiation from a binary system moving on a circular orbit, for which we have $[2/(n+2)]\lambda_{\rm N}\leqslant\lambda_{n\,\rm PN}\leqslant 2\lambda_{\rm N}$. Therefore, if *n* is large enough, say $n\gtrsim 2\lambda_{\rm N}/a$, we expect that there will be some part of the radiation whose frequency is too high for the exterior near zone to exist. What we want to say is that the formulas we shall obtain for the post-Newtonian field of a source "up to any order" are indeed physically valid, strictly speaking, only up to some finite post-Newtonian order $\sim 2\lambda_{\rm N}/a$, where *a* is the size of the source, but that, if we consider a source which is less relativistic, for instance which is obtained by "slowing down" our source so that its Newtonian wavelength gets twice its original value (say), the *same* post-Newtonian formulas can then be used for the new source up to approximately twice the previous post-Newtonian order.

The plan of this paper is as follows. In Sec. II, we recall the construction in Ref. [36] of the multipole expansion of the external field, and we obtain thanks to a result of Ref. [32] the near-zone expansion of that external field ready for subsequent matching. In Sec. III, we implement the post-Newtonian iteration of the inner field inside the matter source, and we find the far-zone (multipolar) expansion of that post-Newtonian solution, also ready for matching. In Sec. IV, we show that the matching works up to any post-Newtonian order, and permits the determination of all the unknowns, in both the external and inner fields. Finally, in Sec. V we check that our post-Newtonian solution satisfies the harmonic-coordinate condition as a consequence of the equations of motion of the source. The technical proofs are relegated to Appendixes A, B, and C.

B. Notation for the Einstein field equations

For the problem at hand, let us introduce an asymptotically Minkowskian coordinate system for which the basic gravitational-wave amplitude, $h^{\mu\nu}=\sqrt{-g}g^{\mu\nu}-\eta^{\mu\nu}$, is divergenceless, i.e., satisfies the de Donder or harmonic gauge condition $\partial_\mu h^{\mu\nu}=0$. Here, $g^{\mu\nu}$ denotes the contravariant metric (satisfying $g^{\mu\rho}g_{\rho\nu}=\delta^\mu_\nu$), g is the determinant of the covariant metric, $g=\det(g_{\mu\nu})$, and $\eta^{\mu\nu}$ represents an auxiliary Minkowskian metric with signature $+2$. With these definitions the Einstein field equations can be recast into the d'Alembertian equation

$$\Box h^{\mu\nu}=\frac{16\pi G}{c^4}\tau^{\mu\nu}, \tag{1.1}$$

where $\Box=\eta^{\mu\nu}\partial_\mu\partial_\nu=-1/c^2\partial^2/\partial t^2+\Delta$ is the (flat space-time) d'Alembertian operator. The source term, $\tau^{\mu\nu}$, can rightly be interpreted as the "effective" stress-energy pseudotensor of the matter and gravitational fields in harmonic coordinates. It is conserved in the usual sense, and that is equivalent to the condition of harmonic coordinates:

$$\partial_\mu h^{\mu\nu}=0\Leftrightarrow\partial_\mu\tau^{\mu\nu}=0. \tag{1.2}$$

The pseudotensor $\tau^{\mu\nu}$ is made of the contribution of the matter fields, described by a stress-energy tensor $T^{\mu\nu}$, and the one due to the gravitational field, given by the gravitational source term $\Lambda^{\mu\nu}$; thus,

$$\tau^{\mu\nu}=|g|T^{\mu\nu}+\frac{c^4}{16\pi G}\Lambda^{\mu\nu}. \tag{1.3}$$

The conservation property (1.2) is equivalent to the conservation, in the covariant sense, of the matter tensor: $\nabla_\mu T^{\mu\nu}=0$. The exact expression of $\Lambda^{\mu\nu}$, taking into account all the nonlinearities of the Einstein field equations, reads

$$\begin{aligned}\Lambda^{\mu\nu}=&-h^{\rho\sigma}\partial^2_{\rho\sigma}h^{\mu\nu}+\partial_\rho h^{\mu\sigma}\partial_\sigma h^{\nu\rho}+\frac{1}{2}g^{\mu\nu}g_{\rho\sigma}\partial_\lambda h^{\rho\tau}\partial_\tau h^{\sigma\lambda}\\&-g^{\mu\rho}g_{\sigma\tau}\partial_\lambda h^{\nu\tau}\partial_\rho h^{\sigma\lambda}-g^{\nu\rho}g_{\sigma\tau}\partial_\lambda h^{\mu\tau}\partial_\rho h^{\sigma\lambda}\\&+g_{\rho\sigma}g^{\lambda\tau}\partial_\lambda h^{\mu\rho}\partial_\tau h^{\nu\sigma}+\frac{1}{8}(2g^{\mu\rho}g^{\nu\sigma}-g^{\mu\nu}g^{\rho\sigma})\\&\times(2g_{\lambda\tau}g_{\epsilon\pi}-g_{\tau\epsilon}g_{\lambda\pi})\partial_\rho h^{\lambda\pi}\partial_\sigma h^{\tau\epsilon}.\end{aligned} \tag{1.4}$$

It is clear from this expression that $\Lambda^{\mu\nu}$ is made of terms which are at least quadratic in the gravitational-field strength $h^{\mu\nu}$ and its first and second space-time derivatives.

In this paper, we look for the solutions of the field equations (1.1)–(1.4) under the following hypotheses. First, we assume that the matter tensor $T^{\mu\nu}$ has a spatially compact support, i.e., can be enclosed into some timelike world tube, say $r \leqslant a$, where $r = |\mathbf{x}|$ is the harmonic-coordinate radial distance. Second, we assume that the matter distribution inside the source is smooth: i.e., $T^{\mu\nu}(\mathbf{x},t) \in C^{\infty}(\mathbb{R}^4)$. We have in mind a smooth hydrodynamical "fluid" system, without any singularities or shocks (*a priori*), that is described by some Eulerian-type equations including high relativistic (post-Newtonian) corrections. In particular, we exclude from the start any sources containing black holes. Notice, however, that it makes sense to apply the formulas derived *a priori* only for smooth matter distributions to systems containing compact objects (including black holes), described by some sort of point-particle singularities; see, e.g., Refs. [41–45]. Finally, in order to select the physically sensible solution of the field equations, we choose some boundary conditions at infinity corresponding to the famous no-incoming-radiation condition. In this paper, we shall rely on a specific construction of the metric outside the domain of the source ($r > a$), which was achieved in Ref. [36] under the assumption that the gravitational field has been independent of time (stationary) in some remote past, in the sense that $t \leqslant -\mathcal{T} \Rightarrow \partial/\partial t[h^{\mu\nu}(\mathbf{x},t)] = 0$. This condition is a means to impose, by brute force, the no-incoming-radiation condition.[3]

II. EXTERIOR FIELD

A. Multipolar expansion of the nonlinear vacuum field

In this section, we review some material from Ref. [36] concerning the construction of *vacuum* metrics by means of mixed multipolar and post-Minkowskian (MPM) expansions. The so-called MPM metrics aim at describing the gravitational field in the region exterior to a general isolated system. In fact they are mathematically defined in the open domain $\mathbb{R}^3_* \times \mathbb{R}$, i.e., $\mathbb{R}^4$ deprived from the spatial origin $r \equiv |\mathbf{x}| = 0$, but of course they do not agree physically with the real solution when $0 < r < a$, since they are vacuum solutions. For our present purpose the point is that the most general physically admissible solution of the vacuum field equations has been obtained in Ref. [36] by a specific construction of the post-Minkowskian solution, say

$$h^{\mu\nu}_{\mathrm{ext}} = \sum_{m=1}^{+\infty} G^m h^{\mu\nu}_{(m)}, \tag{2.1}$$

whose coefficients are in the form of multipolar series, or equivalently decompositions in symmetric-trace-free (STF) products of unit vectors $\hat{n}_L$, that are equivalent to the usual decomposition in spherical harmonics [49]:

$$\forall m \geqslant 1, \quad h^{\mu\nu}_{(m)}(\mathbf{x},t) = \sum_{l=0}^{+\infty} \hat{n}_L(\theta,\phi) h^{\mu\nu}_{(m)L}(r,t). \tag{2.2}$$

The $h^{\mu\nu}_{(m)L}$'s are certain functions of the radial coordinate r and of time t. Inserting the MPM expansion (2.1) and (2.2) into the vacuum field equations (1.1) and (1.2) we obtain, at any post-Minkowskian order m,

$$\Box h^{\mu\nu}_{(m)} = \Lambda^{\mu\nu}_{(m)}[h_{(1)}, \ldots, h_{(m-1)}], \tag{2.3}$$

$$\partial_\mu h^{\mu\nu}_{(m)} = 0, \tag{2.4}$$

where $\Lambda^{\mu\nu}_{(m)}$ denotes the mth post-Minkowskian piece of the gravitational source term defined by Eq. (1.4), i.e., in which we have inserted the previous post-Minkowskian iterations up to the previous order $m-1$ (with the convention that $\Lambda^{\mu\nu}_{(1)} = 0$). Because Eq. (1.4) is at least quadratic in nonlinearities, it is clear that only the preceding iterations, $\leqslant m-1$, are necessary at any post-Minkowskian order m.

Now the solution that was obtained in Ref. [36] has two main characteristics. The first one is related to its particular near-zone structure, which will play a fundamental role in the present paper. Namely, it was proved that each one of the multipolar–post-Minkowskian coefficients $h^{\mu\nu}_{(m)}$ in Eq. (2.1)—that we recall are only defined when $r > 0$—admits a singular near-zone expansion, i.e., when $r \to 0$, with the following structure:

$$\forall N \in \mathbb{N},$$

$$h^{\mu\nu}_{(m)}(\mathbf{x},t) = \sum_{l,a,p} \hat{n}_L r^a (\ln r)^p F^{\mu\nu}_{(m)L,a,p}(t) + R^{\mu\nu}_{(m)N}(\mathbf{x},t), \tag{2.5}$$

where the multipolar order $l \in \mathbb{N}$, where the powers of r are such that $a \in \mathbb{Z}$ with $a_{\min} \leqslant a \leqslant N$ (with $a_{\min}$ a negative integer), and where the powers of $\ln r$ are $p \in \mathbb{N}$ with $p \leqslant m-1$. The maximal divergence when $r \to 0$ occurs for $a_{\min}$, which depends on the post-Minkowskian order m, and satisfies $a_{\min}(m) \to -\infty$ when $m \to +\infty$. Similarly, the maximal power of the logarithms, $p_{\max}(m) = m-1$, tends to infinity with m. The functions $F^{\mu\nu}_{(m)L,a,p}(t)$ are smooth functions of time, $F^{\mu\nu}_{(m)L,a,p} \in C^{\infty}(\mathbb{R})$, which are to be computed by means of the algorithm proposed in Ref. [36], and appear as complicated nonlinear functionals of some more elementary functions parametrizing the linearized ($m=1$) approximation. The remainder term in Eq. (2.5) is such that

$$R^{\mu\nu}_{(m)N}(\mathbf{x},t) = O(r^N) \quad \text{when} \quad r \to 0 \quad \text{and} \quad t = \text{const}. \tag{2.6}$$

The Landau O-symbol takes its usual meaning. This remainder admits also some specific differentiability properties (refer to [36] for the details). The gravitational source term $\Lambda^{\mu\nu}_{(m)}$ admits exactly the same near-zone structure as in Eq. (2.5) with the exception that $p_{\max} = m-2$ in this case (that is, the maximal power of the logarithms increases by one unit when going from the source to the solution).

[3] However, the condition of stationarity in the past, though much weaker than the actual no-incoming radiation condition, does not seem to entail any physical restriction on the applicability of the formalism, even in the case of sources which have always been radiating.

The second important characteristic of the MPM solution concerns the constructive formula which defines it. We find that each one of the post-Minkowskian coefficients $h^{\mu\nu}_{(m)}$ is explicitly constructed by means of the following [36]:

$$h^{\mu\nu}_{(m)} = \underset{B=0}{\mathrm{FP}}\, \Box^{-1}_{\mathrm{Ret}}[\widetilde{r}^B \Lambda^{\mu\nu}_{(m)}] + \sum_{l=0}^{+\infty} \hat{\partial}_L \left\{ \frac{1}{r} X^{\mu\nu}_{(m)L}\left(t - \frac{r}{c}\right)\right\}. \tag{2.7}$$

The first term involves a special type of generalized inverse d'Alembertian operator, built on the standard retarded integral,

$$\Box^{-1}_{\mathrm{Ret}}[\widetilde{r}^B \Lambda^{\mu\nu}_{(m)}](\mathbf{x},t) \equiv -\frac{1}{4\pi}\int_{\mathbb{R}^3} \frac{d^3\mathbf{y}}{|\mathbf{x}-\mathbf{y}|} |\widetilde{\mathbf{y}}|^B \Lambda^{\mu\nu}_{(m)}(\mathbf{y}, t - |\mathbf{x}-\mathbf{y}|/c), \tag{2.8}$$

which extends over the whole three-dimensional space, but inside which a regularization factor has been ''artificially'' introduced, namely

$$\widetilde{r}^B \equiv \left(\frac{r}{r_0}\right)^B, \tag{2.9}$$

where B denotes a complex number, $B \in \mathbb{C}$, and r_0 represents an arbitrary constant length scale. The indication $\underset{B=0}{\mathrm{FP}}$ stands for the *finite part* at $B=0$, and means that one should first compute the Laurent expansion when $B \to 0$ of (the analytic continuation of) the B-dependent integral (2.8), and, second, pick up the finite part at $B=0$ in that expansion, i.e., the coefficient of the zeroth power of B. The main property of this generalized retarded operator, which we shall from now on abbreviate as

$$\widetilde{\Box^{-1}_{\mathrm{Ret}}}[\Lambda^{\mu\nu}_{(m)}] \equiv \underset{B=0}{\mathrm{FP}}\, \Box^{-1}_{\mathrm{Ret}}[\widetilde{r}^B \Lambda^{\mu\nu}_{(m)}], \tag{2.10}$$

is that, for source terms $\Lambda^{\mu\nu}_{(m)}$ admitting a near-zone structure of the type (2.5),

$$\Box[\widetilde{\Box^{-1}_{\mathrm{Ret}}} \Lambda^{\mu\nu}_{(m)}] = \Lambda^{\mu\nu}_{(m)}. \tag{2.11}$$

Because the second term in Eq. (2.7) is a retarded solution of the *source-free* wave equation, we see therefore that $h^{\mu\nu}_{(m)}$ represents indeed a solution of the wave equation we had to solve: $\Box h^{\mu\nu}_{(m)} = \Lambda^{\mu\nu}_{(m)}$. However, this is not sufficient because we have also to solve the harmonic-coordinate condition (1.2). We shall refer to [36] for the definition of an algorithm which permits us to compute, simply from the algebraic and differential structure of the vacuum field equations, the necessary form of the second term in Eq. (2.7), in such a way that the harmonic-coordinate condition will be satisfied: $\partial_\mu h^{\mu\nu}_{(m)} = 0$. In fact, we shall not need, in the following, to be more precise about the latter term; simply we keep it in the form of a general retarded solution of the source-free wave equation, parametrized by some tensorial functions $X^{\mu\nu}_{(m)L}(t)$. We assume that these functions are STF with respect to the multi-index L: i.e., $X^{\mu\nu}_{(m)L} \equiv \hat{X}^{\mu\nu}_{(m)L}$, so the multiderivative $\hat{\partial}_L$ in Eq. (2.7) is a STF one (see Ref. [49] for the notation). The latter construction represents the most general physical solution of the field equations outside the source [36].

Let us now proceed with the formal resummation of the post-Minkowskian series. That is, once the results (2.5)–(2.7) have been established for any order m, we sum them from $m=1$ up to infinity. In this way, we obtain some formulas which are valid formally for the complete post-Minkowskian series, and, presumably, could hold true in a more rigorous context of exact solutions. After summation we shall ''forget'' about the post-Minkowskian expansion, and consider that the exterior field $h^{\mu\nu}_{\mathrm{ext}}$ represents merely the *multipole* decomposition of the actual field $h^{\mu\nu}$ outside the compact support of the source (nevertheless, it is wise to keep in mind that the solution came from a formal post-Minkowskian summation). We denote the multipole decomposition by means of the calligraphic letter $\mathcal{M}$. Therefore, our *definition* is that the multipole expansion $\mathcal{M}(h^{\mu\nu})$ of the field outside the isolated source is merely the external solution constructed previously by means of the MPM method, and resummed over the post-Minkowskian index m:

$$\mathcal{M}(h^{\mu\nu}) \equiv h^{\mu\nu}_{\mathrm{ext}}. \tag{2.12}$$

This definition is quite legitimate (and rather obvious) because we know that the MPM metric constitutes the most general solution for the exterior field. Thus, $\mathcal{M}(h^{\mu\nu})$ is a solution of the vacuum field equations, now considered outside the physical domain of the source, $r > a$ (while $h^{\mu\nu}_{\mathrm{ext}}$ had been constructed for any $r > 0$). In that domain, we have evidently the numerical equality

$$\mathcal{M}(h^{\mu\nu}) = h^{\mu\nu} \quad (\text{when} \quad r > a). \tag{2.13}$$

After summation of Eqs. (2.5) and (2.6) over m, we get the near-zone structure

$$\forall N \in \mathbb{N}, \quad \mathcal{M}(h^{\mu\nu}) = \sum \hat{n}_L r^a (\ln r)^p F^{\mu\nu}_{L,a,p}(t) + O(r^N), \tag{2.14}$$

in which the functions $F^{\mu\nu}_{L,a,p}(t) = \Sigma_{m=1}^{+\infty} G^m F^{\mu\nu}_{(m)L,a,p}(t)$, and where $a \leqslant N$ and $p \geqslant 0$. Notice that there is no lower bound for a because $a_{\min}(m) \to -\infty$ when $m \to +\infty$; similarly there is no upper bound for p. Secondly, coming to the constructive formula (2.7) we obtain

$$\mathcal{M}(h^{\mu\nu}) = \widetilde{\Box^{-1}_{\mathrm{Ret}}}[\mathcal{M}(\Lambda^{\mu\nu})] + \sum_{l=0}^{+\infty} \hat{\partial}_L \left\{ \frac{1}{r} X^{\mu\nu}_L\left(t - \frac{r}{c}\right)\right\}, \tag{2.15}$$

where $X^{\mu\nu}_L(t) = \Sigma_{m=1}^{+\infty} G^m X^{\mu\nu}_{(m)L}(t)$. In the following, we shall regard the STF functions $X^{\mu\nu}_L(t)$ as the ''multipole moments'' of the source, because they describe the physics of the source as seen from the exterior. We do not need to be more precise at this point. Let us simply comment that by imposing the harmonic-gauge condition (1.2) we find that there are only six components of these functions which are independent, and this yields the definition of six independent STF source multipole moments $I_L(t),\ J_L(t),\cdots$ (see Ref. [40] for the precise definition). Furthermore, the multipole-moment functions $X_L^{\mu\nu}(t)$ have already been calculated in terms of the stress-energy tensor of a post-Newtonian source in Ref. [40]. However, we prefer to leave these functions undetermined because we shall recover their expressions by means of a somewhat different method, and the agreement we shall find with the result of Ref. [40] will constitute a crucial check of our computation.

B. Near-zone expansion of the multipole decomposition

In anticipation of the matching we consider next the infinite near-zone reexpansion, when $r\to 0$, of the multipole expansion $\mathcal{M}(h^{\mu\nu})$ determined in Eq. (2.15). We have already obtained the general structure of that expansion, given by Eq. (2.14). Let us denote with the help of some overline the *infinite* near-zone expansion (without remainder), whose structure is therefore given by

$$\overline{\mathcal{M}(h^{\mu\nu})}=\sum \hat{n}_L r^a (\ln r)^p F_{L,a,p}^{\mu\nu}(t), \tag{2.16}$$

where $a\in\mathbb{Z}$ and $p\in\mathbb{N}$ (and, of course, the multipolar index $l\in\mathbb{N}$). We must be careful to distinguish the fully fledged multipole decomposition $\mathcal{M}(h^{\mu\nu})$, which is defined as soon as $r>0$ and numerically agrees with the exact solution wherever $r>a$ (in particular when $r\to+\infty$), from its formal near-zone reexpansion $\overline{\mathcal{M}(h^{\mu\nu})}$. Later we shall indicate the post-Newtonian expansion by means of the same overline notation. Indeed, the near-zone expansion is really an expansion when $r/\lambda\to 0$, which is equivalent to an expansion when $c\to+\infty$, since the wavelength of waves is $\lambda=cP$ (with P a typical period of the internal motion). From the result (2.15) we can write

$$\overline{\mathcal{M}(h^{\mu\nu})}=\overline{\widetilde{\Box^{-1}_{\mathrm{Ret}}}[\mathcal{M}(\Lambda^{\mu\nu})]}+\sum_{l=0}^{+\infty}\hat{\partial}_L\left\{\frac{\overline{X_L^{\mu\nu}(t-r/c)}}{r}\right\}. \tag{2.17}$$

The overline in the second term means that one should expand the retardations $t-r/c$ when $r/c\to 0$. More explicitly, we have

$$\hat{\partial}_L\left\{\frac{\overline{X_L^{\mu\nu}(t-r/c)}}{r}\right\}=\sum_{j=0}^{+\infty}\frac{(-)^j}{c^j j!}\hat{\partial}_L(r^{j-1})\overset{(j)}{X}{}_L^{\mu\nu}(t), \tag{2.18}$$

where the superscript (*j*) indicates *j* successive time derivations. The main problem is how to treat the first term in Eq. (2.17). What we essentially want is to know how one can "commute" the operations of taking the near-zone expansion and of applying the retarded integral. In fact, the problem has already been solved in Ref. [32], which succeeded in writing the first term in Eq. (2.17) as the sum of an "instantaneous" operator, acting on the near-zone expansion of the source, and of a particular "antisymmetric" wave (i.e., retarded minus advanced) solution of the source-free d'Alembertian equation. The result of Ref. [32], Eq. (3.2), reads

$$\overline{\widetilde{\Box^{-1}_{\mathrm{Ret}}}[\mathcal{M}(\Lambda^{\mu\nu})]}=\widetilde{\mathcal{I}^{-1}}[\overline{\mathcal{M}(\Lambda^{\mu\nu})}]-\frac{4G}{c^4}\sum_{l=0}^{+\infty}\frac{(-)^l}{l!}\hat{\partial}_L\left\{\frac{\overline{\mathcal{R}_L^{\mu\nu}(t-r/c)-\mathcal{R}_L^{\mu\nu}(t+r/c)}}{2r}\right\}. \tag{2.19}$$

For completeness we present in Appendix A the proof of this result—a version of it which is somewhat improved with respect to that given in Ref. [32]. The first term in Eq. (2.19) involves an operator $\widetilde{\mathcal{I}^{-1}}$, acting on each of the individual terms of the formal near-zone expansion whose structure is given by Eq. (2.16), and which is essentially defined by the solution of the wave equation that is obtained by iterated use of inverse Laplace operators, and regularized by means of our *B*-dependent finite part procedure. Thus,

$$\widetilde{\mathcal{I}^{-1}}[\overline{\mathcal{M}(\Lambda^{\mu\nu})}]=\underset{B=0}{\mathrm{FP}}\sum_{k=0}^{+\infty}\left(\frac{\partial}{c\,\partial t}\right)^{2k}\Delta^{-k-1}[\tilde{r}^B\overline{\mathcal{M}(\Lambda^{\mu\nu})}], \tag{2.20}$$

where $\Delta^{-k-1}=(\Delta^{-1})^{k+1}$, and the action of the inverse Laplacian on the generic term of Eq. (2.16) follows from

$$\Delta^{-1}[\hat{n}_L r^{B+a}(\ln r)^p]=\left(\frac{d}{dB}\right)^p\left[\frac{\hat{n}_L r^{B+a+2}}{(B+a+2-l)(B+a+3+l)}\right] \tag{2.21}$$

[see also Eq. (A16) in Appendix A]. The operator $\widetilde{\mathcal{I}^{-1}}$ plays the central role in the present paper. It can be regarded as (the regularization of) the formal post-Newtonian expansion, when $c\to+\infty$, of the inverse d'Alembert operator, say $\mathcal{I}^{-1}=\overline{1/\Box}=\overline{1/[\Delta-(1/c^2)\partial_t^2]}$. We can refer to $\mathcal{I}^{-1}$ as the operator of the instantaneous potentials, because it acts on the time variable *t* only through time derivations, instead of involving a full integration as for the operator of the retarded potentials $\Box^{-1}_{\mathrm{Ret}}$. Notice that $\mathcal{I}^{-1}$ is closely related to the operator of the symmetric potentials, $\frac{1}{2}[\Box^{-1}_{\mathrm{Ret}}+\Box^{-1}_{\mathrm{Adv}}]$; see Ref. [32] for discussion and the precise relation between these operators. As for the second term in Eq. (2.19), it is made of an "antisymmetric" wave, which represents in fact a solution of the d'Alembertian equation that is regular in a neighborhood of the origin $r=0$. Its near-zone expansion $r/c\to 0$ is composed only of terms containing some odd powers of $1/c$:

$$\hat{\partial}_L\left\{\frac{\overline{\mathcal{R}_L^{\mu\nu}(t-r/c)-\mathcal{R}_L^{\mu\nu}(t+r/c)}}{2r}\right\}$$
$$=-\sum_{i=l}^{+\infty}\frac{\hat{\partial}_L(r^{2i})}{(2i+1)!}\frac{\overset{(2i+1)}{\mathcal{R}_L^{\mu\nu}}(t)}{c^{2i+1}}$$
$$=-\frac{1}{(2l+1)!!}\sum_{k=0}^{+\infty}\widetilde{\Delta^{-k}}(\hat{x}_L)\frac{\overset{(2k+2l+1)}{\mathcal{R}_L^{\mu\nu}}(t)}{c^{2k+2l+1}}. \quad (2.22)$$

See also Eqs. (2.4)–(2.7) in Ref. [32] for alternative forms of the antisymmetric wave. In the second of Eqs. (2.22), we have introduced the useful object

$$\widetilde{\Delta^{-k}}(\hat{x}_L)=\frac{(2l+1)!!}{(2k)!!(2l+2k+1)!!}r^{2k}\hat{x}_L, \quad (2.23)$$

which represents the iterated Laplacian operator $\widetilde{\Delta^{-k}}$, regularized by means of the $\mathrm{FP}_{B=0}$ process, acting on $\hat{x}_L$, which denotes the STF projection of the product $x_L\equiv x_{i_1}\cdots x_{i_l}$ [49]. [See also Eq. (C18) for an alternative expression of the same object.] From Ref. [32], or from Eq. (A11) in Appendix A, we get the expression of the functions parametrizing the antisymmetric waves,

$$\mathcal{R}_L^{\mu\nu}(t)=\mathrm{FP}_{B=0}\int d^3\mathbf{x}|\tilde{\mathbf{x}}|^B\hat{x}_L\int_1^{+\infty}dz\,\gamma_l(z)$$
$$\times\mathcal{M}(\tau^{\mu\nu})(\mathbf{x},t-z|\mathbf{x}|/c), \quad (2.24)$$

where $|\tilde{\mathbf{x}}|=|\mathbf{x}|/r_0$ [see Eq. (2.9)]. These functions depend on the whole past history of the source [50]. The z integration involves the weighting function defined by

$$\gamma_l(z)=(-)^{l+1}\frac{(2l+1)!!}{2^l l!}(z^2-1)^l. \quad (2.25)$$

This function is normalized so that $\int_1^{+\infty}dz\,\gamma_l(z)=1$, where the value of the integral is obtained by analytic continuation for $l\in\mathbb{C}$ (see Appendix A). As shown in Ref. [32] (see notably Sec. III D there), the antisymmetric waves in Eq. (2.19) are associated with gravitational radiation reaction effects of a nonlinear origin. In particular, they contain the contribution of wave tails in the radiation reaction force, which appears at the 1.5 PN order relative to the lowest-order radiation damping, i.e., 4 PN order in the equations of motion [31]. To summarize this subsection, we have obtained the near-zone reexpansion of the multipole expansion $\mathcal{M}(h^{\mu\nu})$ as

$$\overline{\mathcal{M}(h^{\mu\nu})}=\widetilde{\mathcal{I}^{-1}}[\overline{\mathcal{M}(\Lambda^{\mu\nu})}]$$
$$-\frac{4G}{c^4}\sum_{l=0}^{+\infty}\frac{(-)^l}{l!}\hat{\partial}_L\left\{\frac{\overline{\mathcal{R}_L^{\mu\nu}(t-r/c)-\mathcal{R}_L^{\mu\nu}(t+r/c)}}{2r}\right\}$$
$$+\sum_{l=0}^{+\infty}\hat{\partial}_L\left\{\frac{\overline{X_L^{\mu\nu}(t-r/c)}}{r}\right\}. \quad (2.26)$$

The functions $\mathcal{R}_L^{\mu\nu}(t)$ are known from Eq. (2.24), but the multipole moments $X_L^{\mu\nu}(t)$ have not yet been specified at this stage (though they have already been calculated in Ref. [40]). Therefore, we have succeeded in computing the near-zone expansion $\overline{\mathcal{M}(h^{\mu\nu})}$ as a functional of the sole unknown constituted by the $X_L^{\mu\nu}$'s; only by matching can these functions be determined.

III. INTERIOR FIELD

A. Post-Newtonian expansion of the near-zone field

Up to now, we have solved the Einstein field equations in the vacuum outside an isolated source ($r>a$), without any reference to the stress-energy tensor $T^{\mu\nu}$ of the matter source. Our next task is to investigate the field equations inside and in the vicinity of the matter source, and more precisely in the so-called near zone, or the region for which $r\ll\lambda$, where λ is the typical wavelength of the emitted waves. From now on we restrict our attention to a post-Newtonian source, whose radius is $a\ll\lambda$. For post-Newtonian sources, the near zone overlaps with the external region in what we shall refer to as the matching region, for which $a<r\ll\lambda$. In the matching region, both the multipolar expansion of the exterior field and the post-Newtonian expansion of the inner field are legitimate.

Let us denote by means of an overline the formal (infinite) post-Newtonian expansion of the field inside the source's near zone, $\overline{h}^{\mu\nu}$, which is of the form

$$\overline{h}^{\mu\nu}(\mathbf{x},t,c)=\sum_{n=2}^{+\infty}\frac{1}{c^n}\underset{n}{\overline{h}}{}^{\mu\nu}(\mathbf{x},t,\ln c). \quad (3.1)$$

By definition, the nth post-Newtonian coefficient $\underset{n}{\overline{h}}{}^{\mu\nu}$ is the factor of the nth power of $1/c$; however, we know from the structure of the near-zone expansion of the exterior field [see Eq. (2.16)] that the post-Newtonian expansion will involve also, besides the usual powers of $1/c$, some logarithms of c (in fact, when stating this we are anticipating the result of the matching). So the coefficients $\underset{n}{\overline{h}}{}^{\mu\nu}$ still depend on c via the logarithm of c, and from Eq. (2.16) we infer that they are in fact some power series in $\ln c$. The first appearance of $\ln c$ is at the 4 PN order (i.e., corresponding to a term $\sim\ln c/c^8$ in the equations of motion) and is associated with the physical effect of wave tails [31]. In Eq. (3.1), we have indicated that the expansion starts at the level $1/c^2$, but we could be more precise because the $0i$ component of $\overline{h}^{\mu\nu}$ starts only at the level $1/c^3$, while the ij component is at least of order $1/c^4$. This does not matter for our purpose; simply in our iteration we include these post-Newtonian coefficients as zero: $\underset{2}{\overline{h}}{}^{0i}=0$ and $\underset{2}{\overline{h}}{}^{ij}=\underset{3}{\overline{h}}{}^{ij}=0$. For the total stress-energy pseudotensor (1.3) we have the same type of expansion,

$$\overline{\tau}^{\mu\nu}(\mathbf{x},t,c)=\sum_{n=-2}^{+\infty}\frac{1}{c^n}\underset{n}{\overline{\tau}}{}^{\mu\nu}(\mathbf{x},t,\ln c). \quad (3.2)$$

The expansion starts with a term of order c^2 corresponding to the rest mass energy of the source ($\overline{\tau}^{\mu\nu}$ has the dimension of an energy density). Here we shall always understand the infinite sums such as Eqs. (3.1) and (3.2) in the sense of *formal* power series, i.e., merely as an ordered collection of coefficients: e.g., $(\underset{n}{\overline{h}}{}^{\mu\nu})_{n\in\mathbb{N}}$. We do not attempt to control the mathematical nature of these series.

In this paper, we make two important assumptions. First, we assume that the post-Newtonian coefficients $\underset{n}{\overline{h}}{}^{\mu\nu}$ (and similarly $\underset{n}{\overline{\tau}}{}^{\mu\nu}$) are *smooth* functions of space-time,

$$\underset{n}{\overline{h}}{}^{\mu\nu}(\mathbf{x},t)\in C^\infty(\mathbb{R}^4). \tag{3.3}$$

Evidently this comes from our consideration of regular (smooth) extended matter distributions, described by $T^{\mu\nu}(\mathbf{x},t)\in C^\infty(\mathbb{R}^4)$, *a priori* excluding black holes or point-particle singularities. Second, we assume that the structure of the expansion at *spatial infinity*, i.e., $r\to+\infty$ with $t=$const, is of the type

$$\forall N\in\mathbb{N},\quad \underset{n}{\overline{h}}{}^{\mu\nu}=\sum \hat{n}_L r^a(\ln r)^p F^{\mu\nu}_{L,n,a,p}(t)+O\left(\frac{1}{r^N}\right) \tag{3.4}$$

(and similarly for each $\underset{n}{\overline{\tau}}{}^{\mu\nu}$). We have purposely written an expansion which is very similar to the one in Eq. (2.14), because as we shall see the functions $F_{L,n,a,p}(t)$ will be equal to the post-Newtonian coefficients of the functions $F_{L,a,p}(t)$ appearing in Eq. (2.14). However, it is important to realize that in contrast to Eq. (2.14), which is a near-zone expansion [cf. the remainder $O(r^N)$], the expansion written in Eq. (3.4) is a *far-zone* one, with remainder $O(1/r^N)$. It would have been clearer to write the latter expansion with some $(\ln r)^p/r^b$ with $b=-a$, but since we are going to show, from the method of matched asymptotic expansions, that the infinite far-zone expansion (ignoring the remainder) is actually the *same* as the infinite near-zone expansion, it is better to write it in this form, with the range of the powers of r in Eq. (3.4) being $-a\leqslant N$ instead of $a\leqslant N$ in Eq. (2.14). In doing so, we are again anticipating the result of the matching. Finally, we assume that, at any given post-Newtonian order n, the maximal divergency of the far-zone expansion (3.4) is finite, i.e., there exists some $a_{\max}(n)\in\mathbb{N}$ such that $a\leqslant a_{\max}(n)$.

Next we perform the iteration of the post-Newtonian field (3.1) up to any order. Our strategy consists of finding the general post-Newtonian solution of the relaxed Einstein field equation (1.1). This solution will depend on some arbitrary ‘‘homogeneous’’ solutions, in the form of harmonic solutions solving the source-free d'Alembertian equation (in a perturbative post-Newtonian sense). In a second stage, we shall obtain these harmonic solutions by imposing the matching to the external multipolar field obtained in Sec. II. Finally, we shall check that our post-Newtonian solution is divergenceless, i.e., it satisfies the harmonic-coordinate condition (1.2), as a consequence of the conservation of the stress-energy pseudotensor $\tau^{\mu\nu}$. Notice that we do not try to incorporate into the post-Newtonian series the boundary conditions at infinity (viz. the no-incoming-radiation condition). Indeed, this is impossible at the level of the post-Newtonian expansion considered alone, because its validity is limited to the near zone. Even if we define an ‘‘improved’’ post-Newtonian series by considering some *retarded* integrals that are formally expanded when $c\to+\infty$ as series of Poisson-like integrals [11], we ultimately end up with an inconsistency, because the Poisson-like integrals are some local-in-time functionals, depending on the source only at the current time t, and we know that the post-Newtonian field starts to depend on the whole past history of the source from the 4 PN order [31–33]. Therefore, we do not follow this route in the present paper, and, instead, we incorporate into the post-Newtonian series the boundary conditions concerning the wave field at infinity by means of the matching equation.

We insert the post-Newtonian ansatz (3.1),(3.2) into the ‘‘relaxed’’ Einstein field equation (1.1), and equate together the powers of $1/c$. The result is an infinite set of Poisson-type equations:

$$\forall n\geqslant 2,\quad \Delta\underset{n}{\overline{h}}{}^{\mu\nu}=16\pi G\underset{n-4}{\overline{\tau}}{}^{\mu\nu}+\partial_t^2\underset{n-2}{\overline{h}}{}^{\mu\nu}. \tag{3.5}$$

Evidently, the second term comes from the split of the d'Alembertian operator into a Laplacian and a second time derivative: $\Box=\Delta-(1/c^2)\partial_t^2$; the time derivative $\partial_0=(1/c)\partial_t$ is smaller than the spatial gradient ∂_i by a factor $1/c$—this is the basic tenet of the approximation. When $n=2$ and $n=3$, the second term in Eq. (3.5) is zero, which we take into account by assuming that $\underset{0}{\overline{h}}{}^{\mu\nu}=\underset{1}{\overline{h}}{}^{\mu\nu}=0$. We proceed by induction, i.e., we fix some post-Newtonian order n, assume that we succeeded in constructing the sequence of previous coefficients $\underset{2}{\overline{h}}{}^{\mu\nu},\ldots,\underset{n-1}{\overline{h}}{}^{\mu\nu}$, and from this we infer the next-order coefficient $\underset{n}{\overline{h}}{}^{\mu\nu}$. The most general solution consists of the sum of a particular solution and of the most general admissible solution of the homogeneous equation, which is simply the source-free Laplace equation. Let us first find a particular solution. We recalled in the Introduction that the usual Poisson integral cannot be used to define a solution, because the bound at infinity becomes rapidly divergent when going to higher and higher post-Newtonian orders. Fortunately, thanks to our two assumptions (3.3) and (3.4), we shall be able to define a *generalized* notion of a Poisson integral, in a way similar to our previous definition of a retarded integral operator in Eq. (2.10). That generalized Poisson integral will constitute an appropriate solution of the post-Newtonian equation. For any source term like $\underset{n}{\overline{\tau}}{}^{\mu\nu}$ which is at once smooth, Eq. (3.3), and admits a far-zone expansion of the type (3.4) [note that Eqs. (3.3) and (3.4) hold for $\underset{n}{\overline{h}}{}^{\mu\nu}$ as well as for $\underset{n}{\overline{\tau}}{}^{\mu\nu}$], we multiply it by the same regularization factor as in Eq. (2.10), and then apply the standard Poisson integral. The result,

$$\Delta^{-1}[\tilde{r}^B \underset{n}{\overline{\tau}}{}^{\mu\nu}](\mathbf{x},t) = -\frac{1}{4\pi}\int_{\mathbb{R}^3}\frac{d^3\mathbf{y}}{|\mathbf{x}-\mathbf{y}|}|\tilde{\mathbf{y}}|^B \underset{n}{\overline{\tau}}{}^{\mu\nu}(\mathbf{y},t), \tag{3.6}$$

where $|\tilde{\mathbf{y}}|^B \equiv |\mathbf{y}/r_0|^B$, defines a certain function of $B\in\mathbb{C}$. The definiteness of that integral relies heavily on the behavior at the bound at infinity. There is no problem with the vicinity of the origin because of the smoothness of the integrand. From the asymptotic expansion (3.4), with $a \leqslant a_{\max}$ [recall that $a_{\max}=a_{\max}(n)$], we find that the integral converges at infinity when $\mathrm{Re}(B) < -a_{\max}-2$. Next we can prove that the latter function of B generates a (unique) analytic continuation down to a neighborhood of the origin $B=0$, except at $B=0$ itself, at which value it admits a Laurent expansion with multiple poles up to some finite order. More details are given in Appendix B. Then, we consider the Laurent expansion of that function when $B\to 0$ and pick up the finite part, or coefficient of the zeroth power of B, of that expansion. This defines our generalized Poisson integral:

$$\widetilde{\Delta^{-1}}[\underset{n}{\overline{\tau}}{}^{\mu\nu}] \equiv \underset{B=0}{\mathrm{FP}}\,\Delta^{-1}[\tilde{r}^B \underset{n}{\overline{\tau}}{}^{\mu\nu}]. \tag{3.7}$$

The finite-part symbol $\mathrm{FP}_{B=0}$ has exactly the same meaning as in Eq. (2.10). However, notice that in contrast to Eq. (2.10) where the regularization factor $\tilde{r}^B$ dealt with the singularity when $r\to 0$, and hence supposes initially that $\mathrm{Re}(B)$ is a large positive number, in Eq. (3.7) the regularization concerns the behavior of the integral when $r\to+\infty$, and so one must start with the situation where $\mathrm{Re}(B)$ is a large *negative* number. The main properties of our generalized Poisson operator are that it solves the Poisson equation,

$$\Delta[\widetilde{\Delta^{-1}}\underset{n}{\overline{\tau}}{}^{\mu\nu}] = \underset{n}{\overline{\tau}}{}^{\mu\nu}, \tag{3.8}$$

and that the solution $\widetilde{\Delta^{-1}}\underset{n}{\overline{\tau}}{}^{\mu\nu}$ has the same properties as the source $\underset{n}{\overline{\tau}}{}^{\mu\nu}$, i.e., the smoothness, Eq. (3.3), and the particular far-zone expansion given by Eq. (3.4). These facts are proved in Appendix B. Therefore, we have found a *particular* solution of the Poisson equation, and, furthermore, this solution can be iterated at will, because the operator $\widetilde{\Delta^{-1}}$ keeps the same properties from the source to the corresponding solution. Quite naturally we denote the iterated Poisson operator $\widetilde{\Delta^{-k-1}} \equiv (\widetilde{\Delta^{-1}})^{k+1}$; it is not difficult to show that

$$\widetilde{\Delta^{-k-1}}[\underset{n}{\overline{\tau}}{}^{\mu\nu}](\mathbf{x},t) = -\frac{1}{4\pi}\underset{B=0}{\mathrm{FP}}\int_{\mathbb{R}^3} d^3\mathbf{y}\frac{|\mathbf{x}-\mathbf{y}|^{2k-1}}{(2k)!}|\tilde{\mathbf{y}}|^B \underset{n}{\overline{\tau}}{}^{\mu\nu}(\mathbf{y},t). \tag{3.9}$$

From that integral we obtain the operator of the "instantaneous" potentials exactly in the same way as in Eq. (2.20), but now acting on post-Newtonian coefficients such as $\underset{n}{\overline{\tau}}{}^{\mu\nu}$, i.e., satisfying both Eqs. (3.3) and (3.4):

$$\widetilde{\mathcal{I}^{-1}}[\underset{n}{\overline{\tau}}{}^{\mu\nu}] = \sum_{k=0}^{+\infty}\left(\frac{\partial}{c\,\partial t}\right)^{2k}\widetilde{\Delta^{-k-1}}[\underset{n}{\overline{\tau}}{}^{\mu\nu}]. \tag{3.10}$$

It is clear that we have a particular solution of d'Alembert's equation:

$$\Box[\widetilde{\mathcal{I}^{-1}}\underset{n}{\overline{\tau}}{}^{\mu\nu}] = \underset{n}{\overline{\tau}}{}^{\mu\nu}. \tag{3.11}$$

We can check that the definition we have proposed in Eqs. (2.20) and (2.21) is a particular case of the more general definition (3.9) and (3.10). Indeed, if we apply the formulas (3.9) and (3.10) to one of the terms composing the "far-zone" expansion of the post-Newtonian coefficient, i.e., $\hat{n}_L r^a (\ln r)^p F(t)$, we get the same result as the one resulting from Eqs. (2.20) and (2.21).

By means of the Poisson operator $\widetilde{\Delta^{-1}}$ so constructed, we first find a *particular* solution of Eq. (3.5):

$$\left(\underset{n}{\overline{h}}{}^{\mu\nu}\right)_{\mathrm{part}} = 16\pi G\,\widetilde{\Delta^{-1}}\underset{n-4}{\overline{\tau}}{}^{\mu\nu} + \partial_t^2\widetilde{\Delta^{-1}}\underset{n-2}{\overline{h}}{}^{\mu\nu}. \tag{3.12}$$

To this solution we add the most general solution of the homogeneous Laplace equation. It can be written, using the STF language, as the sum of two multipolar series, one of them being of the type $\hat{x}_L$, that is regular at the origin $r=0$ and the other one being like $\hat{\partial}_L(1/r)$, i.e., regular "at infinity" $r\to+\infty$ (see Ref. [49] for the notation). Imposing the smoothness condition (3.3) for the post-Newtonian field, we discard the second type $\sim\hat{\partial}_L(1/r)$ and retain as the only admissible homogeneous solution the first type $\sim\hat{x}_L$. Therefore, we find that there must exist some STF-tensorial functions of time, say $\underset{n}{B}{}_L^{\mu\nu}(t)$, such that

$$\left(\underset{n}{\overline{h}}{}^{\mu\nu}\right)_{\mathrm{hom}} = \sum_{l=0}^{+\infty}\underset{n}{B}{}_L^{\mu\nu}(t)\hat{x}_L. \tag{3.13}$$

The functions $\underset{n}{B}{}_L^{\mu\nu}(t)$ will be associated with the reaction of the field onto the source, and will depend on which boundary conditions are to be imposed on the gravitational field at infinity. The most general solution for the nth post-Newtonian coefficient thus reads

$$\underset{n}{\overline{h}}{}^{\mu\nu} = \left(\underset{n}{\overline{h}}{}^{\mu\nu}\right)_{\mathrm{part}} + \left(\underset{n}{\overline{h}}{}^{\mu\nu}\right)_{\mathrm{hom}}. \tag{3.14}$$

It is now trivial to iterate the process. We substitute for $\underset{n-2}{\overline{h}}{}^{\mu\nu}$ on the right-hand side of Eq. (3.12) the same expression but with n replaced by $n-2$, and similarly descend until we stop at either one of the coefficients $\underset{0}{\overline{h}^{\mu\nu}}=0$ or $\underset{1}{\overline{h}^{\mu\nu}}=0$. At this point, $\underset{n}{\overline{h}^{\mu\nu}}$ is expressed in terms of the "previous" $\underset{m}{\overline{\tau}^{\mu\nu}}$'s and $\underset{m}{B_L^{\mu\nu}}$'s, with $m \leqslant n-2$, i.e.,

$$\underset{n}{\overline{h}^{\mu\nu}} = 16\pi G \sum_{k=0}^{[n/2]-1} \partial_t^{2k} \widetilde{\Delta^{-k-1}}[\underset{n-4-2k}{\overline{\tau}^{\mu\nu}}] + \sum_{l=0}^{+\infty} \sum_{k=0}^{[n/2]-1} \underset{n-2k}{\overset{(2k)}{B_L^{\mu\nu}}}(t) \widetilde{\Delta^{-k}}(\hat{x}_L). \tag{3.15}$$

Here $[n/2]$ denotes the integer part of $n/2$; ∂_t^{2k} means the $2k$th partial time derivative $(\partial/\partial t)^{2k}$ and the superscript $(2k)$ the $2k$th total time derivative; the operator $\widetilde{\Delta^{-k-1}}$ is the one defined by Eq. (3.9); the object $\widetilde{\Delta^{-k}}(\hat{x}_L)$ has already been introduced in Eq. (2.23). Once we have the result (3.15), we "resum" it from $n=2$ up to infinity. After commuting the summations over n and k, we arrive at

$$\overline{h}^{\mu\nu} = \frac{16\pi G}{c^4} \widetilde{\mathcal{I}^{-1}}[\overline{\tau}^{\mu\nu}] + \sum_{l=0}^{+\infty} \sum_{k=0}^{+\infty} \frac{1}{c^{2k}} \overset{(2k)}{B_L^{\mu\nu}}(t) \widetilde{\Delta^{-k}}(\hat{x}_L), \tag{3.16}$$

where we have recognized the operator of the "instantaneous" potentials as defined by Eq. (3.10), and where the functions $B_L^{\mu\nu}$ read

$$B_L^{\mu\nu}(t) = \sum_{n=2}^{+\infty} \frac{1}{c^n} \underset{n}{B_L^{\mu\nu}}(t). \tag{3.17}$$

A more compact alternative form is

$$\overline{h}^{\mu\nu} = \frac{16\pi G}{c^4} \widetilde{\mathcal{I}^{-1}}[\overline{\tau}^{\mu\nu}] + \sum_{l=0}^{+\infty} \Delta \widetilde{\mathcal{I}^{-1}}[B_L^{\mu\nu}(t)\hat{x}_L]. \tag{3.18}$$

Actually the latter forms are not the best for our purpose. Since the first term in Eqs. (3.16) or (3.18) is a particular solution of the d'Alembert equation [see Eq. (3.11)], the second term is necessarily equal to (the near-zone reexpansion of) a homogeneous solution of the source-free wave equation, and most importantly a regular solution at it. So it should be in the form of some antisymmetric multipolar waves: retarded minus advanced. Indeed, this readily follows from the second equality in Eq. (2.22). We introduce a new definition $A_L^{\mu\nu}(t)$ by posing

$$B_L^{\mu\nu}(t) = -\frac{\overset{(2l+1)}{A_L^{\mu\nu}}(t)}{c^{2l+1}(2l+1)!!}, \tag{3.19}$$

where the l-dependent factor is chosen to match with Eq. (2.22). [Because of our assumption of stationarity in the past, $t \leqslant -\mathcal{T}$, the relation (3.19) determines $A_L^{\mu\nu}(t)$ up to a constant. However it is clear that this constant will cancel out in the antisymmetric wave in Eq. (3.20).] In terms of this definition, we find the final result of this section,

$$\overline{h}^{\mu\nu} = \frac{16\pi G}{c^4} \widetilde{\mathcal{I}^{-1}}[\overline{\tau}^{\mu\nu}] + \sum_{l=0}^{+\infty} \hat{\partial}_L \left\{ \overline{\frac{A_L^{\mu\nu}(t-r/c) - A_L^{\mu\nu}(t+r/c)}{2r}} \right\}, \tag{3.20}$$

where we recall that the overline means the post-Newtonian or equivalently near-zone expansion [see Eq. (2.22)]. For the time being, we shall refer to the $A_L^{\mu\nu}(t)$'s as the *radiation-reaction* functions.

B. Multipole expansion of the post-Newtonian solution

In the previous section, we obtained the general solution for the post-Newtonian expansion in the form (3.20), and parametrized by some (for the moment) unknown radiation-reaction functions $A_L^{\mu\nu}(t)$. To arrive at this, we made an assumption concerning the particular structure for the *far-zone* expansion, at spatial infinity, of the post-Newtonian coefficients: Eq. (3.4). Here we shall denote the corresponding infinite expansion (without a remainder term) by means of the same calligraphic letter $\mathcal{M}$ as used to denote the multipole expansion, because the far-zone expansion of the post-Newtonian coefficients is equivalent to a multipolar decomposition. From Eq. (3.4) we have

$$\mathcal{M}\left(\underset{n}{\overline{h}^{\mu\nu}}\right) = \sum \hat{n}_L r^a (\ln r)^p F_{L,n,a,p}^{\mu\nu}(t). \tag{3.21}$$

So, summing up the post-Newtonian series,

$$\mathcal{M}(\overline{h}^{\mu\nu}) = \sum \hat{n}_L r^a (\ln r)^p F_{L,a,p}^{\mu\nu}(t), \tag{3.22}$$

where the functions involved are $F_{L,a,p}(t) = \sum_{n=2}^{+\infty}(1/c^n) F_{L,n,a,p}(t)$. As we can see, the far-zone expansion that we have just postulated is exactly the same, with the same functions $F_{L,a,p}(t)$ as the near-zone expansion we had previously written in Eq. (2.16). This equality is already the matching equation between the near-zone expansion of the multipolar field, $\overline{\mathcal{M}(h^{\mu\nu})}$, and the multipolar–far-zone expansion of the post-Newtonian field, $\mathcal{M}(\overline{h}^{\mu\nu})$, whose consequences will be investigated in Sec. IV.

The fundamental result which is needed for computing the far-zone expansion of the post-Newtonian series concerns the expansion of the generalized integral operator $\widetilde{\mathcal{I}^{-1}}$ acting on the post-Newtonian source $\overline{\tau}^{\mu\nu}$. More precisely, we are interested in knowing under which conditions one can commute $\widetilde{\mathcal{I}^{-1}}$ with the operation $\mathcal{M}$ of taking the far-zone expansion. Clearly, the two operations can be commuted at the price of adding some homogeneous solution of the d'Alembert equation. We prove in Appendix C that the latter homogeneous solution is made of multipolar waves of the *symmetric* type, i.e., retarded *plus* advanced. We obtain

$$\mathcal{M}(\widetilde{\mathcal{I}^{-1}}[\overline{\tau}^{\mu\nu}])=\widetilde{\mathcal{I}^{-1}}[\mathcal{M}(\overline{\tau}^{\mu\nu})] - \frac{1}{4\pi}\sum_{l=0}^{+\infty}\frac{(-)^l}{l!}\hat{\partial}_L\left\{\overline{\frac{\mathcal{F}_L^{\mu\nu}(t-r/c)+\mathcal{F}_L^{\mu\nu}(t+r/c)}{2r}}\right\}. \tag{3.23}$$

Here the overline notation has the same meaning as in Sec. II: this is the Taylor expansion when $r\to 0$, but that expansion should be considered, *a posteriori*, as an expansion when $r\to+\infty$. That is, our notation means

$$\hat{\partial}_L\left\{\overline{\frac{\mathcal{F}_L^{\mu\nu}(t-r/c)+\mathcal{F}_L^{\mu\nu}(t+r/c)}{2r}}\right\} =\sum_{i=0}^{+\infty}\frac{\hat{\partial}_L(r^{2i-1})}{(2i)!}\frac{\overset{(2i)}{\mathcal{F}_L^{\mu\nu}}(t)}{c^{2i}}=\sum_{i=0}^{+\infty}\widetilde{\Delta^{-i}}[\hat{\partial}_L(r^{-1})]\frac{\overset{(2i)}{\mathcal{F}_L^{\mu\nu}}(t)}{c^{2i}} \tag{3.24}$$

[see also Eq. (C14)] where the right-hand sides are to be considered as some expansions at spatial infinity, of the general type given by Eq. (3.22). The functions $\mathcal{F}_L^{\mu\nu}(t)$ parametrizing these symmetric waves are STF and explicitly given by

$$\mathcal{F}_L^{\mu\nu}(t)=\sum_{j=0}^{+\infty}\frac{1}{c^{2j}}\underset{B=0}{\text{FP}}\int d^3\mathbf{x}|\tilde{\mathbf{x}}|^B\widetilde{\Delta^{-j}}[\hat{x}_L]\partial_t^{2j}\overline{\tau}^{\mu\nu}(\mathbf{x},t), \tag{3.25}$$

where $\widetilde{\Delta^{-j}}[\hat{x}_L]$ is given by Eq. (2.23). See the proof in Appendix C. An alternative form reads as

$$\mathcal{F}_L^{\mu\nu}(t)=\underset{B=0}{\text{FP}}\int d^3\mathbf{x}|\tilde{\mathbf{x}}|^B\hat{x}_L \times\overline{\int_{-1}^{1}dz\,\delta_l(z)\,\overline{\tau}^{\mu\nu}(\mathbf{x},t-z|\mathbf{x}|/c)}, \tag{3.26}$$

where the integration over the z-dependent cone $t-z|\mathbf{y}|/c$ involves a weighting function $\delta_l(z)$ that is closely related to the function $\gamma_l(z)$ introduced in Eq. (2.25):

$$\delta_l(z)=\frac{(2l+1)!!}{2^{l+1}l!}(1-z^2)^l=-\frac{1}{2}\gamma_l(z), \tag{3.27}$$

and whose integral is normalized to 1: $\int_{-1}^{1}dz\,\delta_l(z)=1$.[4] The function $\delta_l(z)$ approaches the Dirac delta function in the limit of large l: $\lim_{l\to+\infty}\delta_l(z)=\delta(z)$. In Eq. (3.26), we have indicated by means of an overline the fact that this expression is valid only in a sense of post-Newtonian expansion. Note that because the latter post-Newtonian expansion is "even," containing only even powers of $1/c$, one can replace the argument $t-z|\mathbf{y}|/c$ inside $\mathcal{F}_L^{\mu\nu}(t)$ equivalently by $t+z|\mathbf{y}|/c$.

Finally, thanks to Eqs. (3.23)–(3.27), we are in a position to write the infinite multipolar–far-zone expansion of the post-Newtonian solution as

$$\mathcal{M}(\overline{h}^{\mu\nu})=\frac{16\pi G}{c^4}\widetilde{\mathcal{I}^{-1}}[\mathcal{M}(\overline{\tau}^{\mu\nu})] -\frac{4G}{c^4}\sum_{l=0}^{+\infty}\frac{(-)^l}{l!}\hat{\partial}_L\left\{\overline{\frac{\mathcal{F}_L^{\mu\nu}(t-r/c)+\mathcal{F}_L^{\mu\nu}(t+r/c)}{2r}}\right\} +\sum_{l=0}^{+\infty}\hat{\partial}_L\left\{\overline{\frac{A_L^{\mu\nu}(t-r/c)-A_L^{\mu\nu}(t+r/c)}{2r}}\right\}. \tag{3.28}$$

We recall that the radiation-reaction functions $A_L^{\mu\nu}(t)$ are still undetermined at this stage. The symmetric and antisymmetric waves are given by Eqs. (3.24) and (2.22), respectively, considered here as infinite far-zone expansions.

IV. MATCHING

In Sec. II A, we found the most general expression for the multipolar expansion $\mathcal{M}(h^{\mu\nu})$, satisfying the no-incoming-radiation condition, in terms of some unknown "multipole-moment" STF functions $X_L^{\mu\nu}(t)$ [see Eq. (2.15)]. On the other hand, in Sec. III A, we obtained the most general solution for the post-Newtonian expansion $\overline{h}^{\mu\nu}$, as parametrized by a set of unknown "radiation-reaction" STF functions $A_L^{\mu\nu}(t)$ [see Eq. (3.20)]. We are now imposing the matching condition

$$\overline{\mathcal{M}(h^{\mu\nu})}\equiv\mathcal{M}(\overline{h}^{\mu\nu}). \tag{4.1}$$

In fact, we have already postulated this equation when writing that the two formal expansions (2.16) and (3.22) are the same. Recall that the matching equation (4.1) results from the numerical equality $\mathcal{M}(h^{\mu\nu})=\overline{h}^{\mu\nu}$, verified in the exterior near zone: $a<r\ll\lambda$. It is physically justified only for post-Newtonian sources, for which the exterior near zone exists. The matching equation is actually a *functional* identity, i.e., true $\forall(\mathbf{x},t)\in\mathbb{R}^3_*\times\mathbb{R}$; it identifies, *term by term*, two asymptotic singular expansions, each of them being formally taken outside its own domain of validity. In the present context, the matching equation insists that the infinite *near-zone* expansion, $r\to 0$, of the exterior multipolar field is identical to the infinite *far-zone* expansion, $r\to+\infty$, of the inner post-Newtonian field. Let us show now that Eq. (4.1) permits

[4]The normalization for the function $\delta_l(z)$ is consistent with that of the function $\gamma_l(z)$: $\int_1^{+\infty}dz\,\gamma_l(z)=1$, owing to the fact that the integral $\int_{-\infty}^{+\infty}dz(1-z^2)^l$ is zero by complex analytic continuation in $l\in\mathbb{C}$.

determining all the unknowns of the problem: i.e., at once, the multipole moments $X_L^{\mu\nu}$ *and* the radiation-reaction functions $A_L^{\mu\nu}$. In particular, we find that the multipole moments $X_L^{\mu\nu}$ are in agreement with the earlier result derived in Ref. [40].

For the sake of clarity, we restate here the two results we reached for the two sides of Eq. (4.1). The left-hand side was obtained in Eq. (2.26):

$$\overline{\mathcal{M}(h^{\mu\nu})}=\widetilde{\mathcal{I}^{-1}}[\overline{\mathcal{M}(\Lambda^{\mu\nu})}] -\frac{4G}{c^4}\sum_{l=0}^{+\infty}\frac{(-)^l}{l!}\hat{\partial}_L\left\{\frac{\overline{\mathcal{R}_L^{\mu\nu}(t-r/c)-\mathcal{R}_L^{\mu\nu}(t+r/c)}}{2r}\right\} +\sum_{l=0}^{+\infty}\hat{\partial}_L\left\{\frac{\overline{X_L^{\mu\nu}(t-r/c)}}{r}\right\}, \tag{4.2}$$

in which the functions $\mathcal{R}_L^{\mu\nu}$, which come from the nonlinearities of the field equations in vacuum, are known from Eq. (2.24). The right-hand-side of the matching equation was found in Eq. (3.28):

$$\mathcal{M}(\overline{h}^{\mu\nu})=\frac{16\pi G}{c^4}\widetilde{\mathcal{I}^{-1}}[\mathcal{M}(\overline{\tau}^{\mu\nu})] -\frac{4G}{c^4}\sum_{l=0}^{+\infty}\frac{(-)^l}{l!}\hat{\partial}_L\left\{\frac{\overline{\mathcal{F}_L^{\mu\nu}(t-r/c)+\mathcal{F}_L^{\mu\nu}(t+r/c)}}{2r}\right\} +\sum_{l=0}^{+\infty}\hat{\partial}_L\left\{\frac{\overline{A_L^{\mu\nu}(t-r/c)-A_L^{\mu\nu}(t+r/c)}}{2r}\right\}. \tag{4.3}$$

Here, the functions $\mathcal{F}_L^{\mu\nu}$, which depend on the matter and gravitational content of the post-Newtonian source, take the definite expression given by Eqs. (3.25) and (3.26).

Comparing Eqs. (4.2) and (4.3), we readily discover that they share an obvious common term, that is, the first one. Indeed, we manifestly have

$$\widetilde{\mathcal{I}^{-1}}[\overline{\mathcal{M}(\Lambda^{\mu\nu})}]=\widetilde{\mathcal{I}^{-1}}[\mathcal{M}(\overline{\Lambda}^{\mu\nu})]=\frac{16\pi G}{c^4}\widetilde{\mathcal{I}^{-1}}[\mathcal{M}(\overline{\tau}^{\mu\nu})]. \tag{4.4}$$

The first equality comes from the matching equation, as applied to the gravitational source term $\Lambda^{\mu\nu}$, and the second equality comes from the fact that the matter tensor $T^{\mu\nu}$ has a compact support, so that $\mathcal{M}(T^{\mu\nu})=0$. Hence the two first terms in Eqs. (4.2) and (4.3) match together. This is a somewhat remarkable fact, because most of the complexity of the Einstein field equations is actually contained in these terms, either $\mathcal{I}^{-1}[\overline{\mathcal{M}(\Lambda^{\mu\nu})}]$ for the external field or $(16\pi G/c^4)\mathcal{I}^{-1}[\mathcal{M}(\overline{\tau}^{\mu\nu})]$ for the inner one. But for doing the matching, we do not need all this complexity; these two terms match and therefore are to be identified. Notice also that this is a nontrivial result, since the two sides of Eq. (4.1) strongly depend on the yet unknown functions $A_L^{\mu\nu}$ and $X_L^{\mu\nu}$, which enter the latter two terms in a very intricate way, coupled together as they are via the nonlinearities of the field equations. Nevertheless, the matching equation tells us that these terms must be rigorously identical.

As soon as we have noticed that the first terms in Eqs. (4.2) and (4.3) are equal, we can compare the other ones, and because the retarded and advanced waves have some different structures, they must be matched independently, so we get *two* relations to be satisfied. We find that these are solved if and only if the multipole moments in the exterior field *and* the radiation-reaction functions in the inner field are given by

$$X_L^{\mu\nu}(t)=-\frac{4G}{c^4}\frac{(-)^l}{l!}\mathcal{F}_L^{\mu\nu}(t), \tag{4.5}$$

$$A_L^{\mu\nu}(t)=-\frac{4G}{c^4}\frac{(-)^l}{l!}[\mathcal{F}_L^{\mu\nu}(t)+\mathcal{R}_L^{\mu\nu}(t)]. \tag{4.6}$$

Therefore, both the multipole moments and the radiation-reaction terms are determined as some explicit functionals of the pseudotensor $\tau^{\mu\nu}$ and nothing else. [Actually, we could add any constant to the definition of $A_L^{\mu\nu}(t)$, but this is physically irrelevant because the constant disappears from the antisymmetric waves; see also Ref. [50].]

Finally, by way of summary of the results, we take back the latter expressions and fill in the external and inner fields, which are then entirely determined as coming from a unique solution of the Einstein field equations in harmonic coordinates, valid everywhere inside and outside the source. The exterior field is

$$\mathcal{M}(h^{\mu\nu})=\widetilde{\Box^{-1}_{\mathrm{Ret}}}[\mathcal{M}(\Lambda^{\mu\nu})] -\frac{4G}{c^4}\sum_{l=0}^{+\infty}\frac{(-)^l}{l!}\hat{\partial}_L\left\{\frac{\mathcal{F}_L^{\mu\nu}(t-r/c)}{r}\right\}, \tag{4.7}$$

where the multipole moments are given in terms of the post-Newtonian expansion of the stress-energy pseudotensor by

$$\mathcal{F}_L^{\mu\nu}(t)=\underset{B=0}{\mathrm{FP}}\int d^3\mathbf{y}|\tilde{\mathbf{y}}|^B\hat{y}_L\overline{\int_{-1}^{1}dz\,\delta_l(z)\,\overline{\tau}^{\mu\nu}(\mathbf{y},t-z|\mathbf{y}|/c)} =\sum_{j=0}^{+\infty}\frac{1}{c^{2j}}\underset{B=0}{\mathrm{FP}}\int d^3\mathbf{y}|\tilde{\mathbf{y}}|^B\widetilde{\Delta^{-j}}[\hat{y}_L]\partial_t^{2j}\overline{\tau}^{\mu\nu}(\mathbf{y},t). \tag{4.8}$$

This result is in perfect agreement with the multipole decomposition of the exterior field obtained in Ref. [40] [see Eqs. (3.13),(3.14) there]. On the other hand, the inner post-Newtonian field is given by

$$\overline{h}^{\mu\nu}=\frac{16\pi G}{c^4}\widetilde{\mathcal{I}^{-1}}[\overline{\tau}^{\mu\nu}]-\frac{4G}{c^4}\sum_{l=0}^{+\infty}\frac{(-)^l}{l!}\hat{\partial}_L\left\{\frac{\overline{\mathcal{A}_L^{\mu\nu}(t-r/c)-\mathcal{A}_L^{\mu\nu}(t+r/c)}}{2r}\right\}, \tag{4.9}$$

where the radiation-reaction function is composed of two terms:

$$\mathcal{A}_L^{\mu\nu}(t)=\mathcal{F}_L^{\mu\nu}(t)+\mathcal{R}_L^{\mu\nu}(t). \tag{4.10}$$

The first term is merely the exterior multipole moment given by Eq. (4.8), and one can check that it contains the standard radiation-reaction effect at the 2.5 PN order. The $\mathcal{R}_L^{\mu\nu}$ term is defined by Eq. (2.24), or, rather, the post-Newtonian expansion of it, i.e.,

$$\mathcal{R}_L^{\mu\nu}(t)=\underset{B=0}{\mathrm{FP}}\int d^3\mathbf{y}|\widetilde{\mathbf{y}}|^B\hat{y}_L \times\overline{\int_1^{+\infty}dz\,\gamma_l(z)\mathcal{M}(\tau^{\mu\nu})(\mathbf{y},t-z|\mathbf{y}|/c)}. \tag{4.11}$$

This term is quite interesting: it depends on the nonlinearities of the exterior field, described by the gravitational source term $\mathcal{M}(\tau^{\mu\nu})$ (or, more precisely, the nonstationary part of it [50]), which are to be computed by means of the multipolar–post-Minkowskian algorithm of Refs. [36,32] (see, in particular, Sec. III D in Ref. [32] for some detailed computations of this term). Physically, the function $\mathcal{R}_L^{\mu\nu}$ contains the effect of wave tails in the radiation reaction force which arises at the 4 PN order [31–33]. It is not difficult [using notably the formula (5.21) below] to derive the more explicit expression for the contribution of $\mathcal{R}_L^{\mu\nu}$ to the antisymmetric wave in Eq. (4.9):

$$\hat{\partial}_L\left\{\frac{\overline{\mathcal{R}_L^{\mu\nu}(t-r/c)-\mathcal{R}_L^{\mu\nu}(t+r/c)}}{2r}\right\}=\sum_{i=0}^{l}\frac{(-)^l(l+i)!}{2^i i!(l-i)!}\sum_{k=0}^{+\infty}\frac{\widetilde{\Delta^{-k}}(\hat{x}_L)}{c^{2k+l-i}}\underset{B=0}{\mathrm{FP}}\times\int d^3\mathbf{y}|\widetilde{\mathbf{y}}|^B\frac{\hat{y}_L}{|\mathbf{y}|^{l+i+1}}\partial_t^{2k+l-i}\mathcal{M}(\tau^{\mu\nu})(\mathbf{y},t-|\mathbf{y}|/c). \tag{4.12}$$

When they are computed by post-Minkowskian approximations, the remaining integrals will typically yield, after integration over the angles, some "hereditary-like" contributions, depending on the whole integrated past of the matter source (see Ref. [32]).

It is tempting to speculate that the second term in Eq. (4.9), made of the antisymmetric multipolar waves parametrized by the functions $\mathcal{A}_L^{\mu\nu}(t)$, can be regarded as the contribution, in a sense to be made more precise, of the radiation reaction forces at work inside the post-Newtonian source. (Indeed we have checked that these functions contain the known radiation-reaction terms at the dominant 2.5 PN order as well as the dominant contribution of tails at the 4 PN order.) We shall leave for future work the systematic study of this term, as well as the possibility to answer the latter speculation.

V. HARMONIC-COORDINATE CONDITION

The latter solution for the post-Newtonian field, Eqs. (4.9)–(4.12), has been obtained without imposing, in an explicit way, the condition of harmonic coordinates (1.2). Indeed, we have assumed this condition to be true, and we simply matched together the post-Newtonian and multipolar-post-Minkowskian expansions, satisfying the relaxed Einstein field equations (1.1) in their respective domains. We found that the matching determines uniquely the expressions of the multipole moments $X_L^{\mu\nu}(t)$ and radiation-reaction functions $A_L^{\mu\nu}(t)$ as some functionals of the stress-energy pseudotensor $\tau^{\mu\nu}$. However, we never used the harmonic-coordinate condition during the matching; it was not necessary for the formal determination of the unknown parameters $(X_L^{\mu\nu}, A_L^{\mu\nu})$. Therefore, it is quite important to check that our post-Newtonian solution is divergenceless as a consequence of the conservation of the pseudotensor $\tau^{\mu\nu}$ [see Eq. (1.2)], so that we really grasp a solution of the full Einstein field equations.

We check the divergenceless of $\overline{h}^{\mu\nu}$ directly on Eq. (3.16). We apply the ∂_μ operator on each side of the equality:

$$\partial_\mu\overline{h}^{\mu\nu}=\frac{16\pi G}{c^4}\partial_\mu\widetilde{\mathcal{I}^{-1}}[\overline{\tau}^{\mu\nu}]+\partial_\mu\left[\sum_{l,k=0}^{+\infty}\frac{1}{c^{2k}}\overset{(2k)}{B}{}_L^{\mu\nu}(t)\widetilde{\Delta^{-k}}(\hat{x}_L)\right]. \tag{5.1}$$

We must transform the two terms on the right-hand side in order to make explicit the fact that these two terms are exactly opposite. The first term, that is to say, the divergence of the $\widetilde{\mathcal{I}^{-1}}$ operator, is not obvious since, even if time derivatives commute with $\widetilde{\mathcal{I}^{-1}}$, spatial derivatives do not,

$$\partial_\mu\widetilde{\mathcal{I}^{-1}}[\overline{\tau}^{\mu\nu}]=\widetilde{\mathcal{I}^{-1}}[\partial_0\overline{\tau}^{0\nu}]+\partial_i\widetilde{\mathcal{I}^{-1}}[\overline{\tau}^{i\nu}]. \tag{5.2}$$

$\widetilde{\mathcal{I}^{-1}}$ is a sum of $\widetilde{\Delta^{-k-1}}\partial_t^{2k}$, and spatial derivatives do not commute with $\widetilde{\Delta^{-k-1}}$ because of the $|\widetilde{\mathbf{y}}|^B$ factor [see Eq. (3.9) for the exact expression]. To see how to tackle this problem, let us start with the spatial divergence of $\widetilde{\Delta^{-k-1}}$. We assume that $\overline{\tau}(\mathbf{x},t)$ is a function of the "post-Newtonian" type, i.e., satisfies the requirements (3.3) and (3.4). The ∂_i derivative, in Eq. (5.3), applies first to the kth Poisson's kernel, but after having noticed that the x^i derivative of this kernel was equal to minus the y^i derivative of it, we can make an integration by part and distribute the y^i derivative on $|\widetilde{\mathbf{y}}|^B$ and on $\overline{\tau}$ so that

$$\partial_i\widetilde{\Delta^{-k-1}}(\overline{\tau}(\mathbf{x},t))$$
$$=-\frac{1}{4\pi}\underset{B=0}{\mathrm{FP}}\int d^3\mathbf{y}|\widetilde{\mathbf{y}}|^B\partial_{x^i}\left(\frac{|\mathbf{x}-\mathbf{y}|^{2k-1}}{(2k)!}\right)\overline{\tau}(\mathbf{y},t)$$
$$=-\frac{1}{4\pi}\underset{B=0}{\mathrm{FP}}\int d^3\mathbf{y}|\widetilde{\mathbf{y}}|^B\frac{|\mathbf{x}-\mathbf{y}|^{2k-1}}{(2k)!}\partial_i\overline{\tau}(\mathbf{y},t)$$
$$-\frac{1}{4\pi}\underset{B=0}{\mathrm{FP}}\int d^3\mathbf{y}\partial_i(|\widetilde{\mathbf{y}}|^B)\frac{|\mathbf{x}-\mathbf{y}|^{2k-1}}{(2k)!}\widetilde{\tau}(\mathbf{y},t). \tag{5.3}$$

The first term in the last line of the previous equality is equal to $\widetilde{\Delta^{-k-1}}(\partial_i\overline{\tau})$. Now, let us concentrate on the last term in the same line. We can, of course, write $\partial_i(|\widetilde{\mathbf{y}}|^B)=Bn_i|\widetilde{\mathbf{y}}|^{B-1}/r_0$. Moreover, since $\overline{\tau}(\mathbf{y},t)$ is regular at the origin ($|\mathbf{y}|=0$), the integral is always convergent on any neighborhood of the origin. Translating these two remarks in the last integral of Eq. (5.3), and since we take the finite part when $B=0$, this last integral is zero, because of the explicit factor B, when ranging from $|\mathbf{y}|=0$ up to some arbitrary finite value $|\mathbf{y}|=\mathcal{R}$. So, after replacing $|\mathbf{x}-\mathbf{y}|^{2k-1}$ by $(2k)!\widetilde{\Delta^{-k}}(|\mathbf{x}-\mathbf{y}|^{-1})$ we are left, in Eq. (5.4), with one integral ranging over $|\mathbf{y}|>\mathcal{R}$,

$$\underset{B=0}{\mathrm{FP}}\int d^3\mathbf{y}\partial_i(|\widetilde{\mathbf{y}}|^B)\frac{|\mathbf{x}-\mathbf{y}|^{2k-1}}{(2k)!}\overline{\tau}(\mathbf{y},t)$$
$$=\underset{B=0}{\mathrm{FP}}\int_{|\mathbf{y}|>\mathcal{R}}d^3\mathbf{y}\partial_i(|\widetilde{\mathbf{y}}|^B)\mathcal{M}[\widetilde{\Delta^{-k}}(|\mathbf{x}-\mathbf{y}|^{-1})]\mathcal{M}(\overline{\tau})(\mathbf{y},t)$$
$$=\sum_{n=0}^{k}\sum_{l\geqslant 0}\frac{(-)^l}{l!}\widetilde{\Delta^{-k+n}}(\hat{x}_L)$$
$$\times\underset{B=0}{\mathrm{FP}}\int_{|\mathbf{y}|>\mathcal{R}}d^3\mathbf{y}\partial_i(|\widetilde{\mathbf{y}}|^B)\widetilde{\Delta^{-n}}(\hat{\partial}_L(|\mathbf{y}|^{-1}))\mathcal{M}(\overline{\tau})(\mathbf{y},t). \tag{5.4}$$

In the last line of the previous equation, we expanded the kth Poisson's kernel for $|\mathbf{y}|\gg|\mathbf{x}|$ using Eq. (C12). This is possible thanks to the fact that $\mathcal{R}$ is arbitrary and may be chosen such that $\mathcal{R}\gg|\mathbf{x}|$. We also note that $\overline{\tau}$ turned into $\mathcal{M}(\overline{\tau})$ because $\overline{\tau}=\mathcal{M}(\overline{\tau})$ in the far zone. In this way,

$$\partial_i\widetilde{\Delta^{-1-k}}[\overline{\tau}(\mathbf{x},t)]=\widetilde{\Delta^{-1-k}}[\partial_i\overline{\tau}(\mathbf{x},t)]$$
$$-\frac{1}{4\pi}\sum_{n=0}^{k}\sum_{l\geqslant 0}\frac{(-)^l}{l!}\widetilde{\Delta^{-k+n}}(\hat{x}_L)\underset{B=0}{\mathrm{FP}}\int_{|\mathbf{y}|>\mathcal{R}}d^3\mathbf{y}\partial_i(|\widetilde{\mathbf{y}}|^B)$$
$$\times\widetilde{\Delta^{-n}}[\hat{\partial}_L(|\mathbf{y}|^{-1})]\mathcal{M}(\overline{\tau})(\mathbf{y},t), \tag{5.5}$$

and we notice that the commutation of the spatial derivative and the generalized kth Poisson integral depends only on the behavior of $\overline{\tau}(\mathbf{x},t)$ at spatial infinity. This fact was foreseeable since for a function $\overline{\tau}(\mathbf{x},t)$ with compact support the commutation would be trivial. Thanks to the general result given by Eq. (5.5), in which we replace $\overline{\tau}$ by $\overline{\tau}^{i\nu}$, we can determine the spatial divergence of $\widetilde{\mathcal{I}^{-1}}(\overline{\tau}^{i\nu})$. We can then get $\partial_\mu\widetilde{\mathcal{I}^{-1}}[\overline{\tau}^{\mu\nu}]$ that is the sum of $\widetilde{\mathcal{I}^{-1}}[\partial_\mu\overline{\tau}^{\mu\nu}]$ and a nontrivial term. Since $\partial_\mu\overline{\tau}^{\mu\nu}=0$, the result for the first term of Eq. (5.1) reduces to the nontrivial term, that is to say,

$$\frac{16\pi G}{c^4}\partial_\mu\widetilde{\mathcal{I}^{-1}}[\overline{\tau}^{\mu\nu}]=-\frac{4G}{c^4}\sum_{k,l,n\geqslant 0}\frac{(-)^l}{l!}\widetilde{\Delta^{-k}}(\hat{x}_L)$$
$$\times\underset{B=0}{\mathrm{FP}}\int_{|\mathbf{y}|>\mathcal{R}}d^3\mathbf{y}\partial_i(|\widetilde{\mathbf{y}}|^B)\widetilde{\Delta^{-n}}[\hat{\partial}_L(|\mathbf{y}|^{-1})]\mathcal{M}(\overset{(2k+2n)}{\overline{\tau}^{i\nu}})(\mathbf{y},t). \tag{5.6}$$

Now, we want to prove that the second term on the right-hand side of Eq. (5.1) is exactly the opposite. In Eq. (5.7), we expand this last term in its *3+1* form so that we can treat separately terms with time derivative and terms with spatial derivative,

$$\sum_{n,l\geqslant 0}\frac{1}{c^{2n}}\overset{(2n)}{B_L^{i\nu}}(t)\partial_i[\widetilde{\Delta^{-n}}(\hat{x}_L)]+\sum_{n,l\geqslant 0}\frac{1}{c^{2n}}\partial_0\overset{(2n)}{B_L^{0\nu}}(t)\widetilde{\Delta^{-n}}(\hat{x}_L). \tag{5.7}$$

The first term of Eq. (5.7), thanks to a STF formula, can be written without the use of spatial derivative. The index i coming from this derivative is distributed on the multi-index L as

$$\sum_{n,l\geqslant 0}\frac{1}{c^{2n}}\overset{(2n)}{B_L^{i\nu}}(t)\partial_i[\widetilde{\Delta^{-n}}(\hat{x}_L)]$$
$$=\sum_{n,l\geqslant 0}\frac{1}{c^{2n}}\left\{\frac{1}{2l+3}\overset{(2n)}{B_L^{i\nu}}(t)\widetilde{\Delta^{-n+1}}(\hat{x}_{iL})\right.$$
$$\left.+l\overset{(2n)}{B_{iL-1}^{i\nu}}(t)\widetilde{\Delta^{-n}}(\hat{x}_{L-1})\right\}. \tag{5.8}$$

In the second term of Eq. (5.7), we express the function $B_L^{0\nu}$ in terms of $\mathcal{F}_L^{0\nu}$ and $\mathcal{R}_L^{0\nu}$ [cf. Eqs. (3.19) and (4.6)] because the time derivative, ∂_0, will act on the integrand of these two time-varying moments:

$$\widetilde{\Delta^{-n}}(\hat{x}_L)\partial_0B_L^{0\nu}(t)=\frac{4G(-)^l}{c^{5+2l}(2l+1)!!l!}\widetilde{\Delta^{-n}}(\hat{x}_L)\{\partial_0\overset{(2l+1)}{\mathcal{R}_L^{0\nu}}(t)$$
$$+\partial_0\overset{(2l+1)}{\mathcal{F}_L^{0\nu}}(t)\}. \tag{5.9}$$

First, we investigate the case of $\partial_0\mathcal{F}_L^{0\nu}$, using the formula (3.25), where the time derivative acts on $\overline{\tau}^{0\nu}$,

$$\widetilde{\Delta^{-n}}(\hat{x}_L)\partial_0\mathcal{F}_L^{0\nu}(t)$$
$$=\widetilde{\Delta^{-n}}(\hat{x}_L)\sum_{k\geqslant 0}\frac{1}{c^{2k}}\underset{B=0}{\mathrm{FP}}\int d^3\mathbf{y}|\widetilde{\mathbf{y}}|^B\widetilde{\Delta^{-k}}(\hat{y}_L)\partial_0\overset{(2k)}{\overline{\tau}^{0\nu}}(\mathbf{y},t). \tag{5.10}$$

We can replace $\partial_0\overline{\tau}^{0\nu}$ by $-\partial_i\overline{\tau}^{i\nu}$ thanks to the conservation equation of the pseudotensor. After integrating by part we get

$$\widetilde{\Delta^{-n}}(\hat{x}_L)\sum_{k\geqslant 0}\frac{1}{c^{2k}}\underset{B=0}{\mathrm{FP}}\int d^3\mathbf{y}\partial_i(|\widetilde{\mathbf{y}}|^B)\widetilde{\Delta^{-k}}(\hat{y}_L)\overset{(2k)}{\overline{\tau}^{i\nu}}(\mathbf{y},t)$$
$$+\widetilde{\Delta^{-n}}(\hat{x}_L)\sum_{n\geqslant 0}\frac{1}{c^{2k}}\underset{B=0}{\mathrm{FP}}\int d^3\mathbf{y}|\widetilde{\mathbf{y}}|^B\partial_i[\widetilde{\Delta^{-k}}(\hat{y}_L)]\overset{(2k)}{\overline{\tau}^{i\nu}}(\mathbf{y},t). \tag{5.11}$$

The same STF formula as used in Eq. (5.8) enables one to transform the second term of Eq. (5.11) so that, at the end, we get the definitive result

$$\widetilde{\Delta^{-n}}(\hat{x}_L)\partial_0\mathcal{F}_L^{0\nu}(t)=\widetilde{\Delta^{-n}}(\hat{x}_L)\sum_{k\geqslant 0}\frac{1}{c^{2k}}\underset{B=0}{\mathrm{FP}}$$
$$\times\int d^3\mathbf{y}\partial_i(|\widetilde{\mathbf{y}}|^B)\widetilde{\Delta^{-k}}(\hat{y}_L)\overset{(2k)}{\overline{\tau}^{i\nu}}(\mathbf{y},t)$$
$$+l\widetilde{\Delta^{-n}}(\hat{x}_{iL-1})\mathcal{F}_{L-1}^{i\nu}(t)$$
$$+\frac{\widetilde{\Delta^{-n}}(\hat{x}_L)}{c^2(2l+3)}\overset{(2)}{\mathcal{F}_{iL}^{i\nu}}(t). \tag{5.12}$$

We can, now, investigate the case of the first term in Eq. (5.9), which is a little bit more complicated since it involves a retarded integral,

$$\widetilde{\Delta^{-n}}(\hat{x}_L)\partial_0\mathcal{R}_L^{0\nu}(t)=\widetilde{\Delta^{-n}}(\hat{x}_L)\underset{B=0}{\mathrm{FP}}\int d^3\mathbf{y}|\widetilde{\mathbf{y}}|^B\hat{y}_L$$
$$\times\int_1^\infty dz\,\gamma_l(z)\mathcal{M}(\partial_0\tau^{0\nu})(\mathbf{y},t-z|\mathbf{y}|/c), \tag{5.13}$$

where the function $\gamma_l(z)$ is given by Eq. (2.25) (for simplicity's sake we do not write the overline indicating the post-Newtonian expansion). We do the replacement of $\partial_0\tau^{0\nu}$ into $-\partial_i\tau^{i\nu}$. Before integrating by part, we should notice that the partial derivative ∂_i acts on $\tau^{i\nu}$ which is then evaluated at the event $(\mathbf{y},t-z|\mathbf{y}|/c)$; we must be careful about the space dependence of the time variable $t-z|\mathbf{y}|/c$. The last equation then becomes

$$-\widetilde{\Delta^{-n}}(\hat{x}_L)\underset{B=0}{\mathrm{FP}}\int d^3\mathbf{y}|\widetilde{\mathbf{y}}|^B\hat{y}_L$$
$$\times\partial_i\left(\int_1^\infty dz\,\gamma_l(z)\mathcal{M}(\tau^{i\nu})(\mathbf{y},t-z|\mathbf{y}|/c)\right)$$
$$-\widetilde{\Delta^{-n}}(\hat{x}_L)\underset{B=0}{\mathrm{FP}}\int d^3\mathbf{y}|\widetilde{\mathbf{y}}|^B\hat{y}_L n_i$$
$$\times\int_1^\infty dz\frac{z}{c}\gamma_l(z)\mathcal{M}(\overset{(1)}{\tau^{i\nu}})(\mathbf{y},t-z|\mathbf{y}|/c). \tag{5.14}$$

In this way, the first term can be integrated by part straightforwardly, in terms of $d^3\mathbf{y}$ integration, showing up a $\partial_i(|\widetilde{\mathbf{y}}|^B)$ term and a $\partial_i(\hat{y}_L)$ term. The second term will also be integrated by part, in terms of dz integration, using the fact $d/dz[\gamma_{l+1}(z)]=-(2l+3)z\gamma_l(z)$; so we have

$$\widetilde{\Delta^{-n}}(\hat{x}_L)\underset{B=0}{\mathrm{FP}}\int d^3\mathbf{y}\partial_i(|\widetilde{\mathbf{y}}|^B)\hat{y}_L$$
$$\times\int_1^\infty dz\,\gamma_l(z)\mathcal{M}(\tau^{i\nu})(\mathbf{y},t-z|\mathbf{y}|/c)+\widetilde{\Delta^{-n}}(\hat{x}_L)\underset{B=0}{\mathrm{FP}}$$
$$\times\int d^3\mathbf{y}|\widetilde{\mathbf{y}}|^B\partial_i(\hat{y}_L)\int_1^\infty dz\,\gamma_l(z)\mathcal{M}(\tau^{i\nu})(\mathbf{y},t-z|\mathbf{y}|/c)$$
$$+\frac{\widetilde{\Delta^{-n}}(\hat{x}_L)}{c^2(2l+3)}\underset{B=0}{\mathrm{FP}}\int d^3\mathbf{y}|\widetilde{\mathbf{y}}|^B\hat{y}_L y_i$$
$$\times\int_1^\infty dz\,\gamma_{l+1}(z)\mathcal{M}(\overset{(2)}{\tau^{i\nu}})(\mathbf{y},t-z|\mathbf{y}|/c). \tag{5.15}$$

The sum of these three terms can be transformed so that the function $\mathcal{R}_L^{i\nu}$ shows up. Since for any STF tensor $\hat{T}_L\partial_i(\hat{y}_L)=l\hat{T}_{iL-1}\hat{y}_{L-1}$ and $\hat{T}_L\hat{y}_L y_i=\hat{T}_L\hat{y}_{iL}+[l/(2l+1)]\hat{T}_{iL-1}\hat{y}_{L-1}|\mathbf{y}|^2$, and keeping in mind that all the multi-indices L will have to be summed, we can write

$$\widetilde{\Delta^{-n}}(\hat{x}_L)\underset{B=0}{\mathrm{FP}}\int d^3\mathbf{y}\partial_i(|\widetilde{\mathbf{y}}|^B)\hat{y}_L$$
$$\times\int_1^\infty dz\,\gamma_l(z)\mathcal{M}(\tau^{i\nu})(\mathbf{y},t-z|\mathbf{y}|/c)$$
$$+l\widetilde{\Delta^{-n}}(\hat{x}_{iL-1})\underset{B=0}{\mathrm{FP}}\int d^3\mathbf{y}|\widetilde{\mathbf{y}}|^B\hat{y}_{L-1}$$
$$\times\int_1^\infty dz\,\gamma_l(z)\mathcal{M}(\tau^{i\nu})(\mathbf{y},t-z|\mathbf{y}|/c)+\frac{\widetilde{\Delta^{-n}}(\hat{x}_L)}{c^2(2l+3)}\underset{B=0}{\mathrm{FP}}$$
$$\times\int d^3\mathbf{y}|\widetilde{\mathbf{y}}|^B\hat{y}_{iL}\int_1^\infty dz\,\gamma_{l+1}(z)\mathcal{M}(\overset{(2)}{\tau^{i\nu}})(\mathbf{y},t-z|\mathbf{y}|/c)$$
$$+\frac{l\widetilde{\Delta^{-n}}(\hat{x}_{iL-1})}{c^2(2l+1)(2l+3)}\underset{B=0}{\mathrm{FP}}\int d^3\mathbf{y}|\widetilde{\mathbf{y}}|^B|\mathbf{y}|^2\hat{y}_{L-1}$$
$$\times\int_1^\infty dz\,\gamma_{l+1}(z)\mathcal{M}(\overset{(2)}{\tau^{i\nu}})(\mathbf{y},t-z|\mathbf{y}|/c). \tag{5.16}$$

An interesting relation between γ_l functions, $d^2/dz^2[\gamma_{l+1}(z)]=(2l+1)(2l+3)[\gamma_{l-1}(z)-\gamma_l(z)]$, after integrating by part the last integral, in terms of dz integration, allows us to get the more explicit form

$$\widetilde{\Delta^{-n}}(\hat{x}_L)\partial_0\mathcal{R}^{0\nu}_L(t) = \widetilde{\Delta^{-n}}(\hat{x}_L)\,\underset{B=0}{\mathrm{FP}}\int d^3\mathbf{y}\,\partial_i(|\tilde{\mathbf{y}}|^B)\hat{y}_L\int_1^\infty dz\,\gamma_l(z)\mathcal{M}(\tau^{i\nu})(\mathbf{y},t-z|\mathbf{y}|/c)+l\widetilde{\Delta^{-n}}(\hat{x}_{iL-1})\mathcal{R}^{i\nu}_{L-1}(t)+\frac{\widetilde{\Delta^{-n}}(\hat{x}_L)}{c^2(2l+3)}\overset{(2)}{\mathcal{R}}{}^{i\nu}_{iL}(t). \tag{5.17}$$

Summing up Eqs. (5.12) and (5.17), we obtain

$$\sum_{n\geqslant 0}\partial_0\overset{(2n)}{B}{}^{0\nu}_L(t)\widetilde{\Delta^{-n}}(\hat{x}_L) = \frac{4G}{c^4}\sum_{l,n\geqslant 0}\frac{2^l(-)^l}{(2l+1)!}\widetilde{\Delta^{-n}}(\hat{x}_L)\times\Bigg[\sum_{k\geqslant 0}\underset{B=0}{\mathrm{FP}}\int d^3\mathbf{y}\,\partial_i(|\tilde{\mathbf{y}}|^B)\widetilde{\Delta^{-n}}(\hat{y}_L)\frac{\overset{(2n+2k+2l+1)}{\overline{\tau}^{i\nu}}(\mathbf{y},t)}{c^{2n+2k+2l+1}} + \underset{B=0}{\mathrm{FP}}\int d^3\mathbf{y}\,\partial_i(|\tilde{\mathbf{y}}|^B)\hat{y}_L\times\int_1^\infty dz\,\gamma_l(z)\frac{\overset{(2n+2l+1)}{\mathcal{M}(\tau^{i\nu})}}{c^{2n+2l+1}}(\mathbf{y},t-z|\mathbf{y}|/c)\Bigg] - \sum_{n,l\geqslant 0}\frac{1}{c^{2n}}\left\{\frac{1}{2l+3}\overset{(2n)}{B}{}^{i\nu}_L(t)\times\widetilde{\Delta^{-n+1}}(\hat{x}_{iL})+l\overset{(2n)}{B}{}^{i\nu}_{iL-1}(t)\widetilde{\Delta^{-n}}(\hat{x}_{L-1})\right\}. \tag{5.18}$$

The last line cancels out the terms coming from Eq. (5.8).

We can therefore write down the result for the divergence of $\overline{h}^{\mu\nu}$ which, at this stage, depends only on terms with integrals of $\partial_i(|\tilde{\mathbf{y}}|^B)$ and having the spatial structure given by $\widetilde{\Delta^{-n}}(\hat{x}_L)$. After summing Eqs. (5.6), (5.8), and (5.18) we get

$$\partial_\mu\overline{h}^{\mu\nu} = -\frac{4G}{c^4}\sum_{n,l,k\geqslant 0}\frac{(-)^l}{l!}\frac{\widetilde{\Delta^{-n}}(\hat{x}_L)}{c^{2k+2n}}\underset{B=0}{\mathrm{FP}}\times\int_{|\mathbf{y}|>\mathcal{R}}d^3\mathbf{y}\,\partial_i(|\tilde{\mathbf{y}}|^B)\widetilde{\Delta^{-k}}[\hat{\partial}_L(|\mathbf{y}|^{-1})]\overset{(2n+2k)}{\mathcal{M}(\overline{\tau}^{i\nu})}(\mathbf{y},t) + \frac{4G}{c^4}\sum_{n,l,k\geqslant 0}\frac{2^l(-)^l}{(2l+1)!}\frac{\widetilde{\Delta^{-n}}(\hat{x}_L)}{c^{2n+2k+2l+1}}\underset{B=0}{\mathrm{FP}}\times\int_{|\mathbf{y}|>\mathcal{R}}d^3\mathbf{y}\,\partial_i(|\tilde{\mathbf{y}}|^B)\widetilde{\Delta^{-k}}(\hat{y}_L)\overset{(2n+2k+2l+1)}{\mathcal{M}(\overline{\tau}^{i\nu})}(\mathbf{y},t) + \frac{4G}{c^4}\sum_{n,l\geqslant 0}\frac{2^l(-)^l}{(2l+1)!}\frac{\widetilde{\Delta^{-n}}(\hat{x}_L)}{c^{2n+2l+1}}\times\underset{B=0}{\mathrm{FP}}\int d^3\mathbf{y}\,\partial_i(|\tilde{\mathbf{y}}|^B)\hat{y}_L\times\int_1^\infty dz\,\gamma_l(z)\overset{(2n+2l+1)}{\mathcal{M}(\tau^{i\nu})}(\mathbf{y},t-z|\mathbf{y}|/c). \tag{5.19}$$

In the second term we have used the fact that the integral depends only on the values for which $|\mathbf{y}|>\mathcal{R}$ to write $\overline{\tau}^{i\nu}=\mathcal{M}(\overline{\tau}^{i\nu})$ on that domain. The last term of Eq. (5.19) depends on a retarded integral of the multipolar post-Minkowskian expansion $\mathcal{M}(\tau^{i\nu})$. By integrating by part the integral over z one can transform this last term into

$$\frac{4G}{c^4}\sum_{l,n\geqslant 0}\frac{2^l(-)^l}{(2l+1)!}\widetilde{\Delta^{-n}}(\hat{x}_L)\sum_{p\geqslant 0}\frac{1}{c^{2n-p+l}}\underset{B=0}{\mathrm{FP}}\times\int_{|\mathbf{y}|<\mathcal{R}}d^3\mathbf{y}\,\partial_i(|\tilde{\mathbf{y}}|^B)\hat{y}_L|\mathbf{y}|^{-p-l-1}\overset{(p+l)}{\gamma_l}(1)\times\overset{(2n-p+l)}{\mathcal{M}(\tau^{i\nu})}(\mathbf{y},t-|\mathbf{y}|/c). \tag{5.20}$$

The superscript $(p+l)$ on the γ_l function refers to the z differentiation. It is straightforward to show, using the fact that $\overset{(l)}{\gamma_l}(z)=(-)^{l+1}(2l+1)!!P_l(z)$ is directly related to the Legendre polynomial, that

$$\overset{(p+l)}{\gamma_l}(1)=(-)^{l+1}\frac{(2l+1)!!(l+p)!}{2^p p!(l-p)!}. \tag{5.21}$$

Since $\mathcal{M}(\tau^{i\nu})$ is singular at the origin (but regular at infinity), and because of the explicit factor B brought about by the derivative $\partial_i(|\tilde{\mathbf{y}}|^B)$, the integral in Eq. (5.20) ranges over $|\mathbf{y}|<\mathcal{R}$ (and even $|\mathbf{y}|<\epsilon$, where ϵ is an arbitrary small number). We can then expand $\mathcal{M}(\tau^{i\nu})(\mathbf{y},t-|\mathbf{y}|/c)$ when $c\to+\infty$. Furthermore, we can change the integration over $|\mathbf{y}|<\mathcal{R}$ into an integration over $|\mathbf{y}|>\mathcal{R}$ by simply changing the sign in front of the integral. Indeed, this comes from a technical lemma, which plays an important role in Refs. [39,40]; see before Eq. (C6) in Appendix C, and the proof given in Ref. [51]. Thus,

$$-\frac{4G}{c^4}\sum_{n,l,k\geqslant 0}\frac{2^l(-)^l}{(2l+1)!}\widetilde{\Delta^{-n}}(\hat{x}_L)\sum_{p\geqslant 0}\frac{1}{c^{2n-p+l+k}}\times\underset{B=0}{\mathrm{FP}}\int_{|\mathbf{y}|>\mathcal{R}}d^3\mathbf{y}\,\partial_i(|\tilde{\mathbf{y}}|^B)\hat{n}_L\times\frac{(-)^k}{k!}|\mathbf{y}|^{k-p-1}\overset{(p+l)}{\gamma_l}(1)\overset{(2n-p+l+k)}{\mathcal{M}(\tau^{i\nu})}(\mathbf{y},t). \tag{5.22}$$

By changing the label k into $2k+p-l$ and $2k+1+p-l$, in order to cover odd and even numbers, we are able to write the previous expression in terms of some sums of real numbers indexed by p, i.e.,

$$-\frac{4G}{c^4}\sum_{n,l,k\geqslant 0}\frac{2^l(-)^l}{(2l+1)!}\widetilde{\Delta^{-n}}(\hat{x}_L)\left\{\sum_{p=0}^{l}\frac{(-1)^{p+l}\overset{(p+l)}{\gamma_l}(1)}{(p-l+2k)!}\right\}$$
$$\times \underset{B=0}{\mathrm{FP}}\int_{|\mathbf{y}|>\mathcal{R}}d^3\mathbf{y}\,\partial_i(|\tilde{\mathbf{y}}|^B)\hat{n}_L|\mathbf{y}|^{2k-l-1}$$
$$\times \overset{(2n+2k)}{\mathcal{M}(\tau^{i\nu})}(\mathbf{y},t)+\frac{4G}{c^4}\sum_{n,l,k\geqslant 0}\frac{2^l(-)^l}{(2l+1)!}\widetilde{\Delta^{-n}}(\hat{x}_L)$$
$$\times\left\{\sum_{p=0}^{l}\frac{(-1)^{p+l}\overset{(p+l)}{\gamma_l}(1)}{(p-l+2k+1)!}\right\}$$
$$\times \underset{B=0}{\mathrm{FP}}\int_{|\mathbf{y}|>\mathcal{R}}d^3\mathbf{y}\,\partial_i(|\tilde{\mathbf{y}}|^B)\hat{n}_L|\mathbf{y}|^{2k-l}\overset{(2n+2k+1)}{\mathcal{M}(\tau^{i\nu})}(\mathbf{y},t). \quad (5.23)$$

The sums in curly brackets are found to be explicit expressions depending on k and l and some factorial combinations:

$\forall k\geqslant l+1,$

$$\sum_{p=0}^{l}\frac{(-1)^{p+l}\overset{(p+l)}{\gamma_l}(1)}{(p-l+2k)!}=-\frac{(2l+1)!!}{2^k k!(2k-2l-1)!!}, \quad (5.24)$$

$$\forall k\geqslant l,\quad \sum_{p=0}^{l}\frac{(-1)^{p+l}\overset{(p+l)}{\gamma_l}(1)}{(p-l+2k+1)!}=-\frac{(2l+1)!!}{2^{k-l}(k-l)!(2k+1)!!}, \quad (5.25)$$

$$\forall k\leqslant l,\quad \sum_{p=0}^{l}\frac{(-1)^{p+l}\overset{(p+l)}{\gamma_l}(1)}{(p-l+2k)!}=(-1)^{k+l+1}\frac{(2l+1)!!(2l-2k-1)!!}{2^k k!}, \quad (5.26)$$

$$\forall k\leqslant l-1,\quad \sum_{p=0}^{l}\frac{(-1)^{p+l}\overset{(p+l)}{\gamma_l}(1)}{(p-l+2k+1)!}=0. \quad (5.27)$$

Thanks to these formulas, one can transform Eq. (5.23) into

$$\frac{4G}{c^4}\sum_{n,l\geqslant 0}\sum_{k\geqslant l+1}\frac{(-)^l}{l!}\frac{1}{2^k k!(2k-2l-1)!!}\widetilde{\Delta^{-n}}(\hat{x}_L)$$
$$\times \underset{B=0}{\mathrm{FP}}\int_{|\mathbf{y}|>\mathcal{R}}d^3\mathbf{y}\,\partial_i(|\tilde{\mathbf{y}}|^B)\hat{n}_L|\mathbf{y}|^{2k-l-1}\overset{(2n+2k)}{\mathcal{M}(\tau^{i\nu})}(\mathbf{y},t)$$
$$+\frac{4G}{c^4}\sum_{n,l\geqslant 0}\sum_{k\leqslant l}\frac{(-)^k(2l-2k-1)!!}{l!2^k k!}\widetilde{\Delta^{-n}}(\hat{x}_L)$$
$$\times \underset{B=0}{\mathrm{FP}}\int_{|\mathbf{y}|>\mathcal{R}}d^3\mathbf{y}\,\partial_i(|\tilde{\mathbf{y}}|^B)\hat{n}_L|\mathbf{y}|^{2k-l-1}\overset{(2n+2k)}{\mathcal{M}(\tau^{i\nu})}(\mathbf{y},t)$$
$$-\frac{4G}{c^4}\sum_{n,l\geqslant 0}\sum_{k\geqslant l}\frac{(-)^l}{l!}\frac{1}{2^{k-l}(2k+1)!!(k-l)!}\widetilde{\Delta^{-n}}(\hat{x}_L)$$
$$\times \underset{B=0}{\mathrm{FP}}\int_{|\mathbf{y}|>\mathcal{R}}d^3\mathbf{y}\,\partial_i(|\tilde{\mathbf{y}}|^B)\hat{n}_L|\mathbf{y}|^{2k-l}\overset{(2n+2k+1)}{\mathcal{M}(\tau^{i\nu})}(\mathbf{y},t). \quad (5.28)$$

In the latter expression, we can recognize

$\forall k\geqslant l,$

$$\widetilde{\Delta^{-k}}[\hat{\partial}_L(|\mathbf{y}|^{-1})]=\frac{1}{(2k-2l-1)!!2^k k!}\hat{n}_L|\mathbf{y}|^{2k-l-1}, \quad (5.29)$$

$\forall k\leqslant l,$

$$\widetilde{\Delta^{-k}}[\hat{\partial}_L(|\mathbf{y}|^{-1})]=\frac{(-)^{k+l}(2l-2k-1)!!}{2^k k!}\hat{n}_L|\mathbf{y}|^{2k-l-1}, \quad (5.30)$$

$$\widetilde{\Delta^{-k}}(\hat{y}_L)=\frac{(2l+1)!!}{2^k k!(2k+2l+1)!!}\hat{y}_L|\mathbf{y}|^{2k}, \quad (5.31)$$

so that we obtain

$$\frac{4G}{c^4}\sum_{l,n\geqslant 0}\frac{2^l(-)^l}{(2l+1)!}\widetilde{\Delta^{-n}}(\hat{x}_L)\underset{B=0}{\mathrm{FP}}$$
$$\times\int d^3\mathbf{y}\,\partial_i(|\tilde{\mathbf{y}}|^B)\hat{y}_L\int_1^\infty dz\,\gamma_l(z)\overset{(2n+2l+1)}{\mathcal{M}(\tau^{i\nu})}(\mathbf{y},t-z|\mathbf{y}|/c)$$
$$=\frac{4G}{c^4}\sum_{n,l,k\geqslant 0}\frac{(-)^l}{l!}\widetilde{\Delta^{-n}}(\hat{x}_L)$$
$$\times \underset{B=0}{\mathrm{FP}}\int_{|\mathbf{y}|>\mathcal{R}}d^3\mathbf{y}\,\partial_i(|\widetilde{\mathbf{y}}|^B)\widetilde{\Delta^{-k}}[\hat{\partial}_L(|\mathbf{y}|^{-1})]\overset{(2n+2k)}{\mathcal{M}(\overline{\tau^{i\nu}})}(\mathbf{y},t)$$
$$-\frac{4G}{c^4}\sum_{n,l,k\geqslant 0}\frac{2^l(-)^l}{(2l+1)!}\widetilde{\Delta^{-n}}(\hat{x}_L)$$
$$\times \underset{B=0}{\mathrm{FP}}\int_{|\mathbf{y}|>\mathcal{R}}d^3\mathbf{y}\,\partial_i(|\tilde{\mathbf{y}}|^B)\widetilde{\Delta^{-k}}(\hat{y}_L)\overset{(2n+2k+2l+1)}{\mathcal{M}(\overline{\tau^{i\nu}})}(\mathbf{y},t). \quad (5.32)$$

After replacing Eq. (5.32) in Eq. (5.19), at long last we find

$$\partial_\mu\overline{h^{\mu\nu}}=0. \quad (5.33)$$

In this way, we have checked that the post-Newtonian metric, found by matching as a definite functional of the stress-energy pseudotensor $\tau^{\mu\nu}$, satisfies the harmonic-coordinate condition as a consequence of the conservation of this pseudotensor.

APPENDIX A: NEAR-ZONE EXPANSION OF THE RETARDED INTEGRAL

This appendix, provided here for completeness, is an extended, and also somewhat simplified, version of the derivation given in Appendix A of Ref. [32]. We are interested in source functions, say $\mathcal{M}(\tau)(\mathbf{x},t)$, having the form of an exterior multipole-moment decomposition, valid outside the compact-support domain of the source. We employ the same notation as in Sec. II A (except that we do not write the space-time indices): τ denotes the pseudotensor of the source; notably we have $\mathcal{M}(\tau)=(c^4/16\pi G)\mathcal{M}(\Lambda)$, where Λ is the gravitational source term. The two basic properties of the function $\mathcal{M}(\tau)(\mathbf{x},t)$ are that it is smooth on $\mathbb{R}^4$ deprived from the spatial origin $r=0$:

$$\mathcal{M}(\tau)(\mathbf{x},t)\in C^{\infty}(\mathbb{R}^3_{*}\times\mathbb{R}), \tag{A1}$$

and that it admits a near-zone expansion, when $r\to 0$ (with $t=$const), having the appropriate structure [cf. Eq. (2.14)]: i.e., $\forall N\in\mathbb{N}$,

$$\mathcal{M}(\tau)(\mathbf{x},t)=\sum \hat{n}_L r^a(\ln r)^p G_{L,a,p}(t)+O(r^N), \tag{A2}$$

where $a\in\mathbb{Z}$ with $a\leqslant N$ and $p\in\mathbb{N}$. As in Sec. II B, we denote with an overline the formal (infinite) near-zone expansion,

$$\overline{\mathcal{M}(\tau)(\mathbf{x},t)}=\sum \hat{n}_L r^a(\ln r)^p G_{L,a,p}(t). \tag{A3}$$

It is very important to make the distinction between $\mathcal{M}(\tau)$ and its formal near-zone expansion $\overline{\mathcal{M}(\tau)}$. Here we shall investigate the retarded integral of the product $r^B\mathcal{M}(\tau)(\mathbf{x},t)$, where $B\in\mathbb{C}$, by means of analytic continuation (we pose $r_0=1$ in this appendix). For this task we assume at first that the real part of B is large enough so as to "kill" the divergencies, when $r\to 0$, of the expansion (A2), so that the retarded integral is initially well-defined. Therefore, rigorously speaking, we are allowed to do this only if there exists a finite maximal divergency, i.e., some $a_{\min}\leqslant a$ in Eq. (A2) with finite $a_{\min}\in\mathbb{Z}$. We have seen in Sec. II A that such maximal divergency exists at any given post-Minkowskian order m, but no longer exists for the full post-Minkowskian series because $a_{\min}(m)\to -\infty$ when $m\to +\infty$. The consequence is that the analytic continuation is in principle justified only at a given finite post-Minkowskian order. But, as explained in Sec. II A, we sum up systematically all the post-Minkowskian results. In this way, we are entitled to proceed as we do below; simply we have to remember that the end result will be *a priori* true only in a sense of formal post-Minkowskian expansions.

We decompose the source term into multipoles according to

$$\mathcal{M}(\tau)(\mathbf{x},t)=\sum_{l=0}^{+\infty}\hat{n}_L\sigma_L(r,t), \tag{A4}$$

where the σ_L's are STF functions in $L=i_1\cdots i_l$. The inverse formula is

$$\sigma_L(r,t)=\frac{(2l+1)!!}{l!}\int\frac{d\Omega}{4\pi}\hat{n}_L\mathcal{M}(\tau)(\mathbf{x},t), \tag{A5}$$

where $d\Omega$ is the solid-angle element around the unit vector $n_i=n^i=x^i/r$. Then the expression of the retarded integral, in a sense of analytic continuation in B, is given by the following explicit formula, obtained in Ref. [36] [see Eqs. (6.3)–(6.5) there]:

$$\Box^{-1}_{\mathrm{Ret}}[r^B\mathcal{M}(\tau)(\mathbf{x},t)] =\sum_{l=0}^{+\infty}\int_{-\infty}^{t-r}ds\,\hat{\partial}_L\left\{\frac{R_L^B\left(\frac{t-r-s}{2},s\right)-R_L^B\left(\frac{t+r-s}{2},s\right)}{r}\right\} \tag{A6}$$

(we pose $c=1$ and $r_0=1$ in this appendix), where the function $R_L^B(\rho,s)$ reads

$$R_L^B(\rho,s)=\rho^l\int_0^{\rho}dx\frac{(\rho-x)^l}{l!}\left(\frac{2}{x}\right)^{l-1}x^B\sigma_L(x,x+s). \tag{A7}$$

Following the same procedure as in Eqs. (A6) and (A7) in Ref. [32], we are allowed to rewrite the expression (A6) into the alternative form

$$\Box^{-1}_{\mathrm{Ret}}[r^B\mathcal{M}(\tau)(\mathbf{x},t)] =\sum_{l=0}^{+\infty}\int_{-r}^{r}du\,\hat{\partial}_L\left\{\frac{1}{r}R_L^B\left(\frac{u+r}{2},t-u\right)\right\} -\frac{1}{4\pi}\sum_{l=0}^{+\infty}\frac{(-)^l}{l!}\hat{\partial}_L\left[\frac{\mathcal{R}_L^B(t-r)-\mathcal{R}_L^B(t+r)}{2r}\right]. \tag{A8}$$

The "antisymmetric" wave is parametrized by $\mathcal{R}_L^B(t)$, which is related to the function $R_L^B(\rho,s)$ by

$$\mathcal{R}_L^B(t)=8\pi(-)^{l+1}l!\int_{-\infty}^{t}ds\,R_L^B\left(\frac{t-s}{2},s\right). \tag{A9}$$

Inserting Eq. (A7), and performing some change of variables, we obtain

$$\mathcal{R}_L^B(t)=\frac{4\pi l!}{(2l+1)!!}\int_0^{+\infty}dx\,x^{B+l+2}\times\int_1^{+\infty}dz\,\gamma_l(z)\sigma_L(x,t-zx), \tag{A10}$$

and, using the relation (A5), and considering the variable x as the norm of $\mathbf{x}\in\mathbb{R}^3$, we further get

$$\mathcal{R}_L^B(t)=\int d^3\mathbf{x}|\mathbf{x}|^B\hat{x}_L\times\int_1^{+\infty}dz\,\gamma_l(z)\mathcal{M}(\tau)(\mathbf{x},t-z|\mathbf{x}|). \tag{A11}$$

In these expressions the function $\gamma_l(z)$ is defined by

$$\gamma_l(z)=(-)^{l+1}\frac{(2l+1)!!}{2^l l!}(z^2-1)^l, \tag{A12}$$

where the particular l-dependent factor has been chosen in such a way that the integral is normalized to 1 in the following sense (see Ref. [32]). Considering first that l is a complex number such that $-1<\mathrm{Re}(l)<-\frac{1}{2}$, we can compute the integral of $\gamma_l(z)$ by means of the Euler Γ function, with the result

$$\int_1^{+\infty} dz\,\gamma_l(z)=2(-)^{l+1}\frac{\Gamma(2l+2)\Gamma(-2l-1)}{\Gamma(l+1)\Gamma(-l)}. \tag{A13}$$

The right-hand side of this equation can be analytically continued to all values $l\in\mathbb{C}$ except half-integer values, and is found to be equal to 1 when l is an integer:

$$\int_1^{+\infty} dz\,\gamma_l(z)=1 \quad (l\in\mathbb{N}). \tag{A14}$$

Next, let us treat the first term on the right-hand side of Eq. (A8), say

$$J^B(\mathbf{x},t)\equiv\sum_{l=0}^{+\infty}\int_{-r}^{r}du\,\hat{\partial}_L\left\{\frac{1}{r}R_L^B\left(\frac{u+r}{2},t-u\right)\right\}. \tag{A15}$$

This term is a particular solution of the d'Alembertian equation $\Box J^B=r^B\mathcal{M}(\tau)$ [since the second term in Eq. (A8) is a source-free solution]. We shall prove that the (formal) near-zone expansion of that term, i.e., $\overline{J^B(\mathbf{x},t)}$, is given by the integral of the "instantaneous" potentials acting on the near-zone expansion of the source term, i.e., $r^B\overline{\mathcal{M}(\tau)(\mathbf{x},t)}$. For any of the terms composing the multipolar source $r^B\overline{\mathcal{M}(\tau)}$ [see Eq. (A3)], we first define

$$\Delta^{-1}[\hat{n}_L r^{B+a}(\ln r)^p G_{L,a,p}(t)] =\left(\frac{d}{dB}\right)^p\left[\frac{\hat{n}_L r^{B+a+2}G_{L,a,p}(t)}{(B+a+2-l)(B+a+3+l)}\right] \tag{A16}$$

(this being justified by the fact that one gets an identity by applying Δ on both sides). Clearly the previous formula can be iterated and so we can define the operator $\Delta^{-k-1}\equiv(\Delta^{-1})^{k+1}$, applied on each separate terms in Eq. (A3) and therefore on the complete series $r^B\overline{\mathcal{M}(\tau)(\mathbf{x},t)}$. From this we obtain the instantaneous-potentials operator, as the formal expansion series

$$\mathcal{I}^{-1}[r^B\overline{\mathcal{M}(\tau)(\mathbf{x},t)}] =\sum_{k=0}^{+\infty}\left(\frac{\partial}{c\,\partial t}\right)^{2k}\Delta^{-k-1}[r^B\overline{\mathcal{M}(\tau)(\mathbf{x},t)}]. \tag{A17}$$

Notice that this operator $\mathcal{I}^{-1}$ contains only some even powers of $1/c$. An important point for our purpose is that $\mathcal{I}^{-1}[r^B\overline{\mathcal{M}(\tau)}]$ is *proportional* to the regularization factor r^B, and it evidently satisfies $\Box(\mathcal{I}^{-1}[r^B\overline{\mathcal{M}(\tau)}])=r^B\overline{\mathcal{M}(\tau)}$. On the other hand, we have also the equation $\Box\overline{J^B}=r^B\overline{\mathcal{M}(\tau)}$, which comes from applying the overline operation onto $\Box J^B=r^B\mathcal{M}(\tau)$. This shows that $\mathcal{I}^{-1}[r^B\overline{\mathcal{M}(\tau)}]$ and $\overline{J^B}$ must differ by a solution of the homogeneous equation, hence there should exist some functions $C_L^B(t)$ and $D_L^B(t)$ such that

$$\overline{J^B(\mathbf{x},t)}=\mathcal{I}^{-1}[r^B\overline{\mathcal{M}(\tau)(\mathbf{x},t)}] +\sum_{l=0}^{+\infty}\hat{\partial}_L\left\{\frac{\overline{C_L^B(t-r)+D_L^B(t+r)}}{r}\right\}. \tag{A18}$$

Note that the dependence on B of the second term is "hidden" inside the functions C_L^B and D_L^B. Let us now prove that in fact the latter functions must be zero. This is a simple consequence of the expression (A7) for the function $R_L^B(\rho,s)$, from which we deduce that the expansion when $\rho\to 0$ of this function is proportional to ρ^B; in fact, it has the structure $R_L^B\sim\Sigma\rho^{B+b}(\ln\rho)^q$, when $\rho\to 0$. From this knowledge, we easily find that the near-zone expansion of $\overline{J^B}$ is proportional to the factor r^B. Since, as we have remarked, this is also the case of the first term in Eq. (A18), $\mathcal{I}^{-1}[r^B\overline{\mathcal{M}(\tau)}]$, and since it is impossible that (the near-zone expansion of) the second term in Eq. (A18) is itself proportional to r^B—the B's affect only the functions C_L^B and D_L^B but not the structure of the near-zone expansion—we conclude that C_L^B and D_L^B are identically zero. Hence we have proved

$$\overline{J^B(\mathbf{x},t)}=\mathcal{I}^{-1}[r^B\overline{\mathcal{M}(\tau)(\mathbf{x},t)}]. \tag{A19}$$

It suffices now to apply the overline operation (i.e., to take the near-zone expansion) onto Eq. (A8) to get our final result,

$$\overline{\Box^{-1}_{\mathrm{Ret}}[r^B\mathcal{M}(\tau)(\mathbf{x},t)]} =\mathcal{I}^{-1}[r^B\overline{\mathcal{M}(\tau)(\mathbf{x},t)}] -\frac{1}{4\pi}\sum_{l=0}^{+\infty}\frac{(-)^l}{l!}\hat{\partial}_L\left[\frac{\overline{\mathcal{R}_L^B(t-r)-\mathcal{R}_L^B(t+r)}}{2r}\right], \tag{A20}$$

where we recall that the function $\mathcal{R}_L^B(t)$ has been given by Eq. (A11). (The formula used in Sec. II B results from applying the finite part operation $\mathrm{FP}_{B=0}$.)

APPENDIX B: THE GENERALIZED POISSON OPERATOR

In Appendix A, we have been interested in source functions of the multipolar type $\mathcal{M}(\tau)(\mathbf{x},t)$, which are smooth in $\mathbb{R}^3_*\times\mathbb{R}$ and possess a *near-zone* expansion of the type (A3). In the present appendix, we consider some source functions of the post-Newtonian type $\overline{\tau}(\mathbf{x},t)$. These are supposed to be smooth all over $\mathbb{R}^4$,

$$\overline{\tau}(\mathbf{x},t)\in C^{\infty}(\mathbb{R}^4), \tag{B1}$$

and to admit a *far-zone* expansion with structure ($\forall N\in\mathbb{N}$),

$$\overline{\tau}(\mathbf{x},t)=\sum \hat{n}_L r^a(\ln r)^p G_{L,a,p}(t)+S_N(\mathbf{x},t), \tag{B2}$$

where $a\in\mathbb{Z}$, with $-N\leqslant a$, and $p\in\mathbb{N}$. The remainder term is $S_N(\mathbf{x},t)=O(1/r^N)$ when $r\rightarrow+\infty$ with $t=$const.

Let us consider some $B\in\mathbb{C}$, and a radius $\mathcal{R}\in\mathbb{R}$ with $\mathcal{R}>0$. We define two integrals, corresponding to a split of the Poisson integral between "near-zone" and "far-zone" contributions, separated by the radius $\mathcal{R}$:

$$I^B_<(\mathbf{x},t)=-\frac{1}{4\pi}\int_{|\mathbf{y}|<\mathcal{R}}\frac{d^3\mathbf{y}}{|\mathbf{x}-\mathbf{y}|}|\tilde{\mathbf{y}}|^B\overline{\tau}(\mathbf{y},t), \tag{B3}$$

$$I^B_>(\mathbf{x},t)=-\frac{1}{4\pi}\int_{|\mathbf{y}|>\mathcal{R}}\frac{d^3\mathbf{y}}{|\mathbf{x}-\mathbf{y}|}|\tilde{\mathbf{y}}|^B\overline{\tau}(\mathbf{y},t). \tag{B4}$$

The B-dependent regularization factor is $|\tilde{\mathbf{y}}|^B\equiv(|\mathbf{y}|/r_0)^B$. It is easily checked that the near-zone integral $I^B_<(\mathbf{x},t)$ is well-defined (convergent) when $\mathrm{Re}(B)>-3$ and that the far-zone one $I^B_>(\mathbf{x},t)$ is well-defined when $\mathrm{Re}(B)<-a_{\max}-2$, where $a_{\max}$ is the maximal power of r in the expansion (B2). So we have to assume at this stage the existence of some maximal divergency corresponding to some power $a_{\max}$. Strictly speaking, our present investigation is thus valid only at some finite post-Newtonian order. But, in the end, we sum up the results, and we consider the complete post-Newtonian series to hold true in a formal sense.

We want first to check that the integrals (B3) and (B4) can be analytically continued down to a neighborhood of $B=0$ (except at the value $B=0$ itself), say in the open domain $\mathcal{B}_\epsilon$ defined by $0<|B|<\epsilon$ (where $\epsilon<1$). There is no problem with the near-zone integral $I^B_<(\mathbf{x},t)$ which is clearly convergent all over $\mathcal{B}_\epsilon$ and even at the value $B=0$. Concerning the far-zone integral $I^B_>(\mathbf{x},t)$ we replace the function $\overline{\tau}$ inside the integrand by its far-zone expansion (B2):

$$I^B_>(\mathbf{x},t)=-\frac{1}{4\pi}\int_{|\mathbf{y}|>\mathcal{R}}\frac{d^3\mathbf{y}|\tilde{\mathbf{y}}|^B}{|\mathbf{x}-\mathbf{y}|}\times\left\{\sum \hat{n}_L(\mathbf{y})|\mathbf{y}|^a(\ln|\mathbf{y}|)^p G_{L,a,p}(t)+S_N(\mathbf{y},t)\right\}. \tag{B5}$$

When N is large enough, the contribution due to the remainder S_N is convergent all over $\mathcal{B}_\epsilon$ and at $B=0$, with evidently the value at $B=0$ given by

$$\int_{|\mathbf{y}|>\mathcal{R}}\frac{d^3\mathbf{y}|\tilde{\mathbf{y}}|^B}{|\mathbf{x}-\mathbf{y}|}S_N(\mathbf{y},t)=\int_{|\mathbf{y}|>\mathcal{R}}\frac{d^3\mathbf{y}}{|\mathbf{x}-\mathbf{y}|}S_N(\mathbf{y},t)+O(B). \tag{B6}$$

Thus we need only to deal with the other contributions, which consist of a finite sum of terms, say

$$\sum\int_{|\mathbf{y}|>\mathcal{R}}\frac{d^3\mathbf{y}|\tilde{\mathbf{y}}|^B}{|\mathbf{x}-\mathbf{y}|}\hat{n}_L(\mathbf{y})|\mathbf{y}|^a(\ln|\mathbf{y}|)^p. \tag{B7}$$

Let us suppose that the field point $\mathbf{x}$ lies inside the far-zone domain, i.e., $\mathcal{R}<|\mathbf{x}|$. We distinguish between the two cases, where $|\mathbf{y}|<|\mathbf{x}|$ and $|\mathbf{x}|<|\mathbf{y}|$. For each of these two cases we substitute into the integrals the appropriate multipolar expansion of the factor $1/|\mathbf{x}-\mathbf{y}|$, for instance $1/|\mathbf{x}-\mathbf{y}|=\sum_{l=0}^{+\infty}(-)^l/l!\,y^L\hat{\partial}_L(1/|\mathbf{x}|)$ when $|\mathbf{y}|<|\mathbf{x}|$. This leads, after performing the integration over the angles, to some series of radial integrals having the structure (ignoring some unimportant factors)

$$\sum\frac{\hat{x}_L}{|\mathbf{x}|^{2l+1}}\int_{\mathcal{R}}^{|\mathbf{x}|}d|\mathbf{y}||\mathbf{y}|^{B+a+l+2}(\ln|\mathbf{y}|)^p+\sum\hat{x}_L\int_{|\mathbf{x}|}^{+\infty}d|\mathbf{y}||\mathbf{y}|^{B+a-l+1}(\ln|\mathbf{y}|)^p. \tag{B8}$$

When $|\mathbf{x}|<\mathcal{R}$ the reasoning is the same but one simply ignores the first term in Eq. (B8) and takes $\mathcal{R}$ as a lower bound in the second term. Computing each of these integrals, we find

$$\sum\frac{\hat{x}_L}{|\mathbf{x}|^{2l+1}}\left(\frac{d}{dB}\right)^p\left[\frac{|\mathbf{x}|^{B+a+l+3}-\mathcal{R}^{B+a+l+3}}{B+a+l+3}\right]+\sum\hat{x}_L\left(\frac{d}{dB}\right)^p\left[\frac{-|\mathbf{x}|^{B+a-l+2}}{B+a-l+2}\right]. \tag{B9}$$

Each of these terms clearly admits an analytic continuation for any $B\in\mathcal{B}_\epsilon$ and in fact for any $B\in\mathbb{C}$ except at integer values. Furthermore, we see from that expression that the function will admit a Laurent expansion when $B\rightarrow0$, with in general some multiple poles [coming from the differentiation $(d/dB)^p$ of simple poles $\sim1/B$]. Hence our statement.

It is clear that the Laplacians of the two integrals $I^B_<$ and $I^B_>$ satisfy, in the domains of the complex plane where these functions were initially valid,

$$\mathrm{Re}(B)>-3\Rightarrow\Delta I^B_<(\mathbf{x},t)=Y(\mathcal{R}-|\mathbf{x}|)|\tilde{\mathbf{x}}|^B\overline{\tau}(\mathbf{x},t), \tag{B10}$$

$$\mathrm{Re}(B)<-a_{\max}-2\Rightarrow\Delta I^B_>(\mathbf{x},t)=Y(|\mathbf{x}|-\mathcal{R})|\tilde{\mathbf{x}}|^B\overline{\tau}(\mathbf{x},t), \tag{B11}$$

where Y denotes the Heaviside step function. Therefore, if we *define* for any $B\in\mathcal{B}_\epsilon$ the object

$$I^B(\mathbf{x},t)=I^B_<(\mathbf{x},t)+\underset{B\in\mathcal{B}_\epsilon}{\text{analytic continuation}}\{I^B_>(\mathbf{x},t)\}, \tag{B12}$$

we find that it necessarily satisfies, for any $B\in\mathcal{B}_\epsilon$, the B-dependent Poisson equation

$$\Delta I^B(\mathbf{x},t)=|\tilde{\mathbf{x}}|^B\overline{\tau}(\mathbf{x},t). \tag{B13}$$

On the other hand, we have learned from Eq. (B9) that I^B admits when $B\to 0$ a Laurent expansion involving (in general) simple and multiple poles. Now the key idea, as we shall prove, is that the *finite part*, or coefficient of the zeroth power of B in the latter Laurent expansion, represents a particular solution of the Poisson equation that we want to solve. Let the Laurent expansion of I^B be

$$I^B(\mathbf{x},t)=\sum_{k=k_{\min}}^{+\infty} i_k(\mathbf{x},t)B^k, \tag{B14}$$

where $k_{\min}\in\mathbb{Z}$, and where the coefficients i_k depend on the field point $(\mathbf{x},t)$. By applying the Laplacian operator onto both sides of Eq. (B14), and using the result (B13) together with the Taylor expansion of the regularization factor $|\tilde{\mathbf{x}}|^B$, we arrive at

$$k_{\min}\leqslant k\leqslant -1 \Rightarrow \Delta i_k=0, \tag{B15}$$

$$k\geqslant 0 \Rightarrow \Delta i_k=\frac{(\ln|\tilde{\mathbf{x}}|)^k}{k!}\overline{\tau}. \tag{B16}$$

Thus, the case $k=0$ shows that the finite-part coefficient in the expansion (B14), namely i_0, is a particular solution of the required equation: $\Delta i_0=\overline{\tau}$. We shall now forget about the intermediate name i_0, and denote, from now on, the latter solution by $\widetilde{\Delta^{-1}}\overline{\tau}\equiv i_0$, or, in more explicit terms,

$$\widetilde{\Delta^{-1}}\overline{\tau}(\mathbf{x},t)=\underset{B=0}{\mathrm{FP}}\,\Delta^{-1}[|\tilde{\mathbf{x}}|^B\overline{\tau}(\mathbf{x},t)], \tag{B17}$$

where Δ^{-1} refers to the standard Poisson integral, and the finite-part symbol $\mathrm{FP}_{B=0}$ means the previous operations of considering the Laurent expansion when $B\to 0$, and picking up the finite-part coefficient. Thus, we have proved that $\Delta[\widetilde{\Delta^{-1}}\overline{\tau}]=\overline{\tau}$, so the generalized inverse Poisson operator $\widetilde{\Delta^{-1}}$ defines a particular solution of the Poisson equation, which has, by construction, none of the problems of divergencies of Poisson integrals which have so much plagued the standard post-Newtonian approximation [8–19].

Finally, let us prove that our generalized solution $\widetilde{\Delta^{-1}}\overline{\tau}$ owns the same properties (B1) and (B2) as the corresponding source $\overline{\tau}$. This verification is important because it will allow us to iterate any number of times the operator $\widetilde{\Delta^{-1}}$, and to obtain the post-Newtonian expansion up to any post-Newtonian order. The main problem amounts to proving that $\widetilde{\Delta^{-1}}\overline{\tau}$ admits the same type of expansion at infinity $|\mathbf{x}|\to+\infty$ as in Eq. (B2). To do this, we consider again the same split into near-zone and far-zone contributions: $\widetilde{\Delta^{-1}}\overline{\tau}=I_<+I_>$, where

$$I_<(\mathbf{x},t)=-\frac{1}{4\pi}\underset{B=0}{\mathrm{FP}}\int_{|\mathbf{y}|<\mathcal{R}}\frac{d^3\mathbf{y}}{|\mathbf{x}-\mathbf{y}|}|\tilde{\mathbf{y}}|^B\overline{\tau}(\mathbf{y},t), \tag{B18}$$

$$I_>(\mathbf{x},t)=-\frac{1}{4\pi}\underset{B=0}{\mathrm{FP}}\int_{|\mathbf{y}|>\mathcal{R}}\frac{d^3\mathbf{y}}{|\mathbf{x}-\mathbf{y}|}|\tilde{\mathbf{y}}|^B\overline{\tau}(\mathbf{y},t). \tag{B19}$$

The near-zone integral admits an expansion at infinity which is of the required type. Indeed, because the integrand is of compact support, $|\mathbf{y}|<\mathcal{R}$, we can replace in it the factor $1/|\mathbf{x}-\mathbf{y}|$ by its expansion $\Sigma(-)^l/l!\,y^L\hat{\partial}_L(1/|\mathbf{x}|)$ and integrate term by term. So we have, $\forall N\in\mathbb{N}$,

$$I_<(\mathbf{x},t)=-\frac{1}{4\pi}\sum_{l=0}^{N-1}\frac{(-)^l}{l!}\hat{\partial}_L\left(\frac{1}{r}\right)\times\underset{B=0}{\mathrm{FP}}\int_{|\mathbf{y}|<\mathcal{R}}d^3\mathbf{y}|\tilde{\mathbf{y}}|^B y^L\overline{\tau}(\mathbf{y},t)+O\left(\frac{1}{r^N}\right). \tag{B20}$$

The right-hand side has indeed the same structure as in Eq. (B2). The treatment of the far-zone integral is more delicate. We proceed in a way similar to what was done in Eqs. (B5)–(B9). Namely, we replace into it the source $\overline{\tau}$ by its expansion given by Eq. (B2). This yields (the finite part of) Eq. (B5), which for convenience we reproduce here:

$$I_>(\mathbf{x},t)=-\frac{1}{4\pi}\underset{B=0}{\mathrm{FP}}\int_{|\mathbf{y}|>\mathcal{R}}\frac{d^3\mathbf{y}|\tilde{\mathbf{y}}|^B}{|\mathbf{x}-\mathbf{y}|}\times\left\{\sum\hat{n}_L(\mathbf{y})|\mathbf{y}|^a(\ln|\mathbf{y}|)^p G_{L,a,p}(t)+S_N(\mathbf{y},t)\right\}. \tag{B21}$$

There is a contribution of the remainder and a finite sum of terms with known structure. The remainder contribution is simply given by the value at $B=0$ which has been written on the right-hand side of Eq. (B6). Let us write this term in the form

$$\int_{|\mathbf{y}|>\mathcal{R}}\frac{d^3\mathbf{y}}{|\mathbf{x}-\mathbf{y}|}S_N(\mathbf{y},t)=\sum_{l=0}^{N-4}\frac{(-)^l}{l!}\hat{\partial}_L\left(\frac{1}{r}\right)\int_{|\mathbf{y}|>\mathcal{R}}d^3\mathbf{y}\,y^L S_N(\mathbf{y},t)+T_{N-2}(\mathbf{x},t), \tag{B22}$$

where we introduced the $N-4$ first terms of the multipolar expansion of $1/|\mathbf{x}-\mathbf{y}|$ when $r=|\mathbf{x}|\to+\infty$, and where

$$T_{N-2}(\mathbf{x},t)=\int_{|\mathbf{y}|>\mathcal{R}}d^3\mathbf{y}\left[\frac{1}{|\mathbf{x}-\mathbf{y}|}-\sum_{l=0}^{N-4}\frac{(-)^l}{l!}y^L\hat{\partial}_L\left(\frac{1}{r}\right)\right]S_N(\mathbf{y},t). \tag{B23}$$

The maximal order $N-4$ of the expansion is chosen in such a way that all the terms in Eq. (B23) are given by convergent integrals at infinity, owing to the fact that the remainder satisfies $S_N=\mathcal{O}(1/r^N)$. Now we prove that T_{N-2}, defined by Eq. (B23), is also a remainder in the sense that $T_{N-2}=O(\ln r/r^{N-2})$. We split T_{N-2} into two integrals, a near-zone integral $T^{\rm near}_{N-2}$ corresponding to the integration range $|\mathbf{y}| \in]\mathcal{R},|\mathbf{x}|[$, and a far-zone one $T^{\rm far}_{N-2}$ corresponding to $|\mathbf{y}| \in]|\mathbf{x}|,+\infty[$. In the near-zone integral, we can use the bound

$$\left|\frac{1}{|\mathbf{x}-\mathbf{y}|}-\sum_{l=0}^{N-4}\frac{(-)^l}{l!}y^L\hat{\partial}_L\left(\frac{1}{r}\right)\right|\leq C_N\frac{|\mathbf{y}|^{N-3}}{|\mathbf{x}|^{N-2}}, \quad \text{(B24)}$$

where C_N is a constant. On the other hand, because $S_N = O(1/r^N)$, there is also a constant A_N, depending on the value of $\mathcal{R}$, such that the following majoration holds:

$$|S_N(\mathbf{y},t)|\leq\frac{A_N}{|\mathbf{y}|^N}. \quad \text{(B25)}$$

Replacing these results into the near-zone integral, we get

$$|T^{\rm near}_{N-2}(\mathbf{x},t)|\leq 4\pi\frac{A_N C_N}{|\mathbf{x}|^{N-2}}\ln\left(\frac{|\mathbf{x}|}{\mathcal{R}}\right). \quad \text{(B26)}$$

In the far-zone integral, we can no longer apply the bound (B24) but still we can employ the majoration (B25). Then we can easily show the inequality (in which $|\mathbf{y}|=|\mathbf{x}|\lambda$)

$$|T^{\rm far}_{N-2}(\mathbf{x},t)|\leq 4\pi\frac{A_N}{|\mathbf{x}|^{N-2}}\int_1^{+\infty}\frac{d\lambda}{\lambda^{N-2}}\left[\frac{1}{\lambda}+\sum_{l=0}^{N-4}\frac{(2l-1)!!}{l!}\lambda^l\right]. \quad \text{(B27)}$$

The integral is convergent. At last, from Eqs. (B26) and (B27) we have proved that $T_{N-2}=O(\ln r/r^{N-2})$. Still it remains to show that the finite sum of terms in Eq. (B21), i.e., besides the remainder, admits some expansions of the required structure. But this follows from applying the finite part operation $\mathrm{FP}_{B=0}$ onto the result (B9), which tells us immediately that we have an expansion of the correct type $\sim\hat{n}_L(\mathbf{x})|\mathbf{x}|^a(\ln|\mathbf{x}|)^q$.

APPENDIX C: FAR-ZONE EXPANSION OF THE POISSON INTEGRAL

Thanks to the investigation in Appendix B, the far-zone (or multipolar) expansion of the object $\widetilde{\Delta^{-1}}[\overline{\tau}]$ happens to be workable. Recall that controlling the far-zone expansion of the post-Newtonian field is fundamental since it is at the basis of the matching. The operation of taking the far-zone expansion is denoted $\mathcal{M}$ when applied on post-Newtonian objects (see Sec. III B). We therefore want to determine the expression of $\mathcal{M}(\widetilde{\Delta^{-1}}[\overline{\tau}])$. That is, we want to relate it to the expansion of the corresponding source, which has the same structure as in Eq. (3.22):

$$\mathcal{M}(\overline{\tau})(\mathbf{x},t)=\sum\hat{n}_L r^a(\ln r)^p G_{L,a,p}(t). \quad \text{(C1)}$$

By the matching equation we know that this far-zone expansion is identical with the near-zone expansion of the external field [see, e.g., Eq. (A3)]. Let us first apply $\mathcal{M}$ onto $\widetilde{\Delta^{-1}}[\overline{\tau}]$ as expressed as a sum of near-zone and far-zone contributions,

$$\mathcal{M}(\widetilde{\Delta^{-1}}[\overline{\tau}])=\mathcal{M}(I_<)+\mathcal{M}(I_>), \quad \text{(C2)}$$

where $I_<$ and $I_>$ are defined by Eqs. (B18) and (B19). The near-zone integral is quite easy to work with. Indeed, from Eq. (B20) we see that its expansion when $r=|\mathbf{x}|\to+\infty$ is obtained by expanding the factor $1/|\mathbf{x}-\mathbf{y}|$ inside the integrand. Therefore, the infinite far-zone expansion (without remainder) reads

$$\mathcal{M}(I_<)=-\frac{1}{4\pi}\mathop{\mathrm{FP}}_{B=0}\int_{|\mathbf{y}|<\mathcal{R}}d^3\mathbf{y}|\tilde{\mathbf{y}}|^B\mathcal{M}\left(\frac{1}{|\mathbf{x}-\mathbf{y}|}\right)\overline{\tau}(\mathbf{y},t), \quad \text{(C3)}$$

in which we denote

$$\mathcal{M}\left(\frac{1}{|\mathbf{x}-\mathbf{y}|}\right)=\sum_{l=0}^{+\infty}\frac{(-)^l}{l!}y_L\hat{\partial}_L\left(\frac{1}{|\mathbf{x}|}\right). \quad \text{(C4)}$$

On the other hand, the far-zone expansion of the far-zone integral $I_>$ has been obtained in Eqs. (B21),(B22), where we found that it comes from replacing the source term by its far-zone expansion (indeed, when $\mathcal{R}$ is large enough, the integration ranges over the domain of validity of the far-zone expansion). So the infinite far-zone expansion of that term is given by

$$\mathcal{M}(I_>)=-\frac{1}{4\pi}\mathop{\mathrm{FP}}_{B=0}\int_{|\mathbf{y}|>\mathcal{R}}\frac{d^3\mathbf{y}|\tilde{\mathbf{y}}|^B}{|\mathbf{x}-\mathbf{y}|}\mathcal{M}(\overline{\tau}(\mathbf{y},t)), \quad \text{(C5)}$$

where the integrand contains the expansion of the source given by Eq. (C1). Now let us use a technical lemma which is quite important in the present formalism, and has already played a crucial role in Refs. [39,40]. This lemma is based on the remark that any radial integral of the type $\int_0^{+\infty}d|\mathbf{y}||\mathbf{y}|^{B+a}(\ln|\mathbf{y}|)^p$, where $B\in\mathbb{C}$ and a and p are arbitrary real numbers, is identically *zero* by analytic continuation in B. See Ref. [51] for the proof. Our useful lemma, which is trivial to relate to the previous remark (after performing the integration over angles), is

$$\mathop{\mathrm{FP}}_{B=0}\int d^3\mathbf{y}|\tilde{\mathbf{y}}|^B\mathcal{M}\left(\frac{1}{|\mathbf{x}-\mathbf{y}|}\right)\mathcal{M}(\overline{\tau}(\mathbf{y},t))=0. \quad \text{(C6)}$$

The point here is that the integral ranges over the complete three-dimensional space $\mathbb{R}^3$. Now we have the "numerical" equalities $\mathcal{M}(1/|\mathbf{x}-\mathbf{y}|)=1/|\mathbf{x}-\mathbf{y}|$ when $|\mathbf{y}|<|\mathbf{x}|$ and $\mathcal{M}(\overline{\tau})=\overline{\tau}$ when $|\mathbf{y}|>a$, where a is the radius of the compact-support source. From this we deduce that as soon as $\mathcal{R}>a$, which we can always assume right from the beginning, and

$|\mathbf{x}|>\mathcal{R}$, which is not a problem because we are considering the limit $|\mathbf{x}|\to+\infty$, we have the identity

$$\underset{B=0}{\mathrm{FP}}\int_{|\mathbf{y}|<\mathcal{R}}\frac{d^3\mathbf{y}|\tilde{\mathbf{y}}|^B}{|\mathbf{x}-\mathbf{y}|}\mathcal{M}(\overline{\tau}(\mathbf{y},t))+\underset{B=0}{\mathrm{FP}}\int_{|\mathbf{y}|>\mathcal{R}}d^3\mathbf{y}|\tilde{\mathbf{y}}|^B\mathcal{M}\left(\frac{1}{|\mathbf{x}-\mathbf{y}|}\right)\overline{\tau}(\mathbf{y},t)=0. \tag{C7}$$

By means of that identity, we can obtain the requested form of the far-zone expansion as

$$\mathcal{M}(\widetilde{\Delta^{-1}}[\overline{\tau}])=-\frac{1}{4\pi}\underset{B=0}{\mathrm{FP}}\int d^3\mathbf{y}|\tilde{\mathbf{y}}|^B\left[\frac{1}{|\mathbf{x}-\mathbf{y}|}\mathcal{M}(\overline{\tau}(\mathbf{y},t))+\mathcal{M}\left(\frac{1}{|\mathbf{x}-\mathbf{y}|}\right)\overline{\tau}(\mathbf{y},t)\right]. \tag{C8}$$

In this particular form we see that the $\mathcal{M}$ operator is distributed on the two terms like a derivative operator would be. In the first term we recognize the action of the generalized Poisson integral. Actually this Poisson operator has been defined in Appendix A when acting on a near-zone expansion of the type (A3), but by matching that expansion is the same as the present far-zone expansion, so the definition is rigorously the same. Finally, we can rewrite Eq. (C8) into the alternative form

$$\mathcal{M}(\widetilde{\Delta^{-1}}[\overline{\tau}])=\widetilde{\Delta^{-1}}[\mathcal{M}(\overline{\tau})]-\frac{1}{4\pi}\sum_{l=0}^{+\infty}\frac{(-)^l}{l!}\hat{\partial}_L(r^{-1})\times\underset{B=0}{\mathrm{FP}}\int d^3\mathbf{y}|\tilde{\mathbf{y}}|^B\hat{y}_L\overline{\tau}(\mathbf{y},t), \tag{C9}$$

which constitutes the main result of this appendix. Notice that Eq. (C9) is in agreement with the multipole expansion of the retarded integral as given by Eqs. (3.11) and (3.12) in Ref. [40], when specialized to the static case where there is no dependence on time.

Next we derive the analogous result concerning the operator of the "instantaneous" potentials

$$\widetilde{\mathcal{I}^{-1}}=\sum_{k=0}^{+\infty}\frac{1}{c^{2k}}\partial_t^{2k}\widetilde{\Delta^{-k-1}}. \tag{C10}$$

We iterate $k+1$ times the result (C9). There is no problem in doing this; the only point is that we use in a repeated way the easily checked formula telling that we are allowed to "operate by parts" the Poisson integral $\widetilde{\Delta^{-1}}$ as

$$\underset{B=0}{\mathrm{FP}}\int d^3\mathbf{z}|\tilde{\mathbf{z}}|^B\hat{z}_L\widetilde{\Delta^{-1}}\overline{\tau}=\underset{B=0}{\mathrm{FP}}\int d^3\mathbf{y}|\tilde{\mathbf{y}}|^B\widetilde{\Delta^{-1}}[\hat{y}_L]\overline{\tau}. \tag{C11}$$

This formula is a consequence of the fact that $\mathrm{FP}_{B=0}\int d^3\mathbf{z}\times|\tilde{\mathbf{z}}|^B(\hat{z}_L/|\mathbf{z}-\mathbf{y}|)=-4\pi\widetilde{\Delta^{-1}}[\hat{y}_L]=[-2\pi/(2l+3)]|\mathbf{y}|^2\hat{y}_L$; see Eq. (4.10) in Ref. [43]. Therefore, we arrive at

$$\mathcal{M}(\widetilde{\Delta^{-k-1}}[\overline{\tau}])=\widetilde{\Delta^{-k-1}}[\mathcal{M}(\overline{\tau})]-\frac{1}{4\pi}\sum_{l=0}^{+\infty}\frac{(-)^l}{l!}\sum_{i=0}^{k}\widetilde{\Delta^{-i}}[\hat{\partial}_L(r^{-1})]\times\underset{B=0}{\mathrm{FP}}\int d^3\mathbf{y}|\tilde{\mathbf{y}}|^B\widetilde{\Delta^{i-k}}[\hat{y}_L]\overline{\tau}(\mathbf{y},t), \tag{C12}$$

and from this it is very simple to derive the requested expression concerning $\widetilde{\mathcal{I}^{-1}}$. We obtain

$$\mathcal{M}(\widetilde{\mathcal{I}^{-1}}[\overline{\tau}])=\widetilde{\mathcal{I}^{-1}}[\mathcal{M}(\overline{\tau})]-\frac{1}{4\pi}\sum_{l=0}^{+\infty}\frac{(-)^l}{l!}\sum_{i=0}^{+\infty}\widetilde{\Delta^{-i}}[\hat{\partial}_L(r^{-1})]\times\sum_{k=i}^{+\infty}\frac{1}{c^{2k}}\underset{B=0}{\mathrm{FP}}\int d^3\mathbf{y}|\tilde{\mathbf{y}}|^B\widetilde{\Delta^{i-k}}[\hat{y}_L]\partial_t^{2k}\overline{\tau}(\mathbf{y},t). \tag{C13}$$

This expression, though completely explicit, does not constitute our final form. Because the "instantaneous" solution is a particular solution of the d'Alembertian equation, it must be possible to reexpress the second term in Eq. (C13) as a combination of some source-free retarded and advanced multipolar waves. To see this we notice that

$$\widetilde{\Delta^{-i}}[\hat{\partial}_L(r^{-1})]=\hat{\partial}_L\left(\frac{r^{2i-1}}{(2i)!}\right), \tag{C14}$$

which shows that the latter homogeneous solution is actually one of the *symmetric* type, i.e., retarded *plus* advanced. Namely we can rewrite Eq. (C13) into the form

$$\mathcal{M}(\widetilde{\mathcal{I}^{-1}}[\overline{\tau}])=\widetilde{\mathcal{I}^{-1}}[\mathcal{M}(\overline{\tau})]-\frac{1}{4\pi}\sum_{l=0}^{+\infty}\frac{(-)^l}{l!}\hat{\partial}_L\left\{\overline{\frac{\mathcal{F}_L(t-r/c)+\mathcal{F}_L(t+r/c)}{2r}}\right\}, \tag{C15}$$

where the overline notation means taking the Taylor expansion of the symmetric wave when the retardation $r/c\to0$ [the result is displayed in Eq. (3.24)]. Actually, this overline notation is somewhat misleading, because, in keeping with the real meaning of the result (C15), one should *a posteriori* interpret the latter Taylor expansion as a *far-zone* (singular) expansion when $r\to+\infty$. However, in view of the matching, it is more fruitful to employ the same overline notation as for the expansion of the antisymmetric waves occurring in the near-zone metric—indeed when doing the matching one is simply interested in identifying some asymptotic expansions which are of the same form. The ‘‘multipole-moment’’ function $\mathcal{F}_L(t)$ in Eq. (C15) is given by

$$\mathcal{F}_L(t)=\sum_{j=0}^{+\infty}\frac{1}{c^{2j}}\underset{B=0}{\mathrm{FP}}\int d^3\mathbf{y}|\tilde{\mathbf{y}}|^B\widetilde{\Delta^{-j}}[\hat{y}_L]\partial_t^{2j}\overline{\tau}(\mathbf{y},t). \tag{C16}$$

Finally, let us find an alternative, more compact, form for this result. We introduce the *l*-dependent function

$$\delta_l(z)=\frac{(2l+1)!!}{2^{l+1}l!}(1-z^2)^l, \tag{C17}$$

whose integral is normalized to 1: $\int_{-1}^{1}dz\,\delta_l(z)=1$. One can readily show that

$$\widetilde{\Delta^{-j}}[\hat{y}_L]=|\mathbf{y}|^{2j}\hat{y}_L\int_{-1}^{1}dz\frac{z^{2j}}{(2j)!}\delta_l(z), \tag{C18}$$

which permits us to express the function $\mathcal{F}_L$ in a form where the post-Newtonian series is formally resummed as

$$\mathcal{F}_L(t)=\underset{B=0}{\mathrm{FP}}\int d^3\mathbf{y}|\tilde{\mathbf{y}}|^B\hat{y}_L\overline{\int_{-1}^{1}dz\,\delta_l(z)\,\overline{\tau}(\mathbf{y},t\pm z|\mathbf{y}|/c)}. \tag{C19}$$

Under this form, we recognize the multipole-moment function introduced in Eq. (3.14) in Ref. [40] (the function remains unchanged by taking either sign $\pm$ in the time argument of $\overline{\tau}$). This result permits us to fully determine the exterior multipolar field by matching, and to recover the expression already obtained in Ref. [40] by means of a somewhat different method.

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[49] Our notation for STF tensors is the following. $L=i_1i_2\cdots i_l$ denotes a multi-index, made of l (spatial) indices. When summing over multi-indices we never write the l summations over the l indices $i_1,\cdots,i_l$ ranging from 1 to 3. The STF product of unit vectors $n_i=n^i\equiv x^i/r$ is denoted $\hat{n}_L=\mathrm{STF}(n_L)$, where n_L is shorthand for $n_{i_1}\cdots n_{i_l}$. For instance, $\hat{n}_{ij}=n_in_j-\frac{1}{3}\delta_{ij}$. Similarly, we denote $x_L=x_{i_1}\cdots x_{i_l}=r^ln_L$ and $\hat{x}_L=\mathrm{STF}(x_L)$. The derivative operator ∂_L is shorthand for $\partial_{i_1}\cdots\partial_{i_l}$, and we have $\hat{\partial}_L=\mathrm{STF}(\partial_L)$. For instance, $\hat{\partial}_{ij}=\partial_{ij}-\frac{1}{3}\delta_{ij}\Delta$. More generally, a function F_L is said to be STF with respect to the l indices composing L if and only if, for any pair of indices $i_p,i_q\in L$, we have $F_{\dots i_p\dots i_q\dots}=F_{\dots i_q\dots i_p\dots}$ and $\delta_{i_pi_q}F_{\dots i_p\dots i_q\dots}=0$ (see Appendixes A and B in Ref. [36] for reviews about the STF formalism).

[50] It is clear that for stationary sources (independent of time), the antisymmetric waves given by Eqs. (2.22) are zero. Therefore, the only contribution to the function $\mathcal{R}_L^{\mu\nu}(t)$ comes from the nonstationary (or radiative) part of the field, which according to our assumption of stationarity in the past is zero when $t\leqslant-\mathcal{T}$, and for which $\mathcal{R}_L^{\mu\nu}(t)$ is perfectly well-defined. For simplicity, in the notation we do not indicate that $\mathcal{R}_L^{\mu\nu}(t)$ should be computed only from the "radiative" part of the source term $\mathcal{M}(\tau^{\mu\nu})$.

[51] We want to prove that the radial integral $\int_0^{+\infty}d|\mathbf{y}||\mathbf{y}|^{B+a}(\ln|\mathbf{y}|)^p$ is zero by analytic continuation ($\forall B\in\mathbb{C}$). First we can get rid of the logarithms by considering some repeated differentiations with respect to B; thus we need only consider the simpler integral $\int_0^{+\infty}d|\mathbf{y}||\mathbf{y}|^{B+a}$. We split the integral into a near-zone integral $\int_0^{\mathcal{R}}d|\mathbf{y}||\mathbf{y}|^{B+a}$ and a far-zone one $\int_{\mathcal{R}}^{+\infty}d|\mathbf{y}||\mathbf{y}|^{B+a}$, where $\mathcal{R}$ is some constant radius. When $\mathrm{Re}(B)$ is a large enough positive number, the value of the near-zone integral is $\mathcal{R}^{B+a+1}/(B+a+1)$, while when $\mathrm{Re}(B)$ is a large negative number, the far-zone integral reads the opposite, $-\mathcal{R}^{B+a+1}/(B+a+1)$. Both obtained values represent the unique analytic continuations of the near-zone and far-zone integrals for any $B\in\mathbb{C}$ except $-a-1$. The complete integral $\int_0^{+\infty}d|\mathbf{y}||\mathbf{y}|^{B+b}$ is equal to the sum of these analytic continuations, and is therefore identically zero ($\forall B\in\mathbb{C}$, including the value $-a-1$).