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Gravitational radiation reaction and balance equations to post-Newtonian order

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Gravitational radiation reaction forces and balance equations are investigated to 3/2 post-Newtonian (1.5PN) order beyond the quadrupole approximation, corresponding to the 4PN order in the equations of motion of an isolated system. By matching a post-Newtonian solution for the gravitational field inside the system to a post-Minkowskian solution (obtained in a previous work) for the gravitational field exterior to the system, we determine the 1PN relativistic corrections to the ''Newtonian'' radiation reaction potential of Burke and Thorne. The 1PN reaction potential involves both scalar and vectorial components, with the scalar component depending on the mass-type quadrupole and octupole moments of the system, and the vectorial component depending in particular on the current-type quadrupole moment. In the case of binary systems, the 1PN radiation reaction potential has been shown elsewhere to yield consistent results for the 3.5PN approximation in the binary's equations of motion. Adding up the effects of tails, the radiation reaction is then written to 1.5PN order. In this paper, we establish the validity to 1.5PN order, for general systems, of the balance equations relating the losses of energy, linear momentum, and angular momentum in the system to the corresponding fluxes in the radiation field far from the system. [S0556-2821(97)00702-9]

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I. INTRODUCTION

The old idea (in any field theory) of losses of energy and momenta in an isolated system, due to the presence of radiation reaction forces in the equations of motion, is of topical interest in the case of the gravitational field. Notably, gravitational radiation reaction forces play an important role in astrophysical binary systems of compact objects (neutron stars or black holes). The electromagnetic-based observations of the Hulse-Taylor [1] binary pulsar PSR 1913+16 have yielded evidence that the binding energy of the pulsar and its companion decreases because of gravitational radiation reaction [2–5].

Even more relevant to the problem of radiation reaction are the future gravitational-based observations of inspiralling (and then coalescing) compact binaries. The dynamics of these systems is entirely driven by gravitational radiation reaction forces. The future detectors such as LIGO and VIRGO should observe the gravitational waves emitted during the terminal phase, starting about twenty thousands orbital rotations before the coalescence of two neutron stars. Because inspiralling compact binaries are very relativistic, and thanks to the large number of observed rotations, the output of the detectors should be compared with a very precise expectation from general relativity [6–8]. In particular Cutler *et al.* [6] have shown that our *a priori* knowledge of the relativistic (or post-Newtonian) corrections in the radiation reaction forces will play a crucial role in our ability to satisfactorily extract information from the gravitational signals. Basically, the reaction forces inflect the time evolution of the binary's orbital phase, which can be determined very precisely because of the accumulation of observed rotations. The theoretical problem of the phase evolution has been addressed using black-hole perturbation techniques, valid when the mass of one body is small as compared with the other mass [9–14], and using the post-Newtonian theory, valid for arbitrary mass ratios [15–18]. It has been shown [10,11,14] that post-Newtonian corrections in the radiation reaction forces should be known up to at least the third post-Newtonian (3PN) order, or relative order c^{-6} in the velocity of light.

The radiation reaction forces in the equations of motion of a self-gravitating system arise at the 2.5PN order (or c^{-5} order) beyond the Newtonian acceleration. Controlling the nth post-Newtonian corrections in the reaction force means, therefore, controlling the $(n+2.5)$th post-Newtonian corrections in the equations of motion. If $n=3$, this is very demanding, and beyond our present knowledge. A way out of this problem is to *assume* the validity of a balance equation for energy, which permits relating the mechanical energy loss in the system to the corresponding flux of radiation far from the system. Using such a balance equation necessitates the knowledge of the equations of motion up to the nPN order instead of the $(n+2.5)$PN one. The price to be paid for this saving is the computation of the far-zone flux up to the same (relative) nPN order. However, this is in general less demanding than going to $(n+2.5)$PN order in the equations of motion. All the theoretical works on inspiralling binaries compute the phase evolution from the energy balance equation. Black-hole perturbations [11–13] reach $n=4$ in this way, and the post-Newtonian theory [15–18] has $n=2.5$.

An important theoretical problem is therefore to improve the present situation by showing the validity to post-Newtonian order of the balance equations for energy, and also for linear and angular momenta. This problem is equivalent to controlling the radiation reaction forces at the same post-Newtonian order. Arguably, this problem is also important in its own (not only for applications to inspiralling compact binaries).

Radiation reaction forces in general relativity have long been investigated (see [19] for a review of works prior the seventies). In the late sixties, Burke and Thorne [20–22], using a method of matched asymptotic expansions, introduced a quasi-Newtonian reactive potential, proportional to the fifth time-derivative of the Newtonian quadrupole moment of the source. At about the same time, Chandrasekhar and collaborators [23–25], pursuing a systematic post-Newtonian expansion in the case of extended fluid systems, found some reactive terms in the equations of motion at the 2.5PN approximation. The reactive forces are different in the two approaches because of the use of different coordinate systems, but both yield secular losses of energy and angular momentum in agreement with the standard Einstein quadrupole formulas (see however [26–28]). These results, after later confirmation and improvements [29–36], show the validity of the balance equations to *Newtonian* order, and in the case of weakly self-gravitating fluid systems. In the case of binary systems of compact objects (such as the binary pulsar and inspiralling binaries), the Newtonian balance equations are also known to be valid, as the complete dynamics of these systems has been worked out (by Damour and Deruelle [37–39,19]) up to the 2.5PN order where radiation reaction effects appear.

Post-Newtonian corrections in the radiation reaction force can be obtained from first principles using a combination of analytic approximation methods. The methods are (i) a post-Minkowskian or nonlinear expansion method for the field in the weak-field domain of the source (including the regions far from the source), (ii) a multipolar expansion method for each coefficient of the post-Minkowskian expansion in the domain exterior to the source, and (iii) a post-Newtonian expansion method (or expansion when $c\to\infty$) in the near-zone of the source (including its interior). Then an asymptotic matching (in the spirit of [20–22]) permits us to connect the external field to the field inside the source. Notably, the methods (i) and (ii) have been developed by Blanchet and Damour [40–42] on foundations laid by Bonnor and collaborators [43–45], and Thorne [46]. The method (iii) and matching have also been developed within the present approach [47–49].

The post-Newtonian correction that is due to gravitational wave tails in the reaction force was determined first using the latter methods [50]. The tails of waves are produced by the scattering of the linear waves on the static spacetime curvature generated by the total mass of the source (see e.g., [44,45]). Tails appear as nonlocal integrals, depending on the full past history of the system, and modifying its present dynamics by a post-Newtonian correction of 1.5PN order in the radiation reaction force, corresponding to 4PN order in the equations of motion [50]. It has been shown in [42] that the tail contribution in the reaction force is such that the balance equation for energy is verified for this particular effect. This is a strong indication that the balance equations are actually valid beyond the Newtonian order (1.5PN order in this case). For completeness we shall include this result in the present paper.

The methods (i)–(ii) have been implemented in [51] in order to investigate systematically the occurence and structure of the contributions in the exterior field which are expected to yield radiation reaction effects [after application of the method (iii) and the relevant matching]. The present paper is the direct continuation of the paper [51], that we shall refer here to as paper I.

Working first within the linearized theory, we investigated in paper I the ''antisymmetric'' component of the exterior field, a solution of the d'Alembertian equation composed of a retarded (multipolar) wave minus the corresponding advanced wave. Antisymmetric waves in the exterior field are expected to yield radiation reaction effects in the dynamics of the source. Indeed, these waves change sign when we reverse the condition of retarded potentials into the advanced condition (in the linearized theory), and have the property of being regular all over the source (when the radial coordinate $r\to 0$). Thus, by matching, the antisymmetric waves in the exterior field are necessarily present in the interior field as well, and can be interpreted as radiation reaction potentials. In a particular coordinate system suited to the (exterior) near zone of the source (and constructed in paper I), the antisymmetric waves define a radiation reaction *tensor* potential in the linearized theory, generalizing the radiation reaction scalar potential of Burke and Thorne [20–22].

Working to nonlinear orders in the post-Minkowskian approximation, we introduced in paper I a particular decomposition of the retarded integral into the sum of an ''instantaneous'' integral, and an homogeneous solution composed of antisymmetric waves (in the same sense as in the linearized theory). The latter waves are associated with radiation reaction effects of nonlinear origin. For instance, they contain the nonlinear tail contribution obtained previously [50]. At the 1PN order, the nonlinear effects lead simply to a redefinition of the multipole moments which parametrize the linearized radiation reaction potential. However, the radiation reaction potential at 1PN order has been derived only in the external field. Thus, it was emphasized in paper I that in order to meaningfully interpret the physical effects of radiation reaction, it is necessary to complete this derivation by an explicit matching to the field inside the source.

We perform the relevant matching (at 1PN order) in the present paper. Namely, we obtain a solution for the field inside the source (satisfying the nonvacuum field equations), which can be transformed by means of a suitable coordinate transformation (in the exterior near-zone of the source) into the exterior field determined in paper I. The matching yields in particular the multipole moments parametrizing the reaction potential as explicit integrals over the matter fields in the source. As the exterior field satisfies physically sensible boundary conditions at infinity (viz the no-incoming radiation condition imposed at past-null infinity), the 1PN-accurate radiation reaction potentials are, indeed, appropriate for the description of the dynamics of an isolated system.

To 1PN order beyond the Burke-Thorne term, the reaction potential involves a scalar potential, depending on the mass-type quadrupole and octupole moments of the source, and a vectorial potential, depending in particular on the current-type quadrupole moment. The existence of such vectorial component was first noticed in the physically restricting case where the dominant quadrupolar radiation is suppressed [28].

A different approach to the problem of radiation reaction has been proposed by Iyer and Will [52,53] in the case of binary systems of point particles. The expression of the radiation reaction force is deduced, in this approach, from the *assumption* that the balance equations for energy and angular momentum are correct (the angular momentum balance equation being necessary for noncircular orbits). Iyer and

Will determine in this way the 2.5PN and 3.5PN approximations in the equations of motion of the binary, up to exactly the freedom left by the unspecified coordinate system. They also check that the 1PN-accurate radiation reaction potentials of the present paper (and paper I) correspond in their formalism, when specialized to binary systems, to a unique and consistent choice of a coordinate system. This represents a nontrivial check of the validity of the 1PN reaction potentials.

In the present paper we prove that the 1PN-accurate radiation reaction force in the equations of motion of a general system extracts energy, linear momentum, and angular momentum from the system at the same rate as given by the (known) formulas for the corresponding radiation fluxes at infinity. The result is extended to include the tails at 1.5PN order. Thus we prove the validity up to the 1.5PN order of the energy and momenta balance equations (which were previously known to hold at Newtonian order, and for the specific effects of tails at 1.5PN order).

Of particular interest is the loss of linear momentum, which can be viewed as a ''recoil'' of the center of mass of the source in reaction to the wave emission. This effect is purely due to the 1PN corrections in the radiation reaction potential, and notably to its vectorial component (the Newtonian reaction potential predicts no recoil). Numerous authors have obtained this effect by computing the *flux* of linear momentum at infinity, and then by relying on the balance equation to get the actual recoil [54–57]. Peres [58] made a direct computation of the linear momentum loss in the source, but limited to the case of the linearized theory. Here we prove the balance equation for linear momentum in the full nonlinear theory.

The results of this paper apply to a weakly self-gravitating system. The case of a source made of strongly self-gravitating (compact) objects is *a priori* excluded. However, the theoretical works on the Newtonian radiation reaction in the binary system PSR 1913+16 [37–39,19] have shown that some ''effacement'' of the internal structure of the compact bodies is at work in general relativity. Furthermore, the computation of the radiation reaction at 1PN order in the case of two point-masses [52,53] has shown agreement with a formal reduction, by means of δ functions, of the 1PN radiation reaction potentials, initially derived in this paper only in the case of weakly self-gravitating systems. These works give us hope that the results of this paper will remain unchanged in the case of systems containing compact objects. If this is the case, the present derivation of the 1.5PN balance equations constitutes a clear support of the usual way of computing the orbital phase evolution of inspiralling compact binaries [6–18].

The plan of this paper is the following. The next section (II) is devoted to several recalls from paper I which are necessary in order that the present paper be essentially self-contained. In Sec. III we obtain, using the matching procedure, the gravitational field inside the source, including the 1PN reactive contributions. Finally, in Sec. IV, we show that the latter reactive contributions, when substituted into the local equations of motion of the source, yield the expected 1PN and then 1.5PN balance equations for energy and momenta.

II. TIME-ASYMMETRIC STRUCTURE OF GRAVITATIONAL RADIATION

A. Antisymmetric waves in the linearized metric

Let $D_e=\{(\mathbf{x},t),|\mathbf{x}|>r_e\}$ be the domain exterior to the source, defined by $r_e>a$, where a is the radius of the source. We assume that the gravitational field is weak everywhere, inside and outside the source. In particular $a \gg GM/c^2$, where M is the total mass of the source. Let us consider, in D_e, the gravitational field at the first approximation in a nonlinearity expansion. We write the components $h^{\mu\nu}$ of the deviation of the metric density from the Minkowski metric $\eta^{\mu\nu}$ in the form [59]

$$h^{\mu\nu} \equiv \sqrt{-g}g^{\mu\nu}-\eta^{\mu\nu}=Gh^{\mu\nu}_{(1)}+O(G^2), \tag{2.1}$$

where the coefficient of the Newton constant G represents the linearized field $h^{\mu\nu}_{(1)}$, satisfying the vacuum linearized field equations in D_e,

$$\Box h^{\mu\nu}_{(1)}=\partial^\mu\partial_\lambda h^{\lambda\nu}_{(1)}+\partial^\nu\partial_\lambda h^{\lambda\mu}_{(1)}-\eta^{\mu\nu}\partial_\lambda\partial_\sigma h^{\lambda\sigma}_{(1)}. \tag{2.2}$$

We denote by $\Box \equiv \eta^{\mu\nu}\partial_\mu\partial_\nu$ the flat space-time d'Alembertian operator.

The general solution of Eqs. (2.2) in D_e can be parametrized (modulo an arbitrary linearized coordinate transformation) by means of two and only two sets of multipole moments, referred to as the mass-type moments, denoted M_L, and the current-type moments, S_L [46]. The capital letter L represents a multi-index composed of l indices, $L=i_1 i_2 \cdots i_l$ (see [59] for our notation and conventions). The multipolarity of the moments is $l \geq 0$ in the case of the mass moments, and $l \geq 1$ in the case of the current moments. The M_L's and S_L's are symmetric and tracefree (STF) with respect to their l indices. The lowest-order moments M, M_i, and S_i are constant, and equal, respectively, to the total constant mass (including the energy of the radiation to be emitted), to the position of the center of mass times the mass, and to the total angular momentum of the source. The higher-order moments, having $l \geq 2$, are arbitrary functions of time, $M_L(t)$ and $S_L(t)$, which encode all the physical properties of the source as seen in the exterior (linearized) field. In terms of these multipole moments, the ''canonical'' linearized solution of Thorne [46] reads

$$h^{00}_{\mathrm{can}(1)}=-\frac{4}{c^2}\sum_{l\geq 0}\frac{(-)^l}{l!}\partial_L\left[\frac{1}{r}M_L\left(t-\frac{r}{c}\right)\right], \tag{2.3a}$$

$$h^{0i}_{\mathrm{can}(1)}=\frac{4}{c^3}\sum_{l\geq 1}\frac{(-)^l}{l!}\partial_{L-1}\left[\frac{1}{r}M^{(1)}_{iL-1}\left(t-\frac{r}{c}\right)\right]+\frac{4}{c^3}\sum_{l\geq 1}\frac{(-)^l l}{(l+1)!}\varepsilon_{iab}\partial_{aL-1}\left[\frac{1}{r}S_{bL-1}\left(t-\frac{r}{c}\right)\right], \tag{2.3b}$$

$$h^{ij}_{\mathrm{can}(1)}=-\frac{4}{c^4}\sum_{l\geqslant 2}\frac{(-)^l}{l!}\partial_{L-2}\left[\frac{1}{r}M^{(2)}_{ijL-2}\left(t-\frac{r}{c}\right)\right]-\frac{8}{c^4}\sum_{l\geqslant 2}\frac{(-)^l l}{(l+1)!}\partial_{aL-2}\left[\frac{1}{r}\varepsilon_{ab(i}S^{(1)}_{j)bL-2}\left(t-\frac{r}{c}\right)\right]. \tag{2.3c}$$

This solution satisfies Eq. (2.2) and the condition of harmonic coordinates (i.e., $\partial_\nu h^{\mu\nu}_{\mathrm{can}(1)}=0$). Here we impose that the multipole moments $M_L(t)$ and $S_L(t)$ are constant in the remote past, before some finite instant $-\mathcal{T}$ in the past. With this assumption the linearized field in Eq. (2.3) (and all the subsequent nonlinear iterations built on it) is stationary in a neighborhood of past-null infinity and spatial infinity (it satisfies time-asymmetric boundary conditions in spacetime). This ensures that there is no radiation incoming on the system, which would be produced by some sources located at infinity.

In paper I (Ref. [51]), the contribution in Eq. (2.3) which changes sign when we reverse the condition of retarded potentials to the advanced condition was investigated. This contribution is obtained by replacing each retarded wave in Eq. (2.3) by the corresponding antisymmetric wave, half the difference between the retarded wave and the corresponding advanced one. The antisymmetric wave changes sign when we reverse the time evolution of the moments $M_L(t)$ and $S_L(t)$, say $M_L(t)\rightarrow M_L(-t)$, and evaluate afterwards the wave at the reversed time $-t$. Thus, Eq. (2.3) is decomposed as

$$h^{\mu\nu}_{\mathrm{can}(1)}=(h^{\mu\nu}_{\mathrm{can}(1)})_{\mathrm{sym}}+(h^{\mu\nu}_{\mathrm{can}(1)})_{\mathrm{antisym}}. \tag{2.4}$$

The symmetric part is given by

$$(h^{00}_{\mathrm{can}(1)})_{\mathrm{sym}}=-\frac{4}{c^2}\sum_{l\geqslant 0}\frac{(-)^l}{l!}\partial_L\left\{\frac{M_L(t-r/c)+M_L(t+r/c)}{2r}\right\}, \tag{2.5a}$$

$$(h^{0i}_{\mathrm{can}(1)})_{\mathrm{sym}}=\frac{4}{c^3}\sum_{l\geqslant 1}\frac{(-)^l}{l!}\partial_{L-1}\times\left\{\frac{M^{(1)}_{iL-1}(t-r/c)+M^{(1)}_{iL-1}(t+r/c)}{2r}\right\}+\frac{4}{c^3}\sum_{l\geqslant 1}\frac{(-)^l l}{(l+1)!}\varepsilon_{iab}\partial_{aL-1}\times\left\{\frac{S_{bL-1}(t-r/c)+S_{bL-1}(t+r/c)}{2r}\right\}, \tag{2.5b}$$

$$(h^{ij}_{\mathrm{can}(1)})_{\mathrm{sym}}=-\frac{4}{c^4}\sum_{l\geqslant 2}\frac{(-)^l}{l!}\partial_{L-2}\times\left\{\frac{M^{(2)}_{ijL-2}(t-r/c)+M^{(2)}_{ijL-2}(t+r/c)}{2r}\right\}-\frac{8}{c^4}\sum_{l\geqslant 2}\frac{(-)^l l}{(l+1)!}\partial_{aL-2}\times\left\{\varepsilon_{ab(i}\frac{S^{(1)}_{j)bL-2}(t-r/c)+S^{(1)}_{j)bL-2}(t+r/c)}{2r}\right\}. \tag{2.5c}$$

The antisymmetric part is given similarly. However, as shown in paper I, it can be rewritten profitably in the equivalent form

$$(h^{\mu\nu}_{\mathrm{can}(1)})_{\mathrm{antisym}}=-\frac{4}{Gc^{2+s}}V^{\mu\nu}_{\mathrm{reac}}-\partial^\mu\xi^\nu-\partial^\nu\xi^\mu+\eta^{\mu\nu}\partial_\lambda\xi^\lambda. \tag{2.6}$$

The second, third, and fourth terms clearly represent a linear gauge transformation, associated with the gauge vector ξ^μ. This vector is made of antisymmetric waves, and reads

$$\xi^0=\frac{2}{c}\sum_{l\geqslant 2}\frac{(-)^l}{l!}\frac{2l+1}{l(l-1)}\partial_L\times\left\{\frac{M^{(-1)}_L(t-r/c)-M^{(-1)}_L(t+r/c)}{2r}\right\}, \tag{2.7a}$$

$$\xi^i=-2\sum_{l\geqslant 2}\frac{(-)^l}{l!}\frac{(2l+1)(2l+3)}{l(l-1)}\partial_{iL}\times\left\{\frac{M^{(-2)}_L(t-r/c)-M^{(-2)}_L(t+r/c)}{2r}\right\}+\frac{4}{c^2}\sum_{l\geqslant 2}\frac{(-)^l}{l!}\frac{2l+1}{l-1}\partial_{L-1}\times\left\{\frac{M_{iL-1}(t-r/c)-M_{iL-1}(t+r/c)}{2r}\right\}+\frac{4}{c^2}\sum_{l\geqslant 2}\frac{(-)^l l}{(l+1)!}\frac{2l+1}{l-1}\varepsilon_{iab}\partial_{aL-1}\times\left\{\frac{S^{(-1)}_{bL-1}(t-r/c)-S^{(-1)}_{bL-1}(t+r/c)}{2r}\right\}. \tag{2.7b}$$

Note that even though we have introduced first and second time antiderivatives of the multipole moments, denoted e.g., by $M^{(-1)}_L(t)$ and $M^{(-2)}_L(t)$, the dependence of Eq. (2.7) on the multipole moments ranges in fact only in the time interval between $t-r/c$ and $t+r/c$ (see paper I). The first term in Eq. (2.6) defines, for our purpose, a radiation reaction tensor potential $V^{\mu\nu}_{\mathrm{reac}}$ in the linearized theory (in this term s takes the values 0,1,2 according to $\mu\nu=00,0i,ij$). The components of this potential are given by [59]

$$V^{00}_{\rm reac}=G\sum_{l\geqslant 2}\frac{(-)^l}{l!}\frac{(l+1)(l+2)}{l(l-1)}\hat{\partial}_L \times\left\{\frac{M_L(t-r/c)-M_L(t+r/c)}{2r}\right\}, \tag{2.8a}$$

$$V^{0i}_{\rm reac}=-c^2G\sum_{l\geqslant 2}\frac{(-)^l}{l!}\frac{(l+2)(2l+1)}{l(l-1)}\hat{\partial}_{iL} \times\left\{\frac{M_L^{(-1)}(t-r/c)-M_L^{(-1)}(t+r/c)}{2r}\right\} +G\sum_{l\geqslant 2}\frac{(-)^l l}{(l+1)!}\frac{l+2}{l-1}\varepsilon_{iab}\hat{\partial}_{aL-1} \times\left\{\frac{S_{bL-1}(t-r/c)-S_{bL-1}(t+r/c)}{2r}\right\}, \tag{2.8b}$$

$$V^{ij}_{\rm reac}=c^4G\sum_{l\geqslant 2}\frac{(-)^l}{l!}\frac{(2l+1)(2l+3)}{l(l-1)}\hat{\partial}_{ijL} \times\left\{\frac{M_L^{(-2)}(t-r/c)-M_L^{(-2)}(t+r/c)}{2r}\right\} -2c^2G\sum_{l\geqslant 2}\frac{(-)^l l}{(l+1)!}\frac{2l+1}{l-1}\varepsilon_{ab(i}\hat{\partial}_{j)aL-1} \times\left\{\frac{S_{bL-1}^{(-1)}(t-r/c)-S_{bL-1}^{(-1)}(t+r/c)}{2r}\right\} \tag{2.8c}$$

[see Eq. (2.19) of paper I]. By adding the contributions of the gauge terms associated with Eq. (2.7) to the radiation reaction potential (2.8) one reconstructs precisely, as stated by Eq. (2.6), the antisymmetric part $(h^{\mu\nu}_{\rm can\,(1)})_{\rm antisym}$ of the linearized field in Eq. (2.3).

The scalar, vector, and tensor components of Eq. (2.8) generalize, within the linearized theory, the Burke-Thorne [20–22] scalar potential by taking into account all multipolarities of waves, and, in principle, all orders in the post-Newtonian expansion. Actually, a full justification of this assertion would necessitate a matching to the field inside the source, such as the one we perform in this paper at 1PN order. At the ''Newtonian'' order, the 00 component of the potential reduces to the Burke-Thorne potential,

$$V^{00}_{\rm reac}=-\frac{G}{5c^5}x^ix^jM^{(5)}_{ij}(t)+O\left(\frac{1}{c^7}\right). \tag{2.9}$$

At this order the $0i$ and ij components make negligible contributions. Recall that a well-known property of the Burke-Thorne reactive potential is to yield an energy loss in agreement with the Einstein quadrupole formula, even though it is derived, in this particular coordinate system, within the linearized theory (see [27]). In this paper we shall show that the same property remains essentially true at the 1PN order. (This property is generally false for other reactive potentials valid in other coordinate systems, for which the nonlinear contributions play an important role.) Evaluating the reaction potential $V^{\mu\nu}_{\rm reac}$ at the 1PN order beyond Eq. (2.9), we find that both the 00 and $0i$ components are to be considered, and are given by

$$V^{00}_{\rm reac}=-\frac{G}{5c^5}x^ax^bM^{(5)}_{ab}(t) +\frac{G}{c^7}\left[\frac{1}{189}x^ax^bx^cM^{(7)}_{abc}(t)-\frac{1}{70}r^2x^ax^bM^{(7)}_{ab}(t)\right] +O\left(\frac{1}{c^9}\right), \tag{2.10a}$$

$$V^{0i}_{\rm reac}=\frac{G}{c^5}\left[\frac{1}{21}\hat{x}^{iab}M^{(6)}_{ab}(t)-\frac{4}{45}\varepsilon_{iab}x^ax^cS^{(5)}_{bc}(t)\right]+O\left(\frac{1}{c^7}\right). \tag{2.10b}$$

At this order the ij components of the potential can be neglected.

In the next subsection we address the question of the corrections to the 1PN reaction potential in Eq. (2.10) which arise from the nonlinear contributions to the exterior field. Answering this question means controlling the nonlinear metric at the 3.5PN order.

B. The 3.5PN approximation in the exterior metric

The radiation reaction potential in Eqs. (2.8)–(2.10) represents the antisymmetric part of the linearized metric in a particular coordinate system, obtained from the initial harmonic coordinate system in which Eq. (2.3) holds by applying the gauge transformation associated with Eq. (2.7). In this coordinate system, the new linearized metric reads

$$h^{\mu\nu}_{(1)}=h^{\mu\nu}_{{\rm can}(1)}+\partial^\mu\xi^\nu+\partial^\nu\xi^\mu-\eta^{\mu\nu}\partial_\lambda\xi^\lambda. \tag{2.11}$$

It fulfills, of course, the linearized equations (2.2). Furthermore, since $\Box\xi^\mu=0$, the harmonic coordinate condition is still satisfied. However, we recall from paper I that the latter new harmonic coordinate system is not completely satisfying for general purposes, and should be replaced by a certain modified (nonharmonic) coordinate system. The reason is that the gauge vector defined by Eq. (2.7) is made of antisymmetric waves, and consequently the metric (2.11) contains both retarded *and* advanced waves. In particular, the metric is no longer stationary in the remote past (before the instant $-\mathcal{T}$ where the moments are assumed to be constant). Of course, the advanced waves which have been introduced are pure gauge. Nevertheless, the nonstationarity of the linearized metric in the remote past breaks one of our initial assumptions, and this can be a source of problems when performing the nonlinear iterations of the metric by this method. Therefore, it was found necessary in paper I to replace the gauge vector (2.7) by a modified gauge vector, such that the modified coordinate system has two properties. First, it reduces in the near-zone of the source (the domain D_i defined in Sec. III), to the unmodified coordinate system given by Eq. (2.7), with a given but *arbitrary* post-Newtonian precision. Second, it reduces *exactly* to the initial harmonic coordinate system in which the linearized metric reads (2.3), in a domain exterior to some *timelike* world tube surrounding the source (in fact, a future-oriented timelike cone whose vertex is at the event $t=-\mathcal{T}$, $\mathbf{x}=\mathbf{0}$). This modified coordinate system is nonharmonic. It has been suggested to the author by T. Damour in a private communication, and is defined in Sec. II C of paper I. By the first property, we see that if we are interested to the field in the near zone of the source, and work at some finite post-Newtonian order (like the 1PN order investigated in this paper), we can make all computations using the unmodified gauge vector (2.7). Indeed, it suffices to adjust a certain constant, denoted K in paper I, so that the modified gauge vector agrees with Eq. (2.7) with a higher post-Newtonian precision. Thus, in the present paper, we shall not use explicitly the modified coordinate system. By the second property, we see that the standard falloff behavior of the metric at the various infinities (notably the standard no-incoming radiation condition at past-null infinity) are preserved in the modified coordinate system. By these two properties, one can argue that what only matters is the *existence* of such a modified coordinate system, which permits us to make the connection with the field at infinity, but that in all practical computations of the metric in the near zone one can use the unmodified coordinate system defined by Eq. (2.7).

Based on the linearized metric (2.11) [or, rather, on the modified linearized metric (2.29) of paper I] we built a full nonlinear expansion,

$$h^{\mu\nu}=Gh^{\mu\nu}_{(1)}+G^2h^{\mu\nu}_{(2)}+G^3h^{\mu\nu}_{(3)}+\cdots, \tag{2.12}$$

satisfying the vacuum field equations in a perturbative sense (equating term by term the coefficients of equal powers of G in both sides of the equations). The nonlinear coefficients $h^{\mu\nu}_{(2)}$, $h^{\mu\nu}_{(3)}$, ... are like $h^{\mu\nu}_{(1)}$ in the form of multipole expansions parametrized by M_L and S_L. The construction of the nonlinear metric is based on the method of [40]. As we are considering multipole expansions valid in D_e (and singular at the spatial origin $r=0$), we need to use at each nonlinear iterations of the field equations a special operator generalizing the usual retarded integral operator when acting on multipole expansions. We denote this operator by $\mathcal{F}\Box_R^{-1}$ to mean the "finite part of the retarded integral operator" (see [40] for its precise definition). The nonlinear coefficients $h^{\mu\nu}_{(2)}$, $h^{\mu\nu}_{(3)}$, ... are given by

$$h^{\mu\nu}_{(2)}=\mathcal{F}\Box_R^{-1}\Lambda^{\mu\nu}_{(2)}(h_{(1)})+q^{\mu\nu}_{(2)}, \tag{2.13a}$$

$$h^{\mu\nu}_{(3)}=\mathcal{F}\Box_R^{-1}\Lambda^{\mu\nu}_{(3)}(h_{(1)},h_{(2)})+q^{\mu\nu}_{(3)},\ \ldots, \tag{2.13b}$$

where the nonlinear source terms $\Lambda^{\mu\nu}_{(2)}$, $\Lambda^{\mu\nu}_{(3)}$, ..., represent the field nonlinearities in vacuum, and depend, at each nonlinear order, on the coefficients of the previous orders. The second terms $q^{\mu\nu}_{(2)}$, $q^{\mu\nu}_{(3)}$, ..., ensure the satisfaction of the harmonic coordinate condition at each nonlinear order (see [40]).

When investigating the 3.5PN approximation, we can disregard purely nonlinear effects, such as the tail effect, which give irreducibly nonlocal contributions in the metric inside the source. These effects arise at the 4PN approximation (see [50] and Sec. IV below). Still there are some nonlinear contributions in the metric at the 3.5PN approximation, which are contained in the first two nonlinear coefficients $h^{\mu\nu}_{(2)}$ and $h^{\mu\nu}_{(3)}$ given by Eq. (2.13). These contributions involve some nonlocal integrals, but which ultimately do not enter the inner metric (after matching). As shown in paper I, the contributions due to $h^{\mu\nu}_{(2)}$ and $h^{\mu\nu}_{(3)}$ in the 1PN radiation reaction potential imply only a modification of the multipole moments M_L and S_L parametrizing the potential. We define two new sets of multipole moments,

$$\widetilde{M}_L(t)=M_L(t)+\left\{\begin{array}{l}\dfrac{G}{c^7}m(t)\quad\text{for}\ \ l=0\\[2ex] \dfrac{G}{c^5}m_i(t)\quad\text{for}\ \ l=1\\[2ex] 0\quad\text{for}\ \ l\geqslant 2\end{array}\right\}+\frac{G}{c^7}T_L(t)$$
$$+O\left(\frac{1}{c^8}\right), \tag{2.14a}$$

$$\widetilde{S}_L(t)=S_L(t)+\left\{\begin{array}{l}\dfrac{G}{c^5}s_i(t)\quad\text{for}\ \ l=1\\[2ex] 0\quad\text{for}\ \ l\geqslant 2\end{array}\right\}+O\left(\frac{1}{c^6}\right), \tag{2.14b}$$

where the functions m, m_i, and s_i are given by the nonlocal expressions

$$m(t)=-\frac{1}{5}\int_{-\infty}^{t}dv\, M^{(3)}_{ab}(v)M^{(3)}_{ab}(v)+F(t), \tag{2.15a}$$

$$m_i(t)=-\frac{2}{5}M_aM^{(3)}_{ia}(t)-\frac{2}{21c^2}\int_{-\infty}^{t}dv\, M^{(3)}_{iab}(v)M^{(3)}_{ab}(v)$$
$$+\frac{1}{c^2}\int_{-\infty}^{t}dv\int_{-\infty}^{v}dw\left[-\frac{2}{63}M^{(4)}_{iab}(w)M^{(3)}_{ab}(w)\right.$$
$$\left.-\frac{16}{45}\varepsilon_{iab}M^{(3)}_{ac}(w)S^{(3)}_{bc}(w)\right]+\frac{1}{c^2}G_i(t), \tag{2.15b}$$

$$s_i(t)=-\frac{2}{5}\varepsilon_{iab}\int_{-\infty}^{t}dv\, M^{(2)}_{ac}(v)M^{(3)}_{bc}(v)+H_i(t). \tag{2.15c}$$

The function $T_L(t)$ in Eq. (2.14a), and the functions $F(t)$, $G_i(t)$, and $H_i(t)$ in Eq. (2.15), are some local (or instantaneous) functions, which do not play a very important role physically (they are computed in paper I). Then the radiation reaction potential at the 1PN order, in the nonlinear theory, is given by the same expression as in Eq. (2.10), but expressed in terms of the new multipole moments, say $\widetilde{\mathcal{M}}\equiv\{\widetilde{M}_L,\widetilde{S}_L\}$,

$$V^{\rm reac}[\widetilde{\mathcal{M}}]=-\frac{G}{5c^5}x^a x^b \widetilde{M}^{(5)}_{ab}(t)+\frac{G}{c^7}\left[\frac{1}{189}x^{abc}\widetilde{M}^{(7)}_{abc}(t)-\frac{1}{70}\,r^2 x^{ab}\widetilde{M}^{(7)}_{ab}(t)\right]+O\left(\frac{1}{c^8}\right), \tag{2.16a}$$

$$V_i^{\rm reac}[\widetilde{\mathcal{M}}]=-\frac{G}{5c^5}\left[\frac{1}{21}\,\hat{x}^{iab}\widetilde{M}^{(6)}_{ab}(t)-\frac{4}{45}\,\varepsilon_{iab}x^{ac}\widetilde{S}^{(5)}_{bc}(t)\right]+O\left(\frac{1}{c^6}\right) \tag{2.16b}$$

[see Eq. (3.53) in paper I]. In the considered coordinate system, the metric, accurate to 1PN order as concerns both the usual nonradiative effects and the radiation reaction effects, reads (coming back to the usual covariant metric $g^{\rm ext}_{\mu\nu}$)

$$g^{\rm ext}_{00}=-1+\frac{2}{c^2}(V^{\rm ext}[\widetilde{\mathcal{M}}]+V^{\rm reac}[\widetilde{\mathcal{M}}]) -\frac{2}{c^4}(V^{\rm ext}[\widetilde{\mathcal{M}}]+V^{\rm reac}[\widetilde{\mathcal{M}}])^2 +\frac{1}{c^6}\,{}_6g^{\rm ext}_{00}+\frac{1}{c^8}\,{}_8g^{\rm ext}_{00}+O\left(\frac{1}{c^{10}}\right), \tag{2.17a}$$

$$g^{\rm ext}_{0i}=-\frac{4}{c^3}(V_i^{\rm ext}[\widetilde{\mathcal{M}}]+V_i^{\rm reac}[\widetilde{\mathcal{M}}])+\frac{1}{c^5}\,{}_5g^{\rm ext}_{0i} +\frac{1}{c^7}\,{}_7g^{\rm ext}_{0i}+O\left(\frac{1}{c^9}\right), \tag{2.17b}$$

$$g^{\rm ext}_{ij}=\delta_{ij}\left[1+\frac{2}{c^2}(V^{\rm ext}[\widetilde{\mathcal{M}}]+V^{\rm reac}[\widetilde{\mathcal{M}}])\right] +\frac{1}{c^4}\,{}_4g^{\rm ext}_{ij}+\frac{1}{c^6}\,{}_6g^{\rm ext}_{ij}+O\left(\frac{1}{c^8}\right). \tag{2.17c}$$

The superscript ext is to remember that the metric is valid in the exterior domain D_e, and will differ from the inner metric by a coordinate transformation (see Sec. III). The Newtonian and 1PN approximations are entirely contained in the externals potentials $V^{\rm ext}$ and $V_i^{\rm ext}$, given by the multipole expansions of *symmetric* waves,

$$V^{\rm ext}[\widetilde{\mathcal{M}}]=G\sum_{l\geqslant 0}\frac{(-)^l}{l!}\partial_L\left\{\frac{\widetilde{M}_L\left(t-\frac{r}{c}\right)+\widetilde{M}_L\left(t+\frac{r}{c}\right)}{2r}\right\}, \tag{2.18a}$$

$$V_i^{\rm ext}[\widetilde{\mathcal{M}}]=-G\sum_{l\geqslant 1}\frac{(-)^l}{l!}\partial_{L-1}\times\left\{\frac{\widetilde{M}^{(1)}_{iL-1}\left(t-\frac{r}{c}\right)+\widetilde{M}^{(1)}_{iL-1}\left(t+\frac{r}{c}\right)}{2r}\right\} -G\sum_{l\geqslant 1}\frac{(-)^l l}{(l+1)!}\varepsilon_{iab}\partial_{aL-1}\times\left\{\frac{\widetilde{S}_{bL-1}\left(t-\frac{r}{c}\right)+\widetilde{S}_{bL-1}\left(t+\frac{r}{c}\right)}{2r}\right\}. \tag{2.18b}$$

The 2PN and 3PN approximations are not controlled at this stage; they are symbolized in Eq. (2.17) by the terms $c^{-n}\,{}_ng^{\rm ext}_{\mu\nu}$. However, these approximations are nonradiative (or nondissipative), as are the Newtonian and 1PN approximations (the 1PN, 2PN, and 3PN terms are ''even'' in the sense that they yield only even powers of $1/c$ in the equations of motion). A discussion of the 4PN approximation in the exterior metric can be found in Sec. III D of paper I.

As it stands, the metric (2.17) is disconnected from the actual source of radiation. The multipole moments $\widetilde{M}_L$ and $\widetilde{S}_L$ are left as some unspecified functions of time. Therefore, in order to determine the radiation reaction potentials (2.16) as some *explicit* functionals of the matter variables, we need to relate the exterior metric (2.17) to a metric valid inside the source, solution of the nonvacuum Einstein field equations. We perform the relevant computation in the next section, and obtain the multipole moments $\widetilde{M}_L$ and $\widetilde{S}_L$ as integrals over the source.

III. THE 1PN-ACCURATE RADIATION REACTION POTENTIALS

A. The inner gravitational field

The near zone of the source is defined in the usual way as being an inner domain $D_i=\{(\mathbf{x},t),|\mathbf{x}|<r_i\}$, whose radius r_i satisfies $r_i>a$ (D_i covers entirely the source), and $r_i\ll\lambda$ (the domain D_i is of small extent as compared with one wavelength of the radiation). These two demands are possible simultaneously when the source is slowly moving, i.e., when there exists a small parameter of the order of $1/c$ when $c\to\infty$, in which case we can assume $r_i/\lambda = O(1/c)$. Furthermore, we can adjust r_e and r_i so that $a<r_e<r_i$ (where r_e defines the external domain D_e in Sec. II).

In this subsection we present the result of the expression of the metric in D_i, to the relevant approximation. In the next subsection we prove that this metric matches to the exterior metric reviewed in Sec. II. The accuracy of the inner metric is 1PN for the usual nonradiative approximations, and 1PN for the dominant radiation reaction. Thus, the metric enables one to control, in the equations of motion, the Newtonian acceleration followed by the first relativistic correction, which is of order c^{-2} or 1PN, then the dominant ''Newtonian'' radiation reaction, of order c^{-5} or 2.5PN, and finally the first relativistic 1PN correction in the reaction, c^{-7} or 3.5PN. The intermediate approximations c^{-4} and c^{-6} (2PN and 3PN) are left undetermined, like in the exterior metric (2.17). The inner metric, in D_i, reads

$$g^{\rm in}_{00} = -1 + \frac{2}{c^2}\mathcal{V}^{\rm in} - \frac{2}{c^4}(\mathcal{V}^{\rm in})^2 + \frac{1}{c^6}\,{}_6g^{\rm in}_{00} + \frac{1}{c^8}\,{}_8g^{\rm in}_{00} + O\left(\frac{1}{c^{10}}\right), \tag{3.1a}$$

$$g^{\rm in}_{0i} = -\frac{4}{c^3}\mathcal{V}^{\rm in}_i + \frac{1}{c^5}\,{}_5g^{\rm in}_{0i} + \frac{1}{c^7}\,{}_7g^{\rm in}_{0i} + O\left(\frac{1}{c^9}\right), \tag{3.1b}$$

$$g^{\rm in}_{ij} = \delta_{ij}\left(1 + \frac{2}{c^2}\mathcal{V}^{\rm in}\right) + \frac{1}{c^4}\,{}_4g^{\rm in}_{ij} + \frac{1}{c^6}\,{}_6g^{\rm in}_{ij} + O\left(\frac{1}{c^8}\right). \tag{3.1c}$$

It is valid in a particular Cartesian coordinate system $(\mathbf{x},t)$, which is to be determined by matching. As in Eq. (2.17), the terms $c^{-n}{}_ng^{\rm in}_{\mu\nu}$ represent the 2PN and 3PN approximations. Note that these terms depend functionally on the source's variables through some *spatial* integrals, extending over the whole three-dimensional space, but that they do not involve any nonlocal integral in time. These terms are ''instantaneous'' (in the terminology of [50]) and ''even,'' so they remain invariant in a time reversal, and do not yield any radiation reaction effects. The remainder terms in Eq. (3.1) represent the 4PN and higher approximations. Note also that some logarithms of c arise starting at the 4PN approximation. For simplicity we do not indicate in the remainders the dependence on $\ln c$. The potentials $\mathcal{V}^{\rm in}$ and $\mathcal{V}^{\rm in}_i$ introduced in Eq. (3.1) are given, as in Eq. (2.17), as the linear combination of two types of potentials. With the notation $\mathcal{V}^{\rm in}_\mu \equiv (\mathcal{V}^{\rm in},\mathcal{V}^{\rm in}_i)$, where the index μ takes the values $0,i$ and where $\mathcal{V}^{\rm in}_0 \equiv \mathcal{V}^{\rm in}$, we have

$$\mathcal{V}^{\rm in}_\mu = V^{\rm in}_\mu[\sigma_\nu] + V^{\rm reac}_\mu[\mathcal{I}]. \tag{3.2}$$

The first type of potential, $V^{\rm in}_\mu$, is given by an integral of the *symmetric* potentials, i.e., by the half-sum of the retarded integral and of the corresponding advanced integral. Our terminology, which means here something slightly different from Sec. II, should be clear from the context. We are referring here to the formal structure of the integral, made of the sum of the retarded and advanced integrals. However, the real behavior of the symmetric integral under a time-reversal operation may be more complicated than a simple invariance. The mass and current densities $\sigma_\mu \equiv (\sigma,\sigma_i)$ of the source are defined by

$$\sigma \equiv \frac{T^{00}+T^{kk}}{c^2}, \tag{3.3a}$$

$$\sigma_i \equiv \frac{T^{0i}}{c}, \tag{3.3b}$$

where $T^{\mu\nu}$ denotes the usual stress-energy tensor of the matter fields (with T^{kk} the spatial trace $\Sigma\,\delta_{jk}T^{jk}$). The powers of $1/c$ in Eq. (3.3) are such that σ_μ admits a finite nonzero limit when $c\to+\infty$. The potentials $V^{\rm in}_\mu$ are given by

$$V^{\rm in}_\mu(\mathbf{x},t) = \frac{G}{2}\int \frac{d^3\mathbf{x}'}{|\mathbf{x}-\mathbf{x}'|}\left[\sigma_\mu\left(\mathbf{x}',t-\frac{1}{c}|\mathbf{x}-\mathbf{x}'|\right) + \sigma_\mu\left(\mathbf{x}',t+\frac{1}{c}|\mathbf{x}-\mathbf{x}'|\right)\right]. \tag{3.4}$$

To lowest order when $c\to+\infty$, $V^{\rm in}$ reduces to the usual Newtonian potential, and $V^{\rm in}_i$ to the usual gravitomagnetic potential. It was noticed in [47] that when using the mass density σ given by Eq. (3.3a), the first (nonradiative) post-Newtonian approximation takes a very simple form, involving simply the square of the potential in the 00 component of the metric. See also Eq. (4.5) below, where we use the post-Newtonian expansion $V^{\rm in} = U + \partial_t^2 X/2c^2 + O(c^{-4})$.

The fact that the inner metric contains some symmetric integrals, and therefore some advanced integrals, does not mean that the field violates the condition of retarded potentials. The metric (3.1) is in the form of a post-Newtonian expansion, which is valid only in the near zone D_i. It is well known that the coefficients of the powers of $1/c$ in a post-Newtonian expansion typically diverge at spatial infinity. This is no concern of ours because the expansion is not valid at infinity (it would give poor results when compared to an exact solution). Thus, the symmetric integral (3.4) should more properly be replaced by its formal post-Newtonian expansion, readily obtained from expanding by means of Taylor's formula the retarded and advanced arguments when $c\to+\infty$. Denoting $\partial_t^{2p} \equiv (\partial/\partial t)^{2p}$ we have

$$V^{\rm in}_\mu(\mathbf{x},t) = G\sum_{p=0}^{+\infty}\frac{1}{(2p)!\,c^{2p}}\int d^3\mathbf{x}'\,|\mathbf{x}-\mathbf{x}'|^{2p-1}\partial_t^{2p}\sigma_\mu(\mathbf{x}',t). \tag{3.5}$$

This expansion could be limited to the precision indicated in Eq. (3.1). It involves (explicitly) only *even* powers of c^{-1}, and is thus expected to yield essentially nondissipative effects. However, the dependence of $V^{\rm in}_\mu$ on c^{-1} is more complicated than indicated in Eq. (3.5). Indeed, the mass and current densities σ_μ depend on the metric (3.1), and thus depend on c^{-1} starting at the post-Newtonian level. Even more, the densities σ_μ and their time derivatives do contain, through the contribution of the reactive potentials $V^{\rm reac}_\mu$ (see below), some odd powers of c^{-1} which are associated (*a priori*) to radiation reaction effects. These ''odd'' contributions in $V^{\rm in}_\mu$ form an integral part of the equations of motion of binary systems at the 3.5PN approximation [53], but we shall prove in Sec. IV that they do not participate to the losses of energy and momenta by radiation at the 1PN order, as they enter the balance equations only in the form of some total time derivatives. The secular losses of energy and momenta are driven by the radiation reaction potentials $V_\mu^{\rm reac}$, to which we now turn.

The 1PN-accurate reaction potentials $V_\mu^{\rm reac}\equiv(V^{\rm reac},V_i^{\rm reac})$ involve dominantly some odd powers of c^{-1}, which correspond in the metric (3.1) to the 2.5PN and 3.5PN approximations c^{-5} and c^{-7} taking place between the (nondissipative) 2PN and 3PN approximations. Since the reactive potentials are added linearly to the potentials $V_\mu^{\rm in}$, the simple form of the 1PN nonradiative approximation mentioned above holds also for the 1PN radiative approximation, in this coordinate system. The $V_\mu^{\rm reac}$'s are given by exactly the same expressions as obtained in paper I for the exterior metric [see Eq. (2.16) in Sec. II], but they depend on some specific ''source'' multipole moments $\mathcal{I}\equiv\{I_L,J_L\}$ instead of the unknown multipole moments $\widetilde{\mathcal{M}}$. Namely,

$$V^{\rm reac}(\mathbf{x},t)=-\frac{G}{5c^5}x_{ij}I_{ij}^{(5)}(t)+\frac{G}{c^7}\left[\frac{1}{189}x_{ijk}I_{ijk}^{(7)}(t)-\frac{1}{70}\mathbf{x}^2x_{ij}I_{ij}^{(7)}(t)\right]+O\left(\frac{1}{c^8}\right), \tag{3.6a}$$

$$V_i^{\rm reac}(\mathbf{x},t)=\frac{G}{c^5}\left[\frac{1}{21}\hat{x}_{ijk}I_{jk}^{(6)}(t)-\frac{4}{45}\varepsilon_{ijk}x_{jm}J_{km}^{(5)}(t)\right]+O\left(\frac{1}{c^6}\right), \tag{3.6b}$$

where we recall our notation $\hat{x}_{ijk}=x_{ijk}-\frac{1}{5}\mathbf{x}^2(\delta_{ij}x_k+\delta_{ik}x_j+\delta_{jk}x_i)$ [59]. The multipole moments $I_{ij}(t)$, $I_{ijk}(t)$, and $J_{ij}(t)$ are some explicit functionals of the densities σ_μ. Only the mass quadrupole $I_{ij}(t)$ in the first term of $V^{\rm reac}$ needs to be given at 1PN order. The relevant expression is

$$I_{ij}=\int d^3\mathbf{x}\left\{\hat{x}_{ij}\sigma+\frac{1}{14c^2}\mathbf{x}^2\hat{x}_{ij}\partial_t^2\sigma-\frac{20}{21c^2}\hat{x}_{ijk}\partial_t\sigma_k\right\} \tag{3.7}$$

[see Eq. (3.21a) for the general expression of I_L]. The mass octupole and current quadrupole $I_{ijk}(t)$ and $J_{ij}(t)$ take their standard Newtonian expressions

$$I_{ijk}=\int d^3\mathbf{x}\hat{x}_{ijk}\sigma+O\left(\frac{1}{c^2}\right), \tag{3.8a}$$

$$J_{ij}=\int d^3\mathbf{x}\varepsilon_{km\langle i}\hat{x}_{j\rangle k}\sigma_m. \tag{3.8b}$$

The potentials $V^{\rm reac}$ and $V_i^{\rm reac}$ generalize to 1PN order the scalar reactive potential of Burke and Thorne [20–22], whose form is that of the first term in Eq. (3.6a). The vectorial potential $V_i^{\rm reac}$ enters the equations of motion at the same 3.5PN order as the 1PN corrections in $V^{\rm reac}$. The first term which is neglected in $V^{\rm reac}$, of order c^{-8} or 1.5PN, is due to the tails of waves (see Sec. IV C).

B. Matching to the exterior field

In this subsection we prove that the inner metric presented above (i) satisfies the Einstein field equations within the source (in the near-zone D_i), and (ii) matches the exterior metric (2.17) in the intersecting region between D_i and the exterior zone D_e (exterior near-zone $D_i\cap D_e$).

Note that during proof (i) we do not check any boundary conditions satisfied by the metric at infinity. Simply, we prove that the metric satisfies the field equations term by term in the post-Newtonian expansion, but at this stage the metric could be made of a mixture of retarded and advanced solutions. Only during proof (ii) does one check that the metric comes from the re-expansion when $c\to\infty$ of a solution of the Einstein field equations satisfying some relevant time-asymmetric boundary conditions at infinity. Indeed, the exterior metric has been constructed in paper I by means of a post-Minkowskian algorithm valid all over the exterior region D_e, and having a no-incoming radiation condition built into it (indeed, the exterior metric was assumed to be stationary in the remote past; see Sec. II).

The proof that Eq. (3.1) is a solution admissible in D_i follows immediately from the particular form taken by the Einstein field equations when developed to 1PN order [47]:

$$\Box\ln(-g_{00}^{\rm in})=\frac{8\pi G}{c^2}\sigma+O_{\rm even}\left(\frac{1}{c^6}\right), \tag{3.9a}$$

$$\Box g_{0i}^{\rm in}=\frac{16\pi G}{c^3}\sigma_i+O_{\rm even}\left(\frac{1}{c^5}\right), \tag{3.9b}$$

$$\Box g_{ij}^{\rm in}=-\frac{8\pi G}{c^2}\delta_{ij}\sigma+O_{\rm even}\left(\frac{1}{c^4}\right). \tag{3.9c}$$

The point is that with the introduction of the logarithm of $-g_{00}^{\rm in}$ as a new variable in Eq. (3.9a), the equations at the 1PN order take the form of *linear* wave equations. The other point is that the neglected post-Newtonian terms in Eq. (3.9) are ''even,'' in the sense that the *explicit* powers of c^{-1} they contain, which come from the differentiations of the metric with respect to the time coordinate $x^0=ct$, correspond formally to integer post-Newtonian approximations (to remember this we have added the subscript ''even'' on the O-symbols). This feature is simply the consequence of the time symmetry of the field equations, implying that to each solution of the equations is associated another solution obtained from it by a time reversal.

Because the potentials $V_\mu^{\rm in}$ satisfy exactly $\Box V_\mu^{\rm in}=-4\pi G\sigma_\mu$, a consistent solution of Eq. (3.9) is easily seen to be given by Eq. (3.1), where the reactive potentials $V_\mu^{\rm reac}$ are set to zero. Now the equations (3.9) are linear wave equations, so we can add linearly to $V_\mu^{\rm in}$ any homogeneous solution of the wave equation which is regular in D_i. One can check from their definition Eq. (3.6) that the reactive potentials $V_\mu^{\rm reac}$ form such a homogeneous solution, as they satisfy

$$\Box V^{\rm reac}=O\left(\frac{1}{c^8}\right), \tag{3.10a}$$

$$\Box V_i^{\rm reac}=O\left(\frac{1}{c^6}\right). \tag{3.10b}$$

So we can add $V_\mu^{\rm reac}$ to $V_\mu^{\rm in}$, defining an equally consistent solution of Eq. (3.9), which is precisely Eq. (3.1), modulo the error terms coming from Eq. (3.10) and which correspond to the neglected 4PN approximation. [Note that $V_\mu^{\rm reac}$ comes from the expansion of the tensor potential (2.8) of the linearized theory, which satisfies *exactly* the source-free wave equation. It would be possible to define $V_\mu^{\rm reac}$ in such a way that there are no error terms in Eq. (3.10). See Eq. (4.33) for more precise expressions of $V_\mu^{\rm reac}$, satisfying more precisely the wave equation.]

With proof (i) done, we undertake proof (ii). More precisely, we show that Eq. (3.1) differs from the exterior metric (2.17) by a mere coordinate transformation in $D_i \cap D_e$. This will be true only if the multipole moments $\widetilde{M}_L$ and $\widetilde{S}_L$ parametrizing Eq. (2.17) agree, to the relevant order, with some source multipole moments I_L and J_L. Fulfilling these matching conditions will ensure (in this approximate framework) the existence and consistency of a solution of the field equations valid everywhere in D_i and D_e. As recalled in the introduction, this is part of the method to work out first the exterior metric leaving the multipole moments arbitrary (paper I), and then to obtain by matching the expressions of these moments as integrals over the source (this paper).

To implement the matching we expand the inner metric (3.1) into multipole moments outside the compact support of the source. The comparison can then be made with Eq. (2.17), which is already in the form of a multipole expansion. Only the potentials $V_\mu^{\rm in}$ need to be expanded into multipoles, as the reactive potentials $V_\mu^{\rm reac}$ are already in the required form. The multipole expansion of the retarded integral of a compact-supported source is well known. For example, the formula has been obtained in the Appendix B of [47] using the STF formalism for spherical harmonics. The multipole expansion corresponding to an advanced integral follows simply from the replacement $c \to -c$ in the formula. The script letter $\mathcal{M}$ will be used to denote the multipole expansion. $\mathcal{M}(V_\mu^{\rm in})$ reads as

$$\mathcal{M}(V^{\rm in}) = G \sum_{l \geqslant 0} \frac{(-)^l}{l!} \partial_L \left\{ \frac{F_L(t-r/c) + F_L(t+r/c)}{2r} \right\}, \tag{3.11a}$$

$$\mathcal{M}(V_i^{\rm in}) = G \sum_{l \geqslant 0} \frac{(-)^l}{l!} \partial_L \left\{ \frac{G_{iL}(t-r/c) + G_{iL}(t+r/c)}{2r} \right\}, \tag{3.11b}$$

where $F_L(t)$ and $G_{iL}(t)$ are some tensorial functions of time given by the integrals

$$F_L(t) = \int d^3\mathbf{x} \hat{x}_L \int_{-1}^{1} dz\, \delta_l(z) \sigma(\mathbf{x}, t + z|\mathbf{x}|/c), \tag{3.12a}$$

$$G_{iL}(t) = \int d^3\mathbf{x} \hat{x}_L \int_{-1}^{1} dz\, \delta_l(z) \sigma_i(\mathbf{x}, t + z|\mathbf{x}|/c). \tag{3.12b}$$

The function $\delta_l(z)$ appearing here takes into account the delays in the propagation of the waves inside the source. It reads

$$\delta_l(z) = \frac{(2l+1)!!}{2^{l+1} l!} (1-z^2)^l; \quad \int_{-1}^{1} dz\, \delta_l(z) = 1 \tag{3.13}$$

(see Eq. (B12) in [47]). Note that the same functions $F_L(t)$ and $G_{iL}(t)$ parametrize both the retarded and the corresponding advanced waves in Eq. (3.11). Indeed $\delta_l(z)$ is an even function of its variable z, so the integrals (3.12) are invariant under the replacement $c \to -c$.

Using an approach similar to the one employed in [47], we perform an irreducible decomposition of the tensorial function G_{iL} (which is STF with respect to its l indices L but not with respect to its $l+1$ indices iL), as a sum of STF tensors of multipolarities $l+1$, l, and $l-1$. The equation (2.17) in [47] gives this decomposition as

$$G_{iL} = C_{iL} - \frac{l}{l+1} \varepsilon_{ai\langle i_l} D_{L-1\rangle a} + \frac{2l-1}{2l+1} \delta_{i\langle i_l} E_{L-1\rangle}, \tag{3.14}$$

where the tensors C_{L+1}, D_L and E_{L-1} (which are STF with respect to all their indices) are given by

$$C_{L+1}(t) = \int d^3\mathbf{x} \int_{-1}^{1} dz\, \delta_l(z) \hat{x}_{<L} \sigma_{i_{l+1}>}(\mathbf{x}, t + z|\mathbf{x}|/c), \tag{3.15a}$$

$$D_L(t) = \int d^3\mathbf{x} \int_{-1}^{1} dz\, \delta_l(z) \varepsilon_{ab\langle i_l} \hat{x}_{L-1\rangle a} \sigma_b(\mathbf{x}, t + z|\mathbf{x}|/c), \tag{3.15b}$$

$$E_{L-1}(t) = \int d^3\mathbf{x} \int_{-1}^{1} dz\, \delta_l(z) \hat{x}_{aL-1} \sigma_a(\mathbf{x}, t + z|\mathbf{x}|/c). \tag{3.15c}$$

Then by introducing the new definitions of STF tensors,

$$A_L = F_L - \frac{4(2l+1)}{c^2(l+1)(2l+3)} E_L^{(1)}, \tag{3.16a}$$

$$B_L = l C_L - \frac{l}{c^2(l+1)(2l+3)} E_L^{(2)}, \tag{3.16b}$$

and by using standard manipulations on STF tensors, we can rewrite the multipole expansions (3.11) in the new form

$$\mathcal{M}(V^{\rm in}) = -c\,\partial_t \phi^0 + G \sum_{l \geqslant 0} \frac{(-)^l}{l!} \partial_L \times \left\{ \frac{A_L(t-r/c) + A_L(t+r/c)}{2r} \right\}, \tag{3.17a}$$

$$\mathcal{M}(V_i^{\rm in})=\frac{c^3}{4}\partial_i\phi^0-G\sum_{l\geqslant 1}\frac{(-)^l}{l!}\partial_{L-1}\times\left\{\frac{B_{iL-1}(t-r/c)+B_{iL-1}(t+r/c)}{2r}\right\}-G\sum_{l\geqslant 1}\frac{(-)^l}{l!}\frac{l}{l+1}\varepsilon_{iab}\partial_{aL-1}\times\left\{\frac{D_{bL-1}(t-r/c)+D_{bL-1}(t+r/c)}{2r}\right\},\tag{3.17b}$$

where we denote

$$\phi^0=-\frac{4G}{c^3}\sum_{l\geqslant 0}\frac{(-)^l}{(l+1)!}\frac{2l+1}{2l+3}\partial_L\times\left\{\frac{E_L(t-r/c)+E_L(t+r/c)}{2r}\right\}.\tag{3.18}$$

Next the moment A_L is expanded when $c\to\infty$ to 1PN order, and the moments B_L, D_L to Newtonian order. The required formula is Eq. (B14) in [47], which immediately gives

$$A_L=\int d^3\mathbf{x}\left\{\hat{x}_L\sigma+\frac{1}{2c^2(2l+3)}\mathbf{x}^2\hat{x}_L\partial_t^2\sigma-\frac{4(2l+1)}{c^2(l+1)(2l+3)}\hat{x}_{iL}\partial_t\sigma_i\right\}+O_{\rm even}\left(\frac{1}{c^4}\right),\tag{3.19a}$$

$$B_L=l\int d^3\mathbf{x}\hat{x}_{\langle L-1}\sigma_{i_l\rangle}+O_{\rm even}\left(\frac{1}{c^2}\right),\tag{3.19b}$$

$$D_L=\int d^3\mathbf{x}\varepsilon_{ab\langle i_l}\hat{x}_{L-1\rangle a}\sigma_b+O_{\rm even}\left(\frac{1}{c^2}\right).\tag{3.19c}$$

Here the notation $O_{\rm even}(c^{-n})$ for the post-Newtonian remainders simply indicates that the whole post-Newtonian expansion is composed only of even powers of c^{-1}, like in Eq. (3.5) (the source densities σ_μ being considered to be independent of c^{-1}), as clear from Eq. (B14) in [47]. We now transform the leading-order term in the equation for B_L using the equation of continuity for the mass density σ. The Newtonian equation of continuity does the needed transformation, but one must be careful about the higher-order post-Newtonian corrections which involve some reactive contributions. It can be checked that these reactive contributions arise only at the order $O(c^{-7})$, so that the equation of continuity reads, with evident notation, $\partial_t\sigma+\partial_i\sigma_i=O_{\rm even}(c^{-2})+O(c^{-7})$. From this one deduces

$$B_L=\frac{d}{dt}\left\{\int d^3\mathbf{x}\hat{x}_L\sigma\right\}+O_{\rm even}\left(\frac{1}{c^2}\right)+O\left(\frac{1}{c^7}\right).\tag{3.20}$$

All elements are now in hands in order to compare, in the exterior near-zone $D_i\cap D_e$, the metrics (3.1) and (2.17). The "source" multipole moments $\mathcal{I}=\{I_L,J_L\}$ are defined by the dominant terms in Eqs. (3.19a) and (3.19c),

$$I_L\equiv\int d^3\mathbf{x}\left\{\hat{x}_L\sigma+\frac{1}{2c^2(2l+3)}\mathbf{x}^2\hat{x}_L\partial_t^2\sigma-\frac{4(2l+1)}{c^2(l+1)(2l+3)}\hat{x}_{iL}\partial_t\sigma_i\right\},\tag{3.21a}$$

$$J_L\equiv\int d^3\mathbf{x}\varepsilon_{ab\langle i_l}\hat{x}_{L-1\rangle a}\sigma_b.\tag{3.21b}$$

The mass-type moment I_L includes 1PN corrections, while the current-type moment J_L is Newtonian. The mass moment I_L was obtained in [47], where it was shown to parametrize the asymptotic metric generated by the source at the 1PN order. When $l=2$ and $l=3$ we recover the moments introduced in Eqs. (3.7)–(3.8). With Eqs. (3.19)–(3.21), the multipole expansions (3.17) become

$$\mathcal{M}(V^{\rm in})=-c\,\partial_t\phi^0+G\sum_{l\geqslant 0}\frac{(-)^l}{l!}\partial_L\times\left\{\frac{I_L(t-r/c)+I_L(t+r/c)}{2r}\right\}+O_{\rm even}\left(\frac{1}{c^4}\right),\tag{3.22a}$$

$$\mathcal{M}(V_i^{\rm in})=\frac{c^3}{4}\partial_i\phi^0-G\sum_{l\geqslant 1}\frac{(-)^l}{l!}\partial_{L-1}\times\left\{\frac{I^{(1)}{}_{iL-1}(t-r/c)+I^{(1)}{}_{iL-1}(t+r/c)}{2r}\right\}-G\sum_{l\geqslant 1}\frac{(-)^l}{l!}\frac{l}{l+1}\varepsilon_{iab}\partial_{aL-1}\times\left\{\frac{J_{bL-1}(t-r/c)+J_{bL-1}(t+r/c)}{2r}\right\}+O_{\rm even}\left(\frac{1}{c^2}\right)+O\left(\frac{1}{c^7}\right).\tag{3.22b}$$

Thus, from the definition [Eq. (2.18)] of the external potentials $V_\mu^{\rm ext}$, we obtain the relationships

$$\mathcal{M}(V^{\rm in})=-c\,\partial_t\phi^0+V^{\rm ext}[\mathcal{I}]+O_{\rm even}\left(\frac{1}{c^4}\right),\tag{3.23a}$$

$$\mathcal{M}(V_i^{\rm in})=\frac{c^3}{4}\partial_i\phi^0+V_i^{\rm ext}[\mathcal{I}]+O_{\rm even}\left(\frac{1}{c^2}\right)+O\left(\frac{1}{c^7}\right),\tag{3.23b}$$

from which we readily infer that the multipole expansion $\mathcal{M}(g_{\mu\nu}^{\rm in})$ of the metric (3.1) reads, in $D_i\cap D_e$,

$$\mathcal{M}(g_{00}^{\rm in})+\frac{2}{c}\partial_t\phi^0=-1+\frac{2}{c^2}(V^{\rm ext}[\mathcal{I}]+V^{\rm reac}[\mathcal{I}])-\frac{2}{c^4}(V^{\rm ext}[\mathcal{I}]+V^{\rm reac}[\mathcal{I}])^2+\frac{1}{c^6}\,{}_6\overline{g_{00}^{\rm in}}+\frac{1}{c^8}\,{}_8\overline{g_{00}^{\rm in}}+O\left(\frac{1}{c^{10}}\right),\tag{3.24a}$$

$$\mathcal{M}(g_{0i}^{\rm in})+\partial_i\phi^0=-\frac{4}{c^3}(V_i^{\rm ext}[\mathcal{I}]+V_i^{\rm reac}[\mathcal{I}])+\frac{1}{c^5}\,{}_5\overline{g}_{0i}^{\rm in}+\frac{1}{c^7}\,{}_7\overline{g}_{0i}^{\rm in}+O\left(\frac{1}{c^9}\right), \tag{3.24b}$$

$$\mathcal{M}(g_{ij}^{\rm in})=\delta_{ij}\left[1+\frac{2}{c^2}(V^{\rm ext}[\mathcal{I}]+V^{\rm reac}[\mathcal{I}])\right]+\frac{1}{c^4}\,{}_4\overline{g}_{ij}^{\rm in}+\frac{1}{c^6}\,{}_6\overline{g}_{ij}^{\rm in}+O\left(\frac{1}{c^8}\right). \tag{3.24c}$$

Clearly the terms depending on ϕ^0 have the form of an infinitesimal gauge transformation of the time coordinate. We can check that the corresponding coordinate transformation can be treated, to the considered order, in a linearized way [recall from Eq. (3.18) that ϕ^0 is of order c^{-3}]. Finally, in the ''exterior'' coordinates

$$x_{\rm ext}^0=x^0+\phi^0(x^\nu)+O_{\rm even}\left(\frac{1}{c^5}\right)+O\left(\frac{1}{c^9}\right), \tag{3.25a}$$

$$x_{\rm ext}^i=x^i+O_{\rm even}\left(\frac{1}{c^4}\right)+O\left(\frac{1}{c^8}\right), \tag{3.25b}$$

the metric (3.24) is transformed into the ''exterior'' metric

$$g_{00}^{\rm ext}=-1+\frac{2}{c^2}(V^{\rm ext}[\mathcal{I}]+V^{\rm reac}[\mathcal{I}])-\frac{2}{c^4}(V^{\rm ext}[\mathcal{I}]+V^{\rm reac}[\mathcal{I}])^2+\frac{1}{c^6}\,{}_6\overline{g}_{00}^{\rm ext}+\frac{1}{c^8}\,{}_8\overline{g}_{00}^{\rm ext}+O\left(\frac{1}{c^{10}}\right), \tag{3.26a}$$

$$g_{0i}^{\rm ext}=-\frac{4}{c^3}(V_i^{\rm ext}[\mathcal{I}]+V_i^{\rm reac}[\mathcal{I}])+\frac{1}{c^5}\,{}_5\overline{g}_{0i}^{\rm ext}+\frac{1}{c^7}\,{}_7\overline{g}_{0i}^{\rm ext}+O\left(\frac{1}{c^9}\right), \tag{3.26b}$$

$$g_{ij}^{\rm ext}=\delta_{ij}\left[1+\frac{2}{c^2}(V^{\rm ext}[\mathcal{I}]+V^{\rm reac}[\mathcal{I}])\right]+\frac{1}{c^4}\,{}_4\overline{g}_{ij}^{\rm ext}+\frac{1}{c^6}\,{}_6\overline{g}_{ij}^{\rm ext}+O\left(\frac{1}{c^8}\right). \tag{3.26c}$$

This metric is exactly identical, as concerns the 1PN, 2.5PN, and 3.5PN approximations, to the exterior metric (2.17) obtained in paper I, except that here the metric is parametrized by the known multipole moments $\mathcal{I}$ instead of the arbitrary moments $\widetilde{\mathcal{M}}$. Thus, we conclude that the two metrics (3.1) and (2.17) match in the overlapping region $D_i \cap D_e$ if (and only if) there is agreement between both types of multipole moments. This determines $\widetilde{M}_L$ and $\widetilde{S}_L$. More precisely, we find that $\widetilde{M}_L$ and $\widetilde{S}_L$ must be related to I_L and J_L given in Eq. (3.21) by

$$\widetilde{M}_L=I_L+O_{\rm even}\left(\frac{1}{c^4}\right)+O\left(\frac{1}{c^8}\right), \tag{3.27a}$$

$$\widetilde{S}_L=J_L+O_{\rm even}\left(\frac{1}{c^2}\right)+O\left(\frac{1}{c^6}\right), \tag{3.27b}$$

where as usual the relation for $\widetilde{M}_L$ is accurate to 1PN order, and the relation for $\widetilde{S}_L$ is Newtonian (we do also control the parity of some neglected terms). Satisfying the latter matching solves the problem at hand, by showing that the inner metric (3.1)–(3.8) results from the post-Newtonian expansion of a solution of the (nonlinear) field equations and a condition of no incoming radiation.

We emphasize the dependence of the result on the coordinate system. Of course, the metric (3.1), which contains the reactive potentials (3.6)–(3.8), is valid only in its own coordinate system. It is a well-known consequence of the equivalence principle that radiation reaction forces in general relativity are inherently dependent on the coordinate system (see e.g., [60] for a comparison between various expressions of the radiation reaction force at the Newtonian order). The coordinate system in which the reactive potentials (3.6)–(3.8) are valid is defined as follows. We start from the particular coordinate system in which the linearized metric is given by Eq. (2.3). Then we apply two successive coordinate transformations. The first one is associated with the gauge vector ξ^μ given by Eq. (2.7), and the second one is associated with ϕ^μ whose only needed component is ϕ^0 given by Eq. (3.18). The resulting coordinate system is the one in which $V_\mu^{\rm reac}$ is valid. (Actually, the gauge vector ξ^μ should be modified according to the procedure defined in Sec. II C of paper I, so that the good falloff properties of the metric at infinity are preserved.)

IV. THE BALANCE EQUATIONS TO POST-NEWTONIAN ORDER

A. Conservation laws for energy and momenta at 1PN order

Up to the second post-Newtonian approximation of general relativity, an isolated system admits some conserved energy, linear momentum, and angular momentum. These have been obtained, in the case of weakly self-gravitating fluid systems, by Chandrasekhar and Nutku [24]. The less accurate 1PN-conserved quantities were obtained before, notably by Fock [61]. In this subsection we rederive, within the present framework [using in particular the mass density σ defined in (3.3a)], the 1PN-conserved energy and momenta of the system. The 1PN energy and momenta are needed in the next subsection, in which we establish their laws of variation during the emission of radiation at 1PN order (hence we do not need the more accurate 2PN-conserved quantities).

To 1PN order, the metric (3.1) reduces to

$$g_{00}^{\rm in}=-1+\frac{2}{c^2}V^{\rm in}-\frac{2}{c^4}(V^{\rm in})^2+O\left(\frac{1}{c^6}\right), \tag{4.1a}$$

$$g_{0i}^{\rm in}=-\frac{4}{c^3}V_i^{\rm in}+O\left(\frac{1}{c^5}\right), \tag{4.1b}$$

$$g_{ij}^{\rm in}=\delta_{ij}\left(1+\frac{2}{c^2}V^{\rm in}\right)+O\left(\frac{1}{c^4}\right), \tag{4.1c}$$

where $V^{\rm in}$ and $V_i^{\rm in}$ are given by Eq. (3.4). In fact, $V^{\rm in}$ and $V_i^{\rm in}$ are given by their post-Newtonian expansions (3.5), which can be limited here to the terms

$$V^{\rm in}=U+\frac{1}{2c^2}\partial_t^2 X+O\left(\frac{1}{c^4}\right), \tag{4.2a}$$

$$V_i^{\rm in}=U_i+O\left(\frac{1}{c^2}\right), \tag{4.2b}$$

where the instantaneous (Poisson-like) potentials U, X, and U_i are defined by

$$U(\mathbf{x},t)=G\int\frac{d^3\mathbf{x}'}{|\mathbf{x}-\mathbf{x}'|}\sigma(\mathbf{x}',t), \tag{4.3a}$$

$$X(\mathbf{x},t)=G\int d^3\mathbf{x}'|\mathbf{x}-\mathbf{x}'|\sigma(\mathbf{x}',t), \tag{4.3b}$$

$$U_i(\mathbf{x},t)=G\int\frac{d^3\mathbf{x}'}{|\mathbf{x}-\mathbf{x}'|}\sigma_i(\mathbf{x}',t). \tag{4.3c}$$

Since $V^{\rm in}$ is a symmetric integral, there are no terms of order c^{-3} in Eq. (4.2a) (such a term would be a simple function of time in the case of a retarded integral). We shall need (only in this subsection) a metric whose space-space components ij are more accurate than in Eq. (4.1c), taking into account the next-order correction term. We introduce an instantaneous potential whose source is the sum of the matter stresses, say $\sigma_{ij}=T^{ij}$, and the (Newtonian) gravitational stresses,

$$P_{ij}(\mathbf{x},t)=G\int\frac{d^3\mathbf{x}'}{|\mathbf{x}-\mathbf{x}'|}\left[\sigma_{ij}+\frac{1}{4\pi G}\times\left(\partial_i U\partial_j U-\frac{1}{2}\delta_{ij}\partial_k U\partial_k U\right)\right](\mathbf{x}',t). \tag{4.4a}$$

The spatial trace $P\equiv P_{ii}$ is

$$P(\mathbf{x},t)=G\int\frac{d^3\mathbf{x}'}{|\mathbf{x}-\mathbf{x}'|}\left[\sigma_{ii}-\frac{1}{2}\sigma U\right](\mathbf{x}',t)+\frac{U^2}{4}. \tag{4.4b}$$

The metric which is accurate enough for our purpose reads, in terms of the instantaneous potentials (4.3) and (4.4),

$$g_{00}^{\rm in}=-1+\frac{2}{c^2}U+\frac{1}{c^4}[\partial_t^2 X-2U^2]+O\left(\frac{1}{c^6}\right), \tag{4.5a}$$

$$g_{0i}^{\rm in}=-\frac{4}{c^3}U_i+O\left(\frac{1}{c^5}\right), \tag{4.5b}$$

$$g_{ij}^{\rm in}=\delta_{ij}\left(1+\frac{2}{c^2}U+\frac{1}{c^4}[\partial_t^2 X+2U^2]\right)+\frac{4}{c^4}[P_{ij}-\delta_{ij}P]+O\left(\frac{1}{c^6}\right). \tag{4.5c}$$

The square root of (minus) the determinant of the metric is

$$\sqrt{-g^{\rm in}}=1+\frac{2}{c^2}U+\frac{1}{c^4}[\partial_t^2 X+2U^2-4P]+O\left(\frac{1}{c^6}\right). \tag{4.5d}$$

Consider the local equations of motion of the source, which state the conservation in the covariant sense of the stress-energy tensor $T^{\mu\nu}$ (i.e., $\nabla_\mu T^\mu_\alpha=0$). These equations, written in a form adequate for our purpose, are

$$\partial_\mu \Pi^\mu_\alpha=\mathcal{F}_\alpha, \tag{4.6}$$

where the left-hand-side is the divergence in the ordinary sense of the material stress-energy density

$$\Pi^\mu_\alpha\equiv\sqrt{-g^{\rm in}}g^{\rm in}_{\alpha\nu}T^{\mu\nu}, \tag{4.7}$$

and where the right-hand side can be viewed as the four-force density

$$\mathcal{F}_\alpha\equiv\tfrac{1}{2}\sqrt{-g^{\rm in}}T^{\mu\nu}\partial_\alpha g^{\rm in}_{\mu\nu}. \tag{4.8}$$

The 1PN-conserved energy and momenta follow from integration of these equations over the ordinary three-dimensional space, which yields the following three laws (using the Gauss theorem to discard some divergences of compact-supported terms)

$$\frac{d}{dt}\left\{-\int d^3\mathbf{x}\Pi^0_0\right\}=-c\int d^3\mathbf{x}\mathcal{F}_0, \tag{4.9a}$$

$$\frac{d}{dt}\left\{\frac{1}{c}\int d^3\mathbf{x}\Pi^0_i\right\}=\int d^3\mathbf{x}\mathcal{F}_i, \tag{4.9b}$$

$$\frac{d}{dt}\left\{\frac{1}{c}\varepsilon_{ijk}\int d^3\mathbf{x}x_j\Pi^0_k\right\}=\varepsilon_{ijk}\int d^3\mathbf{x}(x_j\mathcal{F}_k+\Pi^j_k). \tag{4.9c}$$

The quantities Π^μ_α and $\mathcal{F}_\alpha$ are then determined. With Eq. (4.5) we obtain, for the various components of Π^μ_α,

$$\Pi^0_0=-\sigma c^2+\sigma_{ii}+\frac{4}{c^2}[\sigma P-\sigma_i U_i]+O\left(\frac{1}{c^4}\right), \tag{4.10a}$$

$$\Pi^0_i=c\sigma_i\left(1+4\frac{U}{c^2}\right)-\frac{4}{c}\sigma U_i+O\left(\frac{1}{c^3}\right), \tag{4.10b}$$

$$\Pi^i_0=-c\sigma_i\left(1-4\frac{P}{c^4}\right)-\frac{4}{c^3}\sigma_{ij}U_j+O\left(\frac{1}{c^5}\right), \tag{4.10c}$$

$$\Pi^i_j=\sigma_{ij}\left(1+4\frac{U}{c^2}\right)-\frac{4}{c^2}\sigma_i U_j+O\left(\frac{1}{c^4}\right). \tag{4.10d}$$

Note that Π^i_0 is determined with a better precision than Π^0_i (but we shall not need this higher precision and give it for completeness). For the components of $\mathcal{F}_\alpha$, we find

$$\mathcal{F}_0=\frac{1}{c}\sigma\partial_t\left(U+\frac{1}{2c^2}\partial_t^2 X\right)-\frac{4}{c^3}\sigma_j\partial_t U_j+O\left(\frac{1}{c^5}\right), \tag{4.11a}$$

$$\mathcal{F}_i=\sigma\partial_i\left(U+\frac{1}{2c^2}\partial_t^2X\right)-\frac{4}{c^2}\sigma_j\partial_iU_j+O\left(\frac{1}{c^4}\right). \tag{4.11b}$$

The Π_α^μ's and $\mathcal{F}_\alpha$'s can now be substituted into the integrals on both sides of Eq. (4.9). This is correct because the support of the integrals is the compact support of the source, which is, for a slowly-moving source, entirely located within the source's near zone D_i, where the post-Newtonian expansion is valid. Straightforward computations permit us to re-express the right-hand-sides of Eq. (4.9) into the form of total time derivatives. We do not detail here this computation which is well known (at 1PN order), but we present in Sec. IV B a somewhat general formula which can be used to reach elegantly the result [see Eq. (4.23)]. By transfering the total time derivatives to the left-hand-sides of Eq. (4.9), one obtains the looked-for conservation laws at 1PN order, namely

$$\frac{dE^{1\mathrm{PN}}}{dt}=O\left(\frac{1}{c^4}\right), \tag{4.12}$$

$$\frac{dP_i^{1\mathrm{PN}}}{dt}=O\left(\frac{1}{c^4}\right), \tag{4.13}$$

$$\frac{dS_i^{1\mathrm{PN}}}{dt}=O\left(\frac{1}{c^4}\right), \tag{4.14}$$

where the 1PN energy $E^{1\mathrm{PN}}$, linear momentum $P_i^{1\mathrm{PN}}$, and angular momentum $S_i^{1\mathrm{PN}}$ are given by the integrals over the source

$$E^{1\mathrm{PN}}=\int d^3\mathbf{x}\left\{\sigma c^2+\frac{1}{2}\sigma U-\sigma_{ii}+\frac{1}{c^2}\left[-4\sigma P+2\sigma_iU_i+\frac{1}{2}\sigma\partial_t^2X-\frac{1}{4}\partial_t\sigma\partial_tX\right]\right\}, \tag{4.15}$$

$$P_i^{1\mathrm{PN}}=\int d^3\mathbf{x}\left\{\sigma_i-\frac{1}{2c^2}\sigma\partial_i\partial_tX\right\}, \tag{4.16}$$

$$S_i^{1\mathrm{PN}}=\varepsilon_{ijk}\int d^3\mathbf{x}x_j\left\{\sigma_k+\frac{1}{c^2}\left[4\sigma_kU-4\sigma U_k-\frac{1}{2}\sigma\partial_k\partial_tX\right]\right\}. \tag{4.17}$$

The 1PN energy $E^{1\mathrm{PN}}$ can also be written as [62]

$$E^{1\mathrm{PN}}=\int d^3\mathbf{x}\left\{\sigma c^2+\frac{1}{2}\sigma U-\sigma_{ii}+\frac{1}{c^2}\left[\sigma U^2-4\sigma_{ii}U+2\sigma_iU_i+\frac{1}{2}\sigma\partial_t^2X-\frac{1}{4}\partial_t\sigma\partial_tX\right]\right\}. \tag{4.18}$$

A similar but more precise computation would yield the 2PN-conserved quantities [24].

B. Secular losses of the 1PN-accurate energy and momenta

As the reactive potentials V_μ^{reac} manifestly change sign in a time reversal, they are expected to yield dissipative effects in the dynamics of the system, i.e., secular losses of its total energy, angular momentum and linear momentum. The ''Newtonian'' radiation reaction force is known to extract energy in the system at the same rate as given by the Einstein quadrupole formula, both in the case of weakly self-gravitating systems [20–36] and compact binary systems [37–39]. Similarly the reaction force extracts angular momentum in the system. As concerns linear momentum the Newtonian reaction force is not precise enough, and one needs to go to 1PN order.

In this subsection, we prove that the 1PN-accurate reactive potentials V_μ^{reac} lead to decreases of the 1PN-accurate energy and momenta [computed in Eqs. (4.15)–(4.18)] which are in perfect agreement with the corresponding far-zone fluxes, known from the works [63,46,47] in the case of the energy and angular momentum, and from the works [55–57,46] in the case of linear momentum.

We start again from the equations of motion (4.6)–(4.8), which imply, after spatial integration, the laws (4.9) that we recopy here:

$$\frac{d}{dt}\left\{-\int d^3\mathbf{x}\Pi_0^0\right\}=-c\int d^3\mathbf{x}\mathcal{F}_0, \tag{4.19a}$$

$$\frac{d}{dt}\left\{\frac{1}{c}\int d^3\mathbf{x}\Pi_i^0\right\}=\int d^3\mathbf{x}\mathcal{F}_i, \tag{4.19b}$$

$$\frac{d}{dt}\left\{\frac{1}{c}\varepsilon_{ijk}\int d^3\mathbf{x}x_j\Pi_k^0\right\}=\varepsilon_{ijk}\int d^3\mathbf{x}(x_j\mathcal{F}_k+\Pi_k^j). \tag{4.19c}$$

The left-hand sides are in the form of total time derivatives. To 1PN order, we have seen that the right-hand-sides can be transformed into total time derivatives, which combine with the left-hand-sides to give the 1PN-conserved energy and momenta. Here we shall prove that the contributions due to the reactive potentials in the right-hand-sides cannot be transformed entirely into total time derivatives, and that the remaining terms yield precisely the corresponding 1PN fluxes. The balance equations then follow (modulo a slight assumption and a general argument stated below).

The right-hand sides of Eq. (4.19) are evaluated by substituting the metric (3.1), involving the potentials $\mathcal{V}_\mu^{\mathrm{in}}=V_\mu^{\mathrm{in}}+V_\mu^{\mathrm{reac}}$. The components of the force density (4.8) are found to be

$$\mathcal{F}_0=\frac{\sigma}{c}\partial_t\mathcal{V}^{\mathrm{in}}-\frac{4}{c^3}\sigma_j\partial_t\mathcal{V}_j^{\mathrm{in}}+\frac{1}{c^5}\,{}_5\mathcal{F}_0+\frac{1}{c^7}\,{}_7\mathcal{F}_0+O\left(\frac{1}{c^9}\right), \tag{4.20a}$$

$$\mathcal{F}_i=\sigma\partial_i\mathcal{V}^{\mathrm{in}}-\frac{4}{c^2}\sigma_j\partial_i\mathcal{V}_j^{\mathrm{in}}+\frac{1}{c^4}\,{}_4\mathcal{F}_i+\frac{1}{c^6}\,{}_6\mathcal{F}_i+O\left(\frac{1}{c^8}\right), \tag{4.20b}$$

where we have been careful at handling correctly the uncontrolled 2PN and 3PN approximations, which lead to the terms symbolized by the $c^{-n}{}_n\mathcal{F}_\mu$'s. The equations (4.20) reduce to Eq. (4.11) at the 1PN approximation. They give the components of the force as linear functionals of $\mathcal{V}^{\rm in}$ and $\mathcal{V}_i^{\rm in}$. The remainders are 4PN at least. The same computation using the stress-energy density (4.7) yields the term which is further needed in Eq. (4.19c),

$$\varepsilon_{ijk}\Pi_k^j = -\frac{4}{c^2}\varepsilon_{ijk}\sigma_j\mathcal{V}_k^{\rm in} + \frac{1}{c^4}\,{}_4\mathcal{T}_i + \frac{1}{c^6}\,{}_6\mathcal{T}_i + O\left(\frac{1}{c^8}\right). \tag{4.21}$$

The ${}_n\mathcal{T}_i$'s represent the 2PN and 3PN approximations. Thanks to Eqs. (4.20) and (4.21) one can now transform the laws (4.19) into

$$\frac{d}{dt}\left\{-\int d^3\mathbf{x}\Pi_0^0\right\} = \int d^3\mathbf{x}\left\{-\sigma\partial_t\mathcal{V}^{\rm in} + \frac{4}{c^2}\sigma_j\partial_t\mathcal{V}_j^{\rm in}\right\} + \frac{1}{c^4}\,{}_4X + \frac{1}{c^6}\,{}_6X + O\left(\frac{1}{c^8}\right), \tag{4.22a}$$

$$\frac{d}{dt}\left\{\frac{1}{c}\int d^3\mathbf{x}\Pi_i^0\right\} = \int d^3\mathbf{x}\left\{\sigma\partial_i\mathcal{V}^{\rm in} - \frac{4}{c^2}\sigma_j\partial_i\mathcal{V}_j^{\rm in}\right\} + \frac{1}{c^4}\,{}_4Y_i + \frac{1}{c^6}\,{}_6Y_i + O\left(\frac{1}{c^8}\right), \tag{4.22b}$$

$$\frac{d}{dt}\left\{\frac{1}{c}\varepsilon_{ijk}\int d^3\mathbf{x}x_j\Pi_k^0\right\} = \varepsilon_{ijk}\int d^3\mathbf{x}\left\{\sigma x_j\partial_k\mathcal{V}^{\rm in} - \frac{4}{c^2}\sigma_m x_j\partial_k\mathcal{V}_m^{\rm in} - \frac{4}{c^2}\sigma_j\mathcal{V}_k^{\rm in}\right\} + \frac{1}{c^4}\,{}_4Z_i + \frac{1}{c^6}\,{}_6Z_i + O\left(\frac{1}{c^8}\right), \tag{4.22c}$$

where ${}_nX$, ${}_nY_i$, and ${}_nZ_i$ denote some spatial integrals of the 2PN and 3PN terms in Eqs. (4.20) and (4.21).

Consider first the piece in $\mathcal{V}_\mu^{\rm in}$ which is composed of the potential $V_\mu^{\rm in}$, given by the symmetric integral (3.4) or by the Taylor expansion (3.5). To 1PN order $V_\mu^{\rm in}$ contributes to the laws (4.22) only in the form of total time derivatives (see Sec. IV A). Here we present a more general proof of this result, valid formally up to any post-Newtonian order. This proof shows that the result is due to the very structure of the symmetric potential $V_\mu^{\rm in}$ as given by Eq. (3.5). The technical formula sustaining the proof is

$$\sigma_\mu(x)\partial_t^N\sigma_\nu(x') + (-)^{N+1}\sigma_\nu(x')\partial_t^N\sigma_\mu(x) = \frac{d}{dt}\left\{\sum_{q=0}^{N-1}(-)^q\partial_t^q\sigma_\mu(x)\partial_t^{N-q-1}\sigma_\nu(x')\right\}, \tag{4.23}$$

where $x\equiv(\mathbf{x},t)$ and $x'\equiv(\mathbf{x}',t)$ denote two field points (located in the same hypersurface $t\equiv x^0/c=$const), and where N is some integer [we recall the notation $\sigma_\mu\equiv(\sigma,\sigma_i)$]. The contributions of the symmetric potential in the energy and linear momentum laws (4.22a) and (4.22b) are all of the same type, involving the spatial integral of $\sigma_\mu(x)\partial_\alpha V_\mu^{\rm in}(x)$ (with no summation on μ and $\alpha=0,i$). One replaces into this spatial integral the potential $V_\mu^{\rm in}$ by its Taylor expansion when $c\to\infty$ as given by Eq. (3.5). This yields a series of terms involving when the index $\alpha=0$ the double spatial integral of $|\mathbf{x}-\mathbf{x}'|^{2p-1}\sigma_\mu(x)\partial_t^{2p+1}\sigma_\mu(x')$, and when $\alpha=i$ the double integral of $\partial_i|\mathbf{x}-\mathbf{x}'|^{2p-1}\sigma_\mu(x)\partial_t^{2p}\sigma_\mu(x')$. By symmetrizing the integrand under the exchange $\mathbf{x}\leftrightarrow\mathbf{x}'$ and by using the formula (4.23) where $N=2p+1$ when $\alpha=0$ and $N=2p$ when $\alpha=1,2,3$, one finds that the integral is indeed a total time derivative. The same is true for the symmetric contributions in the angular momentum law (4.22c), which yield a series of integrals of $\varepsilon_{ijk}x_j\partial_k|\mathbf{x}-\mathbf{x}'|^{2p-1}\sigma_\mu(x)\partial_t^{2p}\sigma_\mu(x')$ and $\varepsilon_{ijk}|\mathbf{x}-\mathbf{x}'|^{2p-1}\sigma_j(x)\partial_t^{2p}\times\sigma_k(x')$, on which one uses the formula (4.23) where $N=2p$.

Thus the symmetric (inner) potentials $V_\mu^{\rm in}$ contribute to the right-hand sides of Eq. (4.22) only in the form of total time derivatives. This is true even though, as noticed earlier, the potentials $V_\mu^{\rm in}$ contain some reactive (''time-odd'') terms, through the contributions of the source densities σ_μ. As shown here, these time-odd terms combine with the other time-odd terms present in the σ_μ's appearing explicitly in Eq. (4.22) to form time derivatives. Such time-odd terms will not participate ultimately to the balance equations, but they do participate to the complete 3.5PN approximation in the equations of motion of the system. This fact has been noticed, and these time-odd terms computed for binary systems, by Iyer and Will [53] [see their Eqs. (3.8) and (3.9)].

The numerous time derivatives resulting from the symmetric potentials are then transferred to the left-hand sides of Eq. (4.22). To 1PN order these time derivatives permit reconstructing the 1PN-conserved energy and momenta $E^{\rm 1PN}$, $P_i^{\rm 1PN}$ and $S_i^{\rm 1PN}$. We include also the time derivatives of higher order but they will be negligible in the balance equations (see below). Therefore, we have proved that the laws (4.22) can be rewritten as

$$\frac{d}{dt}\left[E^{\rm 1PN} + \overline{O}\left(\frac{1}{c^4}\right)\right] = \int d^3\mathbf{x}\left\{-\sigma\partial_t V^{\rm reac} + \frac{4}{c^2}\sigma_j\partial_t V_j^{\rm reac}\right\} + \frac{1}{c^4}\,{}_4X + \frac{1}{c^6}\,{}_6X + O\left(\frac{1}{c^8}\right), \tag{4.24a}$$

$$\frac{d}{dt}\left[P_i^{\rm 1PN} + \overline{O}\left(\frac{1}{c^4}\right)\right] = \int d^3\mathbf{x}\left\{\sigma\partial_i V^{\rm reac} - \frac{4}{c^2}\sigma_j\partial_i V_j^{\rm reac}\right\} + \frac{1}{c^4}\,{}_4Y_i + \frac{1}{c^6}\,{}_6Y_i + O\left(\frac{1}{c^8}\right), \tag{4.24b}$$

$$\frac{d}{dt}\left[S_i^{\rm 1PN} + \overline{O}\left(\frac{1}{c^4}\right)\right] = \varepsilon_{ijk}\int d^3\mathbf{x}\left\{\sigma x_j\partial_k V^{\rm reac} - \frac{4}{c^2}\sigma_m x_j\partial_k V_m^{\rm reac} - \frac{4}{c^2}\sigma_j V_k^{\rm reac}\right\} + \frac{1}{c^4}\,{}_4Z_i + \frac{1}{c^6}\,{}_6Z_i + O\left(\frac{1}{c^8}\right). \tag{4.24c}$$

The O-symbols $\overline{O}(c^{-4})$ denote the terms, coming in particular from the symmetric potentials in the right-hand-sides, which are of higher order than 1PN. We add an overbar on these remainder terms to distinguish them from other terms introduced below.

Now recall that the 2PN and 3PN approximations, including in particular the terms ${}_nX$, ${}_nY_i$, and ${}_nZ_i$ in Eq. (4.24), are nonradiative (nondissipative). Indeed they correspond to ''even'' approximations, and depend instantaneously on the parameters of the source. In the case of the 2PN approximation, Chandrasekhar and Nutku [24] have proved explicitly that ${}_4X$, ${}_4Y_i$, and ${}_4Z_i$ can be transformed into total time derivatives, leading to the expressions of the 2PN-conserved energy and momenta. Here we shall assume that the same property holds for the 3PN approximation, namely that the terms ${}_6X$, ${}_6Y_i$, and ${}_6Z_i$ can also be transformed into time derivatives. This assumption is almost certainly correct. The 3PN approximation is not expected to yield any secular decrease of quasi-conserved quantities. It can be argued, in fact, that the 3PN approximation is the last approximation which is purely nondissipative. Under this (slight) assumption we can now transfer the terms ${}_nX$, ${}_nY_i$, and ${}_nZ_i$ to the left-hand sides, where they modify the remainder terms $\overline{O}(c^{-4})$. Thus,

$$\frac{d}{dt}\left[E^{1\mathrm{PN}}+\widetilde{O}\left(\frac{1}{c^4}\right)\right]=\int d^3\mathbf{x}\left\{-\sigma\partial_t V^{\mathrm{reac}}+\frac{4}{c^2}\sigma_j\partial_t V_j^{\mathrm{reac}}\right\}+O\left(\frac{1}{c^8}\right), \tag{4.25a}$$

$$\frac{d}{dt}\left[P_i^{1\mathrm{PN}}+\widetilde{O}\left(\frac{1}{c^4}\right)\right]=\int d^3\mathbf{x}\left\{\sigma\partial_i V^{\mathrm{reac}}-\frac{4}{c^2}\sigma_j\partial_i V_j^{\mathrm{reac}}\right\}+O\left(\frac{1}{c^8}\right), \tag{4.25b}$$

$$\frac{d}{dt}\left[S_i^{1\mathrm{PN}}+\widetilde{O}\left(\frac{1}{c^4}\right)\right]=\varepsilon_{ijk}\int d^3\mathbf{x}\left\{\sigma x_j\partial_k V^{\mathrm{reac}}-\frac{4}{c^2}\sigma_m x_j\partial_k V_m^{\mathrm{reac}}-\frac{4}{c^2}\sigma_j V_k^{\mathrm{reac}}\right\}+O\left(\frac{1}{c^8}\right), \tag{4.25c}$$

where $\widetilde{O}(c^{-4})$ denotes the modified remainder terms, which satisfy, for instance, $E^{1\mathrm{PN}}+\widetilde{O}(c^{-4})=E^{2\mathrm{PN}}+O(c^{-5})$.

The equations (4.25) clarify the way the losses of energy and momenta are driven by the radiation reaction potentials. However, these equations are still to be transformed using the explicit expressions (3.6)–(3.8). When inserting these expressions into the right-hand sides of Eq. (4.25) one is left with numerous terms. All these terms have to be transformed and combined together modulo total time derivatives. Thus, numerous operations by parts on the time variable are performed [i.e., $A\partial_t B=\partial_t(AB)-B\partial_t A$], thereby producing many time derivatives which are transferred as before to the left-hand sides of the equations, where they modify the $\widetilde{O}(c^{-4})$'s by some contributions of order c^{-5} at least (since this is the order of the reactive terms). During the transformation of the laws (4.25a) and (4.25c) for the energy and angular momentum, it is crucial to recognize among the terms the expression of the 1PN-accurate mass quadrupole moment I_{ij} given by Eq. (3.7) [or Eq. (3.21a) with $l=2$], namely

$$I_{ij}=\int d^3\mathbf{x}\left\{\hat{x}_{ij}\sigma+\frac{1}{14c^2}\mathbf{x}^2\hat{x}_{ij}\partial_t^2\sigma-\frac{20}{21c^2}\hat{x}_{ijk}\partial_t\sigma_k\right\}. \tag{4.26}$$

And during the transformation of the law (4.25b) for linear momentum, the important point is to remember that the 1PN-accurate mass dipole moment I_i, whose second time derivative is zero as a consequence of the equations of motion $[d^2I_i/dt^2=O(c^{-4})]$, reads

$$I_i=\int d^3\mathbf{x}\left\{x_i\sigma+\frac{1}{10c^2}\mathbf{x}^2x_i\partial_t^2\sigma-\frac{6}{5c^2}\hat{x}_{ij}\partial_t\sigma_j\right\}. \tag{4.27a}$$

This moment is also a particular case, when $l=1$, of the general formula (3.21a). An alternative expression of the dipole moment is

$$I_i=\int d^3\mathbf{x}x_i\left\{\sigma+\frac{1}{c^2}\left(\frac{1}{2}\sigma U-\sigma_{jj}\right)\right\}+O\left(\frac{1}{c^4}\right), \tag{4.27b}$$

which involves the Newtonian conserved mass density [given also by the three first terms in Eq. (4.15)]. Finally, the end results of the computations are some laws involving in the right-hand sides some quadratic products of derivatives of multipole moments, most importantly the 1PN quadrupole moment (4.26) (the other moments being Newtonian), namely

$$\frac{d}{dt}\left[E^{1\mathrm{PN}}+\hat{O}\left(\frac{1}{c^4}\right)\right]=-\frac{G}{c^5}\left\{\frac{1}{5}I_{ij}^{(3)}I_{ij}^{(3)}+\frac{1}{c^2}\left[\frac{1}{189}I_{ijk}^{(4)}I_{ijk}^{(4)}+\frac{16}{45}J_{ij}^{(3)}J_{ij}^{(3)}\right]\right\}+O\left(\frac{1}{c^8}\right), \tag{4.28a}$$

$$\frac{d}{dt}\left[P_i^{1\mathrm{PN}}+\hat{O}\left(\frac{1}{c^4}\right)\right]=-\frac{G}{c^7}\left\{\frac{2}{63}I_{ijk}^{(4)}I_{jk}^{(3)}+\frac{16}{45}\varepsilon_{ijk}I_{jm}^{(3)}J_{km}^{(3)}\right\}+O\left(\frac{1}{c^9}\right), \tag{4.28b}$$

$$\frac{d}{dt}\left[S_i^{1\mathrm{PN}}+\hat{O}\left(\frac{1}{c^4}\right)\right]=-\frac{G}{c^5}\varepsilon_{ijk}\left\{\frac{2}{5}I_{jm}^{(2)}I_{km}^{(3)}+\frac{1}{c^2}\left[\frac{1}{63}I_{jmn}^{(3)}I_{kmn}^{(4)}+\frac{32}{45}J_{jm}^{(2)}J_{km}^{(3)}\right]\right\}+O\left(\frac{1}{c^8}\right). \tag{4.28c}$$

The remainders in the left-hand sides are such that $\hat{O}(c^{-4})=\widetilde{O}(c^{-4})+O(c^{-5})$. The remainders in the right-hand sides are $O(c^{-8})$ in the cases of energy and angular momentum because of tail contributions (see Sec. IV C), but is $O(c^{-9})$ in the case of the linear momentum.

The last step is to argue that the unknown terms in the left-hand sides, namely the total time derivatives of the remainders $\hat{O}(c^{-4})$, are negligible as compared to the controlled terms in the right-hand sides, despite their larger formal post-Newtonian order (c^{-4} vs c^{-5} and c^{-7}). When computing, for instance, the time evolution of the orbital phase of inspiralling compact binaries [6–18], one uses in the left-hand side of the balance equation the energy valid at the *same* post-Newtonian order as the energy flux in the right-hand side. Because the difference between the orders of magnitude of the two sides of the equations is c^{-5}, we need to show that the time derivative increases the formal post-Newtonian order by a factor c^{-5}. In Eq. (4.27) this means $d\hat{O}(c^{-4})/dt = O(c^{-9})$ [actually, $O(c^{-8})$ would be sufficient in Eq. (4.28), but $O(c^{-9})$ will be necessary in Sec. IV C]. In the case of inspiralling compact binaries, such an equation is clearly true, because the terms $\hat{O}(c^{-4})$ depend only on the orbital separation between the two bodies (the orbit being circular), and thus depend only on the energy which is conserved at 2PN order (for noncircular orbits one would have also a dependence on the angular momentum). Thus the time derivative adds, by the law of composition of derivatives, an extra factor c^{-5} coming from the time derivative of the energy itself. More generally, this would be true for any system whose 2PN dynamics can be parametrized by the 2PN-conserved energy and angular momentum. This argument could perhaps be extended to systems whose 2PN dynamics is integrable, in the sense that the solutions are parametrized by some finite set of integrals of motion, including the integral of energy. Another argument, which is often presented (see e.g., [34]), is that the terms $d\hat{O}(c^{-4})/dt$ are negligible when taken in average for quasiperiodic systems, for instance a binary system moving on a quasi-Keplerian orbit. The time average of a total time derivative is clearly numerically small for such systems, but it seems difficult to quantify precisely the gain in order of magnitude which is achieved in this way, for general systems. The most general argument, valid for any system, is that the terms $d\hat{O}(c^{-4})/dt$ are numerically small when one looks at the evolution of the system over long time scales, for instance $\Delta t \gg \hat{O}(c^{-4}) \times (dE^{1\mathrm{PN}}/dt)^{-1}$ (see Thorne [64], p. 46).

Adopting here $d\hat{O}(c^{-4})/dt = O(c^{-9})$ and the latter general argument, we can neglect the terms $\hat{O}(c^{-4})$ and arrive to the 1PN energy-momenta balance equations

$$\frac{dE^{1\mathrm{PN}}}{dt} = -\frac{G}{c^5}\left\{\frac{1}{5}I^{(3)}_{ij}I^{(3)}_{ij} + \frac{1}{c^2}\left[\frac{1}{189}I^{(4)}_{ijk}I^{(4)}_{ijk} + \frac{16}{45}J^{(3)}_{ij}J^{(3)}_{ij}\right]\right\} + O\left(\frac{1}{c^8}\right), \tag{4.29}$$

$$\frac{dP_i^{1\mathrm{PN}}}{dt} = -\frac{G}{c^7}\left\{\frac{2}{63}I^{(4)}_{ijk}I^{(3)}_{jk} + \frac{16}{45}\varepsilon_{ijk}I^{(3)}_{jm}J^{(3)}_{km}\right\} + O\left(\frac{1}{c^9}\right), \tag{4.30}$$

$$\frac{dS_i^{1\mathrm{PN}}}{dt} = -\frac{G}{c^5}\varepsilon_{ijk}\left\{\frac{2}{5}I^{(2)}_{jm}I^{(3)}_{km} + \frac{1}{c^2}\left[\frac{1}{63}I^{(3)}_{jmn}I^{(4)}_{kmn} + \frac{32}{45}J^{(2)}_{jm}J^{(3)}_{km}\right]\right\} + O\left(\frac{1}{c^8}\right), \tag{4.31}$$

relating the 1PN-conserved energy and momenta, given by explicit integrals over the source [Eqs. (4.15)–(4.18)], to some combinations of derivatives of multipole moments, also given by explicit integrals over the source [see Eqs. (3.7)–(3.8)]. Note that at this order both sides of the equations are in the form of compact-support integrals. The right-hand-sides of Eqs. (4.29)–(4.31) agree exactly with (minus) the fluxes of energy and momenta as computed in the wave zone of the system. See for instance the equations (4.16'), (4.20'), and (4.23') in [46], when truncated to 1PN order [and recalling that the quadrupole moment which enters the 1PN fluxes is precisely the one given by Eq. (4.26)]. Thus, we can conclude on the validity of the balance equations at 1PN order, for weakly self-gravitating systems.

These equations could also be recovered, in principle, from the relations (2.14)–(2.15) (which were obtained in paper I). Indeed, Eqs. (2.14)–(2.15) involve, besides some instantaneous contributions such as $T_L(t)$, some nonlocal (or hereditary) contributions contained in the functions $m(t)$, $m_i(t)$, and $s_i(t)$. These contributions modify the constant monopole and dipole moments M, M_i, and S_i by some expressions which correspond exactly to the emitted fluxes. The balance equations could be recovered (with, though, less precision than obtained in this paper) by using the constancy of the monopole and dipoles M, M_i, and S_i in the equations (2.14) written for $l=0$ and $l=1$, and by using the matching equations obtained in Eq. (3.27), also written for $l=0$ and $l=1$. Related to this, notice the term involving a single time antiderivative in the function $m_i(t)$ of Eq. (2.15b), and which is associated with a secular displacement of the center of mass position.

C. Tail effects at 1.5PN order

To 1.5PN order in the radiation reaction force appears a hereditary integral (i.e., an integral extending on the whole past history of the source), which is associated physically with the effects of gravitational-wave tails. More precisely, it is shown in [50], using the same combination of approximation methods as used in paper I and this paper, that the dominant hereditary contribution in the inner post-Newtonian metric $g^{\mathrm{in}}_{\mu\nu}$ (valid all over D_i) arises at the 4PN order. At this order, the dynamics of a self-gravitating system is thus intrinsically dependent on the full past evolution of the system.

In a particular gauge (defined in [50]), the 4PN-hereditary contribution in $g^{\mathrm{in}}_{\mu\nu}$ is entirely located in the 00 component of the metric, and reads

$$g^{\mathrm{in}}_{00}\big|_{\mathrm{hereditary}} = -\frac{8G^2M}{5c^{10}}x^ix^j\int_0^{+\infty} d\lambda \ln\left(\frac{\lambda}{2}\right) I^{(7)}_{ij}(t-\lambda) + O\left(\frac{1}{c^{11}}\right). \tag{4.32}$$

The hereditary contributions in the other components of the metric ($0i$ and ij) arise at higher order. Note that the hereditary (tail) integral in Eq. (4.32) involves a logarithmic kernel. *A priori*, one should include in the logarithm a constant time scale P in order to adimensionalize the integration variable λ, say $\ln(\lambda/2P)$. However, $\ln P$ would actually be in factor of an instantaneous term [depending only on the current instant t through the sixth time derivative $I^{(6)}_{ij}(t)$], so Eq. (4.32) is in fact independent of the choice of time scale. In Eq. (4.32) we have chosen for simplicity $P=1$ sec. The presence of the tail integral (4.32) in the metric implies a modification of the radiation reaction force at the relative 1.5PN order [50]. The other 4PN terms are not controlled at this stage, but are instantaneous and thus do not yield any radiation reaction effects (indeed the 4PN approximation is ''even'' in the post-Newtonian sense). It was further shown [42] that the 1.5PN tail integral in the radiation reaction is such that there is exact energy balance with a corresponding integral present in the far-zone flux. Here we recover this fact and add it up to the results obtained previously.

As the gauge transformation yielding (4.32) in [50] deals only with 4PN terms, it can be applied to the inner metric $g^{\rm in}_{\mu\nu}$ given by Eq. (3.1) without modifying any of the known terms at the 1PN nonradiative and reactive approximations. It is clear from Eq. (4.32) and the reactive potentials (3.6) that after gauge transformation, the inner metric takes the same form as Eq. (3.1), except that the reactive potentials are now more accurate, and given by

$$V^{\rm reac}(\mathbf{x},t)=-\frac{G}{5c^5}x_{ij}I^{(5)}_{ij}(t)+\frac{G}{c^7}\left[\frac{1}{189}x_{ijk}I^{(7)}_{ijk}(t)-\frac{1}{70}\mathbf{x}^2x_{ij}I^{(7)}_{ij}(t)\right]-\frac{4G^2M}{5c^8}x_{ij}\int_0^{+\infty}d\lambda\ \ln\left(\frac{\lambda}{2}\right)\ I^{(7)}_{ij}(t-\lambda)+O\left(\frac{1}{c^9}\right), \tag{4.33a}$$

$$V^{\rm reac}_i(\mathbf{x},t)=\frac{G}{c^5}\left[\frac{1}{21}\hat{x}_{ijk}I^{(6)}_{jk}(t)-\frac{4}{45}\varepsilon_{ijk}x_{jm}J^{(5)}_{km}(t)\right]+O\left(\frac{1}{c^7}\right). \tag{4.33b}$$

Still there remain in the metric some uncontrolled (even) 4PN terms, but these are made of *instantaneous* spatial integrals over the source variables, exactly like the uncontrolled 2PN and 3PN terms. [The expressions (4.33) can be recovered also from Sec. III D of paper I and a matching similar to the one performed in this paper.] With Eq. (4.33) in hand, one readily extends the balance equations to 1.5PN order. First one obtains Eq. (4.24), but where the reactive potentials are given more accurately by Eq. (4.33), and there are some instantaneous 4PN terms ${}_8X$, ${}_8Y_i$, and ${}_8Z_i$ in the right-hand sides. Extending the (slight) assumption made before concerning the similar 3PN terms, we can transform ${}_8X$, ${}_8Y_i$, and ${}_8Z_i$ into time derivatives and transfer them to the left-hand sides. This yields Eq. (4.25), except that the remainders in the right-hand sides are $O(c^{-9})$ instead of $O(c^{-8})$. Using Eq. (4.33), we then obtain (working modulo total time derivatives) the laws (4.28) augmented by the tail contributions arising at order c^{-8} in the right-hand sides. The remainders in the left-hand sides are of the order $d\hat{O}(c^{-4})/dt=O(c^{-9})$ (arguing as previously), and therefore are negligible as compared to the tail contributions at c^{-8}. In the case of energy the 1.5PN balance equation is obtained as

$$\frac{dE^{\rm 1PN}}{dt}=-\frac{G}{5c^5}I^{(3)}_{ij}I^{(3)}_{ij}-\frac{G}{c^7}\left[\frac{1}{189}I^{(4)}_{ijk}I^{(4)}_{ijk}+\frac{16}{45}J^{(3)}_{ij}J^{(3)}_{ij}\right]-\frac{4G^2M}{5c^8}I^{(3)}_{ij}(t)\int_0^{+\infty}d\lambda\ln\left(\frac{\lambda}{2}\right)\ I^{(5)}_{ij}(t-\lambda)+O\left(\frac{1}{c^9}\right). \tag{4.34}$$

Because there are no terms of order c^{-3} in the internal energy of the system (see Sec. IV A), the energy $E^{\rm 1PN}$ appearing in the left-hand side is in fact valid at the 1.5PN order. Finally, to the required order, one can rewrite Eq. (4.34) equivalently in a form where (minus) the right-hand side is manifestly positive-definite,

$$\frac{dE^{\rm 1PN}}{dt}=-\frac{G}{5c^5}\left[I^{(3)}_{ij}(t)+\frac{2GM}{c^3}\int_0^{+\infty}d\lambda\ \ln\left(\frac{\lambda}{2}\right)I^{(5)}_{ij}(t-\lambda)\right]^2-\frac{G}{c^7}\left[\frac{1}{189}(I^{(4)}_{ijk})^2+\frac{16}{45}(J^{(3)}_{ij})^2\right]+O\left(\frac{1}{c^9}\right). \tag{4.35}$$

Under the latter form one recognizes in the right-hand side the known energy flux at 1.5PN order. Indeed the effective quadrupole moment which appears in the parenthesis agrees with the tail-modified *radiative* quadrupole moment parametrizing the field in the far zone [see Eq. (3.10) in [42]]. [The term associated with the (gauge-dependent) constant 11/12 in the radiative quadrupole moment [42] yields a total time derivative in the energy flux (as would yield any time scale P in the logarithm), and can be neglected in Eq. (4.35).] The 1.5PN balance equation for angular momentum is proved similarly (it involves as required the same tail-modified radiative quadrupole moment). The balance equation for linear momentum does not include any tail contribution at 1.5PN order, and simply remains in the form of Eq. (4.30).

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[59] Our conventions and notation are the following: signature $-+++$; greek indices $=0,1,2,3$; latin indices $=1,2,3$; $g=\det(g_{\mu\nu})$; $\eta_{\alpha\beta}=\eta^{\alpha\beta}$ = flat metric = $\mathrm{diag}(-1,1,1,1)$; $r=|\mathbf{x}|=(x_1^2+x_2^2+x_3^2)^{1/2}$; $n^i=n_i=x^i/r$; $\partial_i=\partial/\partial x^i$; $x^L=x_L=x_{i_1}x_{i_2}\ldots x_{i_l}$ and $\partial_L=\partial_{i_1}\partial_{i_2}\ldots\partial_{i_l}$, where $L=i_1i_2\ldots i_l$ is a multi-index with l indices; $x_{L-1}=x_{i_1}\ldots x_{i_{l-1}}$, $x_{aL-1}=x_a x_{i_1}\ldots x_{i_{l-1}}$, etc...; $\hat{x}_L$ and $\hat{\partial}_L$ are the (symmetric) and trace-free parts of x_L and ∂_L, for instance $\hat{x}_{ij}=x_ix_j-\frac{1}{3}\delta_{ij}r^2$; $\hat{x}_L$ is denoted also $x_{\langle L\rangle}$; $T_{(ij)}=\frac{1}{2}(T_{ij}+T_{ji})$; the superscript (n) denotes n time derivatives.
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