Innermost circular orbit of binary black holes at the third post-Newtonian approximation

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The equations of motion of two point masses have recently been derived at the third post-Newtonian (3PN) approximation of general relativity. From that work we determine the location of the innermost circular orbit (ICO), defined by the minimum of the binary's 3PN energy as a function of the orbital frequency for circular orbits. We find that the post-Newtonian series converges well for equal masses. Spin effects appropriate to corotational black-hole binaries are included. We compare the result with a recent numerical calculation of the ICO in the case of two black holes moving on exactly circular orbits (helical symmetry). The agreement is remarkably good, indicating that the 3PN approximation is adequate to accurately locate the ICO of two black holes with comparable masses. This conclusion is reached with the post-Newtonian expansion expressed in the standard Taylor form, without using resummation techniques such as Padé approximants and/or effective-one-body methods.

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The aim of this article is to compute the innermost circular orbit (ICO) of point-particle binaries in post-Newtonian approximations, and to compare the result with numerical simulations. For the present purpose, the ICO is defined as the minimum, when it exists, of the binary's energy function $E(\Omega)$ of circular orbits, where Ω denotes the orbital frequency. The definition is motivated by our comparison with the numerical work, because this is precisely that minimum which is computed numerically.[1] The energy function is given by the invariant quantity associated with the *conservative* part of the post-Newtonian dynamics, i.e. ignoring the radiation reaction effects. It can be argued, because the radiation reaction damping is neglected, that the importance of the ICO does not lie so much on its strong physical significance, but on the fact that it represents a very useful reference point on the definition of which the post-Newtonian and numerical methods agree.

The question of the conservative dynamics of compact binary systems has been resolved in recent years at the third post-Newtonian (3PN) approximation, corresponding to the order $1/c^6$ beyond the Newtonian force. After the previous work of Refs. [1,2], Jaranowski and Schäfer [3] and Damour, Jaranowski and Schäfer [4] have applied at 3PN order the Arnowitt-Deser-Misner- (ADM-)Hamiltonian formalism of general relativity. On the other hand, extending the method of Ref. [5], Blanchet and Faye [6,7] performed a 3PN iteration of the equations of motion (instead of a Hamiltonian) in harmonic coordinates. In the latter approaches, the compact objects are modelled by point particles, described solely by two mass parameters m_1 and m_2. The end results are physically equivalent in the sense that there exists a unique transformation of the particle's dynamical variables that changes the 3PN harmonic coordinates Lagrangian of de Andrade, Blanchet and Faye [8] into another Lagrangian, whose Legendre transform is identical with the 3PN ADM-coordinates Hamiltonian of Damour, Jaranowski and Schäfer [4].

[1]In particular, we do not define the ICO as a point of dynamical (general-relativistic) unstability.

Post-Newtonian computations of the motion of point particles face the problem of the regularization of the infinite self-field of the particles. The regularization scheme of Hadamard, in "standard" form, was originally adopted in the ADM-Hamiltonian approach [3]. Then an "improved" version of this regularization was defined in Refs. [7] and applied to the computation of the harmonic-coordinates equations of motion [6]. Unfortunately, it has been shown that the Hadamard regularization, either in standard or improved form, leaves unspecified one and only one numerical coefficient in the 3PN equations of motion, ω_{static} in the ADM-Hamiltonian approach, and λ in the harmonic-coordinates formalism. The parameter ω_{static} can be seen as due to some "ambiguity" of the standard Hadamard regularization, while λ appears rather like a parameter of "incompleteness" in the improved version [7] of this regularization. However, these constants turned out to be equivalent, in the sense that [6,4,8]

$$\lambda = -\frac{3}{11}\omega_{\mathrm{static}} - \frac{1987}{3080}. \tag{1}$$

It has been argued in Ref. [9] that the numerical value of ω_{static} could be $\simeq -9$, because for such a value some different "resummation" techniques, when they are implemented at the 3PN order, give approximately the same numerical result for the ICO. Even more, it was suggested [9] that ω_{static} might be precisely equal to $\omega^*_{\mathrm{static}} = -\frac{47}{3} + \frac{41}{64}\pi^2 \simeq -9.34$ (corresponding to $\lambda^* \simeq 1.90$). But, more recently, a computation of ω_{static} has been performed by means of a dimensional regularization, instead of the Hadamard regularization, within the ADM-Hamiltonian formalism [10], with the result

$$\omega_{\mathrm{static}} = 0 \Leftrightarrow \lambda = -\frac{1987}{3080} \simeq -0.64. \tag{2}$$

We adopt in this article the latter value as our preferred one, but in fact it is convenient to keep the ambiguity parameter unspecified, and to investigate the behavior of the solutions for different values of λ or ω_{static}. For instance, we shall keep an eye on the values $\omega^*_{\rm static} \simeq -9.34$ and also $\lambda = 0 \Leftrightarrow \omega_{\rm static} \simeq -2.37$. The latter case corresponds to the special instance where certain logarithmic constants associated with the Hadamard regularization in harmonic coordinates do not depend on the masses [6]. Notice that the result (2) is quite different from $\omega^*_{\rm static} \simeq -9.34$: this suggests, according to Ref. [9], that different resummation techniques, viz. Padé approximants [11] and effective-one-body methods [12], which are designed to "accelerate" the convergence of the post-Newtonian series, do not in fact converge toward the same "exact" solution (or, at least, not as fast as expected).

Let us now compute the ICO of two point particles (modeling black holes) at the 3PN order thanks to the previous body of works [1–8]. The circular-orbit binding energy E (in the center-of-mass frame), and angular momentum J, are deduced either from the 3PN harmonic-coordinates Lagrangian [8] or, equivalently, from the 3PN ADM-coordinates Hamiltonian [4] (we neglect the 2.5PN radiation damping). These functions are expressed in invariant form (the same in different coordinate systems), i.e. with the help of the angular orbital frequency Ω. The 3PN energy (per unit of total mass M), describing "irrotational" circular-orbit binaries, is

$$\frac{E(\Omega)}{M} = -\frac{\nu}{2}(M\Omega)^{2/3}\left\{1 + \left(-\frac{3}{4} - \frac{\nu}{12}\right)(M\Omega)^{2/3} + \left(-\frac{27}{8} + \frac{19}{8}\nu - \frac{\nu^2}{24}\right)(M\Omega)^{4/3} + \left(-\frac{675}{64} + \left[\frac{209323}{4032} - \frac{205}{96}\pi^2 - \frac{110}{9}\lambda\right]\nu - \frac{155}{96}\nu^2 - \frac{35}{5184}\nu^3\right)(M\Omega)^2\right\}. \quad (3)$$

All over this paper we pose $G = c = 1$. Mass parameters are $M = m_1 + m_2$, and the symmetric mass ratio $\nu = m_1 m_2 / M^2$ such that $0 < \nu \leqslant \frac{1}{4}$, with $\nu = \frac{1}{4}$ in the equal-mass case and $\nu \to 0$ in the test-mass limit for one of the bodies. The 3PN angular momentum, scaled by M^2, reads

$$\frac{J(\Omega)}{M^2} = \nu(M\Omega)^{-1/3}\left\{1 + \left(\frac{3}{2} + \frac{\nu}{6}\right)(M\Omega)^{2/3} + \left(\frac{27}{8} - \frac{19}{8}\nu + \frac{\nu^2}{24}\right)(M\Omega)^{4/3} + \left(\frac{135}{16} + \left[-\frac{209323}{5040} + \frac{41}{24}\pi^2 + \frac{88}{9}\lambda\right]\nu + \frac{31}{24}\nu^2 + \frac{7}{1296}\nu^3\right)(M\Omega)^2\right\}. \quad (4)$$

The variations of the energy and angular momentum of the binary on the circular orbit during the inspiral phase obey the evolutionary (or "thermodynamic") law

$$\frac{dE}{d\Omega} = \Omega \frac{dJ}{d\Omega}, \quad (5)$$

TABLE I. Numerical values of the sequence of coefficients of the post-Newtonian series composing the energy function (3).

	Newtonian	1 PN	2PN	3PN
$\nu = \frac{1}{4}$ $\omega_{\rm static} = 0$	1	-0.77	-2.78	-0.97
$\nu = 0$	1	-0.75	-3.37	-10.55

which is equivalent, via the energy and angular-momentum balance equations, to the same relation but between the corresponding gravitational-wave fluxes at infinity. From Eq. (5), we see that the points of extremum for E and J are the same. In the limit $\nu \to 0$, Eqs. (3) and (4) reduce to the 3PN approximations of the known energy and angular momentum of a test particle in the Schwarzschild background:

$$\frac{E^{\rm Sch}(\Omega)}{M} = \nu\{[1 - 2(M\Omega)^{2/3}][1 - 3(M\Omega)^{2/3}]^{-1/2} - 1\}, \quad (6a)$$

$$\frac{J^{\rm Sch}(\Omega)}{M^2} = \nu(M\Omega)^{-1/3}[1 - 3(M\Omega)^{2/3}]^{-1/2}. \quad (6b)$$

We recall that in this case the location of the ICO is given by $M\Omega^{\rm Sch}_{\rm ICO} = 6^{-3/2}$, with $E^{\rm Sch}_{\rm ICO} = \nu M(\sqrt{8/9} - 1)$ and $J^{\rm Sch}_{\rm ICO} = \nu M^2 \sqrt{12}$.

The straightforward post-Newtonian method we follow in this article can be justified by the following arguments. At the location of the ICO we shall find that $M\Omega_{\rm ICO}$ is of the order of 10%. Therefore, we expect that the 1PN approximation will grossly correspond to a relative modification of the binding energy of the order of $v^2 \sim (M\Omega_{\rm ICO})^{2/3}$, i.e. 20%; and similarly that the 2PN and 3PN approximations will yield some effects of magnitude about 5% and 1%, respectively. Consequently the post-Newtonian method should be adequate in the regime of the ICO, provided that it is implemented up to the 3PN order, so as to be accurate enough. On the other hand, we see that the 1PN order should yield a rather poor estimate of the position of the ICO.

Let us now confirm these estimates with the numerical values for the post-Newtonian coefficients in the energy function (3). As we see from Table I, in the case of comparable masses and of our preferred value (2) for the ambiguity parameter, the absolute values of the post-Newtonian coefficients are roughly of the order of one (they do not apparently increase with the order of approximation). This means that the previous estimates are essentially correct. In particular the 3PN approximation should be close to the "exact" value for the ICO. The post-Newtonian series seems to "converge well" (in the case where $\nu = \frac{1}{4}$ and $\omega_{\rm static} = 0$), with a "convergence radius" of the order of one, i.e. at a much higher frequency than the frequency of the ICO.[2] By contrast, we

[2]Actually the post-Newtonian series could be only asymptotic (hence divergent), but nevertheless it should give good results provided that the series is truncated near some optimal order of approximation. In this article we assume that 3PN is not too far from that optimum.

recover in Table I the well-known result (see, e.g. [13,14]) that in the perturbative case $\nu \to 0$ the post-Newtonian series converges slowly: the coefficients increase roughly by a factor 3 at each post-Newtonian order, reflecting the fact that the radius of convergence of the series is $\frac{1}{3}$. This is clear from the exact expression (6a), in which the pole at the value $\frac{1}{3}$ corresponds to the light ring of the Schwarzschild metric. Thus the post-Newtonian method is not very appropriate to the case $\nu=0$, where even the 3PN order would rather poorly approximate the ICO. The situation is therefore the following: in the case of comparable masses, we do not have the exact solution, but fortunately the straightforward post-Newtonian approach is expected to be accurate; in the perturbative limit $\nu=0$, the post-Newtonian series is poorly convergent, but gladly this does not matter because we know the exact results (6).

Having thus justified the validity of our approximation, we look for the point at which both $E(\Omega)$ and $J(\Omega)$ take some minimal values $E_{\mathrm{ICO}}=E(\Omega_{\mathrm{ICO}})$ and $J_{\mathrm{ICO}}=J(\Omega_{\mathrm{ICO}})$. As we see from Eq. (3), at the 3PN order $E(\Omega)$ is a polynomial of the fourth degree in the frequency parameter $x \equiv (M\Omega)^{2/3}$. Therefore, the value of the minimum, $x_{\mathrm{ICO}} = (M\Omega_{\mathrm{ICO}})^{2/3}$, must be a real positive solution of an algebraic equation of the third degree (in general):

$$1+\alpha x+\beta x^2+\gamma x^3=0. \tag{7}$$

The coefficients are straightforwardly obtained from Eq. (3) as

$$\alpha(\nu)=-\frac{3}{2}-\frac{\nu}{6}, \tag{8a}$$

$$\beta(\nu)=-\frac{81}{8}+\frac{57}{8}\nu-\frac{\nu^2}{8}, \tag{8b}$$

$$\gamma(\nu,\lambda)=-\frac{675}{16}+\left[\frac{209323}{1008}-\frac{205}{24}\pi^2-\frac{440}{9}\lambda\right]\nu - \frac{155}{24}\nu^2-\frac{35}{1296}\nu^3. \tag{8c}$$

The regularization constant λ enters only the third-degree monomial (3PN order). Let us describe, in a qualitative way, the existence of solutions of Eq. (7). We find that the equation does not always admit a unique real positive solution, nor even several of them. This depends, for a given choice of the mass ratio ν, on the constant λ. When λ happens to be smaller that some "critical" value $\lambda_0(\nu)$, depending on ν, there is *no* (real positive) solution, and therefore there is no ICO at the 3PN order. When λ is between $\lambda_0(\nu)$ and another "critical" value $\lambda_1(\nu)$, also depending on ν, we obtain *two* real positive solutions. In this case, the energy function admits two extrema, a minimum and a maximum. The maximum occurs at a higher frequency than the minimum of the ICO, and is to be discarded on physical grounds (the corresponding frequency is generally too high, e.g. higher than M^{-1}, for being of physical interest). Finally, when λ is larger than $\lambda_1(\nu)$, there is one and only one real positive solution: x_{ICO}, and this is a minimum of the energy. The latter regime, where the circular-orbit energy admits a unique extremum, which is a minimum (like for the Schwarzschild metric), is the simplest on the physical point of view. The interesting values of λ are located in the regime where $\lambda \geqslant \lambda_1(\nu)$ (for irrotational binaries). We summarize our discussion in Fig. 1.

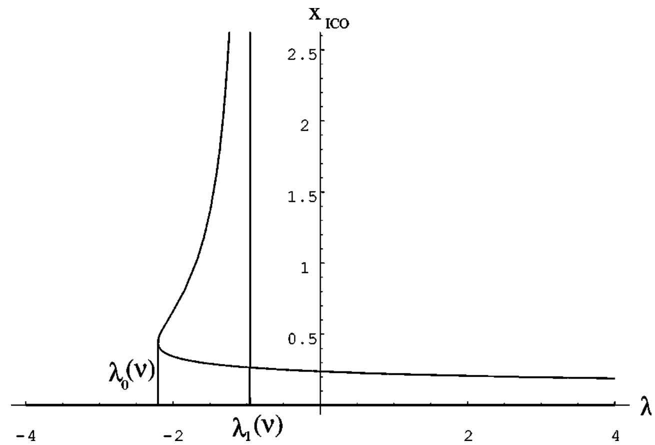

FIG. 1. The possible solutions as a function of the regularization constant λ.There is no solution when $\lambda<\lambda_0(\nu)$, two possible solutions when $\lambda_0(\nu)\leqslant\lambda<\lambda_1(\nu)$ [which become degenerate at $\lambda=\lambda_0(\nu)$], and a unique solution when $\lambda_1(\nu)\leqslant\lambda$. The upper branch, existing between $\lambda_0(\nu)$ and the vertical asymptote at $\lambda=\lambda_1(\nu)$, is actually a maximum of the energy.

It is not difficult to determine analytically the functions $\lambda_0(\nu)$ and $\lambda_1(\nu)$. Indeed, $\lambda_0(\nu)$ represents simply the minimal value of the function $x_{\mathrm{ICO}} \to \lambda(\nu, x_{\mathrm{ICO}})$ (see Fig. 1). Using also Eq. (7), we readily find the mathematical relation defining $\lambda_0(\nu)$:

$$\lambda=\lambda_0(\nu) \Leftrightarrow \gamma(\nu,\lambda) = \frac{2}{27}\left[[\alpha^2(\nu)-3\beta(\nu)]^{3/2}-\alpha^3(\nu)+\frac{9}{2}\alpha(\nu)\beta(\nu)\right], \tag{9}$$

from which the explicit expression of $\lambda_0(\nu)$ can be found using Eqs. (8). On the other hand, the function $\lambda_1(\nu)$ is determined by the cancellation of the third-degree coefficient in the equation (7), i.e.

$$\lambda=\lambda_1(\nu) \Leftrightarrow \gamma(\nu,\lambda)=0. \tag{10}$$

The expression of $\lambda_1(\nu)$ then follows from using Eq. (8c). For allowed values of $\nu \in]0,\frac{1}{4}]$, we find that both $\lambda_0(\nu)$ and $\lambda_1(\nu)$ are increasing functions of ν, with maximal values $\lambda_0(\frac{1}{4})\simeq -2.2$ and $\lambda_1(\frac{1}{4})\simeq -0.96$, and satisfy $\lambda_0(\nu)\to -\infty$ and $\lambda_1(\nu)\to -\infty$ when $\nu\to 0$. Furthermore, we always have $\lambda_0(\nu)<\lambda_1(\nu)$. This analysis shows that in the case of our preferred value $\lambda=-\frac{1987}{3080}\simeq -0.64$, as well as in the cases where $\omega_{\mathrm{static}}=-9.34$ and $\lambda=0$, the energy function $E(\Omega)$ given by Eq. (3), for *any* mass ratio ν, admits a unique extremum, which is a minimum, at some Ω_{ICO} (for corotating binaries we shall find a minimum and also a maximum at very high frequency). We show in Fig. 2 the graph of $E(\Omega)$

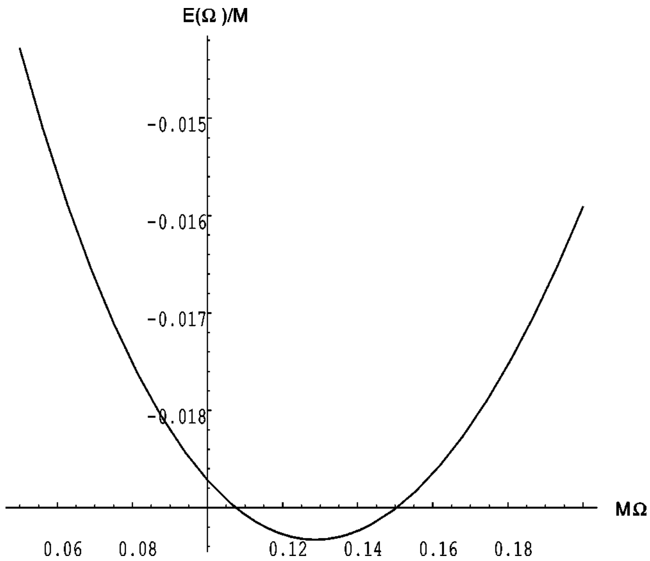

FIG. 2. The 3PN energy function $E(\Omega)$ for equal-mass binaries and $\omega_{\mathrm{static}}=0$.

for equal masses and $\omega_{\mathrm{static}}=0$. Anticipating on our discussion below, it is interesting to compare Fig. 2 with the result of the numerical simulation provided by the Fig. 16 in Ref. [16].

In Table II we present the values of the calculated frequency Ω_{ICO}, the corresponding energy E_{ICO} and angular momentum J_{ICO}, at the 1PN and 2PN orders, and at the 3PN order in the three cases where $\omega_{\mathrm{static}}=0$, $\lambda=0$, and $\omega_{\mathrm{static}}=-9.34$. The 1PN and 2PN approximations are defined by the obvious truncation of Eqs. (3) and (4). Notice how close together already are the 2PN and 3PN approximations (however, the 1PN order seems to be quite inadequate). Let us now show that the 3PN approximation, in standard form (Taylor approximants), appears to be very good to locate the turning point of the ICO, in the sense that the prediction for that point is close to the recent result of numerical relativity.

A novel approach to the problem of the numerical computation of binary black holes in the pre-coalescence stage has been proposed and implemented by Gourgoulhon, Grandclément and Bonazzola [15,16]. This approach uses multi-domain spectral methods [17], and is based on two approximations, the first one is essentially ''technical,'' the other one is ''physical.'' The technical assumption (which could be relaxed in future work) is the conformal flatness of the spatial metric: $\gamma_{ij}=\Psi^4\delta_{ij}$. On the other hand, an imposed ''helical'' symmetry constitutes an important physical restriction to binary systems moving on *exactly* circular orbits. By helical symmetry we mean that the space-time is endowed with a Killing vector field of the type $l^\mu=(\partial/\partial t)+\Omega(\partial/\partial\varphi)$, where $\partial/\partial t$ and $\partial/\partial\varphi$ denote, respectively, the time-like and space-like vectors that coincide asymptotically with the coordinate vectors of an asymptotically inertial observer. A crucial advantage of the helical symmetry, especially in view of the comparison we want to make with the post-Newtonian calculation, is that the orbital frequency Ω is unambiguously defined as the rotation rate of the Killing vector. Thanks to these approximations, Gourgoulhon, Grandclément and Bonazzola [15,16] were able to obtain numerically the energy and angular momentum along the binary's evolutionary sequence, i.e. maintaining Eq. (5) along the sequence, and to determine the minimum of these functions or ICO.

TABLE II. Parameters for the ICO of equal-mass ($\nu=\frac{1}{4}$) binary systems.

	$M\Omega_{\mathrm{ICO}}$	$\frac{E_{\mathrm{ICO}}}{M}$	$\frac{J_{\mathrm{ICO}}}{M^2}$
1PN	0.522	-0.0405	0.621
2PN	0.137	-0.0199	0.779
3PN $\omega_{\mathrm{static}}=0$	0.129	-0.0193	0.786
3PN $\lambda=0$	0.116	-0.0184	0.798
3PN $\omega_{\mathrm{static}}=-9.34$	0.095	-0.0166	0.824

The numerical calculation reported in Refs. [15,16] has been performed in the case of *corotating* black holes, which are spinning with the orbital angular velocity Ω. We must therefore include within our post-Newtonian treatment the effect of spins,[3] appropriate to two Kerr black holes rotating at the orbital rate Ω. By combining the formula of Christodoulou and Ruffini: $m^2=m_{\mathrm{irr}}^2+S^2/(4m_{\mathrm{irr}}^2)$, with the known relation between the black-hole spin and its angular velocity[4]: $S=2m^3\Omega[1+\sqrt{1-(S^2/m^4)}]$, we obtain the total mass m and spin S of each of the corotating black holes in terms of their irreducible mass m_{irr},

$$m=\frac{m_{\mathrm{irr}}}{\sqrt{1-4(m_{\mathrm{irr}}\Omega)^2}}\simeq m_{\mathrm{irr}}+2m_{\mathrm{irr}}^3\Omega^2, \tag{11a}$$

$$S=\frac{4m_{\mathrm{irr}}^3\Omega}{\sqrt{1-4(m_{\mathrm{irr}}\Omega)^2}}\simeq 4m_{\mathrm{irr}}^3\Omega. \tag{11b}$$

The irreducible masses are precisely the ones which are held constant along the evolutionary sequences calculated numerically in Refs. [15,16]. Therefore our first task is to replace all the masses parametrizing the sum $M+E$, where $M=m_1+m_2$ is the total rest mass energy and E is the 3PN binding energy given by Eq. (3), by their equivalent expressions, following Eq. (11a), in terms of the two irreducible masses. It is clear that the leading contribution is that of the kinetic energy of the spins and will come from the replacement of the rest mass energy M; from Eq. (11a) we see that this effect will be of order Ω^2 in the case of corotating binaries, which means by comparison with Eq. (3) that it is equivalent to an ''orbital'' effect at the 2PN order. Higher-order corrections in Eq. (11a) will behave at least like Ω^4

[3]The importance of the effect of spins in corotating systems of neutron stars, for which the ICO is usually determined by the hydrodynamical instability rather than by the effect of general relativity, is well known [18].

[4]More precisely the angular velocity is defined as the one of the outgoing photons that remain forever at the location of the horizon; see Eq. (33.42b) in Ref. [19].

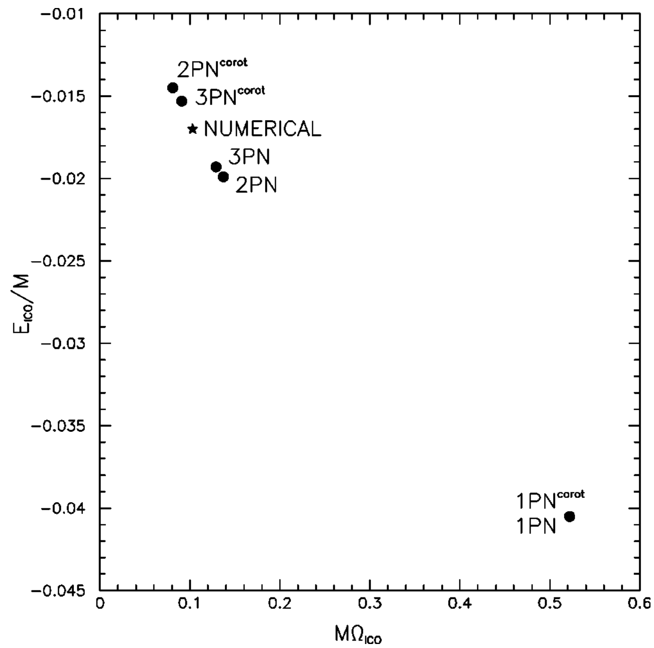

FIG. 3. Results for E_{ICO} versus Ω_{ICO} in the equal-mass case. The asterisk marks the result calculated by numerical relativity. The points indicated by 1PN, 2PN and 3PN are computed from Eq. (3), and correspond to irrotational binaries. The points denoted by $1\mathrm{PN}^{\mathrm{corot}}$, $2\mathrm{PN}^{\mathrm{corot}}$ and $3\mathrm{PN}^{\mathrm{corot}}$ come from the sum of Eqs. (3) and (13), and describe corotational binaries. Both 3PN are $3\mathrm{PN}^{\mathrm{corot}}$ are shown for $\omega_{\mathrm{static}}=0$.

and correspond to the 5PN order at least, negligible for the present purpose. In addition there will be a subdominant contribution, of order $\Omega^{8/3}$ or 3PN, coming from the replacement of the masses into the "Newtonian" part, $\propto\Omega^{2/3}$, of the binding energy E [see Eq. (3)]. At the 3PN approximation we do not need to replace the masses into the post-Newtonian corrections in E. Our second task is to include the relativistic spin-orbit (S.O.) interaction. In the case of spins S_1 and S_2 aligned parallel to the orbital angular momentum (and right-handed with respect to the sense of motion) the S.O. contribution to the energy reads [20,21]

$$E_{\mathrm{S.O.}}=-\nu M(M\Omega)^{5/3}\left[\left(\frac{4}{3}\frac{m_1^2}{M^2}+\nu\right)\frac{S_1}{m_1^2}+\left(\frac{4}{3}\frac{m_2^2}{M^2}+\nu\right)\frac{S_2}{m_2^2}\right]. \tag{12}$$

As can immediately be infered from $S\simeq 4m^3\Omega$, which is deduced from Eq. (11b),[5] in the case of corotating black holes the S.O. effect is of order 3PN and therefore must be retained at the present accuracy [with this approximation, the masses in Eq. (12) can be chosen to be the irreducible ones]. By contrast, the spin-spin (S.S.) interaction turns out to be much smaller, equivalent to the 5PN order for corotating systems. Considering all the contributions present with the 3PN accuracy, we thus obtain three terms: $(2-6\nu)(M\Omega)^2$ coming from the kinetic energy of the corotating spins; $(-\frac{2}{3}\nu+\nu^2)(M\Omega)^{8/3}$ due to a coupling between the spin kinetic energy and the orbital energy; $(-\frac{16}{3}\nu+12\nu^2)(M\Omega)^{8/3}$ due to the S.O. interaction (12). Numerically the kinetic energy of the spins will dominate the other effects. Hence the supplementary energy that is due specifically to the corotation reads

$$\frac{E^{\mathrm{corot}}(\Omega)}{M}=(2-6\nu)(M\Omega)^2+\left(-\frac{18}{3}\nu+13\nu^2\right)(M\Omega)^{8/3}. \tag{13}$$

TABLE III. Parameters for the ICO of corotational equal-mass binary systems.

	$M\Omega_{\mathrm{ICO}}$	$\frac{E_{\mathrm{ICO}}}{M}$
$1\mathrm{PN}^{\mathrm{corot}}$	0.522	−0.0405
$2\mathrm{PN}^{\mathrm{corot}}$	0.081	−0.0145
$3\mathrm{PN}^{\mathrm{corot}}$ $\omega_{\mathrm{static}}=0$	0.091	−0.0153

The total binding energy of the corotating binary is the sum of Eqs. (3) and (13). Notice that we must now understand all the masses in Eqs. (3) and (13) as being the irreducible masses (we no longer indicate the superscripts "irr"), which stay constant when the binary evolves following Eq. (5).

In Table III we present our results for E_{ICO} and Ω_{ICO} of a corotational binary. Since E^{corot}, given by Eq. (13), is at least of order 2PN, the result for $1\mathrm{PN}^{\mathrm{corot}}$ is the same as for 1PN in the irrotational case; then, obviously, $2\mathrm{PN}^{\mathrm{corot}}$ takes into account only the leading 2PN corotation effect (i.e. the kinetic energy of the spins), while $3\mathrm{PN}^{\mathrm{corot}}$ involves also, in particular, the corotational S.O. coupling at 3PN order. In Fig. 3 we plot E_{ICO} versus Ω_{ICO}, computed with and without the corotation effect, and compare the values with the result obtained by numerical relativity under the assumption of helical symmetry [16]. As we can see the 3PN points, and even the 2PN ones, are rather close to the numerical value. As expected, the best agreement is for the 3PN approximation and in the case of corotation:[6] i.e. the point $3\mathrm{PN}^{\mathrm{corot}}$. However, the 1PN approximation is clearly not precise enough, but this is not very surprising in this highly relativistic regime where the orbital velocity reaches $v\sim(M\Omega_{\mathrm{ICO}})^{1/3}\sim 0.5$. Summarizing, we find that the location of the ICO computed by numerical relativity, under the helical-symmetry approximation, is in good agreement with post-Newtonian predictions. This was already pointed out in Ref. [16] from the comparison with Padé and effective-one-body (EOB) methods. This constitutes an appreciable improve-

[5]The moment of inertia of the Kerr black hole in the limit of slow rotations is $I=4m^3$, in accordance with Eq. (2.61) in Ref. [22].

[6]We have checked that our best value, given by $3\mathrm{PN}^{\mathrm{corot}}$, is not significantly modified numerically when we add the higher-order spin effects in Eq. (13) up to the 5PN order, i.e. including, in particular, the S.S. interaction.

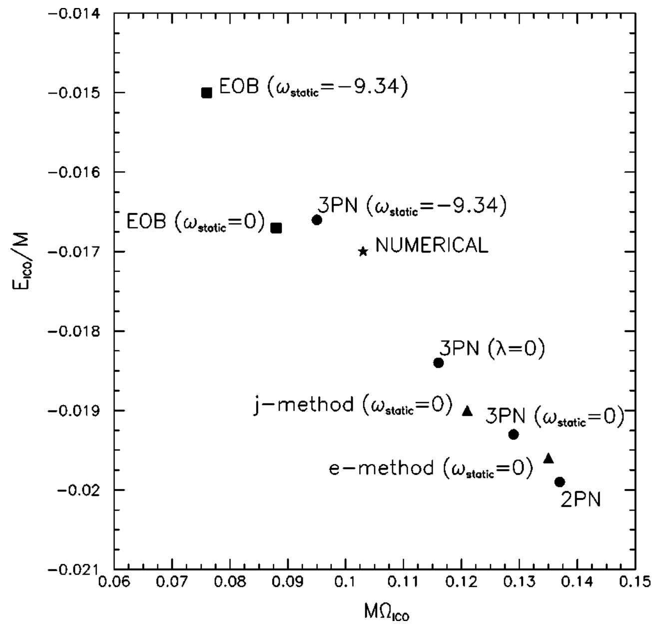

FIG. 4. Results for $E_{\rm ICO}$ in terms of $\Omega_{\rm ICO}$ in the equal-mass case. The *e* and *j* methods are Padé approximants at the 3PN order. EOB refers to the effective-one-body approach at the 3PN order. The points marked by 2PN and 3PN correspond to the standard Taylor post-Newtonian series (this work). The results for Padé, EOB and Taylor are for irrotational binaries.

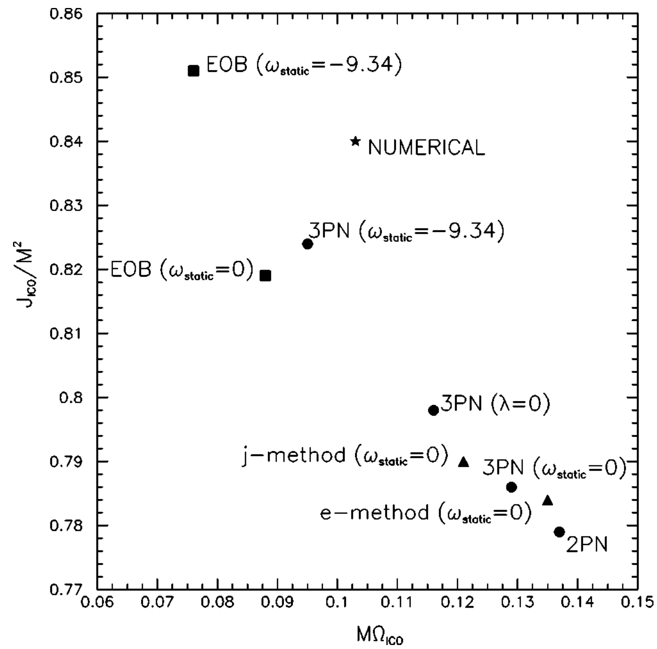

FIG. 5. Same as Fig. 4 but for the angular momentum $J_{\rm ICO}$.

ment of the previous situation, because we recall that the earlier estimates of the ICO in post-Newtonian theory, $M\Omega_{\rm ICO}\simeq 0.06$ and $E_{\rm ICO}/M\simeq -0.009$ [23], and in numerical relativity, $M\Omega_{\rm ICO}\simeq 0.17$ and $E_{\rm ICO}/M\simeq -0.024$ [24,25], strongly disagree with each other, and do not match with the present 3PN results (see Ref. [16] for further discussion).

Let us emphasize that our computation has been based on the standard post-Newtonian approximation, expanded in the usual way as a Taylor series in the frequency-related parameter $x=(M\Omega)^{2/3}$ [see Eqs. (3), (4) and (13)], without using any resummation techniques. In Figs. 4 and 5 we display our Taylor-series-based values for $E_{\rm ICO}$ and $J_{\rm ICO}$ (they are indicated by the marks 2PN and 3PN), and contrast them with some results obtained by means of resummation techniques at the 3PN order: Padé approximants [11,9] and EOB methods [12,9]. All these results agree rather well with each other, and, as we have seen, even the 2PN (Taylor) approximation does well.

A point we make is that the sophisticated Padé approximants give about the same results as the standard post-Newtonian expansion, based on the much simpler Taylor approximants: indeed, see in Figs. 4 and 5 the points referred to as the *e* and *j* methods, which are 3PN Padé resummations built, respectively, on the energy and angular momentum [9]. For the case at hand—equal-mass binaries—there is apparently no improvement from using Padé approximants. Nevertheless, it is true that in the test-mass limit $\nu\to 0$ the Padé series converges rapidly toward the exact result [11]. For instance, the Padé constructed in this case from the 2PN approximation of the energy already coincides with the exact expression for the Schwarzschild metric [given by Eq. (6a)]. But, the results of Figs. 4 and 5 suggest that this interesting feature of the Padé approximants is lost when we turn on ν and consider the equal-mass case $\nu=\frac{1}{4}$. Notice also that the 2PN versions of these Padé, which are given in Table I of Ref. [9], differ much more significantly from the corresponding 3PN ones than in the case of Taylor. For instance, the 2PN *e*-method yields the values $M\Omega_{\rm ICO}\simeq 0.09$ and $E_{\rm ICO}/M\simeq -0.016$, which, respectively, differ by about 36% and 22% with the frequency and energy given by the *e* method at 3PN. In the case of Taylor, the same figures are only 6% and 3%. Thus, on the point of view of the "Cauchy criterium,"[7] the Taylor series seems to converge better that the Padé approximants (for equal masses).

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[7]The Cauchy criterium for the series Σa_n is the fact that $|a_n-a_m|\to 0$ for any n and m.

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