ARTICLES

Gravitational waves from inspiralling compact binaries: Energy loss and waveform to second-post-Newtonian order

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Gravitational waves generated by inspiralling compact binaries are investigated to the second-post-Newtonian (2PN) approximation of general relativity. Using a recently developed 2PN-accurate wave generation formalism, we compute the gravitational waveform and associated energy loss rate from a binary system of point masses moving on a quasicircular orbit. The crucial new input is our computation of the 2PN-accurate "source" quadrupole moment of the binary. Tails in both the waveform and energy loss rate at infinity are explicitly computed. Gravitational radiation reaction effects on the orbital frequency and phase of the binary are deduced from the energy loss. In the limiting case of a very small mass ratio between the two bodies we recover the results obtained by black hole perturbation methods. We find that finite mass ratio effects are very significant as they increase the 2PN contribution to the phase by up to 52%. The results of this paper should be of use when deciphering the signals observed by the future LIGO-VIRGO network of gravitational-wave detectors.

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I. INTRODUCTION

The Laser Interferometric Gravitational Wave Observatory- (LIGO-)VIRGO network of kilometer-size interferometric detectors of gravitational waves is expected to be in operation by the turn of the century [1,2] (see [3] for a recent review). The most promising targets for this network are the gravitational waves emitted during the radiation-reaction-driven inspiral of binary systems of compact objects (neutron stars or black holes). Crucial to the successful detection and deciphering of such waves will be the availability of accurate theoretical templates for the inspiral gravitational waveforms [4–6]. Much theoretical effort is currently being spent on developing improved formalisms tackling the generation of gravitational waves by general material sources and/or on applying existing formalisms to the explicit computation of increasingly accurate inspiral waveforms. Among the existing generation formalisms, the one proposed by us [7–9] can, in principle, be developed to an arbitrarily high accuracy. Recent work by one of us [10] has succeeded in pushing its accuracy to the second-post-Newtonian (2PN) level, i.e., in deriving general expressions for the asymptotic gravitational waveform, as a functional of the matter distribution in the source, which take into account all contributions of fractional order ε^4 beyond the leading ("quadrupole formula") term. Here $\varepsilon \sim v/c \sim (Gm/rc^2)^{1/2}$ denotes the small parameter entering the post-Newtonian expansion appropriate to the description of slowly moving, weakly stressed, weakly self-gravitating systems.

The object of the present paper is to apply the 2PN-accurate generation formalism of Refs. [7–10] to the specific case of an inspiralling compact binary. This is a nontrivial task as the end results of Ref. [10] contain some complicated three-dimensional integrals which are mathematically defined by a procedure of analytic continuation. Note that this contrasts with the end results of our previous 1.5PN-accurate generation formalism [7–9] which contained only integrals extending over the (compact) support of the material source. The higher complexity of the 2PN level is due to the appearance of terms associated with the cubic nonlinearities of Einstein's equations. We shall show below how to explicitly

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compute these terms in the case of binary systems.

Theoretical waveforms such as the one computed below are useful to define parametrized "chirp" templates to be cross correlated with the outputs of the LIGO-VIRGO interferometric detectors. For this technique to be successful, the templates must remain in phase with the exact general-relativistic waveform as long as possible. The phase of the signal is determined by the rate of change of the orbital period resulting from gravitational radiation-reaction effects. Following the usual heuristic approach (which has been validated in detail at the leading order [11]), the effect of gravitational radiation reaction on the orbital period can be computed from the losses of energy (and angular momentum) [12] at infinity. We shall therefore pay special attention to the computation of the 2PN-accurate energy loss from a (quasi) circular compact binary which is obtained here for the first time, together with the resulting orbital phasing of the binary.

Several investigations have recently focused on the computation of the energy loss and waveform in the case of a very small mass ratio between the two bodies [13–16]. Notably the energy loss has been computed in this case both numerically [15] and analytically [16] up to an order going well beyond the 2PN level. The test-mass limit of our result agrees with the 2PN truncation of the results of Refs. [15,16]. However, we find that finite mass effects change very significantly the 2PN numerical contribution to the accumulated orbital phase. Our main results, completed by the contributions due to the spins of the orbiting bodies, have been briefly reported in [17]. Note that our formula for the energy loss has been confirmed by an independent derivation based on a different, albeit less rigorous, method [18].

The organization of the paper is the following. In Sec. II we write down the results of the 2PN generation formalism in a form convenient for the application to inspiralling binaries and we outline our strategy. In Sec. III we compute the 2PN-accurate "source" moments of mass-type (especially the quadrupole moment $\ell = 2$) in the case of a binary system made of two point masses. In particular a crucial cubically nonlinear term is obtained in Sec. III C. The expressions for all the relevant source moments are given in Sec. IV which deals with the explicit computation of the 2PN-accurate waveform and energy loss rate (including relevant tails). The instantaneous orbital phase of the binary is computed at the end of Sec. IV. Technical details are relegated to several appendices: the conserved mass monopole and dipole moments are considered in Appendix A; Appendix B presents an alternative derivation of the cubically nonlinear contribution to the quadrupole moment, which is valid for N-body systems; and Appendix C is a compendium of various formulas for moments.

II. SUMMARY OF THE 2PN-ACCURATE GENERATION FORMALISM

Let us first recall that in a suitable "radiative" coordinate system $X^\mu = (cT, X^i)$ the metric coefficients, say $G_{\alpha\beta}(X^\gamma)$, describing the gravitational field outside an isolated system admit an asymptotic expansion in powers of R^{-1}, when $R = |\mathbf{X}| \to \infty$ with $T - R/c$ and $\mathbf{N} \equiv \mathbf{X}/R$ being fixed ("future null infinity"). The transverse-traceless (TT) projection of the deviation of $G_{\alpha\beta}(X^\gamma)$ from the flat metric (signature $-1, +1, +1, +1$) defines the asymptotic waveform $h^{\rm TT}_{km} \equiv [G_{km}(X) - \delta_{km}]^{\rm TT}$ (latin indices $i, j, k, m, \ldots$ range from 1 to 3). The $1/R$ part of $h^{\rm TT}_{km}$ can be uniquely decomposed into multipoles:

$$h^{\rm TT}_{km}(\mathbf{X}, T) = \frac{4G}{c^2R}\mathcal{P}_{ijkm}(\mathbf{N})\sum_{\ell=2}^{\infty}\frac{1}{c^\ell \ell!}\left\{N_{L-2}U_{ijL-2}(T_R) - \frac{2\ell}{(\ell+1)c}N_{aL-2}\varepsilon_{ab(i}V_{j)bL-2}(T_R)\right\} + O\left(\frac{1}{R^2}\right) . \tag{2.1}$$

The "radiative" multipole moments U_L and V_L (defined for $\ell \geq 2$) denote some functions of the retarded time $T_R \equiv T - R/c$ taking values in the set of symmetric trace-free (STF) three-dimensional Cartesian tensors of order ℓ. Here $L \equiv i_1 \cdots i_\ell$ denotes a spatial multi-index of order ℓ, $N_{L-2} \equiv N_{i_1} \cdots N_{i_{\ell-2}}$, $X_{(ij)} \equiv \frac{1}{2}(X_{ij} + X_{ji})$, and

$$\mathcal{P}_{ijkm}(\mathbf{N}) = (\delta_{ik} - N_iN_k)(\delta_{jm} - N_jN_m) - \frac{1}{2}(\delta_{ij} - N_iN_j)(\delta_{km} - N_kN_m) . \tag{2.2}$$

(For the convenience of the reader we summarize our notation in Ref. [19].) As indicated in Eq. (2.1), for slowly moving systems the multipole order is correlated with the post-Newtonian order. The coefficients in Eq. (2.1) have been chosen so that the moments U_L and V_L reduce, in the nonrelativistic limit $c \to +\infty$ (or $\varepsilon \to 0$), to the ℓth time derivatives of the usual Newtonian mass-type and current-type moments of the source. At the 2PN approximation, i.e., when retaining all terms of fractional order $\varepsilon^4 \sim c^{-4}$ with respect to the leading (Newtonian quadrupole) result, the waveform (2.1) reads

$$h^{\rm TT}_{km} = \frac{2G}{c^4R}\mathcal{P}_{ijkm}\left\{U_{ij} + \frac{1}{c}\left[\frac{1}{3}N_aU_{ija} + \frac{4}{3}\varepsilon_{ab(i}V_{j)a}N_b\right] + \frac{1}{c^2}\left[\frac{1}{12}N_{ab}U_{ijab} + \frac{1}{2}\varepsilon_{ab(i}V_{j)ac}N_{bc}\right]\right.$$
$$\left. + \frac{1}{c^3}\left[\frac{1}{60}N_{abc}U_{ijabc} + \frac{2}{15}\varepsilon_{ab(i}V_{j)acd}N_{bcd}\right] + \frac{1}{c^4}\left[\frac{1}{360}N_{abcd}U_{ijabcd} + \frac{1}{36}\varepsilon_{ab(i}V_{j)acde}N_{bcde}\right] + O(\varepsilon^5)\right\} . \tag{2.3}$$

The rate of decrease of the Bondi energy E_B with respect to the retarded time $T_R \equiv T - R/c$ is related to the waveform by

$$\frac{dE_B}{dT_R} = -\frac{c^3}{32\pi G}\int \left(\frac{\partial h^{\rm TT}_{ij}}{\partial T_R}\right)^2 R^2 d\Omega(\mathbf{N}) \ . \tag{2.4}$$

At the 2PN approximation this yields (with $U^{(n)} \equiv d^n U/dT_R^n$)

$$\frac{dE_B}{dT_R} = -\frac{G}{c^5}\left\{\frac{1}{5}U^{(1)}_{ij}U^{(1)}_{ij} + \frac{1}{c^2}\left[\frac{1}{189}U^{(1)}_{ijk}U^{(1)}_{ijk} + \frac{16}{45}V^{(1)}_{ij}V^{(1)}_{ij}\right] + \frac{1}{c^4}\left[\frac{1}{9072}U^{(1)}_{ijkm}U^{(1)}_{ijkm} + \frac{1}{84}V^{(1)}_{ijk}V^{(1)}_{ijk}\right] + O(\varepsilon^6)\right\} \ . \tag{2.5}$$

A 2PN-accurate gravitational-wave generation formalism is a method allowing one to compute the radiative moments entering Eqs. (2.3) and (2.5) in terms of the source variables with an accuracy sufficient for obtaining the waveform with fractional accuracy $1/c^4$. The latter requirement implies, in view of Eq. (2.3), that one should (at a minimum) compute the mass-type quadrupole radiative moment $U_{i_1i_2}$ with $1/c^4$ accuracy, the mass-type radiative octupole $U_{i_1i_2i_3}$ and the current-type radiative quadrupole $V_{i_1i_2}$ with $1/c^3$ accuracy, $U_{i_1i_2i_3i_4}$ and $V_{i_1i_2i_3}$ with $1/c^2$ accuracy, $U_{i_1i_2i_3i_4i_5}$ and $V_{i_1i_2i_3i_4}$ with $1/c$ accuracy, and $U_{i_1i_2i_3i_4i_5i_6}$ and $V_{i_1i_2i_3i_4i_5}$ with the Newtonian accuracy. Note that these requirements are relaxed if one is only interested in getting the energy loss rate with 2PN accuracy. In that case, Eq. (2.5) shows that one still needs $U_{i_1i_2}$ with $1/c^4$ accuracy, but that it is enough to compute $U_{i_1i_2i_3}$ and $V_{i_1i_2}$ with $1/c^2$ accuracy, and $U_{i_1i_2i_3i_4}$ and $V_{i_1i_2i_3}$ with Newtonian accuracy.

In our generation formalism, the link between the radiative multipoles U_L and V_L and the dynamical state of the material source is obtained in several steps involving as intermediate object a certain vacuum "canonical" metric $g^{\rm can}_{\mu\nu}(x^\lambda_{\rm can})$ expressed in terms of some "canonical" multipoles M_L and S_L (alternatively referred to as algorithmic moments in [8]). On the one hand, the matching of $g^{\rm can}_{\mu\nu}(x^\lambda_{\rm can})$ to a (PN-expanded) near-zone solution of the inhomogeneous Einstein equations allows one to compute M_L and S_L in terms of some suitably defined "source" multipoles I_L(source), J_L(source). On the other hand, the computation of nonlinear effects in the wave zone allows one to compute U_L and V_L as functionals of M_L and S_L. The final result for the 2PN-accurate generation formalism reads (when working in an initially mass-centered coordinate system, i.e., such that the canonical mass dipole M_i vanishes for all times)

$$U_{ij}(T_R) = I^{(2)}_{ij}(T_R) + \frac{2Gm}{c^3}\int_0^{+\infty} d\tau \left[\ln\left(\frac{\tau}{2b}\right) + \frac{11}{12}\right] I^{(4)}_{ij}(T_R-\tau) + O(\varepsilon^5) \ , \tag{2.6a}$$

$$U_{ijk}(T_R) = I^{(3)}_{ijk}(T_R) + \frac{2Gm}{c^3}\int_0^{+\infty} d\tau \left[\ln\left(\frac{\tau}{2b}\right) + \frac{97}{60}\right] I^{(5)}_{ijk}(T_R-\tau) + O(\varepsilon^5) \ , \tag{2.6b}$$

$$V_{ij}(T_R) = J^{(2)}_{ij}(T_R) + \frac{2Gm}{c^3}\int_0^{+\infty} d\tau \left[\ln\left(\frac{\tau}{2b}\right) + \frac{7}{6}\right] J^{(4)}_{ij}(T_R-\tau) + O(\varepsilon^4) \ , \tag{2.6c}$$

for the moments that need to be known beyond the 1PN accuracy, and

$$U_L(T_R) = I^{(\ell)}_L(T_R) + O(\varepsilon^3) \ , \tag{2.7a}$$

$$V_L(T_R) = J^{(\ell)}_L(T_R) + O(\varepsilon^3) \ , \tag{2.7b}$$

for the other ones. Equations (2.6) involve some integrals which are associated with tails; these integrals have in front of them the total mass energy m of the source, and contain a quantity b which is an arbitrary constant (with the dimension of time) parametrizing a certain freedom in the construction of the radiative coordinate system $(T, \mathbf{X})$. More precisely, the link between the (Bondi-type) radiative coordinates $X^\mu = (cT, X^i)$ and the (harmonic) canonical coordinates $x^\mu_{\rm can} = (ct_{\rm can}, x^i_{\rm can})$ reads

$$T_R = t_{\rm can} - \frac{r_{\rm can}}{c} - \frac{2Gm}{c^3}\ln\left(\frac{r_{\rm can}}{cb}\right) + O(\varepsilon^5) + O\left(1/r^2_{\rm can}\right) \ . \tag{2.8}$$

Except for the computation of U_{ij} which requires the knowledge of the mass quadrupole source moment I_{ij} with 2PN accuracy, the computation of the other multipole contributions to the waveform can be obtained from 1PN-accurate expressions of the mass-type and current-type source moments which have been obtained for all values of ℓ in Refs. [7,8], respectively, as explicit integrals extending only on the compact support of the material source. (Note that there are no $1/c^3$ contributions in the *source* moments.) Let us illustrate the structure of the 1PN results by quoting the simpler 1PN mass-type source moments:

$$I_L(t) = \int d^3x \left[\hat{x}_L\sigma(t,\mathbf{x}) + \frac{\hat{x}_L\mathbf{x}^2}{2(2\ell+3)c^2}\frac{\partial^2\sigma(t,\mathbf{x})}{\partial t^2} - \frac{4(2\ell+1)\hat{x}_{iL}}{(\ell+1)(2\ell+3)c^2}\frac{\partial\sigma_i(t,\mathbf{x})}{\partial t}\right] + O(\varepsilon^4) \ . \tag{2.9}$$

The (compact support) matter densities appearing in Eq. (2.9), and their generalizations discussed below, are defined from the contravariant components (in the harmonic, "source" coordinate system x^μ) of the material stress-energy tensor $T^{\mu\nu}$ as

$$\sigma(t,\mathbf{x}) \equiv \frac{T^{00}(t,\mathbf{x}) + T^{ss}(t,\mathbf{x})}{c^2} , \tag{2.10a}$$

$$\sigma_i(t,\mathbf{x}) \equiv \frac{T^{0i}(t,\mathbf{x})}{c} , \tag{2.10b}$$

$$\sigma_{ij}(t,\mathbf{x}) \equiv T^{ij}(t,\mathbf{x}) . \tag{2.10c}$$

The powers of c introduced in Eqs. (2.10) are such that the σ's have a finite nonzero limit as $1/c \to 0$. One associates to the matter densities (2.10) some Newtonian-like potentials, say

$$U(\mathbf{x},t) = G\int \frac{d^3\mathbf{x}'}{|\mathbf{x}-\mathbf{x}'|}\sigma(\mathbf{x}',t) , \tag{2.11a}$$

$$X(\mathbf{x},t) = G\int d^3\mathbf{x}'|\mathbf{x}-\mathbf{x}'|\sigma(\mathbf{x}',t) , \tag{2.11b}$$

$$U_i(\mathbf{x},t) = G\int \frac{d^3\mathbf{x}'}{|\mathbf{x}-\mathbf{x}'|}\sigma_i(\mathbf{x}',t) , \tag{2.11c}$$

$$P_{ij}(\mathbf{x},t) = G\int \frac{d^3\mathbf{x}'}{|\mathbf{x}-\mathbf{x}'|}\left[\sigma_{ij} + \frac{1}{4\pi G}\left(\partial_i U\partial_j U - \frac{1}{2}\delta_{ij}\partial_k U\partial_k U\right)\right](\mathbf{x}',t) . \tag{2.11d}$$

[Note the equation satisfied by the X potential: $\Delta X = 2U$.] The 2PN-accurate source moment $I_L(t)$ has been obtained in Ref. [10] and expressed in terms of the matter densities σ, σ_i, σ_{ij}, the potentials U, U_i, P_{ij} and the trace $P \equiv P_{ss}$. The result [see Eq. (4.21) of Ref. [10]] is

$$\begin{aligned} I_L(t) = \mathrm{FP}_{B=0}\int d^3\mathbf{x}|\mathbf{x}|^B\Bigg\{&\hat{x}_L\left[\sigma + \frac{4}{c^4}(\sigma_{ii}U - \sigma P)\right] + \frac{|\mathbf{x}|^2\hat{x}_L}{2c^2(2\ell+3)}\partial_t^2\sigma \\ &- \frac{4(2\ell+1)\hat{x}_{iL}}{c^2(\ell+1)(2\ell+3)}\partial_t\left[\left(1+\frac{4U}{c^2}\right)\sigma_i + \frac{1}{\pi G c^2}\left(\partial_k U[\partial_i U_k - \partial_k U_i] + \frac{3}{4}\partial_t U\partial_i U\right)\right] \\ &+ \frac{|\mathbf{x}|^4\hat{x}_L}{8c^4(2\ell+3)(2\ell+5)}\partial_t^4\sigma - \frac{2(2\ell+1)|\mathbf{x}|^2\hat{x}_{iL}}{c^4(\ell+1)(2\ell+3)(2\ell+5)}\partial_t^3\sigma_i \\ &+ \frac{2(2\ell+1)}{c^4(\ell+1)(\ell+2)(2\ell+5)}\hat{x}_{ijL}\partial_t^2\left[\sigma_{ij} + \frac{1}{4\pi G}\partial_i U\partial_j U\right] \\ &+ \frac{1}{\pi G c^4}\hat{x}_L\left[-P_{ij}\partial_{ij}U - 2U_i\partial_t\partial_i U + 2\partial_i U_j\partial_j U_i - \frac{3}{2}(\partial_t U)^2 - U\partial_t^2 U\right]\Bigg\} + O(\varepsilon^5) . \end{aligned} \tag{2.12}$$

This expression obviously reduces to Eq. (2.9) at the 1PN level. It was also shown in Eq. (4.13) of Ref. [10] that the current moment J_L, which needs only 1PN accuracy, can be written as

$$\begin{aligned} J_L(t) = \mathrm{FP}_{B=0}\,\varepsilon_{ab<i_\ell}\int d^3\mathbf{x}|\mathbf{x}|^B\Bigg\{&\hat{x}_{L-1>a}\left(1+\frac{4}{c^2}U\right)\sigma_b + \frac{|\mathbf{x}|^2\hat{x}_{L-1>a}}{2c^2(2\ell+3)}\partial_t^2\sigma_b \\ &+ \frac{1}{\pi G c^2}\hat{x}_{L-1>a}\left[\partial_k U(\partial_b U_k - \partial_k U_b) + \frac{3}{4}\partial_t U\partial_b U\right] - \frac{(2\ell+1)\hat{x}_{L-1>ac}}{c^2(\ell+2)(2\ell+3)}\partial_t\left[\sigma_{bc} + \frac{1}{4\pi G}\partial_b U\partial_c U\right]\Bigg\} + O(\varepsilon^4) . \end{aligned} \tag{2.13}$$

This form is equivalent to the result previously derived in Ref. [8] [Eq. (5.18) there]. See below for comments on the symbol $\mathrm{FP}_{B=0}$ in front of Eqs. (2.12) and (2.13).

In this paper, it will be convenient to split the potential P_{ij} of Eq. (2.11d) into a compact-source potential U_{ij} and a nonlinear potential W_{ij} according to

$$P_{ij} = U_{ij} - W_{ij} + \frac{1}{2}\delta_{ij}W_{ss} , \tag{2.14a}$$

where U_{ij} and W_{ij} are defined by

$$U_{ij}(\mathbf{x},t) = G\int \frac{d^3\mathbf{x}'}{|\mathbf{x}-\mathbf{x}'|}\sigma_{ij}(\mathbf{x}',t) , \tag{2.14b}$$

$$W_{ij}(\mathbf{x},t) = -\frac{1}{4\pi}\int \frac{d^3\mathbf{x}'}{|\mathbf{x}-\mathbf{x}'|}[\partial_i U\partial_j U](\mathbf{x}',t) . \tag{2.14c}$$

(We have $\Delta W_{ij} = \partial_i U \partial_j U$.) The compact-source potential U_{ij} should not be confused with the previously defined radiative quadrupole moment U_{ij}. Also note that the notation W_{ij} is the same as used in Ref. [10] to denote a different potential (which is a retarded version of the potential P_{ij}) but which will not be used in this paper. Let us note for future reference that the trace of W_{ij} as defined here satisfies

$$W_{ss} = \frac{1}{2}U^2 - \Phi \ , \tag{2.15a}$$

$$\Phi(\mathbf{x},t) = G\int \frac{d^3\mathbf{x}'}{|\mathbf{x}-\mathbf{x}'|}[\sigma U](\mathbf{x}',t) \ . \tag{2.15b}$$

The potentials introduced above are connected by the approximate differential identities

$$\partial_t U + \partial_i U_i = O(\varepsilon^2) \ , \tag{2.16a}$$

$$\partial_t U_i + \partial_j U_{ij} = \partial_j \left(W_{ij} - \frac{1}{2}\delta_{ij}W_{ss}\right) + O(\varepsilon^2) \ . \tag{2.16b}$$

We replace in Eq. (2.12) the potential P_{ij} by Eqs. (2.14) above, and $\partial_t U$ by the spatial derivative $-\partial_i U_i$. It is also convenient to perform an integration by parts, using $\partial_k U \partial_k U_i \equiv \frac{1}{2}[\Delta(UU_i) - U\Delta U_i - U_i \Delta U]$, which is justified by a reasoning similar to the ones followed in Sec. IV of Ref. [10]. This leads to the form we shall use as the starting point in this work:

$$\begin{aligned} I_L(t) = \mathrm{FP}_{B=0}\int d^3\mathbf{x}|\mathbf{x}|^B \Bigg\{ & \hat{x}_L\left[\sigma - \frac{4}{c^4}\sigma U_{ss} + \frac{4}{c^4}U\sigma_{ss}\right] + \frac{|\mathbf{x}|^2\hat{x}_L}{2c^2(2\ell+3)}\partial_t^2\sigma \\ & - \frac{4(2\ell+1)\hat{x}_{iL}}{c^2(\ell+1)(2\ell+3)}\partial_t\left[\left(1+\frac{2U}{c^2}\right)\sigma_i - \frac{2U_i}{c^2}\sigma + \frac{1}{\pi G c^2}\left(\partial_j U \partial_i U_j - \frac{3}{4}\partial_i U \partial_j U_j\right)\right] \\ & + \frac{|\mathbf{x}|^4\hat{x}_L}{8c^4(2\ell+3)(2\ell+5)}\partial_t^4\sigma - \frac{2(2\ell+1)|\mathbf{x}|^2\hat{x}_{iL}}{c^4(\ell+1)(2\ell+3)(2\ell+5)}\partial_t^3\sigma_i + \frac{2(2\ell+1)\hat{x}_{ijL}}{c^4(\ell+1)(\ell+2)(2\ell+5)}\partial_t^2\left[\sigma_{ij} + \frac{1}{4\pi G}\partial_i U \partial_j U\right] \\ & + \frac{\hat{x}_L}{\pi G c^4}\left[2U_i\partial_{ij}U_j - U_{ij}\partial_{ij}U - \frac{1}{2}(\partial_i U_i)^2 + 2\partial_i U_j \partial_j U_i - \frac{1}{2}\partial_t^2(U^2) + W_{ij}\partial_{ij}U\right]\Bigg\} \ . \end{aligned} \tag{2.17}$$

For simplicity, we henceforth drop most of the post-Newtonian error terms as they are usually evident from the context.

The symbol $\mathrm{FP}_{B=0}$ in Eq. (2.17) stands for "finite part at $B = 0$" and denotes a mathematically well-defined operation of analytic continuation. Let us recall its precise meaning (see [20,21] for details and proofs of the applicability of such a definition): one considers separately two functions of one complex variable B defined by the integrals $I_1(B) \equiv \int_{V_1} d^3\mathbf{x}|\mathbf{x}|^B f(\mathbf{x})$, $I_2(B) \equiv \int_{V_2} d^3\mathbf{x}|\mathbf{x}|^B f(\mathbf{x})$, where V_1 is, say, the ball $0 \leq |\mathbf{x}| \leq r_0$ and V_2 the complementary domain: $|\mathbf{x}| > r_0$. If the function $f(\mathbf{x})$ is, say, continuous in $\mathbb{R}^3$ and has, at most, a polynomial growth $O(|\mathbf{x}|^p)$ when $|\mathbf{x}| \to \infty$, the integral defining $I_1(B)$ is convergent if the real part of B is large enough, say $\mathrm{Re}(B) > -3$, while the integral defining $I_2(B)$ is convergent if $\mathrm{Re}(B) < -p-3$. If moreover the function $f(\mathbf{x})$ admits an asymptotic expansion ("multipolar expansion") of the form $f(\mathbf{x}) \sim \Sigma_{k\leq p} f_{k,L}|\mathbf{x}|^k \hat{n}^L$ as $|\mathbf{x}| \to \infty$ (we do not include logarithms of $|\mathbf{x}|$ to simplify), the function $I_2(B)$ can be analytically continued as a meromorphic function in the complex B plane up to arbitrarily large values of $\mathrm{Re}(B)$ (with possible poles on the real axis). Finally, one considers the sum of $I_1(B)$ and of the analytic continuation of $I_2(B)$ for values of B near 0: the constant term (zeroth power of B) in the Laurent expansion of $I_1(B)$+analytic continuation$[I_2(B)]$ around $B = 0$ defines $\mathrm{FP}_{B=0}\int d^3\mathbf{x}|\mathbf{x}|^B f(\mathbf{x})$. It is easily shown that this definition is independent of the choice of the intermediate radius r_0 used to split $\mathbb{R}^3$ in two regions. Note that, in principle, one should introduce a length scale to a-dimensionalize $|\mathbf{x}|$ before taking its Bth power in (2.17). This is superfluous in our case because the author of [10] has shown that there arise no poles at $B = 0$ and therefore no associated logarithms.

Although the result (2.17) is mathematically well defined (contrary to the results of Refs. [22,23] which are expressed in terms of undefined, divergent integrals), it is a nontrivial task to compute it explicitly in terms of the source variables only. This will be done in the next section, in the case where the source is a binary system of nonrotating compact objects (neutron stars or black holes). To this end, we shall heuristically represent the stress-energy tensor of the material source as a sum of Dirac δ functions. More generally, the stress-energy tensor of a system of N (nonrotating) compact bodies is formally given by

$$T^{\mu\nu}(\mathbf{x},t) = \sum_{A=1}^{N} m_A \frac{dy_A^\mu}{dt}\frac{dy_A^\nu}{dt}\frac{1}{\sqrt{-g}}\frac{dt}{d\tau}\delta(\mathbf{x}-\mathbf{y}_A(t)) \ , \tag{2.18}$$

where m_A denotes the (constant) Schwarzschild mass of the Ath compact body. This yields, for the source variables (2.10),

$$\sigma(\mathbf{x},t) = \sum_{A=1}^{N} \mu_A(t) \left(1 + \frac{\mathbf{v}_A^2}{c^2}\right) \delta(\mathbf{x} - \mathbf{y}_A(t)) \ , \quad (2.19a)$$

$$\sigma_i(\mathbf{x},t) = \sum_{A=1}^{N} \mu_A(t) v_A^i \delta(\mathbf{x} - \mathbf{y}_A(t)) \ , \quad (2.19b)$$

$$\sigma_{ij}(\mathbf{x},t) = \sum_{A=1}^{N} \mu_A(t) v_A^i v_A^j \delta(\mathbf{x} - \mathbf{y}_A(t)) \ , \quad (2.19c)$$

where $v_A^i \equiv dy_A^i/dt$ and

$$\mu_A(t) = m_A[1 + (d_2)_A + (d_4)_A] \ , \quad (2.20a)$$

$$d_2 \equiv \frac{1}{c^2}\left(\frac{\mathbf{v}^2}{2} - V\right) \ , \quad (2.20b)$$

$$d_4 \equiv \frac{1}{c^4}\left(\frac{3}{8}\mathbf{v}^4 + \frac{3}{2}U\mathbf{v}^2 - 4U_i v_i - 2\Phi + \frac{3}{2}U^2 + 4U_{ss}\right) \ , \quad (2.20c)$$

the notation V being a shorthand for the combination

$$V \equiv U + \frac{1}{2c^2}\partial_t^2 X \ , \quad (2.21)$$

which is the potential appearing naturally in the 1PN near-zone metric in harmonic coordinates. The subscript A appearing in Eq. (2.20a) indicates that one must replace the field point $\mathbf{x}$ by the position $\mathbf{y}_A$ of the Ath mass point, while discarding all the ill-defined (formally infinite) terms arising in the limit $\mathbf{x} \to \mathbf{y}_A$. For instance,

$$(U)_A = G \sum_{B \neq A} \frac{\mu_B(t)(1 + \mathbf{v}_B^2/c^2)}{|\mathbf{y}_A - \mathbf{y}_B|} \ . \quad (2.22)$$

[Note that the second time derivative appearing in V, Eq. (2.21), must be explicated before making the replacement $\mathbf{x} \to \mathbf{y}_A(t)$.] Although we do not claim to have verified it by a detailed proof, we feel secure that the formal use of δ functions can be justified at the 2PN accuracy (and even at the 2.5PN accuracy) by combining our generation formalism with the results of Ref. [24]. Indeed, the latter reference showed (by a matching technique) that the metric generated by a system of well-separated strongly self-gravitating bodies was equal, up to the 3PN level, to the metric generated by a mathematically well-defined version of δ functions. See Sec. III C of Ref. [9] for a discussion, at the 1PN level, of how to combine the two formalisms.

III. THE 2PN-ACCURATE MASS MOMENTS OF A COMPACT BINARY

It is convenient to split the starting formula (2.17) into three types of contributions, say

$$I_L = I_L^{(C)} + I_L^{(Y)} + I_L^{(W)} \ . \quad (3.1)$$

Here, $I_L^{(C)}$ ("compact terms") denotes the terms where, because of the explicit presence of a source term $\sigma(\mathbf{x})$, $\sigma_i(\mathbf{x})$, or $\sigma_{ij}(\mathbf{x})$ (or a time derivative thereof), the three-dimensional integral $\int d^3\mathbf{x}$ extends only over the compact support of the material source. The finite part prescription is unnecessary for such terms. [Note that $I_L^{(C)}$ is identical to the 2PN expansion of the exact linearized gravity result given by Eq. (5.33) of [25].] The "Y terms" $I_L^{(Y)}$ [named after the quantity defined in Eq. (3.22) below] denote all the contributions involving the three-dimensional integral of the product of (spatial derivatives of) two Newtonian-like potentials, e.g.,

$$\mathrm{FP}_{B=0} \int d^3\mathbf{x} |\mathbf{x}|^B \hat{x}_{ijL} \partial_t^2 [\partial_i U \partial_j U] = \frac{d^2}{dt^2}\left\{\mathrm{FP}_{B=0} \int d^3\mathbf{x} |\mathbf{x}|^B \hat{x}_{ijL} \partial_i U \partial_j U\right\} \ . \quad (3.2)$$

The time derivatives appearing in the Y terms have all been written in a manner such that they can be factorized in front as total time derivatives acting on (the finite part of) a three-dimensional integral. Finally, $I_L^{(W)}$ denotes the only term of (2.17) involving the three-dimensional integral of a term trilinear in source variables; namely,

$$I_L^{(W)} = \frac{1}{\pi G c^4} \mathrm{FP}_{B=0} \int d^3\mathbf{x} |\mathbf{x}|^B \hat{x}_L W_{ij} \partial_{ij} U \ , \quad (3.3)$$

where we recall that W_{ij}, defined by (2.14c), is a bilinear functional of $\sigma(\mathbf{x}',t)$.

We shall consider in turn the three contributions to I_L. The "compact" and "Y" contributions will be evaluated in the quadrupole case ($\ell = 2$) while the "W" contribution will be calculated for any ℓ. The cases $\ell = 0$ and $\ell = 1$ play a special role as they do not correspond to radiative moments, but to conserved quantities. We check in Appendix A the agreement with known results for these low moments.

A. The compact terms and their explicit form in the circular two-body case

The general, N-extended-body expression for the "compact" contributions to the 2PN mass moments reads

$$I_L^{(C)} = \int d^3\mathbf{x} \Bigg(\hat{x}_L \left[\sigma \left(1 - \frac{4U_{ss}}{c^4}\right) + \frac{4U}{c^4}\sigma_{ss} + \frac{1}{2(2\ell+3)}\left(\frac{\mathbf{x}^2}{c^2}\partial_t^2 \sigma + \frac{1}{4(2\ell+5)}\frac{\mathbf{x}^4}{c^4}\partial_t^4 \sigma\right)\right]$$
$$- \frac{4(2\ell+1)}{(\ell+1)(2\ell+3)c^2}\hat{x}_{iL}\left\{\partial_t\left[\sigma_i\left(1 + \frac{2U}{c^2}\right) - \frac{2U_i}{c^2}\sigma\right] + \frac{1}{2(2\ell+5)}\frac{\mathbf{x}^2}{c^2}\partial_t^3 \sigma_i\right\}$$
$$+ \frac{2(2\ell+1)}{(\ell+1)(\ell+2)(2\ell+5)c^4}\hat{x}_{ijL}\partial_t^2 \sigma_{ij}\Bigg) \ . \quad (3.4)$$

From Eqs. (2.19) we obtain the corresponding point-mass form

$$I_L^{(C)} = \sum_{A=1}^{N} \Bigg(\tilde{\mu}_A \left[1 - \frac{4}{c^4} U_{ss}^A + \frac{4}{c^4} U^A (\mathbf{v}_A)^2\right] \hat{y}_A^L + \frac{1}{2(2\ell+3)c^2} \frac{d^2}{dt^2} (\tilde{\mu}_A \mathbf{y}_A^2 \hat{y}_A^L) + \frac{1}{8(2\ell+3)(2\ell+5)c^4} \frac{d^4}{dt^4} [\tilde{\mu}_A (\mathbf{y}_A^2)^2 \hat{y}_A^L]$$

$$- \frac{4(2\ell+1)}{(\ell+1)(2\ell+3)c^2} \frac{d}{dt} \left\{ \left[\mu_A \left(1 + \frac{2U^A}{c^2}\right) v_A^i - \frac{2U_i^A}{c^2} \tilde{\mu}_A \right] \hat{y}_A^{iL} \right\} - \frac{2(2\ell+1)}{(\ell+1)(2\ell+3)(2\ell+5)c^4} \frac{d^3}{dt^3} (\mu_A v_A^i \mathbf{y}_A^2 \hat{y}_A^{iL})$$

$$+ \frac{2(2\ell+1)}{(\ell+1)(\ell+2)(2\ell+5)c^4} \frac{d^2}{dt^2} (\mu_A v_A^i v_A^j \hat{y}_A^{ijL}) \Bigg), \tag{3.5}$$

in which we have introduced for brevity $\tilde{\mu}_A \equiv \mu_A(1 + \mathbf{v}_A^2/c^2)$. As it is written, the result (3.5) depends (at 2PN order) not only on the positions $\mathbf{y}_A$ and velocities $\mathbf{v}_A$ of the N compact bodies, but also on higher time derivatives thereof, up to $d^4\mathbf{y}_A/dt^4$. To reduce the functional dependence of $I_L^{(C)}$ to a dependence on positions and velocities only, we need to use the post-Newtonian-expanded equations of motion of the N-body system. At this juncture, we restrict ourselves to the simplest case of a binary system evolving on a quasicircular orbit. Consistently with the 2PN accuracy of the generation formalism considered here, we use the 2PN truncation of the 2.5PN equations of motion of Ref. [11]. We use a (harmonic) coordinate system in which the 2PN center-of-mass is at rest, at the origin. In such a coordinate system the 2PN truncation of the equations of motion admit exact circular periodic orbits. Using the 2PN-accurate center-of-mass theorem of Ref. [26], we can express the individual center-of-mass-frame positions of the two bodies in circular orbits in terms of the relative position

$$\mathbf{x} \equiv \mathbf{y}_1 - \mathbf{y}_2 \tag{3.6}$$

as

$$\mathbf{y}_1 = [X_2 + 3\nu\gamma^2(X_1 - X_2)]\mathbf{x} , \tag{3.7a}$$

$$\mathbf{y}_2 = [-X_1 + 3\nu\gamma^2(X_1 - X_2)]\mathbf{x} . \tag{3.7b}$$

[Equations (3.7) are obtained by setting $G_i = 0$ for circular orbits, where G_i is given by Eq. (19) in Ref. [26].] Here, we have denoted

$$m \equiv m_1 + m_2 , \quad X_1 \equiv \frac{m_1}{m} , \quad X_2 \equiv \frac{m_2}{m} \equiv 1 - X_1 , \tag{3.8}$$

and

$$\nu \equiv X_1 X_2 \equiv \frac{m_1 m_2}{m^2} , \tag{3.9}$$

$$\gamma \equiv \frac{Gm}{c^2 r} , \tag{3.10}$$

with $r \equiv |\mathbf{x}| \equiv |\mathbf{y}_1 - \mathbf{y}_2|$ denoting the constant (harmonic) coordinate radius of the relative orbit. One should note in Eq. (3.7) the absence of 1PN corrections to the usual center-of-mass expressions. This is an accident due to the fact that we are restricting our attention to circular orbits. (In the noncircular case there are 1PN corrections to X_2 and $-X_1$ which are proportional to $\mathbf{v}^2 - Gm/r$, see, e.g., [27].) Then the content of the 2PN equations of motion reduces to the knowledge of the 2PN-accurate orbital frequency $\omega_{2\mathrm{PN}}$ given by

$$\omega_{2\mathrm{PN}}^2 \equiv \frac{Gm}{r^3} \left[1 - (3-\nu)\gamma + \left(6 + \frac{41}{4}\nu + \nu^2\right)\gamma^2\right] , \tag{3.11}$$

which is such that

$$\mathbf{v} \equiv \frac{d\mathbf{x}}{dt} , \tag{3.12a}$$

$$\mathbf{a} \equiv \frac{d\mathbf{v}}{dt} \equiv \frac{d^2\mathbf{x}}{dt^2} = -\omega_{2\mathrm{PN}}^2 \mathbf{x} + O(\varepsilon^5) . \tag{3.12b}$$

Let us note that Eqs. (3.12) imply, as usual, that $v \equiv |\mathbf{v}| = \omega_{2\mathrm{PN}}\, r$, so that Eq. (3.11) implies

$$\frac{v^2}{c^2} = \gamma \left[1 - (3-\nu)\gamma + \left(6 + \frac{41}{4}\nu + \nu^2\right)\gamma^2\right] . \tag{3.13}$$

We are now in a position to compute explicitly the "compact" terms $I_L^{(C)}$, and we restrict ourselves to the quadrupole case $\ell = 2$.

Without entering into the details of the calculation of $I_{ij}^{(C)}$, let us mention that there arise many symmetric functions of the two masses which can be straightforwardly expressed in terms of the total mass m and of the quantity $\nu \equiv X_1 X_2$ by using the well-known fact that a symmetric polynomial in X_1 and X_2 can be written in terms of the elementary symmetric combinations $X_1 + X_2 (\equiv 1)$ and $X_1 X_2$. Useful formulas for this reduction are

$$X_1^2 + X_2^2 = 1 - 2\nu , \tag{3.14a}$$

$$X_1^3 + X_2^3 = 1 - 3\nu , \tag{3.14b}$$

$$X_1^4 + X_2^4 = 1 - 4\nu + 2\nu^2 , \tag{3.14c}$$

$$X_1^5 + X_2^5 = 1 - 5\nu + 5\nu^2 . \tag{3.14d}$$

The computation of the quadrupole $I_{ij}^{(C)}$ for two bodies and circular orbits is long and tedious but quite straightforward. We obtain

$$I_{ij}^{(C)} = \mathrm{STF}_{ij}\,\nu m\left[x^{ij} - \frac{\gamma}{42}x^{ij}(1+39\nu) + \frac{11}{21}\frac{r^2}{c^2}v^{ij}(1-3\nu) + \frac{\gamma^2}{1512}x^{ij}(5203-18275\nu-2785\nu^2)\right.$$
$$\left. + \frac{\gamma}{378}\frac{r^2}{c^2}v^{ij}(191-577\nu+109\nu^2)\right] , \tag{3.15}$$

where STF_{ij} denotes the STF projection with respect to the indices ij.

B. The quadratically nonlinear terms

Separating out from Eq. (2.17) the terms involving the three-dimensional integral of a product of (spatial derivatives of) two Newtonian potentials, we define

$$I_L^{(Y)} \equiv \mathrm{FP}_{B=0}\int d^3\mathbf{x}|\mathbf{x}|^B\left\{\frac{\hat{x}_L}{\pi G c^4}\left[2U_i\partial_{ij}U_j - U_{ij}\partial_{ij}U - \frac{1}{2}\partial_i U_i \partial_j U_j + 2\partial_i U_j \partial_j U_i - \frac{1}{2}\partial_t^2(U^2)\right]\right.$$
$$\left. - \frac{4(2\ell+1)\hat{x}_{iL}}{(\ell+1)(2\ell+3)\pi G c^4}\partial_t\left(\partial_j U \partial_i U_j - \frac{3}{4}\partial_i U \partial_j U_j\right) + \frac{(2\ell+1)\hat{x}_{ijL}}{(\ell+1)(\ell+2)(2\ell+5)2\pi G c^4}\partial_t^2(\partial_i U \partial_j U)\right\}. \tag{3.16}$$

These terms can be deduced from the results of Ref. [8]. Denoting $r_1 \equiv |\mathbf{x}-\mathbf{y}_1|$, $r_2 \equiv |\mathbf{x}-\mathbf{y}_2|$, $r_{12} \equiv |\mathbf{y}_1-\mathbf{y}_2|$, the latter reference has introduced the kernel

$$k(\mathbf{x};\mathbf{y}_1,\mathbf{y}_2) \equiv \frac{1}{2}\ln[(r_1+r_2)^2 - r_{12}^2] \tag{3.17}$$

and proven that it satisfies (in the sense of distribution theory) the identity

$$\frac{1}{r_1 r_2} \equiv \Delta_x k - 2\pi\delta_{12} , \tag{3.18}$$

where δ_{12} denotes a distribution supported on the segment joining $\mathbf{y}_1$ to $\mathbf{y}_2$:

$$\delta_{12}(\mathbf{x};\mathbf{y}_1,\mathbf{y}_2) \equiv r_{12}\int_0^1 d\alpha\,\delta(\mathbf{x}-\mathbf{y}_\alpha) , \tag{3.19a}$$

where

$$\mathbf{y}_\alpha \equiv (1-\alpha)\mathbf{y}_1 + \alpha\mathbf{y}_2 . \tag{3.19b}$$

Moreover, the kernel $k(\mathbf{x})$ is such that its multipolar expansion when $|\mathbf{x}| \to \infty$ contains, besides a logarithmic term, $\ln(2|\mathbf{x}|)$, only terms of the type $|\mathbf{x}|^{-\ell-2p}\hat{n}_L$ (with $p \in \mathbb{N}$). The important point, as we are going to see, is that the multipolar expansion of $k(\mathbf{x})$ contains no terms of the type $|\mathbf{x}|^{-\ell-1}\hat{n}_L$, i.e., no homogeneous solutions of the Laplace equation. [This is the feature defining the kernel k by contrast with the kernel g satisfying $\Delta_x g = (r_1 r_2)^{-1}$ everywhere and containing a homogeneous piece $\frac{1}{2}h$ whose distributional source is precisely the $-2\pi\delta_{12}$ term in Eq. (3.18); see [8].] The latter property implies that

$$\mathrm{FP}_{B=0}\int d^3\mathbf{x}|\mathbf{x}|^B\hat{x}_L\Delta_x k(\mathbf{x}) = \mathrm{FP}_{B=0}\int d^3\mathbf{x}\,\Delta_x(|\mathbf{x}|^B\hat{x}_L)k(\mathbf{x})$$
$$= \mathrm{FP}_{B=0}\left\{B(B+2\ell+1)\int d^3\mathbf{x}|\mathbf{x}|^{B+\ell-2}\hat{n}_L k(\mathbf{x})\right\} = 0. \tag{3.20}$$

Here, the first equality is obtained by integrating by parts [the surface term at infinity vanishing by analytic continuation from the case where $\mathrm{Re}(B)$ is large and negative], and the last follows from the fact that $k(\mathbf{x})$ is well behaved at the origin $|\mathbf{x}| = 0$ and contains no "homogeneous" terms $|\mathbf{x}|^{-\ell-1}\hat{n}_L$ in its multipolar expansion at infinity. Indeed, going back to the definition recalled above of analytically continued integrals, as the sum of one integral over the ball $0 \le |\mathbf{x}| \le r_0$ and one over its complement $|\mathbf{x}| > r_0$, we see that the factor B in front will give a zero result, except if the integral near the origin ($0 \le |\mathbf{x}| \le \varepsilon$) or the one near infinity ($|\mathbf{x}| > r_0$, with r_0 arbitrarily large) generates a pole at $B = 0$ in the complex B plane. As $k(\mathbf{x})$ exhibits no powerlike blow up near the origin $\mathbf{x} = 0$ (even when the latter coincides with $\mathbf{y}_1$ or $\mathbf{y}_2$) the integral near the origin is easily seen not to generate any pole at $B = 0$. Concerning the integral near infinity, written as $\int_{r_0}^\infty dr\, r^{B+\ell}(\int_{S_2} d\Omega\,\hat{n}_L k)$, we see from the orthogonality of the $\hat{n}_L$'s over the sphere S_2 that only the presence of terms $\propto \hat{n}_L/r^{\ell+1}$ in the multipolar expansion of $k(\mathbf{x})$ could generate a pole through $\int_{r_0}^\infty dr\, r^{B+\ell}r^{-\ell-1} = -B^{-1}r_0^B$.

By combining the identity (3.18) with the result (3.20) we conclude that

$$\mathrm{FP}_{B=0}\int d^3\mathbf{x}|\mathbf{x}|^B\hat{x}_L\frac{1}{r_1r_2}=-2\pi Y^L(\mathbf{y}_1,\mathbf{y}_2) \tag{3.21}$$

(see also Sec. IV in Ref. [10]) in which, following the notation of [8], we have introduced

$$Y^L(\mathbf{y}_1,\mathbf{y}_2)\equiv\int d^3\mathbf{x}\,\hat{x}_L\delta_{12}(\mathbf{x};\mathbf{y}_1,\mathbf{y}_2)\equiv r_{12}\int_0^1 d\alpha\, y_\alpha^{\langle L\rangle}\ , \tag{3.22}$$

where $y_\alpha^{\langle L\rangle}$ denotes the STF projection of $y_\alpha^{i_1}\cdots y_\alpha^{i_\ell}$ with y_α^i defined by Eq. (3.19b). As the dependence of $y_\alpha^{\langle L\rangle}$ on α is polynomial it is easy to perform the integration over α in (3.22) to get [10]

$$Y^L(\mathbf{y}_1,\mathbf{y}_2)=\frac{|\mathbf{y}_1-\mathbf{y}_2|}{\ell+1}\sum_{p=0}^{\ell}y_1^{\langle L-P}y_2^{P\rangle}\ , \tag{3.23}$$

where $y_2^P=y_2^{i_1}\cdots y_2^{i_p}$, $y_1^{L-P}=y_1^{i_{p+1}}\cdots y_1^{i_\ell}$. By taking derivatives of both sides of (3.21) with respect to y_1^i or y_2^i and integrating over $\mathbf{y}_1$ and $\mathbf{y}_2$ after having weighted the integrand with some source functions $\sigma_\alpha(\mathbf{y}_1)$, $\sigma_\beta(\mathbf{y}_2)$ (where σ_α denote σ, σ_i or σ_{ij}), one can obtain all the terms in (3.16) since these are all bilinear in some Newtonian-like potentials. For instance, the right-hand side of Eq. (3.2) can be written as

$$\frac{d^2}{dt^2}\left\{-2\pi G^2\int d^3\,\mathbf{y}_1\sigma(\mathbf{y}_1,t)\int d^3\mathbf{y}_2\sigma(\mathbf{y}_2,t)\left(-\frac{\partial}{\partial y_1^i}\right)\left(-\frac{\partial}{\partial y_2^j}\right)Y^{ijL}(\mathbf{y}_1,\mathbf{y}_2)\right\}\ , \tag{3.24}$$

where one must be careful about the minus signs appearing in the spatial derivatives due to $\partial r_1^{-1}/\partial x^i=-\partial r_1^{-1}/\partial y_1^i$, $\partial r_2^{-1}/\partial x^i=-\partial r_2^{-1}/\partial y_2^i$, and about keeping the total time derivatives factorized in front of the whole expression. It is convenient to introduce a special notation for the derivatives of Y^L with respect to $\mathbf{y}_1$ and $\mathbf{y}_2$, say

$$_{ij}Y^L\equiv\frac{\partial}{\partial y_1^i}\frac{\partial}{\partial y_1^j}Y^L\ , \tag{3.25a}$$

$$_iY_j^L\equiv\frac{\partial}{\partial y_1^i}\frac{\partial}{\partial y_2^j}Y^L\ , \tag{3.25b}$$

$$Y_{ij}^L\equiv\frac{\partial}{\partial y_2^i}\frac{\partial}{\partial y_2^j}Y^L\ . \tag{3.25c}$$

With this notation in hand, it is easy, from the result (3.21), to obtain the following expression for the "Y-type" contribution to I_L defined in Eq. (3.16):

$$\begin{aligned}I_L^{(Y)}=&-\frac{2G}{c^4}\iint d^3\mathbf{y}_1d^3\mathbf{y}_2\Big\{2\sigma_1^s\sigma_2^kY_{sk}^L-\sigma_1^{sk}\sigma_2Y_{sk}^L-\frac{1}{2}\sigma_1^s\sigma_2^k\ {}_sY_k^L+2\sigma_1^s\sigma_2^k\ {}_kY_s^L-\frac{1}{2}\partial_t^2(\sigma_1\sigma_2Y^L)\\&-\frac{4(2\ell+1)}{(\ell+1)(2\ell+3)}\partial_t\left[\sigma_1\sigma_2^k\ {}_kY_a^{aL}-\frac{3}{4}\sigma_1^s\sigma_2\ {}_sY_a^{aL}\right]+\frac{2\ell+1}{2(\ell+1)(\ell+2)(2\ell+5)}\partial_t^2\left(\sigma_1\sigma_2\ {}_aY_b^{abL}\right)\Big\}\ ,\end{aligned} \tag{3.26}$$

where $\sigma_1\equiv\sigma(\mathbf{y}_1,t)$, $\sigma_2\equiv\sigma(\mathbf{y}_2,t)$, etc.

Finally, the point-mass limit is obtained by inserting Eqs. (2.19) into Eq. (3.26). Note that, because of the overall c^{-4} factor, it is enough to use the Newtonian approximation for the source terms [e.g., $\sigma(\mathbf{x},t)=\Sigma_Am_A\delta(\mathbf{x}-\mathbf{y}_A(t))+O(\varepsilon^2)$]. To increase the readability of the result, it is convenient to introduce a shorthand for the contractions of the derivatives (3.25) with the velocities:

$$Y_{v_Av_C}^L\equiv v_A^iv_C^j\ Y_{ij}^L\ , \tag{3.27a}$$

$$_{v_Av_C}Y^L\equiv v_A^iv_C^j\ {}_{ij}Y^L\ , \tag{3.27b}$$

$$_{v_A}Y_{v_C}^L\equiv v_A^iv_C^j\ {}_iY_j^L\ , \tag{3.27c}$$

where $A,C=1,2$, as well as a mixed notation such as, e.g.,

$$_{v_A}Y_j^L\equiv v_A^i\ {}_iY_j^L\ . \tag{3.27d}$$

This leads, for any ℓ, to the following expression for the Y terms:

$$\begin{aligned}I_L^{(Y)}=&-\frac{2G}{c^4}\sum_{A,C}m_Am_C\Big\{2Y_{v_Av_C}^L-Y_{v_Av_A}^L\\&-\frac{1}{2}\ {}_{v_A}Y_{v_C}^L+2\ {}_{v_C}Y_{v_A}^L-\frac{1}{2}\partial_t^2\left(Y^L\right)\\&-\frac{4(2\ell+1)}{(\ell+1)(2\ell+3)}\partial_t\left[{}_{v_C}Y_s^{sL}-\frac{3}{4}{}_{v_A}Y_s^{sL}\right]\\&+\frac{2\ell+1}{2(\ell+1)(\ell+2)(2\ell+5)}\partial_t^2\left[{}_aY_b^{abL}\right]\Big\}\ ,\end{aligned} \tag{3.28}$$

in which all functions Y^L are evaluated with $\mathbf{y}_A$ in their first argument, and $\mathbf{y}_C$ in their second one, e.g., ${}_{v_A}Y_{v_C}^L=v_A^iv_C^j\ {}_iY_j^L(\mathbf{y}_A,\mathbf{y}_C)$, $Y_{v_Av_A}^L=v_A^iv_A^jY_{ij}^L(\mathbf{y}_A,\mathbf{y}_C)$, and in which all the self-terms $A=C$ must be omitted. Finally, by using expression (3.23) for Y^L, a long calculation yields the following explicit expression for the Y-type mass quadrupole for two bodies in a circular orbit, and considered in the center-of-mass frame:

$$I_{ij}^{(Y)} = -\frac{2m\nu\gamma}{63}\,\mathrm{STF}_{ij}\left[\gamma x^{ij}(55-155\nu-53\nu^2) + \frac{r^2}{c^2}v^{ij}(-118+92\nu-10\nu^2)\right] . \quad (3.29)$$

C. The cubically nonlinear term

Let us now tackle the cubically nonlinear term

$$I_L^{(W)} = \frac{1}{\pi G c^4}\mathrm{FP}_{B=0}\int d^3\mathbf{x}|\mathbf{x}|^B \hat{x}^L W_{ij}\partial_{ij}U , \quad (3.30)$$

with $W_{ij} = \Delta^{-1}(\partial_i U \partial_j U)$. We shall show how to evaluate $I_L^{(W)}$ explicitly, for all values of ℓ, in the case where the source is a binary system. [Note in passing that as (3.30) depends only on the instantaneous mass distribution of the source our result is valid whatever be the orbit (circular or not) of the binary.] To do this for all values of ℓ we need what is essentially a generalization of the method of [8], i.e., a detailed study of some cubically nonlinear kernel. In the particular (and most urgently needed) case of the mass quadrupole, $\ell = 2$, one can evaluate (3.30) by other means, as is shown in Appendix B which succeeds in computing $I_{ij}^{(W)}$ for N-body systems. This gives us an independent check of the results below.

Let us define a function $W(\mathbf{x},t)$ [which is a trilinear, nonlocal functional of $\sigma(\mathbf{x}',t)$] by

$$\Delta W = -\frac{4}{c^4}W_{ij}\partial_{ij}U , \quad (3.31a)$$

$$W(\mathbf{x}) = +\frac{1}{\pi c^4}\int \frac{d^3\mathbf{x}'}{|\mathbf{x}-\mathbf{x}'|}W_{ij}(\mathbf{x}')\partial_{ij}U(\mathbf{x}') , \quad (3.31b)$$

where, for brevity, we suppress the dependence on the time variable which is the same on both sides. Note that the integral (3.31b) is a usual, convergent integral [at infinity $W_{ij} = O(1/r)$, $\partial_{ij}U = O(1/r^3)$]. As usual, W is characterized as being the unique solution (in the sense of distribution theory) of (3.31a) which falls off at spatial infinity.

Our method for computing $I_L^{(W)}$ is similar to the one we used in Eqs. (3.17)–(3.20) above to compute $I_L^{(Y)}$ from the results of [8] on the kernels k and g. Inserting (3.31a) into (3.30) gives

$$I_L^{(W)} = -\frac{1}{4\pi G}\mathrm{FP}_{B=0}\int d^3\mathbf{x}|\mathbf{x}|^B\hat{x}^L\Delta_x W(\mathbf{x}) . \quad (3.32)$$

Integrating by parts the two spatial derivatives of Δ_x [the analytic continuation from a case where $\mathrm{Re}(B)$ is large and negative ensuring the vanishing of any surface term at infinity], using the formula $\Delta_x(|\mathbf{x}|^B\hat{x}^L) = B(B+2\ell+1)r^{B+\ell-2}\hat{n}^L$, where $r \equiv |\mathbf{x}|$, $n^i \equiv x^i/r$, and writing explicitly the definition of the analytically continued integral (in polar coordinates: $d^3\mathbf{x} = r^2 dr d\Omega$) we get

$$I_L^{(W)} = -\frac{1}{4\pi G}\mathrm{FP}_{B=0}\left\{B(B+2\ell+1)\left[\int_0^{r_0}dr\int d\Omega\, r^{B+\ell}\hat{n}^L W + \int_{r_0}^{\infty}dr\int d\Omega\, r^{B+\ell}\hat{n}^L W\right]\right\} . \quad (3.33)$$

Because of the regularity of $W(\mathbf{x})$ near the origin $\mathbf{x} = 0$ [28], the first integral on the right-hand side of (3.33) will continuously depend on $B \in \mathbb{C}$ near $B = 0$, and will therefore not contribute to $I_L^{(W)}$ because of the explicit B factor in front. We are therefore left with (since the second integral can have at most a simple pole)

$$I_L^{(W)} = -\frac{2\ell+1}{4\pi G}\,\mathrm{FP}_{B=0}\left\{B\int_{r_0}^{\infty}dr\, r^{B+\ell}\left[\int d\Omega\,\hat{n}^L W(\mathbf{x})\right]\right\} , \quad (3.34)$$

where r_0 can be taken arbitrarily large, so that expression (3.34) depends only on the asymptotic expansion of $W(\mathbf{x})$ for $|\mathbf{x}| \to \infty$, say

$$W(\mathbf{x}) = \sum_{p\geq 1,\,\ell\geq 0} W_L^p\,\frac{\hat{n}^L}{r^p} . \quad (3.35)$$

Although this is not *a priori* evident from its definition (3.31), the explicit expression we shall derive below for $W(\mathbf{x})$ shows that the multipolar expansion (3.35) proceeds according to the inverse powers of r, without involving logarithms. [Actually, the presence of logarithms in (3.35) would not change the result below.] As was already mentioned above, the only terms in the multipolar expansion (3.35) which can generate a simple pole in B so as to cancel the factor B in front are those which correspond to a *homogeneous* solution of the Laplace equation $\propto \hat{n}^L/r^{\ell+1}$. Let us then *define* the "homogeneous" piece of W by restricting the double sum in (3.35) to the case where $p = \ell+1$, say (with the definition $W_L^{\mathrm{hom}} \equiv W_L^{\ell+1}$)

$$W^{\mathrm{hom}}(\mathbf{x}) \equiv \sum_{\ell\geq 0} W_L^{\mathrm{hom}}\frac{\hat{n}^L}{r^{\ell+1}} . \quad (3.36)$$

Note that this definition is independent of the choice of the origin $\mathbf{x} = 0$ around which one is expanding. This stems from the easily verified fact that under a constant shift $\mathbf{x} \to \mathbf{x}+\mathbf{c}$ the homogeneous and inhomogeneous pieces of W do not mix. Using $\int_{r_0}^{\infty} dr r^{B-1} = -B^{-1}r_0^B = -B^{-1} + O(B^0)$ and $\int d\Omega\,\hat{n}_L\hat{n}_{L'}\hat{T}_{L'} = [4\pi\ell!/(2\ell+1)!!]\hat{T}_L$ [where $n!! \equiv n(n-2)(n-4)\cdots$ (1 or 2)] we get the following link between $I_L^{(W)}$ and the coefficients appearing in the multipolar expansion (3.36) of $W^{\mathrm{hom}}(\mathbf{x})$:

$$G I_L^{(W)} = \frac{\ell!}{(2\ell-1)!!}W_L^{\mathrm{hom}} . \quad (3.37)$$

Inserting back (3.37) into (3.36) shows that the multipolar expansion of $W^{\rm hom}$ can be compactly written in a form which is familiar:

$$W^{\rm hom} = G \sum_{\ell \geq 0} \frac{(-)^\ell}{\ell!} I_L^{(W)} \partial_L \frac{1}{r} \,. \tag{3.38}$$

The form (3.38) is related to another, more convenient, expression for $I_L^{(W)}$ (in terms of the "source" of $W^{\rm hom}$) that we shall derive below. Let us now obtain an explicit expression for $W(\mathbf{x})$ in the point-mass limit and extract its homogeneous part.

Inserting $U = Gm_1/r_1 + Gm_2/r_2$ in the definition of W_{ij} yields [with $n_1^i \equiv (x^i - y_1^i)/r_1$, etc.]

$$\triangle W_{ij} = G^2 \left[m_1^2 \frac{n_1^i n_1^j}{r_1^4} + m_2^2 \frac{n_2^i n_2^j}{r_2^4} + m_1 m_2 \left(\partial_{y_1^i} \partial_{y_2^j} + \partial_{y_1^j} \partial_{y_2^i} \right) \frac{1}{r_1 r_2} \right] , \tag{3.39}$$

whose solution can be written as

$$W_{ij} = G^2 [m_1^2 W_{ij}^{11} + m_2^2 W_{ij}^{22} + m_1 m_2 W_{ij}^{12}] \,, \tag{3.40}$$

with

$$W_{ij}^{11} = \frac{1}{8} \partial_{ij} \ln r_1 + \frac{1}{8} \frac{\delta^{ij}}{r_1^2} \,, \tag{3.41a}$$

$$W_{ij}^{22} = \frac{1}{8} \partial_{ij} \ln r_2 + \frac{1}{8} \frac{\delta^{ij}}{r_2^2} \,, \tag{3.41b}$$

$$W_{ij}^{12} = {}_i g_j + {}_j g_i \,. \tag{3.41c}$$

In Eqs. (3.41a), (3.41b) the derivatives $\partial_i \equiv \partial / \partial x^i$, while in Eq. (3.41c) g is the quadratic kernel [8]

$$g(\mathbf{x}; \mathbf{y}_1, \mathbf{y}_2) \equiv \ln(r_1 + r_2 + r_{12}) \,, \tag{3.42a}$$

$$\Delta_x g = \frac{1}{r_1 r_2} \,, \tag{3.42b}$$

and we have introduced the same abbreviated notation as above:

$${}_i g_j \equiv \frac{\partial}{\partial y_1^i} \frac{\partial}{\partial y_2^j} g \,. \tag{3.43}$$

From (3.40) we get the cubically nonlinear effective source term

$$W_{ij} \partial_{ij} U = G^3 [m_1^3 \mathcal{A}^{111} + m_1^2 m_2 \mathcal{A}^{112} + m_1 m_2^2 \mathcal{A}^{122} + m_2^3 \mathcal{A}^{222}] \tag{3.44}$$

with

$$\mathcal{A}^{111} = W_{ij}^{11} \partial_{ij} \frac{1}{r_1} = \frac{1}{8} \partial_{ij} \ln r_1 \partial_{ij} \frac{1}{r_1} = -\frac{1}{2 r_1^5} , \tag{3.45}$$

$$\mathcal{A}^{112} = W_{ij}^{12} \partial_{ij} \frac{1}{r_1} + W_{ij}^{11} \partial_{ij} \frac{1}{r_2} = 2\, {}_i g_j \partial_{ij} \frac{1}{r_1} + \frac{1}{8} \partial_{ij} \ln r_1 \partial_{ij} \frac{1}{r_2} + \frac{1}{8} \frac{1}{r_1^2} \Delta \frac{1}{r_2} , \tag{3.46}$$

the other terms in (3.44) being obtained by exchanging the roles of $\mathbf{y}_1$ and $\mathbf{y}_2$.

Note that the two $\mathbf{x}$ derivatives acting on $1/r_1$ and $1/r_2$ in (3.46) (which introduce nonlocally integrable singularities in $\mathbf{x}$ space) are left uneffected and must be interpreted in the sense of distribution theory (in $\mathbf{x}$ space). Rigorously speaking, one is not allowed to work within the framework of distribution theory (because one is dealing with the product of a distribution $\partial_{ij} r_1^{-1}$ by a function ${}_i g_j$ which is not smooth at $\mathbf{x} = \mathbf{y}_1$). One should, e.g., use the well-defined analytic continuation procedure of [24] which has been shown to correctly describe the self-gravity effects of compact bodies. [The latter analytic continuation procedure yields in particular an unambiguous treatment of the self-source term $\mathcal{A}^{111}$, Eq. (3.45).] In practice, a technically easier way to deal with this subtlety is to work with the $\mathbf{y}_1$ and $\mathbf{y}_2$ derivatives of quantities which are less singular in $\mathbf{x}$ space (see the first terms in the definitions of H and K below). When doing so, there arises only one term which is not well defined in the sense of distribution theory, and this term, $\propto \mathbf{n}_1 \delta(\mathbf{x} - \mathbf{y}_1)$, clearly vanishes when treated more properly by analytic continuation. Computing $\Delta^{-1}(W_{ij} \partial_{ij} U)$ is easy for $\mathcal{A}^{111}$ [using $\Delta r_1^{-3} = 6 r_1^{-5}$] and the last term in $\mathcal{A}^{112}$ [using $r_1^{-2} \Delta r_2^{-1} = r_{12}^{-2} \Delta r_2^{-1} = \Delta (r_{12}^{-2} r_2^{-1})$]. The other contributions to $\mathcal{A}^{112}$ are much more intricate to deal with. We succeeded in evaluating explicitly $\Delta^{-1} \mathcal{A}^{112}$ by combining the results of Refs. [29,30] which pointed out the usefulness of considering certain $\mathbf{y}_1$ and $\mathbf{y}_2$ derivatives of combinations of g, $\ln r_1$, r_1, r_2 and r_{12}, with the fact (contained in a somewhat roundabout way in Ref. [31]) that the inverse Laplacian of $r_1^{-4} r_2^{-1}$ has a simple expression:

$$\Delta^{-1} \left(\frac{1}{r_1^4 r_2} \right) = \frac{1}{2 r_{12}^2} \frac{r_2}{r_1^2} \,. \tag{3.47}$$

[The latter being most simply obtained from its easily verified "inverse:" $\Delta(r_2/r_1^2) = 2 r_{12}^2 / (r_1^4 r_2)$.] Denoting $\nabla_1 \equiv \partial / \partial \mathbf{y}_1$, $\Delta_1 \equiv \partial / \partial y_1^i \partial / \partial y_1^i$, $\nabla_1 \cdot \nabla_2 \equiv \partial / \partial y_1^i \partial / \partial y_2^i$, and $n_{12}^i \equiv (y_1^i - y_2^i)/r_{12}$, we define the quantities

$$H \equiv \Delta_1 (\nabla_1 \cdot \nabla_2) \left[\frac{r_1 + r_{12}}{2} g \right] - \frac{1}{2} \frac{r_2}{r_{12}^2 r_1^2} + \frac{1}{2} \frac{1}{r_{12} r_1^2} + \frac{\mathbf{n}_{12} \cdot \mathbf{n}_1}{r_{12} r_1^2} + \frac{1}{r_{12}^2 r_1} + \frac{3}{2} \frac{\mathbf{n}_{12} \cdot \mathbf{n}_1}{r_{12}^2 r_1} - \frac{\mathbf{n}_{12}}{r_{12}^2} \cdot \nabla_1 g \,, \tag{3.48}$$

$$K \equiv (\nabla_1 \cdot \nabla_2) \left[\frac{1}{r_2} \ln r_1 - \frac{1}{r_2} \ln r_{12} \right] + \frac{1}{2} \frac{r_2}{r_{12}^2 r_1^2} - \frac{1}{2} \frac{1}{r_1^2 r_2} + \frac{1}{2} \frac{1}{r_{12}^2 r_2} \,, \tag{3.49}$$

which verify

$$\Delta_x H = 2\, {}_i g_j \, \partial_{ij} \frac{1}{r_1} \,, \tag{3.50}$$

$$\Delta_x K = 2\, \partial_{ij} \ln r_1 \, \partial_{ij} \frac{1}{r_2} \,. \tag{3.51}$$

Finally, introducing the combination

$$Q \equiv 4H + \frac{1}{4} K + \frac{1}{2} \frac{1}{r_{12}^2 r_2} \,, \tag{3.52}$$

we find that $W(\mathbf{x})$, the unique solution of (3.31a) falling off at infinity, is given by

$$W = \frac{G^3}{c^4}\left[\frac{m_1^3}{3r_1^3} - m_1^2 m_2 Q + (\text{two other terms obtained by exchanging } 1 \leftrightarrow 2)\right] . \tag{3.53}$$

To project out from $W(\mathbf{x})$ the part W^{hom} whose multipolar expansion, when $|\mathbf{x}| \to \infty$, is purely homogeneous, it is convenient, in addition to using the defining criterion that it contains only terms of the type $\hat{n}^L/|\mathbf{x}|^{\ell+1}$, to notice that W^{hom} must also be a non smooth function of $\mathbf{y}_1$ and $\mathbf{y}_2$ (considered jointly). This can be shown either from the general structure of our generation formalism, or, more simply, by remarking, on dimensional and tensorial grounds, that $I_L^{(W)}$ must be of the form $G^2 m_1^p m_2^{3-p}/(c^4 r_{12}^2)$ times a tensor product of ℓ vectors $\mathbf{y}_1$ or $\mathbf{y}_2$. When using these two criteria in conjunction, one finds that many terms in H and K project out to zero. For instance, by explicating

$$\Delta_1\left[\frac{r_1 + r_{12}}{2} g\right] = \left(\frac{1}{r_1} + \frac{1}{r_{12}}\right) g + \frac{1}{r_1} + \frac{1}{r_{12}} - \frac{r_2}{r_{12} r_1} \tag{3.54}$$

with the decomposition $g = k + \frac{1}{2}h$ (see Ref. [8]) where k is smooth in $(\mathbf{y}_1, \mathbf{y}_2)$ and where

$$h(\mathbf{x}; \mathbf{y}_1, \mathbf{y}_2) \equiv \ln\left(\frac{r_1 + r_2 + r_{12}}{r_1 + r_2 - r_{12}}\right) , \tag{3.55a}$$

$$\Delta_x h = -4\pi\delta_{12} , \tag{3.55b}$$

has a purely homogeneous multipolar expansion, one finds that the first term in H [Eq. (3.48)] projects out to zero. Finally one gets

$$\begin{aligned} W^{\text{hom}} = \frac{G^3 m_1^2 m_2}{c^4} \Bigg[& \frac{1}{4}\nabla_1 \cdot \nabla_2 \left(\frac{\ln r_{12}}{r_2}\right) \\ & + \frac{15}{8 r_{12}^2}\left(\frac{r_2}{r_1^2}\right)^{\text{hom}} - 4\,\frac{\mathbf{n}_{12}}{r_{12}} \cdot \frac{\mathbf{n}_1}{r_1^2} - \frac{4}{r_{12}^2 r_1} \\ & - \frac{5}{8}\frac{1}{r_{12}^2 r_2} + 2\,\frac{\mathbf{n}_{12}}{r_{12}^2} \cdot \nabla_1 h\Bigg] + (1 \leftrightarrow 2) , \end{aligned} \tag{3.56}$$

where $(r_2/r_1^2)^{\text{hom}}$ denotes the homogeneous projection of r_2/r_1^2 and where h was defined in Eqs. (3.55). Let us now evaluate $u \equiv (r_2/r_1^2)^{\text{hom}}$. By expanding $r_2 = |\mathbf{x} - \mathbf{y}_2| = |\mathbf{r}_1 + \mathbf{r}_{12}|$ (with $\mathbf{r}_1 = \mathbf{x} - \mathbf{y}_1$, $\mathbf{r}_{12} = \mathbf{y}_1 - \mathbf{y}_2$) in powers of $\mathbf{r}_{12}$, we get the expansion at infinity of r_2/r_1^2:

$$\frac{r_2}{r_1^2} = \sum_{\ell \geq 0} \frac{1}{\ell!} r_{12}^L \frac{\partial_L r_1}{r_1^2} \tag{3.57}$$

(where, as usual, $r_{12}^L \equiv r_{12}^{i_1} \cdots r_{12}^{i_\ell}$). The projection of $\partial_L r_1/r_1^2$ onto the homogeneous solutions $\propto \hat{n}_1^L/r_1^{\ell+1}$ (using $\mathbf{x} = \mathbf{y}_1$ as origin for the expansion at infinity) is simply obtained by taking the STF projection of the multi-index L. Hence [with $(-1)!! = 1$, $(-3)!! = -1$]

$$\begin{aligned} u \equiv \left(\frac{r_2}{r_1^2}\right)^{\text{hom}} &= \sum_{\ell \geq 0} \frac{1}{\ell!} r_{12}^L \frac{\hat{\partial}_L r_1}{r_1^2} \\ &= \sum_{\ell \geq 0} \frac{(-)^{\ell-1}}{\ell!} (2\ell - 3)!!\, r_{12}^L \frac{\hat{n}_1^L}{r_1^{\ell+1}} . \end{aligned} \tag{3.58}$$

Using formula (A25) from [20] this can be rewritten in terms of Legendre polynomials of $\mathbf{n}_1 \cdot \mathbf{n}_{12}$:

$$u = \frac{1}{r_{12}} \sum_{\ell \geq 0} \frac{(-)^{\ell-1}}{2\ell - 1} P_\ell(\mathbf{n}_1 \cdot \mathbf{n}_{12}) \left(\frac{r_{12}}{r_1}\right)^{\ell+1} . \tag{3.59}$$

It is clear from Eqs. (3.58) and (3.59) that $u(\mathbf{x}, \mathbf{y}_1, \mathbf{y}_2)$ is a solution of $\Delta_x u = 0$ which has an axial symmetry around the straight line joining $\mathbf{y}_1$ to $\mathbf{y}_2$. We can represent u in closed form by introducing a distribution of "charges" along the segment $\mathbf{y}_1 - \mathbf{y}_2$, say

$$u(\mathbf{x}; \mathbf{y}_1, \mathbf{y}_2) = \int_0^1 d\alpha\, \frac{w(\alpha)}{|\mathbf{x} - \mathbf{y}_\alpha|} , \quad \mathbf{y}_\alpha \equiv (1 - \alpha)\mathbf{y}_1 + \alpha \mathbf{y}_2 . \tag{3.60}$$

By identifying (3.60) and (3.59) on the axis of symmetry, we see that the weight $w(\alpha)$ with which the "charges" are distributed on the segment $\mathbf{y}_1 - \mathbf{y}_2$ must satisfy

$$\int_0^1 d\alpha\, w(\alpha) \alpha^\ell = -\frac{1}{2\ell - 1} . \tag{3.61}$$

We find that $w(\alpha)$ must be defined within the framework of distribution theory (rather than that of ordinary, locally integrable functions) as

$$w(\alpha) = \text{Pf}\left[-\frac{1}{2}\alpha^{-3/2}\right] , \tag{3.62}$$

where the symbol Pf denotes Hadamard's partie finie for an integral over the interval $0 \leq \alpha \leq 1$ (see Ref. [32]). Actually, this could also be written in terms of the analytic-continuation finite part operator used in our general formalism, but Hadamard's Pf operator is a simpler object when, as is the case in Eq. (3.62), the integrals to be defined involve noninteger powers. We want also to emphasize by the change of notation that the appearance of the distribution $w(\alpha)$ is quite disconnected from the formalism behind our starting formula (2.17). We note in passing that we could dispense of using Hadamard's partie finie by using $w(\alpha) = (d/d\alpha)(\alpha^{-1/2})$, where the derivative is taken in the sense of distributions, and by integrating (3.60) by parts. However, the form (3.62) is technically more convenient for our purpose.

Summarizing the results so far, we have succeeded in resumming the infinite multipolar series defining W^{hom}, Eq. (3.36), to obtain it as a *finite* sum of derivatives of r_1^{-1} and r_2^{-1} [see Eq. (3.56)], plus two more complicated homogeneous solutions: a $\mathbf{y}_1$ gradient of the function h, Eqs. (3.55), and the function u, which is a distributional superposition of elementary solutions $|\mathbf{x}-\mathbf{y}_\alpha|^{-1}$ on the segment $\mathbf{y}_1-\mathbf{y}_2$. Note that while Eq. (3.36) defined only the asymptotic expansion at infinity of $W^{\rm hom}$, our closed-form result for $W^{\rm hom}(\mathbf{x})$ defines it for all values of $\mathbf{x}$. At this stage, it is convenient to bypass the link (3.37) and (3.38) between the object we seek, $I_L^{(W)}$, and the coefficients of the multipolar expansion of $W^{\rm hom}(\mathbf{x})$ by using our detailed results on $W^{\rm hom}$ for defining the distributional "source" (in $\mathbf{x}$ space) of $W^{\rm hom}$ by

$$S(\mathbf{x}) \equiv -\frac{1}{4\pi}\Delta_x W^{\rm hom}(\mathbf{x}) \ . \tag{3.63}$$

Thanks to our resummation of $W^{\rm hom}$ as a sum of elementary "homogeneous" (in the sense of *functions*, but not of *distributions*) solutions, the definition (3.63) makes sense with $S(\mathbf{x})$ being a distribution (in $\mathbf{x}$ space) whose support is localized on the segment joining $\mathbf{y}_1$ to $\mathbf{y}_2$. More precisely, we have

$$S = \frac{G^3 m_1^2 m_2}{c^4}\left[\frac{1}{4}\nabla_1\cdot\nabla_2[\ln r_{12}\delta(\mathbf{x}-\mathbf{y}_2)] - \frac{15}{16}\frac{1}{r_{12}^2}\mathrm{Pf}\int_0^1 d\alpha\,\alpha^{-3/2}\delta(\mathbf{x}-\mathbf{y}_\alpha) - 4\frac{\mathbf{n}_{12}}{r_{12}}\cdot\nabla_1\delta(\mathbf{x}-\mathbf{y}_1)\right.$$
$$\left. -\frac{4}{r_{12}^2}\delta(\mathbf{x}-\mathbf{y}_1) - \frac{5}{8}\frac{1}{r_{12}^2}\delta(\mathbf{x}-\mathbf{y}_2) + 2\frac{\mathbf{n}_{12}}{r_{12}^2}\cdot\nabla_1\left(r_{12}\int_0^1 d\alpha\,\delta(\mathbf{x}-\mathbf{y}_\alpha)\right)\right] + (1\leftrightarrow 2) \ . \tag{3.64}$$

The introduction of the source S of $W^{\rm hom}$ simplifies our evaluation of the multipole moments $I_L^{(W)}$. Indeed, from the definition (3.36) and the fact that $S(\mathbf{x})$ has a *compact* support we deduce, as usual,

$$W^{\rm hom}(\mathbf{x}) = \int d^3\mathbf{y}\frac{S(\mathbf{y})}{|\mathbf{x}-\mathbf{y}|} = \sum_{\ell\geq 0}\frac{(-)^\ell}{\ell!}\partial_L\frac{1}{|\mathbf{x}|}\int d^3\mathbf{y}\, y^L S(\mathbf{y}) \ . \tag{3.65}$$

By comparing with (3.38), we find

$$G I_L^{(W)} = \int d^3\mathbf{x}\,\hat{x}^L S(\mathbf{x}) \ . \tag{3.66}$$

It is interesting to remark that the work of the present subsection, together with the one of the previous subsection, is analogous to the method used in Ref. [8]. Basically, we have decomposed $-4W_{ij}\partial_{ij}U/c^4$, which was an effective source term for the inner metric (i.e., a cubically nonlinear analogue of the quadratically nonlinear terms obtained from the elementary object $r_1^{-1}r_2^{-1}$) into a piece $\Delta_x W^{\rm hom} = -4\pi S$ which has a compact support (analogue to $\Delta_x h = -4\pi\delta_{12}$, in [8], defining the compact source $\tau_c^{\mu\nu}$) and a complementary piece $\Delta_x W^{\rm inhom}$, with $W^{\rm inhom} \equiv W - W^{\rm hom}$ being the analogue of the kernel k, which does not contribute to (2.17). In other words, the combination of the results of Ref. [10] with the results given here end up by yielding a formula for I_L given *explicitly* by an integral over a *compact support* [after the two replacements $r_1^{-1}r_2^{-1} \to -2\pi\delta_{12}$ and $W_{ij}\partial_{ij}U/c^4 \to \pi S$ in Eq. (2.17)]. From Eq. (3.66) one can read off directly from (3.64) an explicit expression for $I_L^{(W)}$. Note that one can get rid of the logarithm appearing in the first term of this expression by explicating the ∇_1 derivative:

$$\frac{1}{4}(\nabla_1\cdot\nabla_2)(\ln r_{12}\,\hat{y}_2^L) = -\frac{1}{4r_{12}^2}\hat{y}_2^L + \frac{1}{4}\frac{\mathbf{n}_{12}}{r_{12}}\cdot\nabla_2\hat{y}_2^L \ . \tag{3.67}$$

This leads to our (essentially) final result

$$I_L^{(W)} = \frac{G^2 m_1^2 m_2}{c^4}\left[-\frac{4}{r_{12}^2}\hat{y}_1^L - \frac{7}{8r_{12}^2}\hat{y}_2^L - 4\frac{\mathbf{n}_{12}}{r_{12}}\cdot\nabla_1\hat{y}_1^L + \frac{1}{4}\frac{\mathbf{n}_{12}}{r_{12}}\cdot\nabla_2\hat{y}_2^L + 2\frac{\mathbf{n}_{12}}{r_{12}^2}\cdot\nabla_1\left(r_{12}\int_0^1 d\alpha\hat{y}_\alpha^L\right) - \frac{15}{16}\frac{1}{r_{12}^2}\mathrm{Pf}\int_0^1 d\alpha\,\alpha^{-3/2}\hat{y}_\alpha^L\right] + (1\leftrightarrow 2) \ , \tag{3.68}$$

where, as usual, $\hat{y}_\alpha^L \equiv y_\alpha^{\langle i_1}\cdots y_\alpha^{i_\ell\rangle}$, and where, because of the dissymmetry betweeen $\mathbf{y}_1$ and $\mathbf{y}_2$ in the last integral, it is important to recall the definition used here: $\mathbf{y}_\alpha \equiv (1-\alpha)\mathbf{y}_1 + \alpha\mathbf{y}_2$ (which differs from the notation used in [8] by the replacement $\alpha \to 1-\alpha$).

To get a completely explicit expression for $I_L^{(W)}$ in terms of $\mathbf{y}_1$ and $\mathbf{y}_2$ one needs first to compute the integrals appearing in (3.68): the first one has been given in Eq. (3.23) above [actually, one uses here the equivalent expression given by Eq. (5.24) in [8]], the second one is simply obtained by expanding the tensorial power $y_\alpha^L = (\mathbf{y}_1 - \alpha\mathbf{y}_{12})^{\otimes L}$ in powers of α and by using the elementary integrals (3.61). Then the derivatives with respect to $\mathbf{y}_1$ and $\mathbf{y}_2$ must be effected. One can drastically cut down the whole calculation by using the fact that $I_L^{(W)}$ must [as is clear from (3.66)] transform under global spatial shifts of the coordinate system, $\mathbf{x}\to\mathbf{x}+\boldsymbol{\varepsilon}$, $\mathbf{y}_A \to \mathbf{y}_A + \boldsymbol{\varepsilon}$ according to $\delta_{\boldsymbol{\varepsilon}} I_L^{(W)} = \ell\varepsilon_{\langle i_\ell}I_{L-1\rangle}^{(W)} + O(\varepsilon^2)$ (see also Appendix B of [8]). Defining for brevity some ℓ-dependent coefficients,

$$C_\ell \equiv 2\delta_\ell^0 + 4\delta_\ell^1 + (-)^\ell\frac{\ell^2+3\ell+2}{2(2\ell-1)} \tag{3.69a}$$

(where $\delta_\ell^0 = 1$ if $\ell = 0$ and 0 otherwise, and similarly for δ_ℓ^1), we obtain finally

$$I_L^{(W)} = -\frac{G^2 m_1^2 m_2}{c^4 r_{12}^2} \sum_{p=0}^{\ell} \binom{\ell}{p} C_p \, y_1^{\langle L-P} y_{12}^{P\rangle} + (1 \leftrightarrow 2) \,, \quad \binom{\ell}{p} \equiv \frac{\ell!}{p!(\ell-p)!} \,, \tag{3.69b}$$

where the multi-indices are such that there are $\ell - p$ indices on $\mathbf{y}_1$ and p on $\mathbf{y}_{12} = \mathbf{y}_1 - \mathbf{y}_2$. Writing out explicitly the low orders in ℓ we find

$$I_L^{(W)} = -\frac{G^2 m_1^2 m_2}{c^4 r_{12}^2} \overline{Q}_L + (1 \leftrightarrow 2) \,, \tag{3.70}$$

with

$$\overline{Q} = 1 \,, \tag{3.71a}$$
$$\overline{Q}_i = y_1^i + y_{12}^i \,, \tag{3.71b}$$
$$\overline{Q}_{ij} = y_1^{\langle ij\rangle} + 2 y_1^{\langle i} y_{12}^{j\rangle} + 2 y_{12}^{\langle ij\rangle} \,, \tag{3.71c}$$
$$\overline{Q}_{ijk} = y_1^{\langle ijk\rangle} + 3 y_1^{\langle ij} y_{12}^{k\rangle} + 6 y_1^{\langle i} y_{12}^{jk\rangle} - 2 y_{12}^{\langle ijk\rangle} \,, \tag{3.71d}$$
$$\overline{Q}_{ijkl} = y_1^{\langle ijkl\rangle} + 4 y_1^{\langle ijk} y_{12}^{l\rangle} + 12 y_1^{\langle ij} y_{12}^{kl\rangle} - 8 y_1^{\langle i} y_{12}^{jkl\rangle} + \frac{15}{7} y_{12}^{\langle ijkl\rangle} \,. \tag{3.71e}$$

One should note that in all the formulas for W and $I_L^{(W)}$ given above, starting with Eq. (3.53), one must complement the explicitly written results by adding similar terms, proportional to $m_1 m_2^2$, obtained by exchanging the labels 1 and 2 (remember that $y_{21}^i \equiv y_2^i - y_1^i = -y_{12}^i$). For instance, the complete expression of the quadrupole reads

$$I_{ij}^{(W)} = -\frac{G^2 m_1 m_2}{c^4 r_{12}^2} \mathrm{STF}_{ij} \left[m_1 y_1^{ij} + m_2 y_2^{ij} + 2(m_1 y_1^i - m_2 y_2^i) y_{12}^j + 2(m_1 + m_2) y_{12}^{ij} \right] . \tag{3.72}$$

An independent, direct derivation of the quadrupole (3.72) (without using the decomposition of W into homogeneous and inhomogeneous parts) is given in Appendix B. In the center-of-mass frame, we finally get, for the cubically nonlinear contribution to the quadrupole,

$$I_{ij}^{(W)} = -\nu m \gamma^2 (2 + 5\nu) \hat{x}^{ij} \,. \tag{3.73}$$

Summing up the explicit results obtained for the three pieces of the 2PN-accurate quadrupole (considered for circular orbits, in the center-of-mass frame) we get

$$\begin{aligned} I_{ij} = I_{ij}^{(C)} + I_{ij}^{(Y)} + I_{ij}^{(W)} = \mathrm{STF}_{ij}\, \nu m \Bigg\{ & x^{ij} - \frac{\gamma}{42}(1 + 39\nu) x^{ij} + \frac{11}{21}\frac{r^2}{c^2} v^{ij} (1 - 3\nu) \\ & - \frac{\gamma^2 x^{ij}}{1512}(461 + 18395\nu + 241\nu^2) + \frac{\gamma r^2 v^{ij}}{378 c^2}(1607 - 1681\nu + 229\nu^2) \Bigg\} \,. \end{aligned} \tag{3.74}$$

IV. THE 2PN-ACCURATE WAVEFORM AND ENERGY LOSS

A. The waveform including its tail contribution

The 2PN-accurate waveform is given by Eq. (2.3) in terms of the "radiative" multipole moments U_L and V_L which are in turn linked to the source moments I_L and J_L by Eqs. (2.6) and (2.7). The latter equations involve some tail integrals and therefore yield a natural decomposition of the waveform into two pieces, one which depends on the state of the binary at the retarded instant $T_R \equiv T - R/c$ only (we qualify this piece as "instantaneous"), and one which is *a priori* sensitive to the binary's dynamics at all previous instants $T_R - \tau \leq T_R$ (we refer to this piece as the "tail" contribution). More precisely, we decompose

$$h_{km}^{\mathrm{TT}} = (h_{km}^{\mathrm{TT}})_{\mathrm{inst}} + (h_{km}^{\mathrm{TT}})_{\mathrm{tail}} \,, \tag{4.1}$$

where the "instantaneous" contribution is defined by

$$\begin{aligned} (h_{km}^{\mathrm{TT}})_{\mathrm{inst}} = \frac{2G}{c^4 R} \mathcal{P}_{ijkm} \Bigg\{ & I_{ij}^{(2)} + \frac{1}{c}\left[\frac{1}{3} N_a I_{ija}^{(3)} + \frac{4}{3} \varepsilon_{ab(i} J_{j)a}^{(2)} N_b\right] + \frac{1}{c^2}\left[\frac{1}{12} N_{ab} I_{ijab}^{(4)} + \frac{1}{2} \varepsilon_{ab(i} J_{j)ac}^{(3)} N_{bc}\right] \\ & + \frac{1}{c^3}\left[\frac{1}{60} N_{abc} I_{ijabc}^{(5)} + \frac{2}{15} \varepsilon_{ab(i} J_{j)acd}^{(4)} N_{bcd}\right] + \frac{1}{c^4}\left[\frac{1}{360} N_{abcd} I_{ijabcd}^{(6)} + \frac{1}{36} \varepsilon_{ab(i} J_{j)acde}^{(5)} N_{bcde}\right] \Bigg\}, \end{aligned} \tag{4.2}$$

and where the "tail" contribution reads

$$(h^{\rm TT}_{km})_{\rm tail} = \frac{2G}{c^4R}\frac{2Gm}{c^3}\mathcal{P}_{ijkm}\int_0^{+\infty} d\tau\left\{\ln\left(\frac{\tau}{2b_1}\right) I^{(4)}_{ij}(T_R-\tau) + \frac{1}{3c}\ln\left(\frac{\tau}{2b_2}\right) N_a I^{(5)}_{ija}(T_R-\tau) + \frac{4}{3c}\ln\left(\frac{\tau}{2b_3}\right)\varepsilon_{ab(i}N_b J^{(4)}_{j)a}(T_R-\tau)\right\} . \tag{4.3a}$$

We have used for simplicity the notation

$$b_1 \equiv b\,e^{-11/12}\ ,\quad b_2 \equiv b\,e^{-97/60}\ ,\quad b_3 \equiv b\,e^{-7/6}\ . \tag{4.3b}$$

Note that the decomposition (4.1)–(4.3) into instantaneous and tail contributions is convenient but by no means unique. In particular the logarithms of the constants b_1, b_2, b_3 in Eq. (4.3a) could be as well transferred to the instantaneous side (4.2).

We first study the instantaneous contribution (4.2), which is straightforwardly computed from the 2PN-accurate mass quadrupole moment previously derived [Eq. (3.74)], and from the knowledge of the other moments necessitating the 1PN accuracy at most. The 1PN-accurate source moments have been derived in [7] [see Eq. (2.9) above] and [8]. Note that the result (5.18) of [8] is equivalent to inserting Eq. (3.21) into the result (2.13) above. We list below the relevant mass-type and current-type moments which have to be inserted into the waveform (4.2):

$$I_{ij} = \nu m\,{\rm STF}_{ij}\left\{x^{ij} - \frac{\gamma}{42}(1+39\nu)\,x^{ij} + \frac{11}{21}\frac{r^2}{c^2}v^{ij}(1-3\nu) - \frac{\gamma^2 x^{ij}}{1512}\left(461+18395\nu+241\nu^2\right) + \frac{\gamma r^2 v^{ij}}{378c^2}\left(1607-1681\nu+229\nu^2\right)\right\} , \tag{4.4a}$$

$$I_{ijk} = \nu m\ (X_2-X_1)\,{\rm STF}_{ijk}\left\{x^{ijk} - \gamma\nu x^{ijk} + \frac{r^2}{c^2}\ v^{ij}x^k(1-2\nu)\right\} , \tag{4.4b}$$

$$I_{ijkl} = \nu m\,{\rm STF}_{ijkl}\left\{x^{ijkl}(1-3\nu) + \frac{\gamma}{110}x^{ijkl}\ (3-125\nu+345\nu^2) + \frac{78}{55}\frac{r^2}{c^2}\ v^{ij}x^{kl}(1-5\nu+5\nu^2)\right\} , \tag{4.4c}$$

$$I_{ijklm} = \nu m(1-2\nu)(X_2-X_1)\,\hat{x}^{ijklm}\ , \tag{4.4d}$$

$$I_{ijklmn} = \nu m(1-5\nu+5\nu^2)\,\hat{x}^{ijklmn}\ , \tag{4.4e}$$

$$J_{ij} = \nu m(X_2-X_1)\,{\rm STF}_{ij}\,\varepsilon_{jab}\left\{x^{ai}v^b + \frac{\gamma}{28}\ x^{ai}v^b(67-8\nu)\right\} , \tag{4.4f}$$

$$J_{ijk} = \nu m\,{\rm STF}_{ijk}\,\varepsilon_{kab}\left\{x^{aij}v^b(1-3\nu) + \frac{\gamma x^{aij}v^b}{90}(181-545\nu+65\nu^2) + \frac{7}{45}\frac{r^2}{c^2}\ x^a v^{bij}\ (1-5\nu+5\nu^2)\right\} , \tag{4.4g}$$

$$J_{ijkl} = \nu m(1-2\nu)(X_2-X_1)\,{\rm STF}_{ijkl}\,\varepsilon_{lab}\,x^{aijk}v^b\ , \tag{4.4h}$$

$$J_{ijklm} = \nu m(1-5\nu+5\nu^2)\,{\rm STF}_{ijklm}\,\varepsilon_{mab}\,x^{aijkl}v^b\ . \tag{4.4i}$$

Note that the authors of Ref. [8] gave two different forms of the current-type moments: a "central" form, its Eq. (5.18), equivalent to the form (2.13) above (taken from [10]), and a "potential" form. We found the first form simpler to evaluate when one is interested in computing the moments in the center-of-mass frame. [By using it, we detected an error in [33] which computed the potential form of the octupole J_{ijk}: the coefficient of the twelfth term in Eq. (9) of Ref. [33] should read $-5/2$ instead of $+3/2$ (this twelfth term was the only one not checked by the transformation law of multipole moments under a constant shift of the spatial origin of the coordinates).] The relevant time derivatives of the moments (4.4), as well as all their contractions with the unit direction $\mathbf{N}$ which are needed in the computation of $(h^{\rm TT}_{km})_{\rm inst}$, are relegated to Appendix C. One gets, for the 2PN-accurate instantaneous part of the waveform (in the case of two bodies moving on a circular orbit),

$$(h^{\rm TT}_{km})_{\rm inst} = \frac{2G\nu m}{c^4R}\,\mathcal{P}_{ijkm}\left\{\xi^{(0)}_{ij} + (X_2-X_1)\xi^{(1/2)}_{ij} + \xi^{(1)}_{ij} + (X_2-X_1)\,\xi^{(3/2)}_{ij} + \xi^{(2)}_{ij}\right\} , \tag{4.5a}$$

where the various post-Newtonian pieces are given by

$$\xi^{(0)}_{ij} = 2\left(v^{ij} - \frac{Gm}{r}n^{ij}\right) , \tag{4.5b}$$

$$\xi^{(1/2)}_{ij} = -6\frac{Gm}{r}(Nn)\frac{n^{(i}v^{j)}}{c} - \frac{(vN)}{c}\left\{\frac{Gm}{r}n^{ij} - 2v^{ij}\right\} , \tag{4.5c}$$

$$\xi^{(1)}_{ij} = \frac{1}{3}(1-3\nu)\left[(Nn)^2\gamma\left\{10\frac{Gm}{r}n^{ij} - 14v^{ij}\right\} - 32\,(Nn)(Nv)\gamma n^{(i}v^{j)} + \frac{(Nv)^2}{c^2}\left\{6v^{ij} - 2\frac{Gm}{r}n^{ij}\right\}\right] - \gamma v^{ij}\left(\frac{1}{3}+\nu\right) + \gamma\frac{Gm}{r}n^{ij}\left(\frac{19}{3}-\nu\right) , \tag{4.5d}$$

$$\xi^{(3/2)}_{ij} = (1-2\nu)\left\{\frac{65}{6}(Nn)^3\gamma\frac{Gm}{r}\frac{n^{(i}v^{j)}}{c} - \frac{46}{3}(Nn)(Nv)^2\frac{\gamma}{c}n^{(i}v^{j)} + \gamma(Nn)^2\frac{(Nv)}{c}\left[-\frac{43}{3}v^{ij} + \frac{37}{4}\frac{Gm}{r}n^{ij}\right] + \frac{(Nv)^3}{c^3}\left[-\frac{1}{3}\frac{Gm}{r}n^{ij} + 2v^{ij}\right]\right\} + (Nn)\gamma\left(\frac{95-18\nu}{6}\right)\frac{Gm}{r}\frac{n^{(i}v^{j)}}{c} + \frac{(Nv)}{c}\left[-\frac{2}{3}(1+\nu)\gamma v^{ij} + \frac{81-2\nu}{12}\gamma\frac{Gm}{r}n^{ij}\right], \tag{4.5e}$$

$$\begin{aligned}\xi^{(2)}_{ij} = {} & \gamma^2 n^{ij}\left[-\frac{Gm}{r}\left(\frac{361+65\nu+45\nu^2}{60}\right) + (Nv)^2\left(\frac{101-295\nu-15\nu^2}{15}\right)\right.\\ & \left. -\frac{Gm}{r}(Nn)^2\left(\frac{309-995\nu+195\nu^2}{15}\right) + \frac{86}{5}(Nn)^2(Nv)^2(1-5\nu+5\nu^2) - \frac{94}{15}\frac{Gm}{r}(Nn)^4(1-5\nu+5\nu^2)\right]\\ & + v^{ij}\left[-\gamma^2\left(\frac{419+1325\nu+15\nu^2}{60}\right) - \gamma\frac{(Nv)^2}{c^2}(1-3\nu-\nu^2)\right.\\ & + \gamma^2(Nn)^2\left(\frac{163-545\nu+135\nu^2}{15}\right) + 2\frac{(Nv)^4}{c^4}(1-5\nu+5\nu^2)\\ & \left. + \frac{128}{15}\gamma^2(Nn)^4(1-5\nu+5\nu^2) - 30\gamma\frac{(Nv)^2(Nn)^2}{c^2}(1-5\nu+5\nu^2)\right]\\ & + n^{(i}v^{j)}\gamma\left[\gamma(Nn)(Nv)\left(\frac{176-560\nu+80\nu^2}{5}\right) - 20(nN)\frac{(Nv)^3}{c^2}(1-5\nu+5\nu^2)\right.\\ & \left. + \frac{228}{5}\gamma(Nn)^3(Nv)(1-5\nu+5\nu^2)\right]. \end{aligned} \tag{4.5f}$$

Here $\mathbf{n}$ is shorthand for $\mathbf{x}/r$. Usual Euclidean scalar products are denoted, e.g., by $(Nn) = \mathbf{N}\cdot\mathbf{n}$. We recall that the parameter $\gamma = Gm/(c^2r)$ is related to the relative velocity $\mathbf{v}$ of the bodies by Eq. (3.13). When comparing the waveform (4.5) with the less accurate 1.5PN waveform obtained in Ref. [34], we find several discrepancies between the coefficients entering the 1.5PN piece $\xi^{(3/2)}_{ij}$ [Eq. (4.5e)]. However it has been shown [18] that the difference between $\xi^{(3/2)}_{ij}$ obtained here and the corresponding 1.5PN piece in Ref. [34] exactly vanishes after application of the TT projection operator $\mathcal{P}_{ijkm}$. Therefore the 1.5PN truncation of the waveform (4.5) perfectly agrees with the 1.5PN waveform obtained in Ref. [34]. We have checked directly, by differentiating and squaring the complete 2PN waveform (4.5) for circular orbits, that the correct instantaneous part of the energy loss is recovered [i.e., Eq. (4.12) below].

The tail contribution (4.3) is more difficult to evaluate than the instantaneous one because it involves an integral extending over the whole past evolution (or past "history") of the binary. In another sense, however, it is easier to evaluate because it necessitates at the 2PN level the knowledge of the quadrupole and octupole moments of the binary with *Newtonian* accuracy only. This is clear owing to the explicit powers of $1/c$ appearing in Eq. (4.3a). (Only at the 2.5PN level shall we need to include a post-Newtonian correction into the tail contribution.) In order to compute the tail integrals, we shall follow a procedure which is *a priori* dangerously formal (although most natural), but has been fully justified in Ref. [35] where it was proved to yield the correct numerical value of the integrals, provided that a weak assumption concerning the behavior of the gravitational field in the past ($-\tau \to -\infty$) is satisfied. This assumption is essentially that the ℓth time derivative of a moment of order ℓ tends to a constant when $-\tau \to -\infty$. It serves to preclude, for instance, the emission of a strong burst of radiation in the remote past which would make a nonphysical contribution to the tail integrals (see also Ref. [21] for a discussion). This assumption is satisfied in the case where the binary is formed by capture of two bodies moving on an initial quasihyperbolic orbit with small enough energy. The method consists (i) in substituting into the tail integrals in Eq. (4.3a) the components of the moments calculated for a *fixed* (nondecaying) *circular* orbit whose constant orbital frequency is equal to the *current* value of the frequency at time T_R, i.e., $\omega \equiv \omega_{2\mathrm{PN}} = \omega_{2\mathrm{PN}}(T_R)$. Then it consists (ii) in evaluating each resulting integral by means of the formula

$$\int_0^{+\infty} d\tau \,\ln\left(\frac{\tau}{2b}\right) e^{i\Omega\tau} = \frac{-1}{\Omega}\left[\frac{\pi}{2}\mathrm{sgn}(\Omega) + i[\ln(2|\Omega|b) + C]\right], \tag{4.6}$$

where $C = 0.577\ldots$ is Euler's constant and Ω is the frequency of the radiation [i.e., a real number $\Omega = \pm n\omega$; $\mathrm{sgn}(\Omega) \equiv \pm 1$ and $|\Omega| \equiv n\omega$]. Note that this formula is to be applied even though the left-hand side of (4.6) is an absolutely convergent integral only when Ω possesses a strictly positive imaginary part, $\mathrm{Im}(\Omega) > 0$ (compare Appendix A and Sec. 3 in Ref. [35]). As proved in Appendix B of Ref. [35], methods (i) and (ii) are valid for an orbit which is actually decaying, and thus for which $\omega(T_R - \tau)$ tends formally to zero when $-\tau \to -\infty$. The numerical errors made in following (i) and (ii) have been shown to be of order $O[\xi \ln \xi]$, where ξ is the usual adiabatic small parameter describing the decay of the orbit ($\xi \sim \dot\omega/\omega^2$), taken at the *current* instant T_R, i.e.,

$\xi = \xi(T_R)$. The proof consists in showing that an adequately defined "remote past" contribution to the tail integrals is itself of order $O[\xi(T_R)]$, so that only the "recent" values of the frequency, near the current value $\omega = \omega(T_R)$, contribute to the tail integrals. Although this has not rigorously been shown, the proof could in principle be extended to orbits which had a non-negligible eccentricity in the past, and had in fact whatever behavior in the past which is consistent with the weak assumption made above. Note that the decay of the orbit is driven by radiation reaction effects which are of order $O(\varepsilon^5)$ in the post-Newtonian parameter ε, and so the adiabatic parameter ξ is itself of order $O(\varepsilon^5)$. One can therefore safely neglect in the 2PN waveform all the errors brought about by the above procedure (i) and (ii).

We now calculate the two independent polarizations associated with the tail contribution (4.3) with respect to two unit directions $\mathbf{P}$ and $\mathbf{Q}$ perpendicular to $\mathbf{N}$ and such that $\mathbf{N}$, $\mathbf{P}$, $\mathbf{Q}$ forms a right-handed orthonormal triad. We adopt the usual convention that $\mathbf{P}$ and $\mathbf{Q}$ lie along the major and minor axes of the projection of the circular orbit on the plane of the sky, respectively, and we denote by $c = \cos i$ and $s = \sin i$ the cosine and sine of the angle between the line of sight $\mathbf{N}$ and the normal to the orbital plane [$c = \cos i$ is not to be confused with the speed of light which we denote exceptionally by c_0 in Eqs. (4.7)–(4.10)]. Denoting by ϕ the instantaneous orbital phase of the binary [defined as an angle, oriented in the sense of the motion, such that $\phi = \frac{\pi}{2}$ (mod 2π) when the relative direction of the two bodies is $\mathbf{n} = \mathbf{P}$], and using the relevant time derivatives and contractions of *Newtonian* moments as readily calculated from Appendix C, we follow the items (i) and (ii) above and bring the two "plus" and "cross" polarizations $(h_+)_{\text{tail}} \equiv \frac{1}{2}(P_iP_j - Q_iQ_j)\,(h_{ij}^{\text{TT}})_{\text{tail}}$ and $(h_\times)_{\text{tail}} \equiv \frac{1}{2}(P_iQ_j + Q_iP_j)(h_{ij}^{\text{TT}})_{\text{tail}}$ into the form

$$(h_+)_{\text{tail}} = \frac{4G^4}{c_0^7 R}\frac{\nu m^4}{r^4}\int_0^{+\infty} d\tau \left\{ -4(1+c^2)\ln\left(\frac{\tau}{2b_1}\right)\cos[2\phi(T_R-\tau)] \right. $$
$$+\gamma^{1/2}(X_2-X_1)s\left[\frac{81}{8}(1+c^2)\ln\left(\frac{\tau}{2b_2}\right)\sin[3\phi(T_R-\tau)] \right.$$
$$\left.\left. +\frac{1}{8}(5+c^2)\ln\left(\frac{\tau}{2b_2}\right)\sin[\phi(T_R-\tau)] - \frac{3}{10}\sin[\phi(T_R-\tau)]\right]\right\}\,, \qquad (4.7)$$

and

$$(h_\times)_{\text{tail}} = \frac{4G^4}{c_0^7 R}\frac{\nu m^4}{r^4}\int_0^{+\infty} d\tau \left\{ -8c\ln\left(\frac{\tau}{2b_1}\right)\sin[2\phi(T_R-\tau)] + \gamma^{1/2}(X_2-X_1)sc\left[-\frac{81}{4}\ln\left(\frac{\tau}{2b_2}\right)\cos[3\phi(T_R-\tau)]\right.\right.$$
$$\left.\left. -\frac{3}{4}\ln\left(\frac{\tau}{2b_2}\right)\cos[\phi(T_R-\tau)] - \frac{3}{10}\cos[\phi(T_R-\tau)]\right]\right\}. \qquad (4.8)$$

Still following (i) and (ii) we now compute (4.7) and (4.8) by replacing ϕ by a linear function of time, $\phi(T) = \omega T + \phi_0$ where $\omega = \omega(T_R)$ is the current value of the orbital frequency, and by applying formula (4.6) to each integral. As a result we get

$$(h_+)_{\text{tail}} = \frac{4G^3}{c_0^7 R}\frac{\nu m^3 \omega}{r}\left\{ 2(1+c^2)\left[[\ln(4\omega b_1)+C]\sin 2\phi + \frac{\pi}{2}\cos 2\phi\right] \right.$$
$$+\gamma^{1/2}(X_2-X_1)s\left[\frac{27}{8}(1+c^2)\left[[\ln(6\omega b_2)+C]\cos 3\phi - \frac{\pi}{2}\sin 3\phi\right]\right.$$
$$\left.\left. +\frac{1}{8}(5+c^2)\left[[\ln(2\omega b_2)+C]\cos\phi - \frac{\pi}{2}\sin\phi\right] + \frac{3}{10}\cos\phi\right]\right\} \qquad (4.9)$$

and

$$(h_\times)_{\text{tail}} = \frac{4G^3}{c_0^7 R}\frac{\nu m^3 \omega}{r}\left\{ -4c\left[[\ln(4\omega b_1)+C]\cos 2\phi - \frac{\pi}{2}\sin 2\phi\right]. \right.$$
$$+\gamma^{1/2}(X_2-X_1)sc\left[\frac{27}{4}\left[[\ln(6\omega b_2)+C]\sin 3\phi + \frac{\pi}{2}\cos 3\phi\right].\right.$$
$$\left.\left. +\frac{3}{4}\left[[\ln(2\omega b_2)+C]\sin\phi + \frac{\pi}{2}\cos\phi\right] + \frac{3}{10}\sin\phi\right]\right\}. \qquad (4.10)$$

The orbital frequency ω and phase ϕ in (4.9) and (4.10) denote the current values $\omega = \omega(T_R)$ and $\phi = \phi(T_R)$. Both ω and ϕ will be computed in the next subsection [see Eqs. (4.28)–(4.30)]. [Evidently, though during the computation of Eqs. (4.9) and (4.10), ϕ has been replaced, following Ref. [35], by a linear function of time, its actual time variation is nonlinear.]

B. The energy loss and the associated binary's orbital phasing

The 2PN-accurate energy loss given by Eq. (2.5) is split, similarly to the waveform, into an "instantaneous" contribution and a "tail" one. Let us deal first with the instantaneous contribution, which is defined by

$$\left(\frac{dE_B}{dT_R}\right)_{\rm inst} = -\frac{G}{c^5}\left\{\frac{1}{5}I^{(3)}_{ij}I^{(3)}_{ij} + \frac{1}{c^2}\left[\frac{1}{189}I^{(4)}_{ijk}I^{(4)}_{ijk} + \frac{16}{45}J^{(3)}_{ij}J^{(3)}_{ij}\right] + \frac{1}{c^4}\left[\frac{1}{9072}I^{(5)}_{ijkm}I^{(5)}_{ijkm} + \frac{1}{84}J^{(4)}_{ijk}J^{(4)}_{ijk}\right]\right\} . \tag{4.11}$$

All the needed moments to compute this contribution have been given in Eqs. (4.4). Their relevant time derivatives and "squares" can be found in Appendix C. By a quite straightforward computation we get

$$\left(\frac{dE_B}{dT_R}\right)_{\rm inst} = -\frac{32}{5}\frac{c^5}{G}\nu^2\gamma^5\left\{1 - \gamma\left(\frac{2927}{336} + \frac{5}{4}\nu\right) + \gamma^2\left(\frac{293\,383}{9072} + \frac{380}{9}\nu\right)\right\} . \tag{4.12}$$

As for the tail contribution, it reads (when retaining only the terms which contribute to the 2PN order)

$$\left(\frac{dE_B}{dT_R}\right)_{\rm tail} = -\frac{2G}{5c^5}\frac{2Gm}{c^3}I^{(3)}_{ij}(T_R)\int_0^{+\infty} d\tau\,\ln\left(\frac{\tau}{2b_1}\right) I^{(5)}_{ij}(T_R-\tau) . \tag{4.13}$$

Note that at the 2PN order there appears only the tail integral associated with the mass *quadrupole* moment I_{ij}, which can be replaced by its usual *Newtonian* expression. We now evaluate Eq. (4.13) by the same method (i) and (ii) as used previously for the tails in the waveform. Namely we replace in (4.13) the third and fifth time derivatives of the quadrupole moment by Newtonian quantities valid for circular orbits, and we perform explicitly the contractions between these moments, being careful that one of them is taken at the current instant T_R while the other is taken at the former instant $T_R-\tau$. This readily brings (4.13) into the form

$$\left(\frac{dE_B}{dT_R}\right)_{\rm tail} = \frac{512G^6}{5c^8}\frac{\nu^2 m^7}{r^8}\int_0^{+\infty} d\tau\,\ln\left(\frac{\tau}{2b_1}\right)\cos[2\phi(T_R-\tau)] , \tag{4.14}$$

where for this computation ϕ can be assumed to be a linear phase [35]. Then the use of the integration formula (4.6) immediately yields

$$\left(\frac{dE_B}{dT_R}\right)_{\rm tail} = -\frac{32c^5}{5G}\nu^2\gamma^5\{4\pi\gamma^{3/2}\} . \tag{4.15}$$

There is no dependence on the constant b parametrizing the freedom in constructing the radiative coordinate system. This is not surprising because the constant b (or rather b_1) enters a term which is a total derivative [see Eq. (4.13)] and thus, as we already noticed [12], which vanishes identically for circular orbits.

A central result of this paper, namely the complete 2PN-accurate gravitational energy loss rate from a compact binary moving on a circular orbit, is now obtained by adding Eqs. (4.12) and (4.15):

$$\frac{dE_B}{dT_R} = -\frac{32c^5}{5G}\nu^2\gamma^5\left\{1-\gamma\left(\frac{2927}{336}+\frac{5}{4}\nu\right) + 4\pi\gamma^{3/2} + \gamma^2\left(\frac{293\,383}{9072} + \frac{380}{9}\nu\right)\right\} . \tag{4.16}$$

The γ parameter is $\gamma \equiv Gm/(rc^2)$ where r is the harmonic radial coordinate [see Eqs. (3.10)-(3.13)]. It is to be noticed that in the form (4.16), i.e., when the post-Newtonian expansion is parametrized using γ, there is no term proportional to ν^2 in the relative 2PN contribution. This fact is somewhat surprising because all separate pieces making up the energy loss, i.e., all the different "squares" of moments listed in Appendix C, do contain terms proportional to ν^2. However the final coefficient of ν^2 in the 2PN correction term turns out to be zero. The expression of the energy loss (4.16) completes several previous investigations having obtained either the lower-order PN corrections or the limiting case where the mass of one body is negligible as compared to the other one (limiting case $\nu \to 0$). The 0PN leading term in Eq. (4.16) was known from Ref. [36], and the 1PN correction was added in Refs. [37,38]. The 1.5PN tail correction (4π term) was first computed in the limit $\nu \to 0$ [13] and then shown to be also valid for arbitrary mass ratios in Refs. [35,39] based on Ref. [9]. [The generalization of the energy loss expression for noncircular (eccentric) orbits has been obtained in Ref. [40] for the 0PN leading term and in Ref. [41] for the 1PN corrections. Note that for noncircular orbits the tail term $4\pi\gamma^{3/2}$ has simply to be multiplied by a function $\varphi(e)$ of the eccentricity of the orbit, but unfortunately this function probably does not admit a closed analytical form (see Ref. [35] for the numerical graph of this function).] Finally the 2PN correction term in Eq. (4.16) was known, up to now, only in the limiting case $\nu \to 0$, where it was computed first numerically [15] and then analytically [16]. For comparison with the latter references, and for later convenience, let us reexpress the energy loss (4.16) in terms of a new post-Newtonian parameter defined by $x \equiv (Gm\omega_{2\rm PN}/c^3)^{2/3}$. The γ parameter is related to the x parameter by the inverse of Eq. (3.11): namely,

$$\gamma = x\left[1+\left(1-\frac{1}{3}\nu\right)x+\left(1-\frac{65}{12}\nu\right)x^2\right] \quad (4.17)$$

(which also does not involve ν^2 terms). Inserting (4.17) into (4.16) (and keeping consistently all terms up to the 2PN order) readily yields

$$\frac{dE_B}{dT_R} = -\frac{32c^5}{5G}\nu^2x^5\left\{1+\left(-\frac{1247}{336}-\frac{35}{12}\nu\right)x+4\pi x^{3/2} + \left(-\frac{44\,711}{9072}+\frac{9271}{504}\nu+\frac{65}{18}\nu^2\right)x^2\right\}. \quad (4.18)$$

This expression (which we notice involves now a ν^2 term) relates the two coordinate-independent quantities dE_B/dT_R and $\omega = \omega_{2\mathrm{PN}}$, and is therefore the same in all coordinate systems. It may thus be compared directly with the expression obtained in Refs. [15,16] in the limit $\nu \to 0$. We find that the coefficient $-44\,711/9072$ agrees with the latter references [42].

Let us now denote the 2PN-accurate Bondi energy loss rate (4.18), or total gravitational luminosity, by $\mathcal{L}_B \equiv -dE_B/dT_R$, and let us equate, by a standard argument, the rate of decrease of the *dynamical* energy $\mathcal{E}$ of the binary system to the opposite of $\mathcal{L}_B$, i.e.,

$$\frac{d\mathcal{E}}{dT_R} = -\mathcal{L}_B \ . \quad (4.19)$$

We shall admit here the validity of the balance equation (4.19) to the 2PN order, although it has been validated only to the leading 0PN order and in the case of the binary pulsar [11]. [Note that the 2PN order is the last order at which the binary admits a conserved energy $\mathcal{E}$ (and a conserved angular momentum $\mathcal{J}$).] The balance equation (4.19) drives the variations with time of the instantaneous orbital frequency ω and phase ϕ of the (quasicircular) decaying orbit. These ω and ϕ are the ones which enter the expression of the waveform [see, e.g., Eqs. (4.9) and (4.10)]. To compute ω and ϕ one must evidently specify first what is the left-hand side of Eq. (4.19), i.e., one must know the 2PN-accurate expression of the center-of-mass energy $\mathcal{E}$ for a *fixed* (nondecaying) circular orbit as a function of the parameter x.

Let us recall from Ref. [43] (extending Ref. [27]) that the 2PN motion of a binary system moving on an eccentric orbit admits the representation

$$n(t-t_0) = u - e_t\sin u + \frac{f}{c^4}\sin v + \frac{g}{c^4}(v-u)\ , \quad (4.20a)$$

$$r = a_r(1-e_r\cos u)\ , \quad (4.20b)$$

$$\phi-\phi_0 = K\left\{v+\frac{F}{c^4}\sin 2v+\frac{G}{c^4}\sin 3v\right\}\ , \quad (4.20c)$$

where

$$v = 2\arctan\left[\left(\frac{1+e_\phi}{1-e_\phi}\right)^{1/2}\tan\frac{u}{2}\right]\ . \quad (4.20d)$$

The time t, separation r between the two bodies and polar angle (or phase) ϕ in these equations correspond to harmonic coordinates. The motion is parametrized by some "eccentric" and "real" anomalies u and v, and ten constants enter Eqs. (4.20) in addition to the initial t_0 and ϕ_0: $n \equiv 2\pi/P$ where P is the time of return to the periastron (or period); $K \equiv \Theta/2\pi$ where Θ is the angle of return to the periastron ($K-1$ is the relative periastron advance per rotation); the semimajor axis a_r; three types of eccentricities e_t, e_r, and e_ϕ; and four constants f, g, F, and G entering purely 2PN terms. Most important for our purpose are the expressions of the constants n and K obtained in Ref. [43] [see Eqs. (3.11) and (3.12) there] in terms of the constant center-of-mass energy $\mathcal{E}$ and angular momentum $\mathcal{J}$. Denoting $\mathcal{E} = \nu m\overline{\mathcal{E}}$ and $\mathcal{J} = G\nu m^2 h$ we have

$$n = \frac{(-2\overline{\mathcal{E}})^{3/2}}{Gm}\left\{1+\frac{1}{4}(15-\nu)\frac{\overline{\mathcal{E}}}{c^2}+\frac{3}{32}\left(185+10\nu+\frac{11}{3}\nu^2\right)\frac{\overline{\mathcal{E}}^2}{c^4}-\frac{3}{2}(5-2\nu)\frac{(-2\overline{\mathcal{E}})^{3/2}}{c^4h}\right\}, \quad (4.21)$$

$$K = 1+\frac{3}{c^2h^2}\left\{1+\left(\frac{5}{2}-\nu\right)\frac{\overline{\mathcal{E}}}{c^2}+\left(\frac{35}{4}-\frac{5}{2}\nu\right)\frac{1}{c^2h^2}\right\}. \quad (4.22)$$

The other constants in Eqs. (4.20) have been computed in Ref. [44]; however we shall here only need, in addition to Eqs. (4.21) and (4.22), the fact that in the *circular* orbit case, which is defined by $e_r = 0$, the two other eccentricities e_t and e_ϕ vanish to 2PN order, hence the two anomalies u and v agree to this order, and that the three terms involving the constants f, F, and G also vanish (see [44]). The 2PN-accurate circular motion of the binary is thus simply described by the equations

$$r = a_r\ , \quad (4.23a)$$

$$\phi-\phi_0 = nK(t-t_0)\ , \quad (4.23b)$$

which show that the orbital frequency ω in the circular orbit case is equal to the product nK. Now $\omega = nK$ can straightforwardly be obtained from Eqs. (4.21) and (4.22) and the fact that $\overline{\mathcal{E}}$ and h are related to each other by the 1PN-accurate relation $1+2\overline{\mathcal{E}}h^2 = \frac{1}{2}(9+\nu)\overline{\mathcal{E}}/c^2$ (consequence of $e_r = 0$). We find [45]

$$\omega = \frac{(-2\overline{\mathcal{E}})^{3/2}}{Gm}\left\{1-\frac{1}{4}(9+\nu)\frac{\overline{\mathcal{E}}}{c^2}+\frac{3}{32}\left(297-134\nu+\frac{11}{3}\nu^2\right)\frac{\overline{\mathcal{E}}^2}{c^4}\right\}\ . \quad (4.24)$$

The inversion of Eq. (4.24) then yields our desired relation between the energy $\mathcal{E} = \nu m\overline{\mathcal{E}}$ and the parameter $x = (Gm\omega/c^3)^{2/3}$ for circular orbits:

$$\mathcal{E} = -\frac{c^2}{2}\nu m x\left\{1-\frac{1}{12}(9+\nu)x-\frac{1}{8}\left(27-19\nu+\frac{\nu^2}{3}\right)x^2\right\} . \quad (4.25)$$

With Eqs. (4.18) and (4.25) in hand, it is now a simple matter to transform the energy balance equation (4.19) into the ordinary differential equation

$$d\theta = \frac{1}{64}dx\, x^{-5}\left\{1+\left(\frac{743}{336}+\frac{11}{4}\nu\right)x-4\pi x^{3/2}+\left(\frac{3\,058\,673}{1\,016\,064}+\frac{5429}{1008}\nu+\frac{617}{144}\nu^2\right)x^2\right\} , \quad (4.26)$$

where we have introduced for convenience the adimensional time variable

$$\theta \equiv \frac{c^3\nu}{5Gm}T_R . \quad (4.27)$$

Solving (4.26) leads to the variation in time θ of the parameter x and hence of the instantaneous frequency ω of the quasicircular orbit. Similarly, integrating $d\phi = \omega dT_R = (5/\nu)x^{3/2}d\theta$ leads to the variation of the instantaneous orbital phase ϕ. We finally obtain

$$\frac{Gm\omega}{c^3} = \frac{1}{8}(\theta_c-\theta)^{-3/8}\left\{1+\frac{1}{8}p(\theta_c-\theta)^{-1/4}-\frac{3\pi}{10}(\theta_c-\theta)^{-3/8}+\frac{1}{64}q(\theta_c-\theta)^{-1/2}\right\} , \quad (4.28)$$

$$\phi_c-\phi = \frac{1}{\nu}(\theta_c-\theta)^{5/8}\left\{1+\frac{5}{24}p(\theta_c-\theta)^{-1/4}-\frac{3\pi}{4}(\theta_c-\theta)^{-3/8}+\frac{5}{64}q(\theta_c-\theta)^{-1/2}\right\} , \quad (4.29)$$

where

$$p \equiv \frac{743}{336}+\frac{11}{4}\nu , \quad q \equiv \frac{1\,855\,099}{225\,792}+\frac{56\,975}{4032}\nu+\frac{371}{32}\nu^2 , \quad (4.30)$$

and where ϕ_c and θ_c denote the values of the phase and adimensional time (4.27) at the instant of the coalescence. The frequency and phase (4.28) and (4.29) have to be inserted into the expression of the waveform. Note that the coefficient of the 2PN contribution in Eq. (4.29) (proportional to q), increases by 52% between the test-mass limit ($\nu = 0$) and the equal-mass case ($\nu = 1/4$). This shows that finite mass effects (which cannot be obtained in perturbation calculations of black hole spacetimes) play a very significant role in the definition of 2PN-accurate theoretical waveforms. This proves the importance of post-Newtonian generation formalisms for constructing accurate templates to be used in matched filtering of the data from future gravitational-wave detectors.

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APPENDIX A: THE MASS MONOPOLE AND DIPOLE MOMENTS

Since the expression (2.17) of the 2PN-accurate mass multipole moment is valid for all orders of multipolarity ℓ, it is important to verify that for the lowest orders $\ell = 0$ and $\ell = 1$ it reduces to the expressions of the *conserved* mass monopole and mass dipole moments. This verification has already been done in Appendix B of [10] in the mass monopole case $\ell = 0$. Here we shall check that for two bodies moving on a circular orbit the mass monopole reduces to the expression computed in [26], and the mass dipole is *zero*, as it must be since we are using a mass-centered frame.

The mass monopole I and dipole I_i, which are obtained by setting $\ell = 0$ and $\ell = 1$ in Eq. (2.17), are split into three "compact," "Y," and "W" contributions according to Eq. (3.1) and evaluated separately using the method developed in Sec. III. The "W" contributions have in fact already been computed in Eqs. (3.70), (3.71a), and (3.71b). We quote only the results. For $\ell = 0$ we find

$$I^{(C)} = m\left[1-\frac{1}{2}\nu\gamma-\frac{1}{8}\nu(1-15\nu)\gamma^2\right] , \quad (A1a)$$

$$I^{(Y)} = 2m\nu(1-\nu)\gamma^2 , \quad (A1b)$$

$$I^{(W)} = -m\nu\gamma^2 . \quad (A1c)$$

Adding together these contributions leads to

$$I = m\left[1-\frac{1}{2}\nu\gamma+\frac{1}{8}\nu(7-\nu)\gamma^2\right] , \quad (A2)$$

which is in agreement with Eq. (20) in Ref. [26] when specialized to circular orbits. For $\ell = 1$ we get

$$I_i^{(C)} = (X_2-X_1)m\nu\gamma^2\left(\frac{2}{7}+\frac{29}{35}\nu\right)x^i , \quad (A3a)$$

$$I_i^{(Y)} = (X_2-X_1)m\nu\gamma^2\left(-\frac{9}{7}-\frac{29}{35}\nu\right)x^i , \quad (A3b)$$

$$I_i^{(W)} = (X_2-X_1)m\nu\gamma^2 x^i , \quad (A3c)$$

in which we have used (for the "C" term only) the 2PN-accurate mass-centered frame equation (3.7). The three contributions (A3) sum up to zero,

$$I_i = 0 \ , \tag{A4}$$

as was to be checked.

APPENDIX B: ALTERNATIVE DERIVATION OF THE W TERM

We have computed in Sec. III C of the main text the cubically nonlinear "W" term [defined by Eq. (3.30)] for all values of the order of multipolarity ℓ, but in the special case where the source is a *binary* system. In this appendix we present an alternative derivation of this W term which is valid for a general fluid system (and hence for a system made of N compact bodies), but is limited to the *quadrupole* case $\ell = 2$. When $N = 2$ *and* $\ell = 2$, i.e., when one evaluates the W term for a binary system *and* in the quadrupole case, which is what interests us in this paper, we find that both derivations agree on the result given by Eq. (3.72).

Specializing the definition (3.30) to $\ell = 2$, we thus want to compute

$$I_{ij}^{(W)} = \frac{1}{\pi G c^4} \mathrm{FP}_{B=0} \int d^3\mathbf{x} |\mathbf{x}|^B \hat{x}_{ij} W_{km} \partial_{km} U \ , \tag{B1}$$

where we recall that the potential W_{km} is defined by

$$W_{km}(\mathbf{x},t) = -\frac{1}{4\pi} \int \frac{d^3\mathbf{x}'}{|\mathbf{x}-\mathbf{x}'|} [\partial_k U \partial_m U](\mathbf{x}',t) \ . \tag{B2}$$

We first perform on (B1) two integrations by parts in order to shift the spatial derivatives acting on U to the left side of the integrand. As can be proved thanks to Eq. (4.2) of Ref. [10], all the terms coming from the differentiation of the analytic continuation factor $|\mathbf{x}|^B$ and having explicitly B as a factor are zero. [Indeed, recall that Eq. (4.2) of Ref. [10] permits one to freely integrate by parts all terms in the source moment (2.17) as if the analytic continuation factor and the "finite part" were absent.] The surface terms are also zero by analytic continuation. Hence we can split the W term into two pieces,

$$I_{ij}^{(W)} = I_{ij}^{(W1)} + I_{ij}^{(W2)} \ , \tag{B3}$$

given respectively by

$$I_{ij}^{(W1)} = \frac{2}{\pi G c^4} \mathrm{STF}_{ij} \ \mathrm{FP}_{B=0} \int d^3\mathbf{x} |\mathbf{x}|^B W_{ij} U \ , \tag{B4}$$

$$I_{ij}^{(W2)} = \frac{1}{\pi G c^4} \mathrm{FP}_{B=0} \int d^3\mathbf{x} |\mathbf{x}|^B [4 \partial_k W_{k\langle i} x_{j\rangle} + \partial_{km} W_{km} \hat{x}_{ij}] U \tag{B5}$$

(where the angular brackets $\langle\rangle$ denote the trace-free projection). In (B5) the divergences of the potential W_{ij} are given by

$$\partial_k W_{ik} = \frac{1}{4} \partial_i (U^2) + \frac{G}{2} \int \frac{d^3\mathbf{x}'}{|\mathbf{x}-\mathbf{x}'|} [\sigma \partial_i U - U \partial_i \sigma](\mathbf{x}',t) \ , \tag{B6a}$$

$$\partial_{km} W_{km} = \frac{1}{4} \Delta (U^2) + \frac{G}{2} \int \frac{d^3\mathbf{x}'}{|\mathbf{x}-\mathbf{x}'|} [\sigma \Delta U - U \Delta \sigma](\mathbf{x}',t) \ . \tag{B6b}$$

The most interesting contribution to compute is the first one, i.e., $I_{ij}^{(W1)}$ given by (B4). This contribution has no explicit dependence on $\mathbf{x}$ in the integrand; this fact, which is special to the quadrupolar case, allows the alternative derivation followed in this appendix. To compute $I_{ij}^{(W1)}$ we employ the kernel $g(\mathbf{x};\mathbf{y}_1,\mathbf{y}_2)$ already introduced in Eqs. (3.42),

$$g(\mathbf{x};\mathbf{y}_1,\mathbf{y}_2) = \ln(r_1 + r_2 + r_{12}) \ , \tag{B7}$$

which satisfies

$$\Delta_x g = \frac{1}{r_1 r_2} \tag{B8}$$

in the sense of distribution theory. (We denote $r_1 = |\mathbf{x}-\mathbf{y}_1|$, $r_2 = |\mathbf{x}-\mathbf{y}_2|$, $r_{12} = |\mathbf{y}_1 - \mathbf{y}_2|$.) The essence of the present computation of $I_{ij}^{(W1)}$ is to consider a secondary kernel f defined by

$$f(\mathbf{x};\mathbf{y}_1,\mathbf{y}_2) = \frac{1}{3}(\mathbf{r}_1 \cdot \mathbf{r}_2) \left[\ln(r_1 + r_2 + r_{12}) - \frac{1}{3} \right] + \frac{1}{6}(r_{12} r_1 + r_{12} r_2 - r_1 r_2) \ , \tag{B9}$$

and whose main property is to satisfy

$$\Delta_x f = 2\, g \tag{B10}$$

in the sense of distributions. This property is easily checked on the expression (B9). We now recall that the potential W_{ij} can be expressed in terms of the kernel g as

$$W_{ij}(\mathbf{x},t) = G^2 \iint d^3\mathbf{y}_1 d^3\mathbf{y}_2 \sigma(\mathbf{y}_1,t)\sigma(\mathbf{y}_2,t)\frac{\partial^2}{\partial y_1^i \partial y_2^j}\{g(\mathbf{x};\mathbf{y}_1,\mathbf{y}_2)\} . \tag{B11}$$

Thus we see that a secondary potential defined by the same expression as (B11) but with g replaced by the secondary kernel f, i.e.,

$$w_{ij}(\mathbf{x},t) = G^2 \iint d^3\mathbf{y}_1 d^3\mathbf{y}_2 \sigma(\mathbf{y}_1,t)\sigma(\mathbf{y}_2,t)\frac{\partial^2}{\partial y_1^i \partial y_2^j}\{f(\mathbf{x};\mathbf{y}_1,\mathbf{y}_2)\} , \tag{B12}$$

necessarily satisfies

$$\Delta w_{ij} = 2\, W_{ij} \tag{B13}$$

in the sense of distributions. Substituting (B13) into (B4) leads to

$$I_{ij}^{(W1)} = \frac{1}{\pi G c^4} \,\mathrm{STF}_{ij}\, \mathrm{FP}_{B=0} \int d^3\mathbf{x} |\mathbf{x}|^B U \Delta w_{ij} . \tag{B14}$$

The next step is to integrate by parts the Laplace operator in the integrand of (B14). However, note that it is *a priori* not possible to perform this integration by parts ignoring the analytic continuation factor $|\mathbf{x}|^B$ and the finite part symbol. For instance, the result (4.2) of Ref. [10] invoked above does not apply to this case. We thus integrate by parts the integral (B14) keeping all terms coming from the differentiation of $|\mathbf{x}|^B$. This yields (using $\Delta U = -4\pi G\sigma$)

$$I_{ij}^{(W1)} = -\frac{4}{c^4}\mathrm{STF}_{ij} \int d^3\mathbf{x} \sigma w_{ij} + R_{ij} , \tag{B15}$$

where the first term has a compact support (we have removed on it the analytic continuation factor and the finite part symbol), and where the second term is explicitly given by

$$R_{ij} = \frac{1}{\pi G c^4} \,\mathrm{STF}_{ij}\, \mathrm{FP}_{B=0} \int d^3\mathbf{x} |\mathbf{x}|^{B-2} w_{ij} \Big[B(B+1) + 2Bx^k\partial_k\Big] U . \tag{B16}$$

An equivalent expression for R_{ij} which is convenient for our purpose is easily obtained by substitution of the expression (B12) of the potential w_{ij} and use of $U = \int d^3\mathbf{y}_3 \sigma(\mathbf{y}_3,t)/|\mathbf{x}-\mathbf{y}_3|$. We have

$$R_{ij} = \frac{G^2}{\pi c^4} \,\mathrm{STF}_{ij} \iiint d^3\mathbf{y}_1 d^3\mathbf{y}_2 d^3\mathbf{y}_3\, \sigma(y_1)\sigma(y_2)\sigma(y_3)\, \frac{\partial^2}{\partial y_1^i \partial y_2^j}\{K(\mathbf{y}_1,\mathbf{y}_2;\mathbf{y}_3)\}, \tag{B17}$$

where

$$K(\mathbf{y}_1,\mathbf{y}_2;\mathbf{y}_3) = \mathrm{FP}_{B=0}\left[B(B-1) - 2By_3^k \frac{\partial}{\partial y_3^k}\right]\left\{\int d^3\mathbf{x} |\mathbf{x}|^{B-2} \frac{f(\mathbf{x};\mathbf{y}_1,\mathbf{y}_2)}{|\mathbf{x}-\mathbf{y}_3|}\right\} . \tag{B18}$$

To compute $K(\mathbf{y}_1,\mathbf{y}_2;\mathbf{y}_3)$, we need to control the *pole part* when $B \to 0$ of the integral on the right-hand side of (B18). This is because of the explicit factors B and B^2 in front of the integral. [Note that it is important to keep the factors B *and* B^2 in front since as we shall see the integral involves both a simple *and* a double pole when $B \to 0$.] The pole part of the integral in (B18) depends only on the behavior of the integrand at the upper bound $|\mathbf{x}| \to \infty$, so we need only to consider the asymptotic expansion of the kernel $f(\mathbf{x};\mathbf{y}_1,\mathbf{y}_2)$ when $|\mathbf{x}| \to \infty$, or equivalently when $\mathbf{y}_1, \mathbf{y}_2 \to 0$. It is then easily shown that the only terms in the latter expansion of f which generate poles when $B \to 0$ are either of the type a regular solution of Laplace's equation, i.e., $\hat{x}_L$ times a function of $\mathbf{y}_1$ and $\mathbf{y}_2$, or of the type $\hat{x}_L \ln|\mathbf{x}|$ times a function of $\mathbf{y}_1$, $\mathbf{y}_2$ (note that the expansion of f involves a logarithm of $|\mathbf{x}|$). We shall slightly improperly refer to these terms as the "harmonic" terms in f. Their computation can be greatly simplified by noticing that in the asymptotic expansion of $g = \ln(r_1 + r_2 + r_{12})$ when $\mathbf{y}_1$, $\mathbf{y}_2 \to 0$, only the three leading order contributions, constant, linear, and quadratic in $\mathbf{y}_1$, $\mathbf{y}_2$, can contribute to f_{harmonic}. Indeed, the higher-order contributions, at least cubic in $\mathbf{y}_1$, $\mathbf{y}_2$, necessarily involve a function of $\mathbf{x}$ whose dimension is that of $1/|\mathbf{x}|^n$ with $n \geq 3$ (because g is dimensionless), and thus which will never yield a term of the type $\hat{x}_L$ or $\hat{x}_L \ln|\mathbf{x}|$ when multiplied by $\mathbf{r}_1 \cdot \mathbf{r}_2 = \mathbf{x}^2 - \mathbf{x}\cdot\mathbf{y}_1 - \mathbf{x}\cdot\mathbf{y}_2 + \mathbf{y}_1\cdot\mathbf{y}_2$. Computing the expansion of g when $\mathbf{y}_1$, $\mathbf{y}_2 \to 0$, and then the expansion of f, we obtain

$$f_{\mathrm{harmonic}} = \frac{1}{3}\mathbf{y}_1 \cdot \mathbf{y}_2 \ln(2|\mathbf{x}|) - \frac{1}{3}(\mathbf{x}\cdot\mathbf{y}_1 + \mathbf{x}\cdot\mathbf{y}_2)\left[\ln(2|\mathbf{x}|) - \frac{1}{3}\right] . \tag{B19}$$

(This computation of the "harmonic" terms in f bears a resemblance to the computation in Sec. III C of the "homogeneous" part of the function W.) The pole part of the integral in (B18) is then straightforwardly obtained from the replacements $f \to f_{\mathrm{harmonic}}$ and $|\mathbf{x}-\mathbf{y}_3|^{-1} \to |\mathbf{x}|^{-1}(1 + \mathbf{y}_3\cdot\mathbf{x}/|\mathbf{x}|^2)$ (we do not need to expand $|\mathbf{x}-\mathbf{y}_3|^{-1}$ further since $f_{\rm harmonic}$ involves only terms with multipolarity 0 or 1). We find a simple and a double pole. Then we apply to the latter pole part the operator $B[B-1-2y_3^k\partial/\partial y_3^k]$ present in (B18) (being careful about correctly handling the double pole), and compute the finite part at $B=0$. The result is

$$K(\mathbf{y}_1,\mathbf{y}_2;\mathbf{y}_3)=\frac{1}{3}\mathbf{y}_1\cdot\mathbf{y}_2\left(\ln 2+1\right)+\frac{1}{9}(\mathbf{y}_1\cdot\mathbf{y}_3+\mathbf{y}_2\cdot\mathbf{y}_3)\left(\ln 2-\frac{1}{3}\right)\ . \tag{B20}$$

This is a nonzero result; however the quantity of interest in (B17) is

$$\mathrm{STF}_{ij}\frac{\partial^2}{\partial y_1^i\partial y_2^j}\{K(\mathbf{y}_1,\mathbf{y}_2;\mathbf{y}_3)\}=0 \quad\Longrightarrow\quad R_{ij}=0\ , \tag{B21}$$

which shows that the contribution $I_{ij}^{(W1)}$ given by (B15) reduces to its first term, i.e., to the manifestly compact support integral

$$I_{ij}^{(W1)}=-\frac{4}{c^4}\,\mathrm{STF}_{ij}\int d^3\mathbf{x}\sigma\, w_{ij}\ . \tag{B22}$$

This expression is our main result because the second contribution $I_{ij}^{(W2)}$, given by (B5), is evaluated without problem by substituting into it formulas (B6) giving the divergences of the potential W_{ij} and using at various places the function Y^L of Eq. (3.23). Adding up all the terms constituting $I_{ij}^{(W2)}$ to the first contribution $I_{ij}^{(W1)}$ given by (B22), we arrive finally at the following expression for the cubically nonlinear quadrupole term:

$$\begin{aligned}I_{ij}^{(W)}=&-\frac{4G^2}{c^4}\,\mathrm{STF}_{ij}\iiint d^3\mathbf{y}_1 d^3\mathbf{y}_2 d^3\mathbf{y}_3\sigma(y_1)\sigma(y_2)\sigma(y_3)\\&\times\left\{\frac{\partial^2}{\partial y_1^i\partial y_2^j}\left(\frac{1}{3}\mathbf{r}_{13}\cdot\mathbf{r}_{23}\left[\ln(r_{13}+r_{23}+r_{12})-\frac{1}{3}\right]+\frac{1}{6}(r_{12}r_{13}+r_{12}r_{23}-r_{13}r_{23})\right)\right.\\&+\frac{1}{4r_{13}r_{23}}(y_3^iy_3^j-y_1^iy_1^j-y_2^iy_2^j)-\frac{\mathbf{r}_{12}\cdot\mathbf{r}_{13}}{6r_{13}^3r_{12}}(y_1^iy_1^j+y_1^iy_2^j+y_2^iy_2^j)\\&\left.+\frac{r_{12}}{6r_{13}^3}(4y_1^iy_3^j-4y_1^iy_1^j+5y_2^iy_3^j-5y_1^iy_2^j)\right\}\ .\end{aligned} \tag{B23}$$

This expression is valid for a general fluid system. We can evaluate in an explicit way the differentiations with respect to $\mathbf{y}_1$ and $\mathbf{y}_2$ it contains. In doing so one finds that thanks to the trace-free projection there is in fact no logarithmic term in (B23). After reduction of (B23) to the two-body case (using as usual δ functions and formally discarding all infinite self-energy terms), we get

$$I_{ij}^{(W)}=-\frac{G^2m_1m_2}{c^4r_{12}^2}\,\mathrm{STF}_{ij}\{(5m_1+2m_2)y_1^{ij}+(5m_2+2m_1)y_2^{ij}-6(m_1+m_2)y_1^iy_2^j\}\ . \tag{B24}$$

Changing notation with $y_{12}^i\equiv y_1^i-y_2^i$ finally leads to

$$I_{ij}^{(W)}=-\frac{G^2m_1m_2}{c^4r_{12}^2}\,\mathrm{STF}_{ij}\{m_1y_1^{ij}+m_2y_2^{ij}+2(m_1+m_2)y_{12}^{ij}+2(m_1y_1^i-m_2y_2^i)y_{12}^j\}\ , \tag{B25}$$

in complete agreement with the equation (3.72) derived in the main text.

APPENDIX C: A COMPENDIUM OF FORMULAS FOR MOMENTS

We list below the expressions of the time derivatives of moments which are used in the computation of the waveform and energy loss. For the waveform,

$$\begin{aligned}I_{ij}^{(2)}=&\,2\nu m\,\mathrm{STF}_{ij}\left\{v^{ij}-\frac{Gm}{r^3}x^{ij}+\gamma\,\frac{Gm}{r^3}\,\frac{x^{ij}}{42}\,(149-69\nu)-\frac{\gamma}{42}v^{ij}\,(23-27\nu)\right.\\&\left.+\frac{\gamma^2}{1512}\,x^{ij}\,\frac{Gm}{r^3}(-7043+7837\nu-3703\nu^2)+\frac{\gamma^2}{1512}\,v^{ij}\,(-4513-19\,591\nu+1219\nu^2)\right\},\end{aligned} \tag{C1a}$$

$$I_{ijk}^{(3)}=\nu m(X_2-X_1)\,\mathrm{STF}_{ijk}\left[6v^{ijk}-21\,\frac{Gm}{r^3}\,x^{ij}v^k-\gamma(7-8\nu)v^{ijk}+(83-40\nu)\gamma\,\frac{Gm}{r^3}\,x^{ij}v^k\right]\ , \tag{C1b}$$

$$I^{(4)}_{ijkl} = \nu m\,\mathrm{STF}_{ijkl}\left\{24(1-3\nu)v^{ijkl} - 192(1-3\nu)\frac{Gm}{r^3}\,x^{ij}v^{kl} + 40(1-3\nu)\,\frac{G^2m^2}{r^6}\,x^{ijkl} + \frac{288}{55}\gamma\frac{Gm}{r^3}\,x^{ij}v^{kl}\,(161-585\nu+255\nu^2) + \frac{4}{55}\gamma\frac{G^2m^2}{r^6}\,x^{ijkl}\,(-3909+13\,495\nu-4695\nu^2) + \frac{12}{11}\gamma\,v^{ijkl}\,(-41+183\nu-139\nu^2)\right\}\,, \tag{C1c}$$

$$I^{(5)}_{ijklm} = 5\nu m(1-2\nu)(X_2-X_1)\,\mathrm{STF}_{ijklm}\left[24v^{ijklm} - 360\,\frac{Gm}{r^3}\,x^{ij}v^{klm} + 241\frac{G^2m^2}{r^6}\,x^{ijkl}v^m\right]\,, \tag{C1d}$$

$$I^{(6)}_{ijklmn} = 24\nu m(1-5\nu+5\nu^2)\,\mathrm{STF}_{ijklmn}\left[30v^{ijklmn} - 750\frac{Gm}{r^3}\,x^{ij}v^{klmn} + 1070\frac{G^2m^2}{r^6}\,x^{ijkl}v^{mn} - 94\frac{G^3m^3}{r^9}\,x^{ijklmn}\right]\,, \tag{C1e}$$

$$J^{(2)}_{ij} = -\nu m(X_2-X_1)\,\frac{Gm}{r^3}\,\mathrm{STF}_{ij}\,\varepsilon_{jab}x^{ai}v^b\left\{1-\,\frac{\gamma}{28}\,(17-20\nu)\right\}\,, \tag{C1f}$$

$$J^{(3)}_{ijk} = -8\nu m\frac{Gm}{r^3}\,\mathrm{STF}_{ijk}\,\varepsilon_{kab}x^{ai}v^{bj}\left[1-3\nu+\frac{\gamma}{90}\,(-103+425\nu-275\nu^2)\right]\,, \tag{C1g}$$

$$J^{(4)}_{ijkl} = 3\nu m(1-2\nu)(X_2-X_1)\,\frac{Gm}{r^3}\,\mathrm{STF}_{ijkl}\left\{\varepsilon_{lab}x^av^b\left(-20x^iv^{jk} + 7\,\frac{Gm}{r^3}\,x^{ijk}\right)\right\}\,, \tag{C1h}$$

$$J^{(5)}_{ijklm} = 32\nu m(1-5\nu+5\nu^2)\,\frac{Gm}{r^3}\,\mathrm{STF}_{ijklm}\left\{\varepsilon_{mab}x^av^b\left(-15x^iv^{jkl} + 17\,\frac{Gm}{r^3}\,x^{ijk}v^l\right)\right\}\,. \tag{C1i}$$

For the energy loss,

$$I^{(3)}_{ij} = -8\nu m\,\mathrm{STF}_{ij}\frac{Gm}{r^3}v^ix^j\left\{1-\frac{\gamma}{42}(149-69\nu) + \frac{\gamma^2}{1512}(7043-7837\nu+3703\nu^2)\right\}\,, \tag{C2a}$$

$$I^{(4)}_{ijk} = \nu m(X_2-X_1)\,\mathrm{STF}_{ijk}\frac{Gm}{r^3}\left\{[21-\gamma(146-61\nu)]\,x^{ijk}\frac{Gm}{r^3} - [60-\gamma(241-122\nu)]v^{ij}x^k\right\}\,, \tag{C2b}$$

$$I^{(5)}_{ijkl} = -8\nu m(1-3\nu)\,\mathrm{STF}_{ijkl}\frac{Gm}{r^3}\left\{60v^{ijk}x^l - 68\frac{Gm}{r^3}x^{ijk}v^l\right\}\,, \tag{C2c}$$

$$J^{(3)}_{ij} = \nu m(X_2-X_1)\,\mathrm{STF}_{ij}\frac{Gm}{r^3}\left\{-1+\frac{\gamma}{28}(17-20\nu)\right\}\varepsilon^{jab}x^av^{bi}\,, \tag{C2d}$$

$$J^{(4)}_{ijk} = -8\nu m(1-3\nu)\,\mathrm{STF}_{ijk}\frac{Gm}{r^3}\varepsilon^{iab}\left\{x^av^{bjk} - \frac{Gm}{r^3}x^{ajk}v^b\right\}\,. \tag{C2e}$$

The TT projections of relevant contractions of time derivatives of moments with $\mathbf{N}$ as used in the waveform are

$$\mathcal{P}_{ijkm}I^{(2)}_{ij} = 2\nu m\mathcal{P}_{ijkm}\left\{v^{ij} - \frac{Gm}{r}\,n^{ij} + \frac{\gamma}{42}\,\frac{Gm}{r}\,n^{ij}(149-69\nu) - \frac{\gamma}{42}\,(23-27\nu)v^{ij} + \frac{\gamma^2}{1512}\,\frac{Gm}{r}\,n^{ij}(-7043+7837\nu-3703\nu^2) + \frac{\gamma^2}{1512}v^{ij}(-4513-19\,591\nu+1219\nu^2)\right\}\,, \tag{C3a}$$

$$\mathcal{P}_{ijkm}I^{(3)}_{ija}N_a = \nu m(X_2-X_1)\mathcal{P}_{ijkm}\left\{6(vN)v^{ij} - 7(vN)\frac{Gm}{r}n^{ij} - 14(nN)\frac{Gm}{r}n^iv^j - \gamma(vN)(7-8\nu)v^{ij} + \frac{1}{3}(83-40\nu)\gamma(vN)\frac{Gm}{r}n^{ij} + \frac{2}{3}(83-40\nu)\gamma(nN)\frac{Gm}{r}n^iv^j\right\}\,, \tag{C3b}$$

$$\mathcal{P}_{ijkm}I^{(4)}_{ijab}\,N_{ab} = \nu m\mathcal{P}_{ijkm}\Bigg\{(1-3\nu)\Bigg[24(vN)^2v^{ij}+\frac{8}{7}\,\frac{Gm}{r}\,v^{ij}$$
$$-32\,\frac{Gm}{r}\,(nN)^2v^{ij}-32(vN)^2\,\frac{Gm}{r}\,n^{ij}\;-\;\frac{8}{7}\,\frac{G^2m^2}{r^2}\,n^{ij}$$
$$+40(nN)^2\,\frac{G^2m^2}{r^2}\,n^{ij}-128(nN)(vN)\,\frac{Gm}{r}\,n^iv^j\Bigg]$$
$$+\gamma\Bigg[\frac{12}{385}\,(-109+325\nu+5\nu^2)\,\frac{Gm}{r}\,v^{ij}+\frac{12}{11}\,(-41+183\nu-139\nu^2)(vN)^2v^{ij}$$
$$+\frac{48}{55}\,(161-585\nu+255\nu^2)(nN)^2\,\frac{Gm}{r}\,v^{ij}+\frac{4}{385}\,(657-2075\nu+315\nu^2)\,\frac{G^2m^2}{r^2}\,n^{ij}$$
$$+\frac{48}{55}\,(161-585\nu+255\nu^2)\,\frac{Gm}{r}\,(vN)^2n^{ij}+\frac{4}{55}\,(-3909+13\,495\nu-4695\nu^2)(nN)^2\,\frac{G^2m^2}{r^2}\,n^{ij}$$
$$+\;\frac{192}{55}\,(161-585\nu+255\nu^2)(nN)(vN)\,\frac{Gm}{r}\,n^iv^j\Bigg]\Bigg\}\;, \tag{C3c}$$

$$\mathcal{P}_{ijkm}I^{(5)}_{ijabc}\,N_{abc} = 5\nu m(1-2\nu)(X_2-X_1)\mathcal{P}_{ijkm}\Bigg[24\left\{(Nv)^3v^{ij}-\frac{v^2}{3}(Nv)v^{ij}\right\}$$
$$-36\,\frac{Gm}{r}\Bigg\{n^{ij}(Nv)^3+6n^iv^j(Nn)(Nv)^2+3(Nn)^2(Nv)v^{ij}$$
$$-\,\frac{1}{3}\,(Nv)v^{ij}-\frac{1}{3}v^2(Nv)n^{ij}-\frac{2}{3}(Nn)v^2n^iv^j\Bigg\}$$
$$+\frac{241}{5}\,\frac{G^2m^2}{r^2}\,\left\{3n^{ij}(Nn)^2(Nv)+2n^iv^j(Nn)^3-\,\frac{1}{3}\,n^{ij}(Nv)-\frac{2}{3}n^iv^j(Nn)\right\}\Bigg]\;, \tag{C3d}$$

$$\mathcal{P}_{ijkm}\,I^{(6)}_{ijabcd}\,N_{abcd} = 24\nu m(1-5\nu+5\nu^2)\mathcal{P}_{ijkm}\Bigg[30\left\{v^{ij}(Nv)^4-\frac{6}{11}\,v^2v^{ij}(Nv)^2+\frac{1}{33}\,v^4v^{ij}\right\}$$
$$-\,50\,\frac{Gm}{r}\Bigg\{n^{ij}(Nv)^4+8n^iv^j(Nn)(Nv)^3+6v^{ij}(Nn)^2(Nv)^2$$
$$-\frac{6}{11}\,[v^2(n^{ij}(Nv)^2+4n^iv^j(Nn)(Nv)+v^{ij}(Nn)^2)+v^{ij}(Nv)^2]+\,\frac{1}{33}\,(2v^2v^{ij}+v^4n^{ij})\Bigg\}$$
$$+\,\frac{214}{3}\,\frac{G^2m^2}{r^2}\Bigg\{6n^{ij}(Nn)^2(Nv)^2+(Nn)^4v^{ij}+8n^iv^j(Nn)^3(Nv)$$
$$-\,\frac{6}{11}\,[n^{ij}(Nv)^2+4n^iv^j(Nn)(Nv)+(Nn)^2v^{ij}+v^2n^{ij}(Nn)^2]+\,\frac{1}{33}\,(v^{ij}+2v^2n^{ij})\Bigg\}$$
$$-\;94\,\frac{G^3m^3}{r^3}\,\left[(nN)^4n^{ij}-\frac{6}{11}(Nn)^2n^{ij}+\frac{1}{33}\,n^{ij}\right]\Bigg]\;, \tag{C3e}$$

$$\mathcal{P}_{ijkm}\varepsilon_{abi}J^{(2)}_{ja}\;N_b = -\,\nu m\,(X_2-X_1)\,\frac{Gm}{r}\,\mathcal{P}_{ijkm}\left(1-\,\frac{\gamma}{28}\,(17-20\nu)\right)[(nN)v^in^j-(vN)n^{ij}]\;, \tag{C3f}$$

$$\mathcal{P}_{ijkm}\varepsilon_{abi}\,J^{(3)}_{jac}\;N_{bc} = -\,\frac{4}{3}\,\nu m\,\frac{Gm}{r}\,\mathcal{P}_{ijkm}\Bigg\{(1-3\nu)\,\left[\,\frac{Gm}{r}\,n^{ij}-v^{ij}+3(nN)^2v^{ij}-3(vN)^2n^{ij}\right]$$
$$+\frac{\gamma}{90}\,\Bigg[\frac{Gm}{r}\,n^{ij}(-373+1325\nu-545\nu^2)$$
$$+[3(nN)^2v^{ij}-3(vN)^2n^{ij}-v^{ij}](-103+425\nu-275\nu^2)\Bigg]\Bigg\}\;, \tag{C3g}$$

$$\mathcal{P}_{ijkm}\varepsilon_{abi}\,J^{(4)}_{jacd}N_{bcd} = 3\nu m(1-2\nu)(X_2-X_1)\,\frac{Gm}{r}\,\mathcal{P}_{ijkm}$$
$$\times\,\Bigg[-\frac{5}{3}\Bigg\{-4v^in^j(Nv)^2(Nn)+8v^{ij}(Nn)^2(Nv)$$
$$+\,2(Nv)v^2n^{ij}+2(Nn)v^2n^iv^j-4n^{ij}(Nv)^3-2v^{ij}(Nv)-\;\frac{v^2}{7}[2v^in^j(Nn)-2n^{ij}(Nv)]\Bigg\}$$
$$+\,\frac{7}{4}\,\frac{Gm}{r}\,\Bigg\{4(Nn)^3v^in^j-4(Nv)(Nn)^2n^{ij}$$
$$-\;\;2(Nn)v^in^j-\,\frac{2}{7}[(Nn)v^in^j-(Nv)n^{ij}]\Bigg\}\Bigg]\;, \tag{C3h}$$

$$\mathcal{P}_{ijkm}\varepsilon_{abi}\,J^{(5)}_{jacde}N_{bcde} = 32\nu m(1-5\nu+5\nu^2)\,\frac{Gm}{r}\mathcal{P}_{ijkm}$$
$$\times\Bigg[-\frac{15}{20}\Big\{-10(Nv)^3(Nn)v^in^j + 6v^2(Nn)(Nv)n^iv^j$$
$$+4v^2(Nv)^2n^{ij} - 5(Nv)^4n^{ij} + 15(Nn)^2(Nv)^2v^{ij}$$
$$-\;3(Nv)^2v^{ij} - v^2(Nn)^2v^{ij} - \frac{1}{3}v^4n^{ij} + \frac{1}{3}v^2v^{ij}\Big\}$$
$$+\frac{17}{20}\frac{Gm}{r}\Big\{10(Nn)^3(Nv)v^in^j - 6(Nn)(Nv)v^in^j$$
$$-15(Nn)^2(Nv)^2n^{ij} + 5(Nn)^4v^{ij} - 4(Nn)^2v^{ij} + 3v^2(Nn)^2n^{ij}$$
$$+(Nv)^2n^{ij} - \frac{1}{3}v^2n^{ij} + \frac{1}{3}v^{ij}\Big\}\Bigg]\;. \qquad \text{(C3i)}$$

Finally the "squares" of time derivatives of moments as used in the energy loss are

$$(I^{(3)}_{ij})^2 = 32(\nu m)^2\frac{G^3m^3}{r^5}\left\{1\;-\;\frac{\gamma}{21}(212-90\nu) + \frac{\gamma^2}{2646}(130\,150 - 76\,007\nu + 31\,442\nu^2)\right\}\;, \qquad \text{(C4a)}$$

$$(I^{(4)}_{ijk})^2 = (\nu m)^2\frac{G^2m^2}{r^4}\frac{(1-4\nu)}{15}\left\{126\frac{G^2m^2}{r^2}[21-2\gamma(146-61\nu)]\right.$$
$$\left. +\;480v^4[30-\gamma(241-122\nu)]\;+18\frac{Gm}{r}v^2[420-\gamma(4607-2074\nu)]\right\}\;, \qquad \text{(C4b)}$$

$$(I^{(5)}_{ijkl})^2 = \frac{512}{7}(\nu m)^2(1-3\nu)^2\frac{G^2m^2}{r^4}\left\{450v^6 + 578\frac{G^2m^2}{r^2}v^2 + 765\frac{Gm}{r}v^4\right\}\;, \qquad \text{(C4c)}$$

$$(J^{(3)}_{ij})^2 = (\nu m)^2\frac{G^2m^2}{r^4}v^4\frac{(1-4\nu)}{2}\left\{1-\frac{\gamma}{14}(17-20\nu)\right\}\;, \qquad \text{(C4d)}$$

$$(J^{(4)}_{ijk})^2 = \frac{128}{15}(\nu m)^2(1-3\nu)^2\frac{G^2m^2}{r^4}\left\{2v^6 + 2\frac{G^2m^2}{r^2}v^2 + \frac{Gm}{r}v^4\right\}\;. \qquad \text{(C4e)}$$

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