The third post-Newtonian gravitational wave polarizations and associated spherical harmonic modes for inspiralling compact binaries in quasi-circular orbits

2008 Class. Quantum Grav. 25 165003

(http://iopscience.iop.org/0264-9381/25/16/165003)

IOP Publishing Classical and Quantum Gravity
Class. Quantum Grav. **25** (2008) 165003 (44pp) doi:10.1088/0264-9381/25/16/165003

The third post-Newtonian gravitational wave polarizations and associated spherical harmonic modes for inspiralling compact binaries in quasi-circular orbits

Luc Blanchet[1], **Guillaume Faye**[1], **Bala R Iyer**[2] **and Siddhartha Sinha**[2,3]

[1] $\mathcal{G}\mathbb{R}\varepsilon\mathbb{C}\mathcal{O}$, Institut d'Astrophysique de Paris—CNRS, 98$^{\rm bis}$ Boulevard Arago, 75014 Paris, France
[2] Raman Research Institute, Bangalore-560 080, India
[3] Department of Physics, Indian Institute of Science, Bangalore-560 012, India

E-mail: blanchet@iap.fr, faye@iap.fr, bri@rri.res.in and p_siddhartha@rri.res.in

Received 9 February 2008, in final form 28 May 2008
Published 30 July 2008
Online at stacks.iop.org/CQG/25/165003

Abstract
The gravitational waveform (GWF) generated by inspiralling compact binaries moving in quasi-circular orbits is computed at the third post-Newtonian (3PN) approximation to general relativity. Our motivation is two-fold: (i) to provide accurate templates for the data analysis of gravitational wave inspiral signals in laser interferometric detectors; (ii) to provide the associated spin-weighted spherical harmonic decomposition to facilitate comparison and match of the high post-Newtonian prediction for the inspiral waveform to the numerically-generated waveforms for the merger and ringdown. This extension of the GWF by half a PN order (with respect to previous work at 2.5PN order) is based on the algorithm of the multipolar post-Minkowskian formalism, and mandates the computation of the relations between the radiative, canonical and source multipole moments for general sources at 3PN order. We also obtain the 3PN extension of the source multipole moments in the case of compact binaries, and compute the contributions of hereditary terms (tails, tails-of-tails and memory integrals) up to 3PN order. The end results are given for both the complete plus and cross polarizations and the separate spin-weighted spherical harmonic modes.

PACS numbers: 04.25.Nx, 04.30.−w, 97.60.Jd, 97.60.Lf

1. Introduction

One of the most important sources of gravitational radiation for the laser interferometric detectors LIGO, VIRGO [1, 2] and LISA [3] is the inspiralling and merging compact binary system. Until the late inspiral, prior to merger, the gravitational waves are accurately described by the post-Newtonian (PN) approximation to general relativity [4], while the late inspiral and subsequent merger and ringdown phases are computed by a full-fledged numerical integration of the Einstein field equations [5–8]. A new field has emerged recently consisting of high-accuracy comparisons between the PN predictions and the numerically-generated waveforms. Such comparisons and matching to the PN results have proved currently to be very successful [9–12]. They clearly show the need to include high PN corrections not only for the evolution of the binary's orbital phase but also for the modulation of the gravitational amplitude.

The aim of this paper is to compute the full gravitational waveform generated by inspiralling compact binaries moving in quasi-circular orbits at the third post-Newtonian (3PN) order[4]. By the full waveform (FWF) at a certain PN order, we mean the waveform including all higher-order amplitude corrections and hence all higher-order harmonics of the orbital frequency consistent with that PN order. The FWF is to be contrasted with the so-called restricted waveform (RWF) which retains only the leading-order harmonic at twice the orbital frequency. In applications to data analysis both the FWF and RWF should incorporate the orbital phase evolution up to the maximum available post-Newtonian order which is currently 3.5PN [13–15]. Previous investigations[16–18] have obtained the FWF up to 2.5PN order[5]. Recently, Kidder [19] pointed out that there is already enough information in the existing PN results [17] to control the *dominant* mode of the waveform, in a spin-weighted spherical harmonic decomposition, at the 3PN order. This mode, having $(\ell, m) = (2, 2)$, is the one which is computed in most numerical simulations, and which is therefore primarily needed for comparison with the PN waveforms. In the present paper, we shall extend the works [16–19] by computing all the spin-weighted spherical harmonic modes (ℓ, m) consistent with the 3PN gravitational polarizations.

The data analysis of ground-based and space-based detectors has traditionally been based on the RWF approximation [20–26]. However, the need to consider the FWF as a more powerful template has been emphasized, not only for performing a more accurate parameter estimation [27–30], but also for improving the mass reach and the detection rate [31–33]. Another motivation for considering the FWF instead of the RWF is to perform cosmological measurements of the Hubble parameter and dark energy using supermassive inspiralling black-hole binaries which are known to constitute standard gravitational wave candles (or sirens) in cosmology [34, 35]. Indeed it has been shown that using the FWF in the data analysis of LISA will yield substantial improvements (with respect to the RWF) of the angular resolution and the estimation of the luminosity distance of gravitational wave sirens [36, 37]. This means that LISA may be able to uniquely identify the galaxy cluster in which the supermassive black-hole coalescence took place, and thereby permit the measurement of the red-shift of the source which is crucially needed for investigating the equation of state of dark energy [36].

It turns out that in order to control the FWF at the 3PN order we need to further develop the multipolar post-Minkowskian (MPM) wave generation formalism [38–43]. The MPM formalism describes the radiation field of any isolated post-Newtonian source and constitutes the basis of current PN calculations[6]. In this formalism, the radiation field is first of all parametrized by means of two sets of radiative multipole moments [47]. These moments are then related (by means of an algorithm for solving the nonlinearities of the field equations) to

[4] As usual, we refer to nPN as the order equivalent to terms $\sim(v/c)^{2n}$ in the asymptotic waveform (beyond the Einstein quadrupole formula), where v denotes the binary's orbital velocity and c is the speed of light.

[5] The computation of the FWF is more demanding than that of the phase because it not only requires multipole moments with higher multipolarity but also higher PN accuracy in many of these multipole moments. This is why the FWF is known to a lower PN order than the phase.

[6] An alternative formalism called DIRE has been developed by Will and collaborators [44–46].

the so-called canonical moments which constitute some useful intermediaries for describing the external field of the source. Finally, the canonical moments are expressed in terms of the operational source moments which are given by explicit integrals extending over the matter source and gravitational field. In previous studies [13, 17, 48, 49] most of the required source moments in the case of compact binaries were computed, or techniques were developed to compute them. The important step which remains here is to refine, by applying the MPM framework, the relationships between the radiative and canonical moments—this means taking into account more nonlinear interactions between multipole moments—and between the canonical and source moments. The latter relationship involves controlling the coordinate transformation between two MPM algorithms respectively defined from the sets of canonical and source moments.

The plan of this paper is as follows. In section 2, we recall the basic formulae for defining the FWF in terms of radiative multipole moments. Sections 3 and 4 apply the MPM formalism to obtain general formulae for relating the radiative moments to the source moments via the canonical moments. Section 5 summarizes the results for all the relevant moments parametrizing the FWF at 3PN order. The time derivatives of source moments are investigated in section 6 and the various hereditary contributions are computed in section 7. The complete polarization waveforms at 3PN order are given in section 8 for data analysis applications. Finally, the spin-weighted spherical harmonic modes of the 3PN waveform are provided in section 9 for use in numerical relativity.

For the benefits of readers we provide in the appendix a list of symbols used in the paper together with their main meaning.

2. The polarization waveforms

The full waveform (FWF) propagating in the asymptotic regions of an isolated source, h^{TT}_{ij}, is the transverse-traceless (TT) projection of the metric deviation at the leading order $1/R$ in the distance $R = |\mathbf{X}|$ to the source, in a radiative-type coordinate system $X^\mu = (cT, \mathbf{X})$. The FWF can be uniquely decomposed [47] into radiative multipole components parametrized by symmetric-trace-free (STF) mass-type moments U_L and current-type ones V_L[7]. The radiative moments are functions of the retarded time $T_R = T - R/c$ in radiative coordinates. By definition we have, up to any multipolar order ℓ,

$$h^{\mathrm{TT}}_{ij} = \frac{4G}{c^2R}\mathcal{P}^{\mathrm{TT}}_{ijkl}(\mathbf{N})\sum_{\ell=2}^{+\infty}\frac{1}{c^\ell \ell!}\left\{N_{L-2}U_{klL-2}(T_R) - \frac{2\ell}{c(\ell+1)}N_{aL-2}\varepsilon_{ab(k}V_{l)bL-2}(T_R)\right\} + \mathcal{O}\left(\frac{1}{R^2}\right). \tag{2.1}$$

Here $\mathbf{N} = \mathbf{X}/R = (N_i)$ is the unit vector pointing from the source to the far away detector. The TT projection operator in (2.1) reads $\mathcal{P}^{\mathrm{TT}}_{ijkl} = \mathcal{P}_{ik}\mathcal{P}_{jl} - \frac{1}{2}\mathcal{P}_{ij}\mathcal{P}_{kl}$ where $\mathcal{P}_{ij} = \delta_{ij} - N_iN_j$ is the projector orthogonal to the unit direction $\mathbf{N}$. We introduce two unit polarization vectors $\mathbf{P}$ and $\mathbf{Q}$, orthogonal and transverse to the direction of propagation $\mathbf{N}$ (hence $\mathcal{P}_{ij} = P_iP_j + Q_iQ_j$).

[7] The notation is: $L = i_1\cdots i_\ell$ for a multi-index composed of ℓ multipolar spatial indices $i_1,\ldots,i_\ell$ (ranging from 1 to 3); similarly $L-1 = i_1\cdots i_{\ell-1}$ and $aL-2 = ai_1\cdots i_{\ell-2}$; $N_L = N_{i_1}\cdots N_{i_\ell}$ is the product of ℓ spatial vectors N_i (similarly for $x_L = x_{i_1}\cdots x_{i_\ell}$); $\partial_L = \partial_{i_1}\cdots\partial_{i_\ell}$ and say $\partial_{aL-2} = \partial_a\partial_{i_1}\cdots\partial_{i_{\ell-2}}$ denote the product of partial derivatives $\partial_i = \partial/\partial x^i$; in the case of summed-up (dummy) multi-indices L, we do not write the ℓ summations from 1 to 3 over their indices; the STF projection is indicated using brackets, $T_{\langle L\rangle} = \mathrm{STF}[T_L]$; thus $U_L = U_{\langle L\rangle}$ and $V_L = V_{\langle L\rangle}$ for STF moments; for instance we write $x_{\langle i}v_{j\rangle} = \frac{1}{2}(x_iv_j + x_jv_i) - \frac{1}{3}\delta_{ij}\mathbf{x}\cdot\mathbf{v}$; ε_{abc} is the Levi-Civita antisymmetric symbol such that $\varepsilon_{123} = 1$; time derivatives are denoted with a superscript (n).

Our convention for the choice of **P** and **Q** will be clarified in section 8. Then the two 'plus' and 'cross' polarization states of the FWF are defined by

$$\begin{pmatrix} h_+ \\ h_\times \end{pmatrix} = \frac{4G}{c^2R}\begin{pmatrix} \frac{P_iP_j-Q_iQ_j}{2} \\ \frac{P_iQ_j+P_jQ_i}{2} \end{pmatrix}\sum_{\ell=2}^{+\infty}\frac{1}{c^\ell \ell!}\left\{N_{L-2}U_{ijL-2}(T_R) - \frac{2\ell}{c(\ell+1)}N_{aL-2}\varepsilon_{ab(i}V_{j)bL-2}(T_R)\right\} + \mathcal{O}\left(\frac{1}{R^2}\right). \tag{2.2}$$

Although the multipole decompositions (2.1) and (2.2) are all what we need for our purpose, it will also be important, having in view the ongoing comparisons between the PN and numerical results [9–12], to consider separately the various modes (ℓ, m) of the FWF as defined with respect to a basis of spin-weighted spherical harmonics. To this end we decompose h_+ and $h_\times$ in the standard way as (see, e.g., [9, 19])

$$h_+ - \mathrm{i}h_\times = \sum_{\ell=2}^{+\infty}\sum_{m=-\ell}^{\ell} h^{\ell m} Y^{\ell m}_{-2}(\Theta, \Phi), \tag{2.3}$$

where the spin-weighted spherical harmonics of weight -2 is function of the spherical angles (Θ, Φ) defining the direction of propagation $\mathbf{N}$,[8] and is given by

$$Y^{\ell m}_{-2} = \sqrt{\frac{2\ell+1}{4\pi}}d_2^{\ell m}(\Theta)\,\mathrm{e}^{\mathrm{i}m\Phi}, \tag{2.4a}$$

$$d_2^{\ell m} = \sum_{k=k_1}^{k_2}\frac{(-)^k}{k!}\frac{\sqrt{(\ell+m)!(\ell-m)!(\ell+2)!(\ell-2)!}}{(k-m+2)!(\ell+m-k)!(\ell-k-2)!}\left(\cos\frac{\Theta}{2}\right)^{2\ell+m-2k-2}\left(\sin\frac{\Theta}{2}\right)^{2k-m+2}. \tag{2.4b}$$

Here $k_1 = \max(0, m-2)$ and $k_2 = \min(\ell+m, \ell-2)$. Using the orthonormality properties of these harmonics we obtain the separate modes $h^{\ell m}$ from the surface integral

$$h^{\ell m} = \int \mathrm{d}\Omega[h_+ - \mathrm{i}h_\times]\overline{Y}^{\ell m}_{-2}(\Theta, \Phi), \tag{2.5}$$

where the bar or overline denotes the complex conjugate. On the other hand, we can also, following [19], relate $h^{\ell m}$ directly to the multipole moments U_L and V_L. The result is[9]

$$h^{\ell m} = -\frac{G}{\sqrt{2}Rc^{\ell+2}}\left[U^{\ell m} - \frac{\mathrm{i}}{c}V^{\ell m}\right], \tag{2.6}$$

where $U^{\ell m}$ and $V^{\ell m}$ are the radiative mass and current moments in standard (non-STF) guise [19]. These are related to the STF moments by

$$U^{\ell m} = \frac{4}{\ell!}\sqrt{\frac{(\ell+1)(\ell+2)}{2\ell(\ell-1)}}\alpha_L^{\ell m}U_L, \tag{2.7a}$$

$$V^{\ell m} = -\frac{8}{\ell!}\sqrt{\frac{\ell(\ell+2)}{2(\ell+1)(\ell-1)}}\alpha_L^{\ell m}V_L. \tag{2.7b}$$

Here $\alpha_L^{\ell m}$ denotes the STF tensor connecting together the usual basis of spherical harmonics $Y^{\ell m}$ to the set of STF tensors $N_{\langle L\rangle} = N_{\langle i_1}\cdots N_{i_\ell\rangle}$ (where the brackets indicate the STF

[8] For the data analysis of compact binaries in section 8 the direction of propagation will be defined by the angles $(\Theta, \Phi) = (i, \frac{\pi}{2})$ where i is the inclination angle of the orbit over the plane of the sky.

[9] We have an overall sign difference with [19] due to a different choice for the polarization triad $(\mathbf{N}, \mathbf{P}, \mathbf{Q})$.

projection). Indeed both $Y^{\ell m}$ and $N_{\langle L\rangle}$ are basis of an irreducible representation of weight ℓ of the rotation group. They are related by

$$N_{\langle L\rangle}(\Theta,\Phi) = \sum_{m=-\ell}^{\ell} \alpha_L^{\ell m} Y^{\ell m}(\Theta,\Phi), \tag{2.8a}$$

$$Y^{\ell m}(\Theta,\Phi) = \frac{(2\ell+1)!!}{4\pi l!}\overline{\alpha}_L^{\ell m} N_{\langle L\rangle}(\Theta,\Phi), \tag{2.8b}$$

with the STF tensorial coefficient being[10]

$$\alpha_L^{\ell m} = \int \mathrm{d}\Omega\, N_{\langle L\rangle}\overline{Y}^{\ell m}. \tag{2.9}$$

As observed in [19] this is especially useful if some of the radiative moments are known to higher PN order than others. In this case, the comparison with the numerical calculation for these individual modes can be made at higher PN accuracy.

3. Relation between the radiative and canonical moments

The basis of our computation of the radiative moments is the multipolar-post-Minkowskian (MPM) formalism [38–43] which iterates the general solution of the Einstein field equations outside an isolated matter system in the form of a post-Minkowskian or nonlinearity expansion. The formalism is then supplemented by a matching to the PN gravitational field valid in the near zone of the source. In this section and the next one we sketch the main features of the MPM iteration of the exterior field while limiting ourselves to quadratic nonlinear order because this is what we need for the new terms required in the FWF at 3PN order[11]. We shall work with harmonic coordinates $x^\mu = (ct, \mathbf{x})$, which means that

$$\partial_\mu h^{\alpha\mu} = 0, \tag{3.1}$$

where the 'gothic' metric deviation reads $h^{\alpha\beta} = \sqrt{-g}g^{\alpha\beta} - \eta^{\alpha\beta}$, with g the determinant and $g^{\alpha\beta}$ the inverse of the usual covariant metric, and with $\eta^{\alpha\beta} = \mathrm{diag}(-1,1,1,1)$ being an auxiliary Minkowskian metric[12]. Up to quadratic nonlinear order the vacuum Einstein field equations take the form

$$\Box h^{\alpha\beta} = N_2^{\alpha\beta}(h) + \mathcal{O}(h^3), \tag{3.2}$$

where $\Box = \eta^{\mu\nu}\partial_\mu\partial_\nu$ is the flat spacetime d'Alembertian operator, and where $N_2^{\alpha\beta}$ denotes the quadratic part of the gravitational source term in harmonic coordinates—a quadratic functional of h and its first and second spacetime derivatives given explicitly by

$$\begin{aligned}N_2^{\alpha\beta}(h) = {}& -h^{\mu\nu}\partial_\mu\partial_\nu h^{\alpha\beta} + \tfrac{1}{2}\partial^\alpha h_{\mu\nu}\partial^\beta h^{\mu\nu} - \tfrac{1}{4}\partial^\alpha h\partial^\beta h \\ & - 2\partial^{(\alpha}h_{\mu\nu}\partial^\mu h^{\beta)\nu} + \partial_\nu h^{\alpha\mu}\left(\partial^\nu h^\beta_\mu + \partial_\mu h^{\beta\nu}\right) \\ & + \eta^{\alpha\beta}\left[-\tfrac{1}{4}\partial_\rho h_{\mu\nu}\partial^\rho h^{\mu\nu} + \tfrac{1}{8}\partial_\mu h\partial^\mu h + \tfrac{1}{2}\partial_\mu h_{\nu\rho}\partial^\nu h^{\mu\rho}\right],\end{aligned} \tag{3.3}$$

with $h = \eta^{\mu\nu}h_{\mu\nu}$. The four-divergence of this source term reads

$$\partial_\mu N_2^{\alpha\mu} = -\tfrac{1}{4}\partial^\alpha h\,\Box h + \tfrac{1}{2}\left[\partial^\alpha h_{\mu\nu} - 2\partial_\mu h^\alpha_\nu\right]\Box h^{\mu\nu}. \tag{3.4}$$

[10] The notation used in [19, 47] is related to ours by $\mathcal{Y}_L^{\ell m} = \frac{(2\ell+1)!!}{4\pi\ell!}\overline{\alpha}_L^{\ell m}$.

[11] Cubic nonlinearities do contribute at the 3PN order in the form of 'tail-of-tails' but those have already been computed [50].

[12] Beware of the fact that the TT waveform defined by (2.1) differs by a sign from the spatial components of the gothic metric deviation, $h_{ij}^{\mathrm{TT}} = -\mathcal{P}_{ijkl}^{\mathrm{TT}}h_{kl} + \mathcal{O}(h^2)$.

In this paper, we shall consider two explicit constructions of the quadratic-order external metric following the MPM formalism. The first construction, dealt with in this section, will be parametrized by two (and only two) sets of moments, mass moments M_L and current moments S_L, which are referred to as the *canonical multipole moments*. The canonical moments are crucially distinct from the radiative moments U_L and V_L, and the MPM construction will provide the relations linking them to U_L, V_L. The second construction (in section 4) will deal with the link between M_L, S_L and *six* sets of moments I_L, J_L, W_L, X_L, Y_L and Z_L collectively named the *source multipole moments*. Among these, the moments I_L (mass-type) and J_L (current-type) play the most important role, while for reasons explained below the other moments W_L, X_L, Y_L and Z_L are called the *gauge multipole moments* and will appear to be subdominant.

Armed with such definitions, the computation of the radiative field (2.1) and (2.2) proceeds in a modular way (see section 6 of [43] for further discussion). We start with relating the radiative moments $\{U_L, V_L\}$ to the canonical moments $\{M_L, S_L\}$ which are to be viewed as convenient intermediate constructs relating the radiation field and the matter source. The canonical moments are then in turn connected to the actual multipole moments of the source $\{I_L, J_L, W_L, X_L, Y_L, Z_L\}$. The point of the above strategy is that the source moments admit closed-form expressions as integrals over the stress-energy distribution of the matter and gravitational fields. The expressions of $I_L, \ldots, Z_L$ for general sources are given by (5.15)–(5.20) in [43] and shall not be reproduced here[13]. Note that the above formalism can be applied only to PN sources, which remain confined in their own near zone; the final expressions of the source moments are valid only for sources that are semi-relativistic like inspiralling compact binaries.

Consider the so-called canonical construction of the MPM metric in harmonic coordinates, designated that way because it is based on Thorne's [47] canonical expression for the linearized approximation $h^{\alpha\beta}_{\text{can}\,1}$ (given by (3.6)). The MPM metric is parametrized by the canonical multipole moments M_L and S_L and reads, to quadratic order,

$$h^{\alpha\beta}_{\text{can}} = G h^{\alpha\beta}_{\text{can}\,1}[M_L, S_L] + G^2 h^{\alpha\beta}_{\text{can}\,2}[M_L, S_L] + \mathcal{O}(G^3), \tag{3.5}$$

where the Newton constant G is introduced as a convenient book-keeping parameter for labelling the successive nonlinear approximations. From (3.1) and (3.2) the linearized approximation $h^{\alpha\beta}_{\text{can}\,1}$ obviously satisfies $\partial_\mu h^{\alpha\mu}_{\text{can}\,1} = 0$ together with $\Box h^{\alpha\beta}_{\text{can}\,1} = 0$. Following [38, 39] we adopt the following explicit retarded solution of these equations:

$$h^{00}_{\text{can}\,1} = -\frac{4}{c^2}\sum_{\ell=0}^{\infty}\frac{(-)^\ell}{\ell!}\partial_L[r^{-1}M_L(t-r/c)], \tag{3.6a}$$

$$h^{0i}_{\text{can}\,1} = \frac{4}{c^3}\sum_{\ell=1}^{\infty}\frac{(-)^\ell}{\ell!}\left\{\partial_{L-1}\big[r^{-1}M^{(1)}_{iL-1}(t-r/c)\big] + \frac{\ell}{\ell+1}\varepsilon_{iab}\partial_{aL-1}[r^{-1}S_{bL-1}(t-r/c)]\right\}, \tag{3.6b}$$

$$h^{ij}_{\text{can}\,1} = -\frac{4}{c^4}\sum_{\ell=2}^{\infty}\frac{(-)^\ell}{\ell!}\left\{\partial_{L-2}\big[r^{-1}M^{(2)}_{ijL-2}(t-r/c)\big] + \frac{2\ell}{\ell+1}\partial_{aL-2}\big[r^{-1}\varepsilon_{ab(i}S^{(1)}_{j)bL-2}(t-r/c)\big]\right\}, \tag{3.6c}$$

with $F^{(n)}(t)$ denoting n time derivatives of $F(t)$. These expressions represent the most general solution of the vacuum linearized field equations modulo a change of gauge [47].

[13] Below we give the source moments needed at the 3PN order in a form already reduced to the case of compact binaries in circular orbits.

Next, the quadratically nonlinear term $h^{\alpha\beta}_{\text{can}\,2}$—and subsequently all nonlinear terms $h^{\alpha\beta}_{\text{can}\,n}$—is constructed by the following algorithm. We first define

$$u^{\alpha\beta}_{\text{can}\,2} = \mathop{\text{FP}}_{B=0} \Box^{-1}_{\text{ret}} \left[\left(\frac{r}{r_0} \right)^B N_2^{\alpha\beta}(h_{\text{can}\,1}) \right], \tag{3.7}$$

where $\Box^{-1}_{\text{ret}}$ represents the ordinary (flat) retarded integral operator acting on the source $N_2^{\alpha\beta}(h_{\text{can}\,1})$ which is obtained by insertion of the linearized metric (3.6) into the quadratic source term given by (3.3). The symbol $\text{FP}_{B=0}$ refers to a specific operation of taking the finite part when the complex number B tends to zero. Such a finite part involves the multiplication of the source term by a regularization factor $(r/r_0)^B$, where r_0 represents an arbitrary constant length scale (and $B \in \mathbb{C}$). The finite part is necessary for dealing with multipolar expansions which are singular at the origin $r = 0$ (like in (3.6)). It will not be further detailed here and we refer to [38, 41, 43] for full details. The point is that the object (3.7) obeys the d'Alembertian equation we want to solve, namely

$$\Box\, u^{\alpha\beta}_{\text{can}\,2} = N_2^{\alpha\beta}(h_{\text{can}\,1}). \tag{3.8}$$

However, such a solution is *a priori* not divergenceless and so the harmonic coordinate condition needs not to be satisfied. To obtain a solution which is divergenceless we add to $u^{\alpha\beta}_{\text{can}\,2}$ another piece $v^{\alpha\beta}_{\text{can}\,2}$ defined as follows. Computing the divergence $w^{\alpha}_{\text{can}\,2} = \partial_\mu u^{\alpha\mu}_{\text{can}\,2}$, we readily find

$$w^{\alpha}_{\text{can}\,2} = \mathop{\text{FP}}_{B=0} \Box^{-1}_{\text{ret}} \left[B \left(\frac{r}{r_0} \right)^B \frac{n_i}{r} N_2^{\alpha i}(h_{\text{can}\,1}) \right], \tag{3.9}$$

where we used the fact that the source term of (3.8), as an immediate consequence of (3.4), is divergenceless, $\partial_\beta N^{\alpha\beta}_{\text{can}\,2} = 0$. Again, because the source term is divergenceless, the divergence $w^{\alpha}_{\text{can}\,2}$ must be a (retarded) solution of the source-free d'Alembertian equation, $\Box\, w^{\alpha}_{\text{can}\,2} = 0$. This can also be checked from the fact that there is a factor B explicit in the source of (3.9) (appearing because of the differentiation of the regularization factor r^B), and therefore the finite part at $B = 0$ is actually equal to the residue in the Laurent expansion when $B \to 0$, and is necessarily a retarded solution of the source-free equation [38].

Given any vector of the type $w^{\alpha}_{\text{can}\,2}$, i.e. one which is of the form of a retarded solution of the d'Alembertian equation, we can always find four sets of STF tensors N_L, P_L, Q_L and R_L such that the following decomposition holds:

$$w^{0}_{\text{can}\,2} = \sum_{\ell=0}^{+\infty} \partial_L[r^{-1} N_L(t - r/c)], \tag{3.10a}$$

$$w^{i}_{\text{can}\,2} = \sum_{\ell=0}^{+\infty} \partial_{iL}[r^{-1} P_L(t - r/c)] + \sum_{\ell=1}^{+\infty} \{\partial_{L-1}[r^{-1} Q_{iL-1}(t - r/c)] + \varepsilon_{iab}\partial_{aL-1}[r^{-1} R_{bL-1}(t - r/c)]\}. \tag{3.10b}$$

From this decomposition (which is unique) we define the object $v^{\alpha\beta}_{\text{can}\,2}$ by the formulae[14]

$$v^{00}_{\text{can}\,2} = -r^{-1} N^{(-1)} + \partial_a\big[r^{-1}\big(-N_a^{(-1)} + Q_a^{(-2)} - 3P_a\big)\big], \tag{3.11a}$$

$$v^{0i}_{\text{can}\,2} = r^{-1}\big(-Q_i^{(-1)} + 3P_i^{(1)}\big) - \varepsilon_{iab}\partial_a\big[r^{-1} R_b^{(-1)}\big] - \sum_{\ell=2}^{+\infty} \partial_{L-1}[r^{-1} N_{iL-1}], \tag{3.11b}$$

[14] We are adopting here a modified version of the MPM algorithm (with respect to [38]) as proposed in [51].

$$v^{ij}_{\mathrm{can}\,2} = -\delta_{ij} r^{-1} P + \sum_{\ell=2}^{+\infty} \big\{ 2\delta_{ij}\partial_{L-1}[r^{-1}P_{L-1}] - 6\partial_{L-2(i}[r^{-1}P_{j)L-2}] + \partial_{L-2}\big[r^{-1}\big(N^{(1)}_{ijL-2} + 3P^{(2)}_{ijL-2} - Q_{ijL-2}\big)\big] - 2\partial_{aL-2}[r^{-1}\varepsilon_{ab(i}R_{j)bL-2}]\big\}. \tag{3.11c}$$

The superscript $(-p)$ denotes the time anti-derivatives (i.e. time integrals) of the moments. Such anti-derivatives yield some secular losses of mass and momenta by gravitational radiation which have been checked to agree with the corresponding gravitational radiation fluxes, see, e.g., (4.12) in [51]. The formulae (3.11) have been conceived in such a way that the divergence of the so defined $v^{\alpha\beta}_{\mathrm{can}\,2}$ cancels out the divergence of $u^{\alpha\beta}_{\mathrm{can}\,2}$ which is $w^{\alpha}_{\mathrm{can}\,2}$. In the following, we shall denote by $\mathcal{V}^{\alpha\beta}$ the operation for going from a vector such as (3.10)—a retarded solution of the source-free wave equation—to the tensor (3.11). We therefore pose

$$v^{\alpha\beta}_{\mathrm{can}\,2} = \mathcal{V}^{\alpha\beta}\,[w_{\mathrm{can}\,2}], \tag{3.12}$$

and as mentioned before this tensor immediately satisfies $\Box\, v^{\alpha\beta}_{\mathrm{can}\,2} = 0$ (which is obvious) and also

$$\partial_\mu v^{\alpha\mu}_{\mathrm{can}\,2} = -w^{\alpha}_{\mathrm{can}\,2}. \tag{3.13}$$

This property can be directly checked from (3.11) and (3.10). Finally, it is clear from (3.8) and (3.13) that by posing

$$h^{\alpha\beta}_{\mathrm{can}\,2} = u^{\alpha\beta}_{\mathrm{can}\,2} + v^{\alpha\beta}_{\mathrm{can}\,2}, \tag{3.14}$$

we solve the Einstein vacuum field equations at quadratic order, namely

$$\Box\, h^{\alpha\beta}_{\mathrm{can}\,2} = N^{\alpha\beta}_{\mathrm{can}\,2}, \tag{3.15a}$$

$$\partial_\mu h^{\alpha\mu}_{\mathrm{can}\,2} = 0. \tag{3.15b}$$

The MPM algorithm can be extended to any post-Minkowskian order n.

The structure of the quadratic metric $h^{\alpha\beta}_{\mathrm{can}\,2}$ so constructed has been investigated in previous works [40, 51]. It consists of two types of terms: those which depend on the source moments at a single instant, namely the current retarded time $t - r/c$, referred to as *instantaneous* terms, and the other ones which are sensitive to the entire 'past history' of the source, i.e. which depend on all previous times ($\tau \leqslant t - r/c$), and are referred to as the *hereditary* terms. The hereditary terms are themselves composed of three types of contributions, the tail integrals—made from interaction between the mass of the source M and the time-varying moments M_L and S_L (having $\ell \geqslant 2$)—the memory integrals responsible for the so-called nonlinear memory or Christodoulou effect [52–54] (investigated within the present approach in [40, 51]), and semi-hereditary integrals which are in the form of simple anti-derivatives of instantaneous terms and are associated with the secular variations of the mass, linear momentum and angular momentum. The semi-hereditary integrals are given by the time anti-derivatives present in the formula (3.11).

To obtain the radiative moments we expand the metric at future null infinity in a radiative coordinate system $X^\mu = (cT, X^i)$, which is such that the metric admits an expansion in simple powers of $1/R$ without the logarithms which plague the harmonic coordinate system $x^\mu = (ct, x^i)$ [39]. Up to quadratic order and for all multipole interactions we consider, we find that it is sufficient to define for the radiative coordinates $X^i = x^i$ and (denoting $T_R = T - R/c$)

$$T_R = t - \frac{r}{c} - \frac{2GM}{c^3}\ln\left(\frac{r}{r_0}\right) + \mathcal{O}(G^2), \tag{3.16}$$

where r_0 is the length scale introduced in (3.7). Expanding the metric when $R \to \infty$ with $T_R = \text{const}$, and applying the TT projection we obtain the radiative moments U_L and V_L we are seeking by comparing with their definition in (2.1). At linear order the radiative moments agree with the ℓ th time derivatives of the canonical moments, $M_L^{(\ell)}$ and $S_L^{(\ell)}$. At quadratic order we find that tail and nonlinear memory terms appear; these have already been investigated in [40, 51, 50][15]. Their general structure will also be given in (5.1). Finally, we have numerous instantaneous terms whose determination necessitates the straightforward but long implementation of the MPM algorithm (3.7)–(3.14). This is the work required here: we have implemented the MPM algorithm in a Mathematica program to obtain all the instantaneous terms needed to control the 3PN waveform. The presentation of the results is postponed to section 5.1.

4. Relation between the canonical and source moments

4.1. General method

We next need to connect the canonical moments $\{M_L, S_L\}$ to a convenient choice of moments that are suitably defined to play the role of source moments. As it turns out, the source moments are best represented by six multipole moments $\{I_L, J_L, W_L, X_L, Y_L, Z_L\}$ admitting closed-form expressions in the form of integrals over the source and the gravitational field. To define them we consider a MPM construction which is more general than the one given by (3.5), namely (still up to quadratic order)

$$h^{\alpha\beta}_{\text{gen}} = G h^{\alpha\beta}_{\text{gen}\,1}[I_L, J_L, W_L, \ldots, Z_L] + G^2 h^{\alpha\beta}_{\text{gen}\,2}[I_L, J_L, W_L, \ldots, Z_L] + \mathcal{O}(G^3), \tag{4.1}$$

where the linearized metric $h^{\alpha\beta}_{\text{gen}\,1}$ is defined by the canonical expression $h^{\alpha\beta}_{\text{can}\,1}$ explicitly given in (3.6) but parametrized by $\{I_L, J_L\}$ instead of $\{M_L, S_L\}$, and augmented by a linearized *gauge transformation* associated with some vector φ^α_1 parametrized by the remaining moments $\{W_L, X_L, Y_L, Z_L\}$ which can thus rightly be called the gauge moments. Thus,

$$h^{\alpha\beta}_{\text{gen}\,1} = h^{\alpha\beta}_{\text{can}\,1}[I_L, J_L] + \partial\varphi^{\alpha\beta}_1[W_L, X_L, Y_L, Z_L], \tag{4.2}$$

where for any vector φ^α_1 we denote the gauge transformation by

$$\partial\varphi^{\alpha\beta}_1 = \partial^\alpha\varphi^\beta_1 + \partial^\beta\varphi^\alpha_1 - \eta^{\alpha\beta}\partial_\mu\varphi^\mu_1. \tag{4.3}$$

Note that $\partial_\mu\partial\varphi^{\alpha\mu}_1 = \Box\varphi^\alpha_1$. The expression of φ^α_1 in terms of the gauge moments is

$$\varphi^0_1 = \frac{4}{c^3}\sum_{\ell\geqslant 0}\frac{(-)^\ell}{\ell!}\partial_L[r^{-1}W_L(t-r/c)], \tag{4.4a}$$

$$\varphi^i_1 = -\frac{4}{c^4}\sum_{\ell\geqslant 0}\frac{(-)^\ell}{\ell!}\partial_{iL}[r^{-1}X_L(t-r/c)] \tag{4.4b}$$

$$-\frac{4}{c^4}\sum_{\ell\geqslant 1}\frac{(-)^\ell}{\ell!}\left\{\partial_{L-1}[r^{-1}Y_{iL-1}(t-r/c)] + \frac{\ell}{\ell+1}\varepsilon_{iab}\partial_{aL-1}[r^{-1}Z_{bL-1}(t-r/c)]\right\}. \tag{4.4c}$$

The quadratic metric $h^{\alpha\beta}_{\text{gen}\,2}$ will now be defined by the same algorithm as for the canonical metric in section 3 but starting from the general linearized metric (4.2). The result will be

[15] The semi-hereditary integrals associated with secular gravitational radiation losses do not contribute to the radiative moments.

another MPM metric (both the canonical and general metrics are legitimate to describe the exterior field of any isolated matter source [38]) and we shall look for the relation between $\{M_L, S_L\}$ and $\{I_L, J_L, W_L, X_L, Y_L, Z_L\}$ which is necessary in order that these two metrics differ by a coordinate transformation (at quadratic order), and therefore describe the same physical matter source.

To proceed, we have to define

$$u^{\alpha\beta}_{\text{gen}\,2} = \mathop{\text{FP}}_{B=0} \Box^{-1}_{\text{ret}} \left[\left(\frac{r}{r_0} \right)^B N^{\alpha\beta}_2(h_{\text{gen}\,1}) \right]. \tag{4.5}$$

The only difference with (3.7) is that the quadratic source (3.3) is computed from $h^{\alpha\beta}_{\text{gen}\,1}$ instead of $h^{\alpha\beta}_{\text{can}\,1}$. However, since the two linear metrics $h^{\alpha\beta}_{\text{gen}\,1}$ and $h^{\alpha\beta}_{\text{can}\,1}$ differ by the gauge transformation (4.3) the difference between the corresponding sources must have a specific structure, and we find

$$N^{\alpha\beta}_2(h_{\text{gen}\,1}) = N^{\alpha\beta}_2(h_{\text{can}\,1}) + \Box\,\Omega^{\alpha\beta}_2 + \partial\Delta^{\alpha\beta}_2. \tag{4.6}$$

We employ the notation (4.3) for the gauge term $\partial\Delta^{\alpha\beta}_2$. The expressions of the tensor $\Omega^{\alpha\beta}_2$ and vector Δ^α_2 are determined with the help of (3.3) and read

$$\Omega^{\alpha\beta}_2 = -\partial_\mu\big(\varphi^\mu_1\big[h^{\alpha\beta}_{\text{can}\,1} + \partial\varphi^{\alpha\beta}_1\big]\big) + 2\partial_\mu\varphi^{(\alpha}_1 h^{\beta)\mu}_{\text{can}\,1} + \partial_\mu\varphi^\alpha_1\partial^\mu\varphi^\beta_1 + \tfrac{1}{2}\eta^{\alpha\beta}\big[\partial_\mu\varphi^\nu_1\partial_\nu\varphi^\mu_1 - \partial_\mu\varphi^\mu_1\partial_\nu\varphi^\nu_1\big], \tag{4.7a}$$

$$\Delta^\alpha_2 = -h^{\mu\nu}_{\text{can}\,1}\partial_\mu\partial_\nu\varphi^\alpha_1 + \partial_\mu\big(\varphi^\mu_1\,\Box\,\varphi^\alpha_1\big). \tag{4.7b}$$

As a consequence of (4.7) we easily verify that

$$\partial_\mu\Omega^{\alpha\mu}_2 + \Delta^\alpha_2 = 0. \tag{4.8}$$

This relation is consistent with the fact that the source term $N^{\alpha\beta}_2$ is divergenceless (because of (3.4). Hence we see that the divergence of (4.6) is automatically verified, where we use the fact that $\partial_\mu\partial\Delta^{\alpha\mu}_2 = \Box\,\Delta^\alpha_2$.

Applying our specific finite part of the retarded integral operator on both sides of (4.6) we obtain the relation between $u^{\alpha\beta}_{\text{gen}\,2}$ defined by (4.5) and the corresponding $u^{\alpha\beta}_{\text{can}\,2}$ defined by (3.7) in the canonical algorithm, namely

$$u^{\alpha\beta}_{\text{gen}\,2} = u^{\alpha\beta}_{\text{can}\,2} + \Omega^{\alpha\beta}_2 + \partial\phi^{\alpha\beta}_2 + X^{\alpha\beta}_2 + Y^{\alpha\beta}_2. \tag{4.9}$$

The difference between the two prescriptions is made of various terms. The terms $\Omega^{\alpha\beta}_2$ and $\partial\phi^{\alpha\beta}_2$ represent what we would expect if the operation of taking the finite part of the retarded integral would commute with partial derivatives. Here the gauge transformation is associated with the gauge vector defined by the finite part of the retarded integral of Δ^α_2,

$$\phi^\alpha_2 = \mathop{\text{FP}}_{B=0} \Box^{-1}_{\text{ret}} \left[\left(\frac{r}{r_0} \right)^B \Delta^\alpha_2 \right]. \tag{4.10}$$

The last two terms $X^{\alpha\beta}_2$ and $Y^{\alpha\beta}_2$ come from the non-commutation of the finite part of the retarded integral operator $\text{FP}_{B=0}\,\Box^{-1}_{\text{ret}}(r/r_0)^B$ with the differential operators $\Box$ and ∂ which are present in front of the last two terms of (3.3), respectively. We have

$$X^{\alpha\beta}_2 = \mathop{\text{FP}}_{B=0} \Box^{-1}_{\text{ret}} \left[\left(\frac{r}{r_0} \right)^B \Box\,\Omega^{\alpha\beta}_2 \right] - \Omega^{\alpha\beta}_2, \tag{4.11a}$$

$$Y_2^{\alpha\beta} = \mathop{\mathrm{FP}}_{B=0} \Box^{-1}_{\mathrm{ret}} \left[\left(\frac{r}{r_0}\right)^B \partial \Delta_2^{\alpha\beta} \right] - \partial \phi_2^{\alpha\beta}, \tag{4.11b}$$

which can also be seen more formally as the action of 'commutators' namely

$$X_2^{\alpha\beta} = \left[\mathrm{FP}\,\Box^{-1}_{\mathrm{ret}}, \Box\right]\Omega_2^{\alpha\beta}, \tag{4.12a}$$

$$Y_2^{\alpha\beta} = \left[\mathrm{FP}\,\Box^{-1}_{\mathrm{ret}}, \partial\right]\Delta_2^{\alpha\beta}. \tag{4.12b}$$

Our notation for the commutators involved and for the partial derivative ∂ should be clear. Here $\mathrm{FP}\,\Box^{-1}_{\mathrm{ret}}$ is a short hand for $\mathrm{FP}_{B=0}\,\Box^{-1}_{\mathrm{ret}}(r/r_0)^B$ and we have used the fact that $\Box(\mathrm{FP}\,\Box^{-1}_{\mathrm{ret}} f) = f$. It is evident that the non-commutation of $\mathrm{FP}\,\Box^{-1}_{\mathrm{ret}}$ with partial derivatives comes from the presence of the regularization factor r^B. Thus $X_2^{\alpha\beta}$ and $Y_2^{\alpha\beta}$ are built from the spatial differentiation of r^B, i.e. $\partial_i r^B = B n_i r^{B-1}$, and will involve an explicit factor B in their sources. Their expressions read as

$$X_2^{\alpha\beta} = \mathop{\mathrm{FP}}_{B=0} \Box^{-1}_{\mathrm{ret}} \left[B \left(\frac{r}{r_0}\right)^B \left(-\frac{B+1}{r^2}\Omega_2^{\alpha\beta} - \frac{2}{r}\frac{\partial \Omega_2^{\alpha\beta}}{\partial r} \right) \right], \tag{4.13a}$$

$$Y_2^{\alpha\beta} = \mathop{\mathrm{FP}}_{B=0} \Box^{-1}_{\mathrm{ret}} \left[B \left(\frac{r}{r_0}\right)^B \frac{n_i}{r}\left(-\delta^{i\alpha}\Delta_2^{\beta} - \delta^{i\beta}\Delta_2^{\alpha} + \eta^{\alpha\beta}\Delta_2^{i}\right) \right]. \tag{4.13b}$$

In section 4.2, we shall present a practical method to evaluate $X_2^{\alpha\beta}$ and $Y_2^{\alpha\beta}$ at the lowest PN order, given the general quadratic-type structure for the source terms (4.7*a*) and (4.7*b*).

The first part of the MPM algorithm $u^{\alpha\beta}_{\mathrm{gen}\,2}$ has been obtained in (4.9), and we look now for the second part $v^{\alpha\beta}_{\mathrm{gen}\,2}$. To this end we compute the divergence $w^{\alpha}_{\mathrm{gen}\,2} = \partial_\mu u^{\alpha\mu}_{\mathrm{gen}\,2}$. Using (4.9) and the property (4.8) we readily find that

$$w^{\alpha}_{\mathrm{gen}\,2} = w^{\alpha}_{\mathrm{can}\,2} + \partial_\mu U_2^{\alpha\mu}, \tag{4.14}$$

where we pose for simplicity

$$U_2^{\alpha\beta} = X_2^{\alpha\beta} + Y_2^{\alpha\beta}. \tag{4.15}$$

The structure (4.13) of $X_2^{\alpha\beta}$ and $Y_2^{\alpha\beta}$ involving the retarded integral of a source term containing an explicit factor B implies that $U_2^{\alpha\beta}$ is necessarily a retarded solution of the source-free d'Alembertian equation, $\Box U_2^{\alpha\beta} = 0$. Hence, there must exist ten STF tensors $A_L, B_L, \ldots, L_L$ (functions of the retarded time) parametrizing the ten components of $U_2^{\alpha\beta}$ in such a way that

$$U_2^{00} = \sum_{\ell=0}^{+\infty} \partial_L[r^{-1}A_L(t-r/c)], \tag{4.16a}$$

$$U_2^{0i} = \sum_{\ell=0}^{+\infty} \partial_{iL}[r^{-1}B_L(t-r/c)] + \sum_{\ell=1}^{+\infty}\{\partial_{L-1}[r^{-1}C_{iL-1}(t-r/c)] + \varepsilon_{iab}\partial_{aL-1}[r^{-1}D_{bL-1}(t-r/c)]\}, \tag{4.16b}$$

$$U_2^{ij} = \sum_{\ell=0}^{+\infty}\{\partial_{ijL}[r^{-1}E_L(t-r/c)] + \delta_{ij}\partial_L[r^{-1}F_L(t-r/c)]\}$$
$$+ \sum_{\ell=1}^{+\infty}\{\partial_{L-1(i}[r^{-1}G_{j)L-1}(t-r/c)] + \varepsilon_{ab(i}\partial_{j)aL-1}[r^{-1}H_{bL-1}(t-r/c)]\}$$
$$+ \sum_{\ell=2}^{+\infty}\{\partial_{L-2}[r^{-1}K_{ijL-2}(t-r/c)] + \partial_{aL-2}[r^{-1}\varepsilon_{ab(i}L_{j)bL-2}(t-r/c)]\}. \quad (4.16c)$$

The divergence of this tensor, $W_2^\alpha = \partial_\mu U_2^{\alpha\mu}$, will also be of that form and hence there will exist four STF tensors $N'_L, \ldots, R'_L$ such that

$$W_2^0 = \sum_{\ell=0}^{+\infty}\partial_L[r^{-1}N'_L(t-r/c)], \quad (4.17a)$$

$$W_2^i = \sum_{\ell=0}^{+\infty}\partial_{iL}[r^{-1}P'_L(t-r/c)]$$
$$+ \sum_{\ell=1}^{+\infty}\{\partial_{L-1}[r^{-1}Q'_{iL-1}(t-r/c)] + \varepsilon_{iab}\partial_{aL-1}[r^{-1}R'_{bL-1}(t-r/c)]\}. \quad (4.17b)$$

The four tensors $N'_L, \ldots, R'_L$ play exactly the same role as $N_L, \ldots, R_L$ in (3.10), and we shall apply the same algorithm as the one going from (3.10) to (3.11). Thus, we define from the components of W_2^α a new tensor $V_2^{\alpha\beta}$ by this algorithm, which was denoted by $\mathcal{V}^{\alpha\beta}$ in (3.12). Hence

$$V_2^{\alpha\beta} = \mathcal{V}^{\alpha\beta}[W_2], \quad (4.18)$$

so that in component form this tensor reads

$$V_2^{00} = -r^{-1}N'^{(-1)} + \partial_a\big[r^{-1}\big(-N_a'^{(-1)} + Q_a'^{(-2)} - 3P'_a\big)\big], \quad (4.19a)$$

$$V_2^{0i} = r^{-1}\big(-Q_i'^{(-1)} + 3P_i'^{(1)}\big) - \varepsilon_{iab}\partial_a\big[r^{-1}R_b'^{(-1)}\big] - \sum_{\ell=2}^{+\infty}\partial_{L-1}[r^{-1}N'_{iL-1}], \quad (4.19b)$$

$$V_2^{ij} = -\delta_{ij}r^{-1}P' + \sum_{\ell=2}^{+\infty}\big\{2\delta_{ij}\partial_{L-1}[r^{-1}P'_{L-1}] - 6\partial_{L-2(i}[r^{-1}P'_{j)L-2}]$$
$$+ \partial_{L-2}\big[r^{-1}\big(N_{ijL-2}'^{(1)} + 3P_{ijL-2}'^{(2)} - Q'_{ijL-2}\big)\big] - 2\partial_{aL-2}[r^{-1}\varepsilon_{ab(i}R'_{j)bL-2}]\big\}. \quad (4.19c)$$

However, in the present case the tensors $N'_L, \ldots, R'_L$ can be directly related to those parametrizing (4.16). By computing the divergence $W_2^\alpha = \partial_\mu U_2^{\alpha\mu}$ we readily find

$$N'_L = A_L^{(1)} + B_L^{(2)} + C_L, \quad (4.20a)$$

$$P'_L = E_L^{(2)} + F_L + \tfrac{1}{2}G_L + B_L^{(1)}, \quad (4.20b)$$

$$Q'_L = \tfrac{1}{2}G_L^{(2)} + K_L + C_L^{(2)}, \quad (4.20c)$$

$$R'_L = \tfrac{1}{2}H_L^{(2)} + \tfrac{1}{2}L_L + D_L^{(1)}. \quad (4.20d)$$

Thus $V_2^{\alpha\beta}$ can be expressed directly in terms of $A_L, \ldots, L_L$ by substituting (4.20) into (4.19). In doing so we shall discover that the time anti-derivatives present in (4.19) become in fact 'instantaneous' because they are cancelled by some time derivatives coming from (4.20). By construction of (4.19) we have at once $\Box V_2^{\alpha\beta} = 0$ and $\partial_\mu V_2^{\alpha\mu} = -W_2^\alpha$. Applying the MPM algorithm we therefore find for the second part of the algorithm,

$$v^{\alpha\beta}_{\text{gen}\,2} = v^{\alpha\beta}_{\text{can}\,2} + V_2^{\alpha\beta}. \tag{4.21}$$

Gathering the results (4.9) and (4.21) the complete quadratic-order metric is obtained as

$$h^{\alpha\beta}_{\text{gen}\,2} = h^{\alpha\beta}_{\text{can}\,2} + \Omega_2^{\alpha\beta} + \partial\phi_2^{\alpha\beta} + U_2^{\alpha\beta} + V_2^{\alpha\beta}, \tag{4.22}$$

and it satisfies the vacuum Einstein field equations in harmonic coordinates, i.e.,

$$\Box h^{\alpha\beta}_{\text{gen}\,2} = N^{\alpha\beta}_{\text{gen}\,2}, \tag{4.23a}$$

$$\partial_\mu h^{\alpha\mu}_{\text{gen}\,2} = 0. \tag{4.23b}$$

To find the relation between the source and canonical moments we note that the sum of the last two terms in (4.22) is a solution of the linearized vacuum equations, since it satisfies $\Box\big(U_2^{\alpha\beta} + V_2^{\alpha\beta}\big) = 0$ and also $\partial_\mu\big(U_2^{\alpha\mu} + V_2^{\alpha\mu}\big) = 0$. It must therefore be of the form of the general solution $h^{\alpha\beta}_{\text{gen}\,1}$ of these equations which has been given in (4.2), i.e. there should exist some moments δI_L and δJ_L representing specific corrections to I_L and J_L (necessarily at quadratic order) and some gauge vector ψ_2^α such that

$$U_2^{\alpha\beta} + V_2^{\alpha\beta} = h^{\alpha\beta}_{\text{can}\,1}[\delta I_L, \delta J_L] + \partial\psi_2^{\alpha\beta}. \tag{4.24}$$

Let us prove that the corrections we seek to the moments I_L and J_L that are needed to reproduce the canonical moments are indeed provided by these δI_L and δJ_L, i.e.,

$$M_L = I_L + G\delta I_L + \mathcal{O}(G^2), \tag{4.25a}$$

$$S_L = J_L + G\delta J_L + \mathcal{O}(G^2). \tag{4.25b}$$

To this end we have to check that the general metric $h^{\alpha\beta}_{\text{gen}}[I_L, J_L, W_L, \ldots, Z_L]$ constructed at quadratic order in (4.1) is *isometric*—differs by a coordinate transformation—to the canonical metric $h^{\alpha\beta}_{\text{can}}[M_L, S_L]$ given by (3.5). This immediately follows from (4.22) and (4.24) which permits us to recast the general metric (4.1) into the form

$$h^{\alpha\beta}_{\text{gen}}[I_L, J_L, \ldots] = G\big[h^{\alpha\beta}_{\text{can}\,1}[M_L, S_L] + \partial\varphi_1^{\alpha\beta}\big] + G^2\big[h^{\alpha\beta}_{\text{can}\,2}[M_L, S_L] + \Omega_2^{\alpha\beta} + \partial\varphi_2^{\alpha\beta}\big] + \mathcal{O}(G^3), \tag{4.26}$$

where we have posed $\varphi_2^\alpha = \phi_2^\alpha + \psi_2^\alpha$, and where higher-order powers of G are consistently neglected. From this result we conclude that $h^{\alpha\beta}_{\text{gen}}[I_L, J_L, \ldots]$ and $h^{\alpha\beta}_{\text{can}}[M_L, S_L]$ differ by the coordinate transformation

$$x^\alpha_{\text{gen}} = x^\alpha_{\text{can}} + G\varphi_1^\alpha + G^2\varphi_2^\alpha + \mathcal{O}(G^3), \tag{4.27}$$

as we have recognized that $\Omega_2^{\alpha\beta}$ represents precisely the quadratic nonlinear part of that coordinate transformation, i.e. the term which makes it to differ from a linearized gauge transformation. Hence we have proved that the two sets of moments $\{I_L, J_L, W_L, X_L, Y_L, Z_L\}$ and $\{M_L, S_L\}$ related by (4.25) are physically equivalent—they describe the same physical matter source. Note that the relations (4.25) give the canonical moments as functionals of the full set of source moments $\{I_L, J_L, W_L, X_L, Y_L, Z_L\}$. Consequently, just two moments M_L and S_L are still sufficient to describe the external field of any source [38]. Note also that M_L and S_L are almost equal to I_L and J_L in the sense that the corrections δI_L and δJ_L in (4.25) will turn out to be very small in a PN expansion, being of order 2.5PN [42]. This is of course the result of the fact that the gauge moments $\{W_L, X_L, Y_L, Z_L\}$ do not play any physical role at the linear approximation, where the coordinate transformation reduces to the gauge transformation. However, since the theory is covariant with respect to nonlinear diffeomorphisms and not merely with respect to linear gauge transformations, the moments $\{W_L, X_L, Y_L, Z_L\}$ do play a role at the nonlinear level.

4.2. Practical implementation

Finally, let us sketch our practical method to compute the correction terms δI_L and δJ_L. We remark first that they come via (4.24) from the ten STF tensors $A_L, \ldots, L_L$ parametrizing $U_2^{\alpha\beta}$ as given by (4.16). We can therefore express δI_L and δJ_L directly in terms of $A_L, \ldots, L_L$ by following in details the steps (3.10)–(4.20). The result is

$$\delta I_L = -c^2 \frac{(-)^\ell \ell!}{4} \left[A_L + 4B_L^{(1)} + 3E_L^{(2)} + 3F_L + G_L\right], \tag{4.28a}$$

$$\delta J_L = c^3 \frac{(-)^\ell (\ell+1)!}{4\ell} \left[D_L + \frac{1}{2} H_L^{(1)}\right]. \tag{4.28b}$$

The next problem is to compute the tensors $A_L, \ldots, L_L$ in the PN approximation. These are defined from the two objects $X_2^{\alpha\beta}$ and $Y_2^{\alpha\beta}$ which are given in particular by their commutator form (4.12). We thus need to compute the commutator between the operator $\mathrm{FP}\,\Box^{-1}_{\mathrm{ret}}$ and derivative operators, when applied either on the terms $\Omega_2^{\alpha\beta}$ or $\Delta_2^{\alpha\beta}$. The relevant point for our purpose is that the general structure of these terms at the quadratic order is known. Namely $\Omega_2^{\alpha\beta}$ and $\Delta_2^{\alpha\beta}$ are made of quadratic products of retarded multipolar waves, i.e. are given by sums of terms of the type

$$\mathcal{K}_{PQ} = \partial_{\langle P\rangle}[r^{-1}F(t-r/c)]\partial_{\langle Q\rangle}[r^{-1}G(t-r/c)], \tag{4.29}$$

where the functions F and G stand for some time derivatives of moments in the list $\{I_L, J_L, W_L, X_L, Y_L, Z_L\}$. It is convenient to suppress the indices on these moments and to write only the 'active' indices appearing in the spatial multi-derivatives ∂_P and ∂_Q, composed with the multi-indices $P = a_1 \cdots a_p$ and $Q = b_1 \cdots b_q$ (p and q being the number of partial derivatives in ∂_P and ∂_Q). Furthermore, the multi-derivatives in (4.29) are chosen to be STF (this can always be assumed modulo a possible STF decomposition), hence the brackets $\langle\,\rangle$ surrounding their indices. The problem is therefore reduced to that of evaluating, in the PN approximation, the quantities[16]

$$\mathcal{X}_{PQ} = \left[\mathrm{FP}\,\Box^{-1}_{\mathrm{ret}}, \Box\right]\mathcal{K}_{PQ}, \tag{4.30a}$$

$$\mathcal{Y}^i_{PQ} = \left[\mathrm{FP}\,\Box^{-1}_{\mathrm{ret}}, \partial^i\right]\mathcal{K}_{PQ}. \tag{4.30b}$$

Indeed $X_2^{\alpha\beta}$ and $Y_2^{\alpha\beta}$ are given by some sums of terms of the type $\mathcal{X}_{PQ}$ and $\mathcal{Y}^i_{PQ}$ respectively (and multiplied by appropriate constant tensors involving Kronecker symbols to perform the needed contractions).

The term $\mathcal{X}_{PQ}$ has in fact already been computed at the lowest PN order in the appendix of [42]. The result turned out to be quite simple, namely

$$\mathcal{X}_{PQ} = \frac{1}{c}\partial_{\langle PQ\rangle}[r^{-1}(\delta_{p,0}F^{(1)}G + \delta_{0,q}FG^{(1)})] + \mathcal{O}\left(\frac{1}{c^3}\right). \tag{4.31}$$

[16] In the case of $\mathcal{Y}^i_{PQ}$ we can restrict ourselves to a spatial derivative ∂^i because the time derivative ∂_t commutes with the operator $\mathrm{FP}\,\Box^{-1}_{\mathrm{ret}}$, thus $\mathcal{Y}^0_{PQ} = 0$.

Here the functions are evaluated at retarded time $t - r/c$ (with $F^{(1)}$ and $G^{(1)}$ denoting the time derivatives), and $\delta_{p,0}$ and $\delta_{0,q}$ denote the usual Kronecker symbols. As we see in (4.31) the two STF multi-derivative operators $\partial_{\langle P\rangle}$ and $\partial_{\langle Q\rangle}$ originally present in (4.29) have merged into a single STF derivative operator $\partial_{\langle PQ\rangle}$ with $p+q$ indices. The formula (4.31) constitutes a useful practical lemma for doing computations at the lowest PN order. Because of the factor $1/c$ in front of (4.31) the PN 'parity' of the result (4.31) will be opposite to that of the source term (4.29), which in practice will typically be even. As a consequence we shall find that the PN order of $X_2^{\alpha\beta}$ is dominantly 'odd', starting in fact with 2.5PN.

As for the term $\mathcal{Y}^i_{PQ}$, it was not required in [42] but will play a role here for the waveform at the 3PN order. We have worked out the equivalent of (4.31) for this term, and find, still at the lowest PN order,

$$\mathcal{Y}^i_{PQ} = -\frac{p+q}{c(2p+2q+1)}\left\{\delta_{p,0}\delta^i_{\langle b_q}\partial_{PQ-1\rangle}[r^{-1}F^{(1)}G] + \delta_{0,q}\delta^i_{\langle a_p}\partial_{P-1Q\rangle}[r^{-1}FG^{(1)}]\right\} + \mathcal{O}\left(\frac{1}{c^3}\right). \tag{4.32}$$

Consistent with our notation we write $P-1 = a_1\cdots a_{p-1}$ and $Q-1 = b_1\cdots b_{q-1}$. Again there is a factor $1/c$ and we shall find that the corresponding $Y_2^{\alpha\beta}$ is dominantly 'odd', starting at 2.5PN order. Note that the new lemma (4.32) is not independent from the previous one (4.31) and is actually more general than it. Indeed, by computing the divergence of $\mathcal{Y}^i_{PQ}$ using its definition (4.30*b*), we get

$$\partial_i\mathcal{Y}^i_{PQ} = \left[\mathrm{FP}\,\Box^{-1}_{\mathrm{ret}}, \Box\right]\mathcal{K}_{PQ} - \left[\mathrm{FP}\,\Box^{-1}_{\mathrm{ret}}, \partial^i\right]\partial_i\mathcal{K}_{PQ}, \tag{4.33}$$

which can be used to check the consistency of the two formulae (4.31) and (4.32). The results needed at 3PN order for the relation between the canonical and source moments as obtained by these means—namely formulae (4.28) and lemmas (4.31) and (4.32)—are reported in section 5.2.

5. The moments for 3PN waveform

Using the MPM algorithm of section 3 the radiative moments $\{U_L, V_L\}$ are related to the canonical moments $\{M_L, S_L\}$, and following section 4 the canonical moments are in turn expressed in terms of the source moments $\{I_L, J_L, W_L, X_L, Y_L, Z_L\}$. In the current section, we present the results (skipping some details) of the computation of all the moments needed for controlling the FWF in the case of compact binary systems up to 3PN order.

5.1. The radiative moments for 3PN polarizations

To obtain the gravitational polarizations at 3PN order one must compute: the mass radiative quadrupole U_{ij} with 3PN accuracy; the current radiative quadrupole V_{ij} and mass radiative octupole U_{ijk} with 2.5PN accuracy; mass hexadecapole U_{ijkl} and current octupole V_{ijk} with 2PN precision; U_{ijklm} and V_{ijkl} up to 1.5PN order; U_{ijklmn}, V_{ijklm} at 1PN; $U_{ijklmno}$, V_{ijklmn} at 0.5PN; and finally $U_{ijklmnop}$, $V_{ijklmno}$ to Newtonian order. The relations connecting U_L and V_L to the canonical moments M_L and S_L are first obtained following the MPM method of section 3.[17]

The quadratic contributions to the radiative mass (resp. current) moments are found in the form of sums of terms $\delta_2 U_L(u)$ (resp. $\delta_2 V_L(u)/c$) whose general structure reads

$$\delta_2 U_L(u), \delta_2 V_L(u)/c = \frac{G}{c^{m-\ell+2}}\int_{-\infty}^{u} \mathrm{d}s\, \chi_{LK_1K_2}(u,s) A^{(p_1)}_{K_1}(s) A^{(p_2)}_{K_2}(s). \tag{5.1}$$

[17] We have implemented the MPM algorithm on the algebraic computing software Mathematica using the powerful tensor package xTensor [55].

The power of $1/c$ in front is chosen in such a way that m represents the PN order of our calculation of the waveform, i.e. $m = 6$ at the 3PN order. The capital letter A stands either for M or S, meaning that we are considering in (5.1) interactions between canonical moments of the type $M_{K_1}^{(p_1)} M_{K_2}^{(p_2)}$ or $M_{K_1}^{(p_1)} S_{K_2}^{(p_2)}$ or $S_{K_1}^{(p_1)} S_{K_2}^{(p_2)}$, with the superscript (p) denoting time derivatives, and the multi-indices K_1 and K_2 having length k_1 and k_2 (e.g. $K_1 = a_1 \cdots a_{k_1}$)[18]. The kernel $\chi_{LK_1K_2}$ has itself an algebraic structure made of a sum of products of Kronecker and Levi-Civita symbols. Its physical dimension depends on time only, and each of its three sets of indices, L, K_1 and K_2, is symmetric and trace-free (STF). For instantaneous terms, which are functions of the multipole moments $A_{K_1}^{(p_1)}$, $A_{K_2}^{(p_2)}$ evaluated at the instant of emission $u = t - r/c$, it is proportional to the Dirac function $\delta(s - u)$.

The above structure does not exist generically for an arbitrary pair of multipole moments nor for any arbitrary value of k_1 and k_2. A closer look will actually allow us to reduce the number of terms in the source we shall focus on to a few ones, making our task much easier. As a product $\varepsilon_{ijk}\varepsilon_{abc}$ can always be transformed into a linear combination of $\delta_{ia'}\delta_{jb'}\delta_{kc'}$ with $\{a', b', c'\} = \{a, b, c\}$, the number ϵ of Levi-Civita symbols in each of the individual terms $\delta\delta \cdots \delta\varepsilon\varepsilon \cdots \varepsilon$ composing $\chi_{LK_1K_2}$ may be reduced to 0 or 1. The symmetry of parity implies that this number is the same for all terms. Now, if $\epsilon = 0$, the integer $\ell + k_1 + k_2$ is even (equal to twice the number of Kronecker symbols) and all indices in L must contract with an index of K_1 or K_2. Thus, we must necessarily have $k_1 + k_2 \geqslant \ell$. On the other hand, if $\epsilon = 1$, the Levi-Civita symbol carries one index from each of the three STF sets, so that there remain $\ell - 1$ free indices of type L carried by Kronecker symbols, as well as $k_1 - 1$ indices (resp. $k_2 - 1$) of type K_1 (resp. K_2) involved in the contraction of some δ's with the multipole moments. Then, the same arguments as before show that $(\ell - 1) + (k_1 - 1) + (k_2 - 1)$ must be even with $(k_1 - 1) + (k_2 - 1) \geqslant \ell - 1$. The previous constraints can all be summarized by the single statement that $k_1 + k_2 - \ell - \epsilon$ is always an *even* positive integer.

The structure of the quadratic interactions may be further refined by noticing that only the multipole moments that have dimensions compatible with (5.1) are allowed to enter $\delta_2 U$, $\delta_2 V/c$. Let us pose for later convenience $\left[A_{K_1}^{(p_1)}\right] = [M][L]^{a_1+k_1-p_1}[V]^{\alpha_1+p_1}$ and $\left[A_{K_2}^{(p_2)}\right] = [M][L]^{a_2+k_2-p_2}[V]^{\alpha_2+p_2}$, where $[M]$, $[L]$ and $[V]$ denote the dimension of a mass, a length and a velocity respectively. Equating $[U_L] = [V_L/c]$ and $\left[\mathrm{d}s\, \chi_{LK_1K_2}\right]\left[G/c^{m-\ell+2} A_{K_1}^{(p_1)} A_{K_2}^{(p_2)}\right]$ on the one hand, remembering on the other hand that $\left[\chi_{LK_1K_2}\right]$ is a certain power $q \in \mathbb{Z}$ of the time dimension $[T]$, we find

$$\sum_{i=1,2} (a_i + \alpha_i + k_i) = m - 1, \tag{5.2a}$$

$$\sum_{i=1,2} (\alpha_i + p_i) = q + m + 1. \tag{5.2b}$$

Now, we know that $k_1 + k_2 - \ell - \epsilon \in \mathbb{N}$. Moreover, the number $\epsilon = 0, 1$ of Levi-Civita symbols is itself governed by the parity symmetry. More precisely, defining the integers $\tilde{\alpha}_1$, $\tilde{\alpha}_2$ and $\tilde{\epsilon}$, associated with $A_{K_1}^{(p_1)}$, $A_{K_2}^{(p_2)}$ and $\delta_2 U_L$, $\delta_2 V_L/c$ respectively, to be equal to zero when the latter multipole moments are of mass type or to 1 when they are of current type, the consistency of the transformation of both sides of equation (5.1) under parity imposes that $\tilde{\epsilon} = \tilde{\alpha}_1 + \tilde{\alpha}_2 + \epsilon$ [mod 2]. As a result, the maximum multipolar order $k_1 + k_2 - \epsilon$ of the radiative moments containing a quadratic interaction $A_{K_1}^{(p_1)} A_{K_2}^{(p_2)}$ is given by[19]

[18] The reasoning we shall make can be easily generalized to nth nonlinear terms $\delta_n U_L$, $\delta_n V_L/c$ involving n canonical moments $A_{K_1}^{(p_1)}, A_{K_2}^{(p_2)}, \ldots, A_{K_n}^{(p_n)}$.

[19] The remainder function means the usual division remainder: remainder$[\frac{N}{2}] = 0$ or 1 depending on whether N is an even or odd integer.

$$\ell_{\max}(a_i,\alpha_i,\tilde\alpha_i,\tilde\epsilon) = m-1-\sum_{i=1,2}(a_i+\alpha_i) - \text{remainder}\left[\frac{1}{2}\left(\sum_{i=1,2}\tilde\alpha_i+\tilde\epsilon\right)\right]. \tag{5.3}$$

At the $\frac{m}{2}$PN approximation, such a contribution exists only if $\ell_{\max} \geqslant 2$, or equivalently $\sum_{i=1,2}(a_i+\alpha_i) + \text{remainder}\left[\frac{1}{2}\left(\sum_{i=1,2}\tilde\alpha_i+\tilde\epsilon\right)\right] \leqslant m-3$. Once this necessary condition is fulfilled, the orders of multipolarity possibly affected by the piece of nonlinear correction (5.1) are $\ell_{\max}(a_i,\alpha_i,\tilde\alpha_i,\tilde\epsilon)$, $\ell_{\max}(a_i,\alpha_i,\tilde\alpha_i,\tilde\epsilon)-2,\ldots,2$ or 3, with $\tilde\epsilon=0$ (resp. $\tilde\epsilon=1$) for mass (resp. current) radiative moments. The latter 'selection' rules may be generalized to interactions of any post-Minkowskian order n, in which case $m-1$ must be replaced by $m+5-3n$ in expression (5.3) of $\ell_{\max}$ while all summation ranges become $1\leqslant i\leqslant n$. With the selection rules (5.2) and (5.3) we are able to know beforehand which nonlinear multipole interaction is needed to be computed in the radiative moments U_L, V_L at a given PN order.

The result concerning the 3PN mass quadrupole moment U_{ij} is already known [40, 50, 51] and we simply report it here. Actually, at 3PN order U_{ij} involves a cubically nonlinear term, composed of the so-called tails-of-tails, whose computation necessitates an extension of the MPM algorithm to cubic order G^3 [50]. We have

$$\begin{aligned}
U_{ij}(T_R) &= M^{(2)}_{ij}(T_R) + \frac{2GM}{c^3}\int_{-\infty}^{T_R}\mathrm{d}\tau\left[\ln\left(\frac{T_R-\tau}{2\tau_0}\right)+\frac{11}{12}\right]M^{(4)}_{ij}(\tau)\\
&\quad+\frac{G}{c^5}\left\{-\frac{2}{7}\int_{-\infty}^{T_R}\mathrm{d}\tau\, M^{(3)}_{a\langle i}(\tau)M^{(3)}_{j\rangle a}(\tau)\right.\\
&\quad\left.+\frac{1}{7}M^{(5)}_{a\langle i}M_{j\rangle a}-\frac{5}{7}M^{(4)}_{a\langle i}M^{(1)}_{j\rangle a}-\frac{2}{7}M^{(3)}_{a\langle i}M^{(2)}_{j\rangle a}+\frac{1}{3}\varepsilon_{ab\langle i}M^{(4)}_{j\rangle a}S_b\right\}\\
&\quad+2\left(\frac{GM}{c^3}\right)^2\int_{-\infty}^{T_R}\mathrm{d}\tau\left[\ln^2\left(\frac{T_R-\tau}{2\tau_0}\right)+\frac{57}{70}\ln\left(\frac{T_R-\tau}{2\tau_0}\right)+\frac{124\,627}{44\,100}\right]M^{(5)}_{ij}(\tau)\\
&\quad+\mathcal{O}\left(\frac{1}{c^7}\right).
\end{aligned}\tag{5.4}$$

Note the tail integral at 1.5PN order, the tail-of-tail integral at 3PN order, and the nonlinear memory integral at 2.5PN. In the tail and tail-of-tail integrals, M represents the mass monopole moment or total mass of the binary system. The constant τ_0 in the tail integrals is given by $\tau_0 = r_0/c$, where r_0 is the arbitrary length scale originally introduced in the MPM formalism through (3.7), and appearing also in the relation between the radiative and harmonic coordinates as given by (3.16).

The moments required at 2.5PN order are new with this paper (apart from the tails) and involve some interactions between the mass quadrupole moment and the mass octupole or current quadrupole moments. Which type of interactions is determined by using the selection rules discussed above. These moments are given by[20]

$$\begin{aligned}
U_{ijk}(T_R) &= M^{(3)}_{ijk}(T_R) + \frac{2GM}{c^3}\int_{-\infty}^{T_R}\mathrm{d}\tau\left[\ln\left(\frac{T_R-\tau}{2\tau_0}\right)+\frac{97}{60}\right]M^{(5)}_{ijk}(\tau)\\
&\quad+\frac{G}{c^5}\left\{\int_{-\infty}^{T_R}\mathrm{d}\tau\left[-\frac{1}{3}M^{(3)}_{a\langle i}(\tau)M^{(4)}_{jk\rangle a}(\tau)-\frac{4}{5}\varepsilon_{ab\langle i}M^{(3)}_{ja}(\tau)S^{(3)}_{k\rangle b}(\tau)\right]\right.\\
&\quad-\frac{4}{3}M^{(3)}_{a\langle i}M^{(3)}_{jk\rangle a}-\frac{9}{4}M^{(4)}_{a\langle i}M^{(2)}_{jk\rangle a}+\frac{1}{4}M^{(2)}_{a\langle i}M^{(4)}_{jk\rangle a}-\frac{3}{4}M^{(5)}_{a\langle i}M^{(1)}_{jk\rangle a}+\frac{1}{4}M^{(1)}_{a\langle i}M^{(5)}_{jk\rangle a}
\end{aligned}$$

[20] In all formulae below the STF projection $\langle\rangle$ applies only to the 'free' indices denoted $ijkl\ldots$ carried by the moments themselves. Thus the dummy indices such as $abc\ldots$ are excluded from the STF projection.

$$
+\frac{1}{12}M^{(6)}_{a\langle i}M_{jk\rangle a}+\frac{1}{4}M_{a\langle i}M^{(6)}_{jk\rangle a}+\frac{1}{5}\varepsilon_{ab\langle i}\left[-12S^{(2)}_{ja}M^{(3)}_{k\rangle b}-8M^{(2)}_{ja}S^{(3)}_{k\rangle b}-3S^{(1)}_{ja}M^{(4)}_{k\rangle b}\right.
$$

$$
\left.-27M^{(1)}_{ja}S^{(4)}_{k\rangle b}-S_{ja}M^{(5)}_{k\rangle b}-9M_{ja}S^{(5)}_{k\rangle b}-\frac{9}{4}S_aM^{(5)}_{jk\rangle b}\right]+\frac{12}{5}S_{\langle i}S^{(4)}_{jk\rangle}\Bigg\}
$$

$$
+\mathcal{O}\left(\frac{1}{c^6}\right), \tag{5.5a}
$$

$$
V_{ij}(T_R)=S^{(2)}_{ij}(T_R)+\frac{2GM}{c^3}\int_{-\infty}^{T_R}\mathrm{d}\tau\left[\ln\left(\frac{T_R-\tau}{2\tau_0}\right)+\frac{7}{6}\right]S^{(4)}_{ij}(\tau)
$$

$$
+\frac{G}{7c^5}\left\{4S^{(2)}_{a\langle i}M^{(3)}_{j\rangle a}+8M^{(2)}_{a\langle i}S^{(3)}_{j\rangle a}+17S^{(1)}_{a\langle i}M^{(4)}_{j\rangle a}-3M^{(1)}_{a\langle i}S^{(4)}_{j\rangle a}+9S_{a\langle i}M^{(5)}_{j\rangle a}\right.
$$

$$
-3M_{a\langle i}S^{(5)}_{j\rangle a}-\frac{1}{4}S_aM^{(5)}_{ija}-7\varepsilon_{ab\langle i}S_aS^{(4)}_{j\rangle b}+\frac{1}{2}\varepsilon_{ac\langle i}\left[3M^{(3)}_{ab}M^{(3)}_{j\rangle bc}+\frac{353}{24}M^{(2)}_{j\rangle bc}M^{(4)}_{ab}\right.
$$

$$
\left.\left.-\frac{5}{12}M^{(2)}_{ab}M^{(4)}_{j\rangle bc}+\frac{113}{8}M^{(1)}_{j\rangle bc}M^{(5)}_{ab}-\frac{3}{8}M^{(1)}_{ab}M^{(5)}_{j\rangle bc}+\frac{15}{4}M_{j\rangle bc}M^{(6)}_{ab}+\frac{3}{8}M_{ab}M^{(6)}_{j\rangle bc}\right]\right\}
$$

$$
+\mathcal{O}\left(\frac{1}{c^6}\right). \tag{5.5b}
$$

At 2PN order we have the standard tails and some previously known interactions of the mass quadrupole with itself [51], namely

$$
U_{ijkl}(T_R)=M^{(4)}_{ijkl}(T_R)+\frac{G}{c^3}\left\{2M\int_{-\infty}^{T_R}\mathrm{d}\tau\left[\ln\left(\frac{T_R-\tau}{2\tau_0}\right)+\frac{59}{30}\right]M^{(6)}_{ijkl}(\tau)\right.
$$

$$
\left.+\frac{2}{5}\int_{-\infty}^{T_R}\mathrm{d}\tau\,M^{(3)}_{\langle ij}(\tau)M^{(3)}_{kl\rangle}(\tau)-\frac{21}{5}M^{(5)}_{\langle ij}M_{kl\rangle}-\frac{63}{5}M^{(4)}_{\langle ij}M^{(1)}_{kl\rangle}-\frac{102}{5}M^{(3)}_{\langle ij}M^{(2)}_{kl\rangle}\right\}
$$

$$
+\mathcal{O}\left(\frac{1}{c^5}\right), \tag{5.6a}
$$

$$
V_{ijk}(T_R)=S^{(3)}_{ijk}(T_R)+\frac{G}{c^3}\left\{2M\int_{-\infty}^{T_R}\mathrm{d}\tau\left[\ln\left(\frac{T_R-\tau}{2\tau_0}\right)+\frac{5}{3}\right]S^{(5)}_{ijk}(\tau)\right.
$$

$$
\left.+\frac{1}{10}\varepsilon_{ab\langle i}M^{(5)}_{ja}M_{k\rangle b}-\frac{1}{2}\varepsilon_{ab\langle i}M^{(4)}_{ja}M^{(1)}_{k\rangle b}-2S_{\langle i}M^{(4)}_{jk\rangle}\right\}
$$

$$
+\mathcal{O}\left(\frac{1}{c^5}\right). \tag{5.6b}
$$

At 1.5PN we again have some nonlinear interactions (new with this paper) involving the mass octupole and current quadrupole and given by

$$
U_{ijklm}(T_R)=M^{(5)}_{ijklm}(T_R)+\frac{G}{c^3}\left\{2M\int_{-\infty}^{T_R}\mathrm{d}\tau\left[\ln\left(\frac{T_R-\tau}{2\tau_0}\right)+\frac{232}{105}\right]M^{(7)}_{ijklm}(\tau)\right.
$$

$$
+\frac{20}{21}\int_{-\infty}^{T_R}\mathrm{d}\tau\,M^{(3)}_{\langle ij}(\tau)M^{(4)}_{klm\rangle}(\tau)-\frac{710}{21}M^{(3)}_{\langle ij}M^{(3)}_{klm\rangle}-\frac{265}{7}M^{(2)}_{\langle ijk}M^{(4)}_{lm\rangle}
$$

$$
-\frac{120}{7}M^{(2)}_{\langle ij}M^{(4)}_{klm\rangle}-\frac{155}{7}M^{(1)}_{\langle ijk}M^{(5)}_{lm\rangle}-\frac{41}{7}M^{(1)}_{\langle ij}M^{(5)}_{klm\rangle}-\frac{34}{7}M_{\langle ijk}M^{(6)}_{lm\rangle}
$$

$$
\left.-\frac{15}{7}M_{\langle ij}M^{(6)}_{klm\rangle}\right\}+\mathcal{O}\left(\frac{1}{c^4}\right), \tag{5.7a}
$$

$$
\begin{aligned}
V_{ijkl}(T_R) = {} & S^{(4)}_{ijkl}(T_R) + \frac{G}{c^3}\left\{2M\int_{-\infty}^{T_R}\mathrm{d}\tau\left[\ln\left(\frac{T_R-\tau}{2\tau_0}\right)+\frac{119}{60}\right]S^{(6)}_{ijkl}(\tau)\right.\\
& - \frac{35}{3}S^{(2)}_{\langle ij}M^{(3)}_{kl\rangle} - \frac{25}{3}M^{(2)}_{\langle ij}S^{(3)}_{kl\rangle} - \frac{65}{6}S^{(1)}_{\langle ij}M^{(4)}_{kl\rangle} - \frac{25}{6}M^{(1)}_{\langle ij}S^{(4)}_{kl\rangle} - \frac{19}{6}S_{\langle ij}M^{(5)}_{kl\rangle}\\
& - \frac{11}{6}M_{\langle ij}S^{(5)}_{kl\rangle} - \frac{11}{12}S_{\langle i}M^{(5)}_{jkl\rangle} + \frac{1}{6}\varepsilon_{ab\langle i}\left[-5M^{(3)}_{ja}M^{(3)}_{kl\rangle b} - \frac{11}{2}M^{(4)}_{ja}M^{(2)}_{kl\rangle b}\right.\\
& \left.\left. - \frac{5}{2}M^{(2)}_{ja}M^{(4)}_{kl\rangle b} - \frac{1}{2}M^{(5)}_{ja}M^{(1)}_{kl\rangle b} + \frac{37}{10}M^{(1)}_{ja}M^{(5)}_{kl\rangle b} + \frac{3}{10}M^{(6)}_{ja}M_{kl\rangle b} + \frac{1}{2}M_{ja}M^{(6)}_{kl\rangle b}\right]\right\}\\
& + \mathcal{O}\left(\frac{1}{c^4}\right). \qquad (5.7b)
\end{aligned}
$$

For all the other moments that are required, it is sufficient to assume the agreement between the radiative and canonical moments,

$$U_L(T_R) = M^{(\ell)}_L(T_R) + \mathcal{O}\left(\frac{1}{c^3}\right), \qquad (5.8a)$$

$$V_L(T_R) = S^{(\ell)}_L(T_R) + \mathcal{O}\left(\frac{1}{c^3}\right). \qquad (5.8b)$$

5.2. The canonical moments for 3PN polarizations

Following the investigation of section 4 we now give the canonical moments in terms of source-rooted multipole moments. It turns out that the difference between these two types of moments—which is due to the presence of the gauge moments defined by (4.4)—arises only at the small 2.5PN order. The consequence is that we have to worry about this difference only for the 3PN canonical mass quadrupole moment M_{ij}, the 2.5PN mass octopole moment M_{ijk}, and the 2.5PN current quadrupole moment S_{ij}. For the mass quadrupole moment, the requisite correction has already been used in [17] and is given by[21]

$$M_{ij} = I_{ij} + \frac{4G}{c^5}\left[W^{(2)}I_{ij} - W^{(1)}I^{(1)}_{ij}\right] + \mathcal{O}\left(\frac{1}{c^7}\right), \qquad (5.9)$$

where I_{ij} denotes the source mass quadrupole, and where W is the monopole corresponding to the gauge moments W_L (i.e. W is the moment having $\ell = 0$). At the PN order we are working, W is needed only at Newtonian order and will be provided in section 5.3. Note that the remainder in (5.9) is at order 3.5PN—consistently with the accuracy we aim here. Expression (5.9) is valid in a mass-centred frame defined by the vanishing of the mass dipole moment: $I_i = 0$. Note that a formula generalizing (5.9) to all PN orders (and all multipole interactions) is not possible at present and needs to be investigated anew for specific cases. Thus it is convenient in the present approach to use systematically the source moments $\{I_L, J_L, W_L, X_L, Y_L, Z_L\}$ as the fundamental variables describing the source.

Similarly, the other moments M_{ijk} and S_{ij} will admit some correction terms starting at the 2.5PN order. We have computed these new corrections, together with recomputed and confirmed those in (5.9), by following the method of section 4, i.e. evaluating the STF tensors $A_L, \ldots, L_L$ in (4.16) by means of the two practical lemmas (4.31) and (4.32), then plugging these tensors into (4.28). We also performed an independent calculation by implementing the

[21] Equation (11.7a) in [13] contains a sign error with respect to the original result [42] (with no consequence for any of the results in [13]). The correct sign is reproduced here.

Table 1. Nonlinear corrections in M_{ijk} and S_{ij} involving at least one gauge multipole moment at 3PN order. The first entry indicates, for each interaction, which radiative moment it belongs to, whereas the second entry tells us how many time derivatives are involved. STF symbols are omitted.

Number of ∂_t	$1\,(\geqslant p)$		$2\,(\geqslant p)$				$3\,(\geqslant p)$
In M_{ijk}	$I\times Y^{(1)}_{ijk}$	$I^{(1-p)}_{ij}\times Y^{(p)}_k$	$I^{(2-p)}_{ijk}\times W^{(p)}$	$I\times W^{(2)}_{ijk}$	$I^{(2-p)}_{ij}\times W^{(p)}_k$	–	–
In S_{ij}	$J_i\times Y^{(1)}_j$	–	$I\times Z^{(2)}_{ij}$	$\varepsilon_{iab}I^{(2-p)}_{aj}\times Y^{(p)}_b$	$J_i\times W^{(2)}_j$	$J^{(2-p)}_{ij}\times W^{(p)}$	$\varepsilon_{iab}I^{(3-p)}_{aj}\times W^{(p)}_b$

general MPM algorithm of section 3 starting directly with the general linearized metric (4.2) parametrized by the source moments, instead of the canonical metric (3.6) parametrized by the canonical moments.

In this second approach, which fully confirmed the previous results, we need to know beforehand the relevant multipole interactions and we used the same selection rules (5.2) and (5.3) as before. The only difference is that the quadratic interaction we consider, $A^{(p_1)}_{K_1}\times A^{(p_2)}_{K_2}$, are between any two source multipole moments composed of the main moments $\{I_L, J_L\}$ and the gauge moments $\{W_L, X_L, Y_L, Z_L\}$, i.e. the letter A symbolizes now any of the I, J, W, X, Y or Z. By applying the selection rules (5.2) and (5.3) at 3PN order, i.e. for $m=6$, it is straightforward to check that (i) no gauge multipole moment can enter cubic interactions up to our approximation level, and (ii) all quadratic interactions involving at least one gauge moment have to be instantaneous, meaning that $q=-1$ for them. We can in fact determine all possible contributions by inspection. Their full list is given in table 1. Further rules of selection might be used to discard some candidates, but all the contributions to U_{ijk} and V_{ij} that are presented here have been computed explicitly. By retaining only the interactions that involve the pairs of multipole moments composing the elements of table 1 [22], the source used in our algorithms could indeed be reduced to a finite, reasonably small number of terms. However the detailed calculation of some of these interactions turns out to yield zero; this is the case for instance of the interaction $I\times Z^{(2)}_{ij}$ which does not contribute to S_{ij}.

Finally our explicit results for M_{ijk} and S_{ij} are

$$M_{ijk} = I_{ijk} + \frac{4G}{c^5}\left[W^{(2)}I_{ijk} - W^{(1)}I^{(1)}_{ijk} + 3I_{\langle ij}Y^{(1)}_{k\rangle}\right] + \mathcal{O}\left(\frac{1}{c^6}\right), \tag{5.10a}$$

$$S_{ij} = J_{ij} + \frac{2G}{c^5}\left[\varepsilon_{ab\langle i}\left(-I^{(3)}_{j\rangle b}W_a - 2I_{j\rangle b}Y^{(2)}_a + I^{(1)}_{j\rangle b}Y^{(1)}_a\right) + 3J_{\langle i}Y^{(1)}_{j\rangle} - 2J^{(1)}_{ij}W^{(1)}\right] + \mathcal{O}\left(\frac{1}{c^6}\right), \tag{5.10b}$$

where W_i and Y_i are the dipole moments corresponding to the moments W_L and Y_L. The remainders in (5.10) are consistent with our approximation 3PN for the FWF. Besides the mass quadrupole moment (5.9), and mass octopole and current quadrupole moments (5.10), we can state that, with the required 3PN precision, all the other moments M_L agree with their corresponding I_L, and similarly the S_L agree with J_L, namely

$$M_L = I_L + \mathcal{O}\left(\frac{1}{c^5}\right), \tag{5.11a}$$

$$S_L = J_L + \mathcal{O}\left(\frac{1}{c^5}\right). \tag{5.11b}$$

[22] That is, the interactions $I\times Y_{ijk}$, $I_i\times Y_j$, $I\times W_{ijk}$, $I_{ij}\times W_k$, $J_i\times Y_j$, $I\times Z_{ij}$, $J_i\times W_j$ and $J_{ij}\times W$.

5.3. The source moments for 3PN polarizations

We have finally succeeded in parametrizing the FWF entirely in terms of the source moments $\{I_L, J_L, W_L, X_L, Y_L, Z_L\}$ up to 3PN order. The interest of this construction lies in the fact that the source moments are known for general PN matter systems. They were obtained by matching the external MPM field of the source to the internal PN field valid in the source's near zone [41–43]. The source moments have been worked out in the case of compact binary systems with increasing PN precision [13, 17, 48, 49]. Here we list all the required I_L's and J_L's (and also the few needed gauge moments) for non-spinning compact objects and for circular orbits. We do not enter the details because the derivation of these moments follows exactly the same techniques as in [13, 49].

The only moment needed at the 3PN order is the mass quadrupole moment I_{ij}, first computed for circular orbits in [13] and subsequently extended to general orbits in [49]. We write it as

$$I_{ij} = \nu m\left(A x_{\langle ij\rangle} + B\frac{r^3}{Gm}v_{\langle ij\rangle} + C\sqrt{\frac{r^3}{Gm}}x_{\langle i}v_{j\rangle}\right) + \mathcal{O}\left(\frac{1}{c^7}\right). \tag{5.12}$$

The relative position and velocity of the two bodies in harmonic coordinates are denoted by $x^i = y_1^i - y_2^i$ and $v^i = \mathrm{d}x^i/\mathrm{d}t = v_1^i - v_2^i$ (spatial indices are lowered and raised with the Kronecker metric so that $x_i = x^i$ and $v_i = v^i$). The distance between the two particles in harmonic coordinates is denoted $r = |\mathbf{x}|$. The two masses are m_1 and m_2, the total mass is $m = m_1+m_2$ (not to be confused with the mass monopole moment M), the symmetric mass ratio $\nu = m_1m_2/m^2$ satisfies $0 < \nu \leqslant 1/4$ and the mass difference ratio is $\Delta = (m_1-m_2)/m$ which reads also $\Delta = \pm\sqrt{1-4\nu}$ (according to the sign of $m_1 - m_2$). To express the coefficients A, B and C in (5.12) as PN series we introduce the small post-Newtonian parameter

$$\gamma = \frac{Gm}{rc^2}. \tag{5.13}$$

With these notations we have (in the frame of the 'centre-of-mass' and for circular orbits)

$$\begin{aligned} A = 1 &+ \gamma\left(-\frac{1}{42} - \frac{13}{14}\nu\right) + \gamma^2\left(-\frac{461}{1512} - \frac{18\,395}{1512}\nu - \frac{241}{1512}\nu^2\right) \\ &+ \gamma^3\left(\frac{395\,899}{13\,200} - \frac{428}{105}\ln\left(\frac{r}{r_0}\right) + \left[\frac{3\,304\,319}{166\,320} - \frac{44}{3}\ln\left(\frac{r}{r'_0}\right)\right]\nu \right. \\ &\left. + \frac{162\,539}{16\,632}\nu^2 + \frac{2351}{33\,264}\nu^3\right), \end{aligned} \tag{5.14a}$$

$$\begin{aligned} B = {} &\gamma\left(\frac{11}{21} - \frac{11}{7}\nu\right) + \gamma^2\left(\frac{1607}{378} - \frac{1681}{378}\nu + \frac{229}{378}\nu^2\right) \\ &+ \gamma^3\left(-\frac{357\,761}{19\,800} + \frac{428}{105}\ln\left(\frac{r}{r_0}\right) - \frac{92\,339}{5544}\nu + \frac{35\,759}{924}\nu^2 + \frac{457}{5544}\nu^3\right), \end{aligned} \tag{5.14b}$$

$$C = \frac{48}{7}\gamma^{5/2}\nu. \tag{5.14c}$$

The coefficients A and B correspond to conservative PN orders (which are even), while the coefficient C involves a single term at the odd 2.5PN order due to radiation reaction.

Note the appearance of logarithms in both A and B at the 3PN order. These logarithms have two distinct origins, depending on whether they are scaled with the constant r_0 associated with the finite part prescription in (3.7), or with an alternative constant denoted r'_0. The logarithms with r_0 will combine later with other contributions due to tails and tails-of-tails, and the constant r_0 will be absorbed into some unobservable shift of the binary's orbital phase, as can already be seen from the fact that r_0 is associated with the difference of origin of time between harmonic and radiative coordinates, see (3.16).

The other constant r'_0 is defined by $m \ln r'_0 = m_1 \ln r'_1 + m_2 \ln r'_2$, where r'_1 and r'_2 are two *regularization constants* appearing in a Hadamard self-field regularization scheme for the 3PN equations of motion of point masses in harmonic coordinates [56, 57]. The constant r'_0 is therefore present in the 3PN equations of motion and we shall thus also meet this constant in the 3PN orbital frequency given by (6.4). The regularization constant r'_0 is unobservable, since it can be removed by a coordinate transformation at 3PN order—r'_0 can rightly be called a *gauge constant*. In practice, this means that r'_0 will cancel out when using the 3PN equations of motion to compute the time derivatives of the 3PN quadrupole moment, as will be explicitly verified in section 6 [23].

The list of required moments continues with the 2.5PN order at which we need the mass octupole and current quadrupole given by (with $\Delta = \frac{m_1 - m_2}{m}$)

$$\begin{aligned} I_{ijk} = -\nu m \Delta \Bigg\{ & x_{\langle ijk\rangle}\left[1 - \gamma\nu - \gamma^2\left(\frac{139}{330} + \frac{11\,923}{660}\nu + \frac{29}{110}\nu^2\right)\right] \\ & + \frac{r^2}{c^2} x_{\langle i} v_{jk\rangle}\left[1 - 2\nu - \gamma\left(-\frac{1066}{165} + \frac{1433}{330}\nu - \frac{21}{55}\nu^2\right)\right] \\ & + \frac{196}{15}\frac{r}{c}\gamma^2 \nu x_{\langle ij} v_{k\rangle}\Bigg\} + \mathcal{O}\left(\frac{1}{c^6}\right), \end{aligned} \quad (5.15a)$$

$$\begin{aligned} J_{ij} = -\nu m \Delta \Bigg\{ & \varepsilon_{ab\langle i} x_{j\rangle a} v_b \left[1 + \gamma\left(\frac{67}{28} - \frac{2}{7}\nu\right) + \gamma^2\left(\frac{13}{9} - \frac{4651}{252}\nu - \frac{1}{168}\nu^2\right)\right] \\ & - \frac{484}{105}\frac{r}{c}\gamma^2\nu \varepsilon_{ab\langle i} v_{j\rangle a} x_b \Bigg\} + \mathcal{O}\left(\frac{1}{c^6}\right). \end{aligned} \quad (5.15b)$$

At 2PN order we require

$$\begin{aligned} I_{ijkl} = \nu m \Bigg\{ & x_{\langle ijkl\rangle}\left[1 - 3\nu + \gamma\left(\frac{3}{110} - \frac{25}{22}\nu + \frac{69}{22}\nu^2\right) \right. \\ & \left. + \gamma^2\left(-\frac{126\,901}{200\,200} - \frac{58\,101}{2600}\nu + \frac{204\,153}{2860}\nu^2 + \frac{1149}{1144}\nu^3\right)\right] \\ & + \frac{r^2}{c^2} x_{\langle ij} v_{kl\rangle}\left[\frac{78}{55}(1 - 5\nu + 5\nu^2) \right. \\ & \left. + \gamma\left(\frac{30\,583}{3575} - \frac{107\,039}{3575}\nu + \frac{8792}{715}\nu^2 - \frac{639}{715}\nu^3\right)\right] \\ & + \frac{71}{715}\frac{r^4}{c^4} v_{\langle ijkl\rangle}(1 - 7\nu + 14\nu^2 - 7\nu^3)\Bigg\} + \mathcal{O}\left(\frac{1}{c^5}\right), \end{aligned} \quad (5.16a)$$

$$\begin{aligned} J_{ijk} = \nu m \Bigg\{ & \varepsilon_{ab\langle i} x_{jk\rangle a} v_b \left[1 - 3\nu + \gamma\left(\frac{181}{90} - \frac{109}{18}\nu + \frac{13}{18}\nu^2\right) \right. \\ & \left. + \gamma^2\left(\frac{1469}{3960} - \frac{5681}{264}\nu + \frac{48\,403}{660}\nu^2 - \frac{559}{3960}\nu^3\right)\right] \end{aligned}$$

[23] Note also that the 3PN quadrupole moment [13, 49] depended originally on three constants ξ, κ, ζ (called ambiguity parameters) reflecting some incompleteness of the Hadamard self-field regularization. These constants have been computed by means of the powerful dimensional regularization [15, 58], and we have replaced the result, which was $\xi = -\frac{9871}{9240}$, $\kappa = 0$ and $\zeta = -\frac{7}{33}$, back into (5.14).

$$+ \frac{r^2}{c^2}\varepsilon_{ab\langle i}x_a v_{jk\rangle b}\left[\frac{7}{45}(1-5\nu+5\nu^2)+\gamma\left(\frac{1621}{990}-\frac{4879}{990}\nu+\frac{1084}{495}\nu^2-\frac{259}{990}\nu^3\right)\right]\Bigg\}$$
$$+ \mathcal{O}\left(\frac{1}{c^5}\right). \tag{5.16b}$$

At 1.5PN order:

$$I_{ijklm} = -\nu m\Delta\left\{x_{\langle ijklm\rangle}\left[1-2\nu+\gamma\left(\frac{2}{39}-\frac{47}{39}\nu+\frac{28}{13}\nu^2\right)\right]\right.$$
$$\left.+\frac{70}{39}\frac{r^2}{c^2}x_{\langle ijk}v_{lm\rangle}(1-4\nu+3\nu^2)\right\}+\mathcal{O}\left(\frac{1}{c^4}\right), \tag{5.17a}$$

$$J_{ijkl} = -\nu m\Delta\left\{\varepsilon_{ab\langle i}x_{jkl\rangle a}v_b\left[1-2\nu+\gamma\left(\frac{20}{11}-\frac{155}{44}\nu+\frac{5}{11}\nu^2\right)\right]\right.$$
$$\left.+\frac{4}{11}\frac{r^2}{c^2}\varepsilon_{ab\langle i}x_{ja}v_{kl\rangle b}(1-4\nu+3\nu^2)\right\}+\mathcal{O}\left(\frac{1}{c^4}\right). \tag{5.17b}$$

At 1PN order:

$$I_{ijklmn} = \nu m\left\{x_{\langle ijklmn\rangle}\left[1-5\nu+5\nu^2+\gamma\left(\frac{1}{14}-\frac{3}{2}\nu+6\nu^2-\frac{11}{2}\nu^3\right)\right]\right.$$
$$\left.+\frac{15}{7}\frac{r^2}{c^2}x_{\langle ijkl}v_{mn\rangle}(1-7\nu+14\nu^2-7\nu^3)\right\}+\mathcal{O}\left(\frac{1}{c^4}\right), \tag{5.18a}$$

$$J_{ijklm} = \nu m\left\{\varepsilon_{ab\langle i}x_{jklm\rangle a}v_b\left[1-5\nu+5\nu^2+\gamma\left(\frac{1549}{910}-\frac{1081}{130}\nu+\frac{107}{13}\nu^2-\frac{29}{26}\nu^3\right)\right]\right.$$
$$\left.+\frac{54}{91}\frac{r^2}{c^2}\varepsilon_{ab\langle i}x_{jka}v_{lm\rangle b}(1-7\nu+14\nu^2-7\nu^3)\right\}+\mathcal{O}\left(\frac{1}{c^4}\right). \tag{5.18b}$$

At 0.5PN order:

$$I_{ijklmno} = -\nu m\Delta(1-4\nu+3\nu^2)x_{\langle ijklmno\rangle}+\mathcal{O}\left(\frac{1}{c^2}\right), \tag{5.19a}$$

$$J_{ijklmn} = -\nu m\Delta(1-4\nu+3\nu^2)\varepsilon_{ab\langle i}x_{jklmn\rangle a}v_b+\mathcal{O}\left(\frac{1}{c^2}\right). \tag{5.19b}$$

At Newtonian order:

$$I_{ijklmnop} = \nu m(1-7\nu+14\nu^2-7\nu^3)x_{\langle ijklmnop\rangle}+\mathcal{O}\left(\frac{1}{c^2}\right), \tag{5.20a}$$

$$J_{ijklmno} = \nu m(1-7\nu+14\nu^2-7\nu^3)\varepsilon_{ab\langle i}x_{jklmno\rangle a}v_b+\mathcal{O}\left(\frac{1}{c^2}\right). \tag{5.20b}$$

The 2.5PN correction terms in I_{ijk} and J_{ij}, the 2PN terms in I_{ijkl} and J_{ijk}, and the 1PN terms in I_{ijklm} and J_{ijkl} are new with this paper. The higher-order Newtonian moments $I_{ijklmno}$ and J_{ijklmn} were also not needed before, but Newtonian moments are trivial and are given for general ℓ by

$$I_L = \nu m s_\ell(\nu)x_{\langle L\rangle}+\mathcal{O}\left(\frac{1}{c^2}\right), \tag{5.21a}$$

$$J_{L-1} = \nu m s_\ell(\nu) \varepsilon_{ab\langle i_{\ell-1}} x_{L-2\rangle a} v_b + \mathcal{O}\left(\frac{1}{c^2}\right), \tag{5.21b}$$

in which we pose

$$s_\ell(\nu) = X_2^{\ell-1} + (-)^\ell X_1^{\ell-1}. \tag{5.22}$$

Here we define $X_1 = \frac{m_1}{m} = \frac{1}{2}(1+\Delta)$ and $X_2 = \frac{m_2}{m} = \frac{1}{2}(1-\Delta)$ with $\Delta = \frac{m_1-m_2}{m} = \pm\sqrt{1-4\nu}$, so that $X_1 + X_2 = 1$ and $X_1 X_2 = \nu$.[24]

In addition we shall need the mass monopole I agreeing with its canonical counterpart M which parametrizes the various tail terms in section 5.1. Since the tails arise at 1.5PN order we need M only at the 1.5PN relative order. It is given by

$$I = M = m\left(1 - \frac{\nu}{2}\gamma\right) + \mathcal{O}\left(\frac{1}{c^4}\right). \tag{5.23}$$

We require also the current dipole moment or angular momentum J_i (agreeing with its canonical counterpart S_i) since it appears in some nonlinear terms, for instance in (3.7). It is needed only at Newtonian order,

$$J_i = S_i = \nu m \varepsilon_{iab} x_a v_b + \mathcal{O}\left(\frac{1}{c^2}\right). \tag{5.24}$$

Finally, we have to provide the few gauge moments that enter the relations between canonical and source moments found in (5.9) and (5.10). They are readily computed from the general expressions of all the gauge moments $\{W_L, X_L, Y_L, Z_L\}$ given in (5.15)–(5.20) of [43]. The calculation is quite simple because these moments, namely the monopolar moment W and the two dipole moments W_i and Y_i, are Newtonian. For circular orbits we find

$$W = \mathcal{O}\left(\frac{1}{c^2}\right), \tag{5.25a}$$

$$W_i = \frac{1}{10}\nu m \Delta r^2 v^i + \mathcal{O}\left(\frac{1}{c^2}\right), \tag{5.25b}$$

$$Y_i = \frac{1}{5}\frac{G m^2 \nu}{r}\Delta x^i + \mathcal{O}\left(\frac{1}{c^2}\right). \tag{5.25c}$$

We are done with all the source multipole moments needed to control the 3PN accurate FWF generated by compact binary sources in quasi-circular orbits.

6. Time derivatives of the source multipole moments

For the purpose of computing the time derivatives of the source moments we require the 3PN accurate equations of motion of compact binary sources. Like in the computation of the moments we have to take into account both the conservative effects at 1PN, 2PN and 3PN orders, and the effect of radiation reaction at 2.5PN order.

[24] Note that the coefficient $s_\ell(\nu)$ is equal to the product $\tilde{m} f_k(\nu)$ in the notation of [19]. The equivalence of the two expressions follows from the Waring formulae [59] for $X_1^n + X_2^n$ and $X_1^n - X_2^n$. We find (where $\binom{n}{p}$ is the usual binomial coefficient)

$$s_{2k}(\nu) = \sum_{p=0}^{k-1} (-)^p \frac{2k-1}{2k-1-p} \binom{2k-1-p}{p} \nu^p,$$

$$s_{2k+1}(\nu) = -\Delta \sum_{p=0}^{k-1} (-)^p \binom{2k-1-p}{p} \nu^p.$$

We consider non-spinning objects so the motion takes place in a fixed plane, say the x–y plane. The relative position $\mathbf{x} = \mathbf{y}_1 - \mathbf{y}_2$, velocity $\mathbf{v} = \mathrm{d}\mathbf{x}/\mathrm{d}t$, and acceleration $\mathbf{a} = \mathrm{d}\mathbf{v}/\mathrm{d}t$ are given by

$$\mathbf{x} = r\mathbf{n}, \tag{6.1a}$$

$$\mathbf{v} = \dot{r}\mathbf{n} + r\omega\boldsymbol{\lambda}, \tag{6.1b}$$

$$\mathbf{a} = (\ddot{r} - r\omega^2)\mathbf{n} + (r\dot{\omega} + 2\dot{r}\omega)\boldsymbol{\lambda}. \tag{6.1c}$$

For a while the time derivative will be denoted using an over dot. Here $\boldsymbol{\lambda} = \hat{\mathbf{z}} \times \mathbf{n}$ is perpendicular to the unit vector $\hat{\mathbf{z}}$ along the z-direction orthogonal to the orbital plane, and to the binary's separation direction $\mathbf{n}$. The orbital frequency ω is related in the usual way to the orbital phase ϕ by $\omega = \dot{\phi}$.

Through 3PN order, it is possible to model the motion of the binary as a *quasi-circular* orbit decaying by the effect of radiation reaction at the 2.5PN order. This effect is computed by balancing the change in the orbital energy with the total energy flux radiated by the gravitational waves. At 2.5PN order this yields (see, e.g., [18])

$$\dot{r} = -\frac{64}{5}\sqrt{\frac{Gm}{r}}\nu\gamma^{5/2} + \mathcal{O}\left(\frac{1}{c^7}\right), \tag{6.2a}$$

$$\dot{\omega} = \frac{96}{5}\frac{Gm}{r^3}\nu\gamma^{5/2} + \mathcal{O}\left(\frac{1}{c^7}\right), \tag{6.2b}$$

where γ is given by (5.13). By substituting those expressions into (6.1),[25] we obtain the expressions for the inspiral velocity and acceleration,

$$\mathbf{v} = r\omega\boldsymbol{\lambda} - \frac{64}{5}\sqrt{\frac{Gm}{r}}\nu\gamma^{5/2}\mathbf{n} + \mathcal{O}\left(\frac{1}{c^7}\right), \tag{6.3a}$$

$$\mathbf{a} = -\omega^2\mathbf{x} - \frac{32}{5}\sqrt{\frac{Gm}{r^3}}\nu\gamma^{5/2}\mathbf{v} + \mathcal{O}\left(\frac{1}{c^7}\right). \tag{6.3b}$$

A central result of PN calculations of the equations of motion is the expression of the orbital frequency ω in terms of the binary's separation r up to 3PN order. This result has been obtained in harmonic coordinates in [56–58] and independently in [60–62], and in ADM coordinates in [63–65]. In the present work, r is given in harmonic coordinates and the expression of the 3PN orbital frequency is

$$\omega^2 = \frac{Gm}{r^3}\left\{1 + \gamma(-3+\nu) + \gamma^2\left(6 + \frac{41}{4}\nu + \nu^2\right) + \gamma^3\left(-10 + \left[-\frac{75\,707}{840} + \frac{41}{64}\pi^2 + 22\ln\left(\frac{r}{r'_0}\right)\right]\nu + \frac{19}{2}\nu^2 + \nu^3\right) + \mathcal{O}\left(\frac{1}{c^8}\right)\right\}. \tag{6.4}$$

Note that the logarithm at 3PN order involves the same constant r'_0 as in the source quadrupole moment (5.12)–(5.14). This logarithm comes from a Hadamard self-field regularization scheme and its appearance is specific to harmonic coordinates.

As often convenient we shall use in place of the parameter γ given by (5.13) an alternative parameter x directly linked to the orbital frequency (6.4), namely

$$x = \left(\frac{Gm\omega}{c^3}\right)^{2/3}. \tag{6.5}$$

[25] We note that $\ddot{r} = \mathcal{O}(c^{-10})$ is of the order of the *square* of radiation-reaction effects and is therefore zero with this approximation.

The interest in this parameter stems from its invariant meaning in a large class of coordinate systems including the harmonic and ADM coordinate systems. At 3PN order it is given in terms of x by

$$\gamma = x\Bigg\{1 + x\left(1 - \frac{\nu}{3}\right) + x^2\left(1 - \frac{65}{12}\nu\right) + x^3\left(1 + \left[-\frac{2203}{2520} - \frac{41}{192}\pi^2 - \frac{22}{3}\ln\left(\frac{r}{r'_0}\right)\right]\nu + \frac{229}{36}\nu^2 + \frac{\nu^3}{81}\right) + \mathcal{O}\left(\frac{1}{c^8}\right)\Bigg\}. \tag{6.6}$$

Combining (6.4) with (6.6) we find that the velocity squared $v^2 = r^2\omega^2 + \dot{r}^2 = r^2\omega^2 + \mathcal{O}(c^{-10})$ is related to x by

$$\left(\frac{v}{c}\right)^2 = x\Bigg\{1 + x\left(-2 + \frac{2}{3}\nu\right) + x^2\left(1 + \frac{53}{6}\nu + \frac{\nu^2}{3}\right) + x^3\left(\left[-\frac{36\,227}{1260} + \frac{41}{96}\pi^2 + \frac{44}{3}\ln\left(\frac{r}{r'_0}\right)\right]\nu - \frac{29}{9}\nu^2 + \frac{10}{81}\nu^3\right) + \mathcal{O}\left(\frac{1}{c^8}\right)\Bigg\}. \tag{6.7}$$

During the computation of the time derivatives of the source moments, each time an acceleration is produced the result is consistently *order reduced*, i.e. the acceleration is replaced with (6.3*b*) at the right PN order. Such an order reduction will generate in particular some 2.5PN radiation-reaction terms which are to be taken into account in the 3PN waveform. This occurs when computing the time derivatives of the moments I_{ij}, I_{ijk} and J_{ij} that appear in the FWF at Newtonian and 0.5PN orders. On the other hand, when computing the polarization states following (2.2) we shall meet some scalar products of the polarization vectors **P** and **Q** with the relative velocity **v**. If those scalar products occur at Newtonian and 0.5PN orders (i.e. in multipolar pieces corresponding to the moments I_{ij}, I_{ijk} and J_{ij}) we shall have to take into account the 2.5PN radiation-reaction term coming from the expression of **v** given by (6.3*a*)[26]. However it was shown in [18] that the radiation-reaction terms in the FWF at the 2.5PN order can be absorbed into a modification of the orbital phase, where they appear to constitute in fact a very small phase modulation, comparable with unknown contributions in the phase being at least of order 5PN—negligible here since the phase is known only to 3.5PN order. In the present paper, we have chosen[27] to include all the radiation-reaction terms coming from both (6.3*a*) and (6.3*b*), and to present them as 2.5PN and 3PN amplitude corrections in our final results which will be presented in (8.9), (8.10) and (9.4).

Let us next check that the Hadamard self-field regularization constant r'_0 appearing both in the 3PN orbital frequency (6.4) and in the 3PN quadrupole moment (5.14) [28], is actually a gauge constant. To this end we simply verify that r'_0 will be eliminated when expressing the FWF in terms of the gauge invariant parameter (6.5). From (5.14) we see that the dependence on r'_0 of the 3PN quadrupole moment is

$$I_{ij} = \nu m\left[1 - \frac{44}{3}\gamma^3\nu\ln\left(\frac{r}{r'_0}\right)\right]x_{\langle ij\rangle} + \cdots + \mathcal{O}\left(\frac{1}{c^7}\right). \tag{6.8}$$

We indicate by dots all the terms that are independent of r'_0 (for convenience we also show the Newtonian term). Now the FWF depends on the second time derivative of the quadrupole

[26] Not considering the radiation-reaction contribution in **v** given by (6.3*a*) has been the source of an error in [17] which has been pointed out and corrected in [18].

[27] As usual there are many different ways of presenting PN results at a given order of approximation, and choosing one or another is often a matter of convenience.

[28] The other moments are given at 2.5PN order at most; they do not depend on r'_0 since the appearance of regularization constants is a feature of the 3PN approximation.

moment. For circular orbits this reads (coming back to the superscript notation (n) for time derivatives)

$$I_{ij}^{(2)} = 2\nu m \left[1 - \frac{44}{3}\gamma^3 \nu \ln\left(\frac{r}{r_0'}\right)\right] (v_{\langle ij\rangle} + x_{\langle i} a_{j\rangle}) + \cdots + \mathcal{O}\left(\frac{1}{c^7}\right). \tag{6.9}$$

Replacing v_i and a_i by their values (6.3) we get with the required approximation (still being interested only in the fate of the constant r_0')

$$I_{ij}^{(2)} = 2\nu m v^2 \left[1 - \frac{44}{3}\gamma^3 \nu \ln\left(\frac{r}{r_0'}\right)\right] (\lambda_{\langle ij\rangle} - n_{\langle ij\rangle}) + \cdots + \mathcal{O}\left(\frac{1}{c^7}\right). \tag{6.10}$$

The squared velocity $v^2 = r^2\omega^2 + \mathcal{O}(c^{-10})$ appears in factor. It is now clear that replacing v^2 by its expression in terms of the parameter x following (6.7), we produce another logarithmic term containing r_0', namely

$$v^2 = c^2 x \left[1 + \frac{44}{3} x^3 \nu \ln\left(\frac{r}{r_0'}\right)\right] + \cdots + \mathcal{O}\left(\frac{1}{c^7}\right), \tag{6.11}$$

which will cancel out the dependence of the quadrupole moment on r_0' at 3PN order (using the fact that γ can be replaced by x in a small 3PN term). Thus, finally,

$$I_{ij}^{(2)} = 2\nu m c^2 x (\lambda_{\langle ij\rangle} - n_{\langle ij\rangle}) + \cdots + \mathcal{O}\left(\frac{1}{c^7}\right) \tag{6.12}$$

is independent on r_0', which means that this constant cannot affect any physical result at the 3PN order.

7. Computation of the tail and memory integrals

The results of sections 5 and 6 yield the complete control of the *instantaneous* part of the FWF. We now tackle the computation of the *hereditary* part, which is composed of tails (and tails-of-tails and squared tails) and nonlinear memory terms. The hereditary integrals have been explicitly provided in section 5.1 as contributions to the various radiative moments U_L and V_L given by (5.4)–(5.7). Our computation will basically be a straightforward extension of the computation performed at 2.5PN order in section 4 of [17]. Since we employ exactly the same techniques, we skip most of the details and rely on [17] for justification of the method and proofs.

We first consider the nonlinear memory terms. Up to 3PN order we have the 2.5PN memory integrals in the radiative mass quadrupole moment U_{ij} given by (3.7) and the radiative mass hexadecapole moment U_{ijkl} given by (5.6*a*)—these are the memory terms contributing to the FWF at 2.5PN order [17] — and, in addition, we have the memory integral in the mass octupole moment U_{ijk} given by (5.5*a*) and the one in U_{ijklm} given by (5.7*b*)—these contribute specifically at 3PN order[29]. Like in [17] we obtain the corresponding integrands (i.e. the terms under the integral sign) and compute directly their contributions to the two wave polarizations h_+ and $h_\times$. Indeed it is convenient to perform the relevant contractions of the integrands with the polarization vectors **P** and **Q** (see section 8 for the conventions we adopt) so as to only deal with *scalar* quantities.

We find that the memory integrals in h_+ and $h_\times$ are composed of two types of terms. First there is a term, only present in the plus polarization h_+, which does not depend on the orbital phase and can thus be viewed as a *zero-frequency* (DC) term. Actually, because of the steady inspiral, this term is a steadily varying function of time, with an amplitude increasing

[29] Recall that the nonlinear memory terms occur only in the *mass-type* radiative multipole moments U_L.

like some power law of the time remaining till the coalescence. Strictly speaking, this term is to be regarded as *the* memory contribution because it does depend on the behaviour of the system in the remote past, and therefore must be computed using some model for the evolution of the binary system in the past. In the present paper, we find that the only zero-frequency term up to 3PN order is the one which appeared already at 2.5PN order and was evaluated in [17]—interestingly there are no other terms of this type at the 3PN order. Because of the cumulative effect of integration over the whole past we know that this term, though originating from 2.5PN order, finally contributes in the FWF at the Newtonian level [52–54]. In practice, the computation of this DC term reduces (in the circular orbit case) to the evaluation of the single elementary integral

$$\mathrm{I}(T_R) = \frac{(Gm)^{p-1}}{c^{2p-3}} \int_{-\infty}^{T_R} \frac{\mathrm{d}\tau}{r^p(\tau)}. \tag{7.1}$$

Here $r(\tau)$ denotes the binary's separation at any time $\tau \leqslant T_R$ (where $T_R = T - R/c$ is the current time). The coefficient in front of (7.1) is chosen for convenience to make the integral dimensionless. The integral (7.1) is easily computed using a simplified model of binary evolution in the past in which the orbit is assumed to remain circular apart from the gradual inspiral at any time. In this model the binary separation evolves like $r(\tau) \propto (T_c - \tau)^{1/4}$ where T_c denotes the instant of coalescence (see [17] for more details). In the remote past we thus have $r(\tau) \sim (-\tau)^{1/4}$ so the integral (7.1) converges when $p > 4$ (actually we shall only need the case $p = 5$ like in [17]). The result reads

$$\mathrm{I}(T_R) = \frac{5}{64(p-4)} \frac{x^{p-4}(T_R)}{\nu}, \tag{7.2}$$

where $x(T_R)$ denotes the *current* value (i.e. at the current retarded time T_R) of the parameter x defined by (6.5). Witness the memory effect: the end result (7.2) is of order $x^{p-4} = \mathcal{O}(c^{-2p+8})$ which is a factor c^5 *larger* than the original formal PN order $\mathcal{O}(c^{-2p+3})$ as shown in (7.1). Hence, although the memory term is formally of order 2.5PN, its actual contribution to the waveform is comparable to a Newtonian term. As mentioned above we do not find memory (zero-frequency) contributions originating from the next 3PN order, and therefore finally no DC term at 0.5PN order.

Second there are other terms, present in both polarizations, which depend on the orbital phase, and oscillate like some harmonics of the orbital phase (say $n\phi$). Such phase-dependent, oscillating terms do not exhibit the memory effect, essentially because the oscillations, due to the sequence of orbital cycles in the entire life of the binary system, more or less compensate each other. As a result these terms, in contrast with (7.1) and (7.2), keep on their formal PN order. We recover the 2.5PN terms investigated in [17] and in addition we obtain several other terms at 3PN order. The latter are computed by a slight generalization of the method followed in [17]: instead of (4.18) in [17] we need to consider the integral

$$\mathrm{J}(T_R) = \frac{(Gm)^{p-1}}{c^{2p-3}} \int_{-\infty}^{T_R} \mathrm{d}\tau \frac{\mathrm{e}^{\mathrm{i}n\phi(\tau)}}{r^p(\tau)}, \tag{7.3}$$

where $\phi(\tau)$ is the orbital phase at any time, where n and p range over integer or half-integer values (e.g. $n = 1, 3, 5$ and $p = 11/2$ at 3PN order), and where the coefficient is chosen to make the integral dimensionless. Following the steps (4.18)–(4.23) in [17] we compute this integral using our model of binary's past evolution, and in the adiabatic limit, which means that the *current* value of the adiabatic parameter ξ associated with the binary inspiral is considered to be small and of PN order $\xi(T_R) = \mathcal{O}(c^{-5})$. We then find

$$\mathrm{J}(T_R) = x^{p-\frac{3}{2}}(T_R) \frac{\mathrm{e}^{\mathrm{i}n\phi(T_R)}}{\mathrm{i}n} \left[1 + \mathcal{O}\left(\frac{1}{c^5}\right)\right]. \tag{7.4}$$

This result (valid only if $n \neq 0$) permits one to handle all the phase-dependent oscillating terms coming from the memory integrals.

We next turn to the computation of the tails and tails-of-tails present in the radiative moments (5.4)–(5.7). Again we closely follow the previous investigation [17] on which we refer for more details. The computation of tails reduces to the evaluation of an elementary integral involving a logarithmic kernel,

$$\mathrm{K}(T_R) = \frac{(Gm)^{p-1}}{c^{2p-3}} \int_{-\infty}^{T_R} \mathrm{d}\tau \frac{\mathrm{e}^{\mathrm{i}n\phi(\tau)}}{r^p(\tau)} \ln\left(\frac{T_R-\tau}{T_c-T_R}\right), \tag{7.5}$$

in which the logarithm has been scaled with the constant time $T_c - T_R$, instead of the previous normalization by $2\tau_0$, where T_c is the instant of coalescence in the model of [17]. Such scaling can always be done at the price of adding another term proportional to some integral of the type $\mathrm{J}(T_R)$ computed previously. Following the derivation of this integral in [17], we find that, at dominant order in the adiabatic approximation,

$$\mathrm{K}(T_R) = x^{p-\frac{3}{2}}(T_R)\frac{\mathrm{e}^{\mathrm{i}n\phi(T_R)}}{\mathrm{i}n}\left[\frac{\pi}{2\mathrm{i}} - \ln\left(\frac{n}{\xi(T_R)}\right) - C + \mathcal{O}\left(\frac{\ln c}{c^5}\right)\right]. \tag{7.6}$$

Here $C = 0.577\cdots$ is the Euler constant, and $\xi(T_R)$ denotes the current value of the adiabatic parameter associated with the inspiral, which is defined by $\xi(T_R) = [(T_c - T_R)\omega(T_R)]^{-1}$ in the model of [17]. The adiabatic parameter is related to the PN parameter x by

$$\xi(T_R) = \frac{256\nu}{5} x^{5/2}(T_R). \tag{7.7}$$

The squared tails are computed using the same integral (7.5)–(7.6). Concerning the tails-of-tails we simply have to consider an integral involving a logarithm squared,

$$\mathrm{L}(T_R) = \frac{(Gm)^{p-1}}{c^{2p-3}} \int_{-\infty}^{T_R} \mathrm{d}\tau \frac{\mathrm{e}^{\mathrm{i}n\phi(\tau)}}{r^p(\tau)} \ln^2\left(\frac{T_R-\tau}{T_c-T_R}\right), \tag{7.8}$$

which is computed using the same technique with the result

$$\mathrm{L}(T_R) = x^{p-\frac{3}{2}}(T_R)\frac{\mathrm{e}^{\mathrm{i}n\phi(T_R)}}{in}\left[\frac{\pi^2}{6} + \left(C + \ln\left(\frac{n}{\xi(T_R)}\right) + \frac{\mathrm{i}\pi}{2}\right)^2 + \mathcal{O}\left(\frac{\ln c}{c^5}\right)\right]. \tag{7.9}$$

We are done with the computation of all tails and tails-of-tails in the 3PN waveform.

For completeness let us give also the two technical formulae which enables one to arrive at the results (7.6) and (7.9). Posing $y = (T_R - \tau)/(T_c - T_R)$ and $\lambda = n/\xi$, and working at the leading order in the adiabatic limit $\xi \to 0$ or equivalently when $\lambda \to +\infty$, the formulae express that, for any positive or negative λ (see, e.g., [66, pp 573, 574]),

$$\int_0^1 \mathrm{d}y \ln y \,\mathrm{e}^{-\mathrm{i}\lambda y} = \frac{1}{\lambda}\left[-\frac{\pi}{2}\mathrm{sign}(\lambda) + \mathrm{i}(\ln|\lambda| + C)\right] + \mathcal{O}\left(\frac{1}{\lambda^2}\right), \tag{7.10a}$$

$$\int_0^1 \mathrm{d}y \ln^2 y \,\mathrm{e}^{-\mathrm{i}\lambda y} = \frac{\mathrm{i}}{\lambda}\left(-\frac{\pi^2}{6} + \left[-\frac{\pi}{2}\mathrm{sign}(\lambda) + \mathrm{i}(\ln|\lambda| + C)\right]^2\right) + \mathcal{O}\left(\frac{1}{\lambda^3}\right). \tag{7.10b}$$

Note that we are only interested in the *recent past* contribution to the integrals (7.10), corresponding to the interval $0 \leqslant y \leqslant 1$ equivalent to the time interval $2T_R - T_c \leqslant \tau \leqslant T_R$. The reason is that the *remote past* contribution, given by $1 < y < +\infty$ or equivalently $-\infty < \tau < 2T_R - T_c$, is small in the adiabatic limit. This is a characteristic feature of tails: they die out very rapidly, therefore they depend essentially on the recent past evolution of the matter source [40, 67]. In the case at hand this technically means that the remote-past contributions to the integrals are of order

$$\int_1^{+\infty} \mathrm{d}y \ln y \, \mathrm{e}^{-\mathrm{i}\lambda y} = \mathcal{O}\left(\frac{1}{\lambda^2}\right), \tag{7.11a}$$

$$\int_1^{+\infty} \mathrm{d}y \ln^2 y \, \mathrm{e}^{-\mathrm{i}\lambda y} = \mathcal{O}\left(\frac{1}{\lambda^3}\right), \tag{7.11b}$$

as can easily be verified by using integration by parts.

8. 3PN polarization waveforms for data analysis

We specify our conventions for the orbital phase and polarization vectors defining the polarization waveforms (2.2) in the case of quasi-circular binary systems of non-spinning compact objects. If the orbital plane is chosen to be the x–y plane (like in section 6), with the orbital phase ϕ measuring the direction of the unit vector $\mathbf{n} = \mathbf{x}/r$ along the relative separation vector, then

$$\mathbf{n} = \hat{\mathbf{x}} \cos\phi + \hat{\mathbf{y}} \sin\phi, \tag{8.1}$$

where $\hat{\mathbf{x}}$ and $\hat{\mathbf{y}}$ are the unit directions along x and y. Following [16, 17] we choose the polarization vector $\mathbf{P}$ to lie along the x-axis and the observer to be in the y–z plane with

$$\mathbf{N} = s_i \hat{\mathbf{y}} + c_i \hat{\mathbf{z}}, \tag{8.2}$$

where we pose $c_i = \cos i$ and $s_i = \sin i$, with i being the orbit's inclination angle ($0 \leqslant i \leqslant \pi$). With this choice $\mathbf{P}$ lies along the intersection of the orbital plane with the plane of the sky in the direction of the *ascending node* $\mathcal{N}$, i.e. that point at which the bodies cross the plane of the sky moving towards the observer. The orbital phase ϕ is the angle between the ascending node $\mathcal{N}$ and the direction of body one (say). The rotating orthonormal triad ($\mathbf{n}$, $\boldsymbol{\lambda}$, $\hat{\mathbf{z}}$) describing the motion of the binary (see (6.1)) is then related to the fixed polarization triad ($\mathbf{N}$, $\mathbf{P}$, $\mathbf{Q}$) by

$$\mathbf{n} = \mathbf{P} \cos\phi + (c_i \mathbf{Q} + s_i \mathbf{N}) \sin\phi, \tag{8.3a}$$

$$\boldsymbol{\lambda} = -\mathbf{P} \sin\phi + (c_i \mathbf{Q} + s_i \mathbf{N}) \cos\phi, \tag{8.3b}$$

$$\hat{\mathbf{z}} = -s_i \mathbf{Q} + c_i \mathbf{N}. \tag{8.3c}$$

As in previous works [16, 17] we shall present the wave polarizations (2.2) as expansion series in powers of the gauge-invariant PN parameter x defined by (6.5). With a convenient overall factorization we write them as

$$\begin{pmatrix} h_+ \\ h_\times \end{pmatrix} = \frac{2Gm\nu x}{c^2 R} \begin{pmatrix} H_+ \\ H_\times \end{pmatrix} + \mathcal{O}\left(\frac{1}{R^2}\right), \tag{8.4}$$

with the following PN expansion series:

$$H_{+,\times} = \sum_{n=0}^{+\infty} x^{n/2} H_{+,\times}^{(n/2)}. \tag{8.5}$$

The PN coefficients $H_{+,\times}^{(n/2)}$ will be given as functions of the orbital phase ϕ, and will also be polynomials in the symmetric mass ratio ν and depend on the inclination angle i. In addition they will involve, at high PN order, the logarithm of x as we shall discuss below.

Following [16, 17] it is convenient to perform a change of phase variable, from the actual orbital phase ϕ satisfying $\dot{\phi} = \omega$, to some new variable denoted ψ. Recall that the orbital phase ϕ evolves by gravitational radiation reaction and its expression as a function of time is known from previous work [13–15] up to 3.5PN order. We then pose[30]

$$\psi = \phi - \frac{2GM\omega}{c^3}\ln\left(\frac{\omega}{\omega_0}\right), \tag{8.6}$$

where M is the binary's total mass given by (5.23), and where ω_0 denotes the constant

$$\omega_0 = \frac{\mathrm{e}^{\frac{11}{12}-C}}{4\tau_0}. \tag{8.7}$$

Here $\tau_0 = r_0/c$ is the normalization of logarithms in the tail integrals of the radiative moments (3.7)–(5.7); r_0 is the constant included in the definition of the finite part in (3.7). Like τ_0 the constant ω_0 is arbitrary, because it is linked to the difference of origins of time in the far zone and in the near zone, see (3.16). For instance we can choose $\omega_0 = \pi f_{\mathrm{seismic}}$ where f_{seismic} is the entry frequency of some ground-based interferometric detector. Using (5.23) and the notation (6.5) the new phase variable reads

$$\psi = \phi - 3x^{3/2}\left[1 - \frac{\nu}{2}x\right]\ln\left(\frac{x}{x_0}\right), \tag{8.8}$$

where $x_0 = \left(\frac{Gm\omega_0}{c^3}\right)^{2/3}$.[31] Our modified phase variable (8.6)–(8.8) will be valid up to 3PN order but in fact it turns out to be the same as at the previous 2.5PN order [17].

The logarithmic term in ψ corresponds to some spreading of the different frequency components of the wave along the line of sight from the source to the far-away detector, and expresses physically the tail effect as a small delay in the arrival time of gravitational waves. However, practically speaking, the main interest of this term is to minimize the occurrence of logarithms in the FWF. Indeed we note that the logarithmic term in (8.6), although of formal PN order $\mathcal{O}(c^{-3})$, represents in fact a very small modulation of the orbital phase: compared with the dominant phase evolution whose order is that of the inverse of radiation reaction, i.e. $\phi = \mathcal{O}(\xi^{-1}) = \mathcal{O}(c^5)$, this term is of order $\mathcal{O}(c^{-8})$ namely 4PN in the phase evolution, which can be regarded as negligible to the present accuracy. Thus the logarithms associated with the phase modulation in (8.6) will be 'eliminated' from the FWF at 3PN order. This does not mean that we should ignore them but that the formulation in terms of the small phase modulation (8.6) is quite natural (for the data analysis it is probably better to keep the logarithm as it stands in the definition of the phase variable ψ). However all the logarithms will not be 'removed' by this process, and we shall find that some 'true' logarithms remain starting at the 3PN order. Such logarithms cannot be absorbed into some small modulation of the orbital phase, so 3PN will remain as the true order of magnitude of these logarithms in the FWF.

With those conventions and notation we find for the plus polarization[32]

$$H_+^{(0)} = -\left(1+c_i^2\right)\cos 2\psi - \frac{1}{96}s_i^2\left(17+c_i^2\right), \tag{8.9a}$$

$$H_+^{(0.5)} = -s_i\,\Delta\left[\cos\psi\left(\frac{5}{8}+\frac{1}{8}c_i^2\right) - \cos 3\psi\left(\frac{9}{8}+\frac{9}{8}c_i^2\right)\right], \tag{8.9b}$$

$$H_+^{(1)} = \cos 2\psi\left[\frac{19}{6}+\frac{3}{2}c_i^2-\frac{1}{3}c_i^4+\nu\left(-\frac{19}{6}+\frac{11}{6}c_i^2+c_i^4\right)\right] - \cos 4\psi\left[\frac{4}{3}s_i^2\left(1+c_i^2\right)(1-3\nu)\right], \tag{8.9c}$$

[30] A similar phase variable is also introduced in black-hole perturbation theory [68–70].
[31] We have $\ln x_0 = \frac{11}{18} - \frac{2}{3}C - \frac{4}{3}\ln 2 + \frac{2}{3}\ln\left(\frac{Gm}{c^2 r_0}\right)$ in agreement with equation (68) of [19].
[32] We also requote the previous 2.5PN results [17] taking into account the published erratum [17] and the correcting term associated with radiation reaction and pointed out in [18].

$$
\begin{aligned}
H_+^{(1.5)} =\; & s_i \Delta \cos\psi \left[\frac{19}{64} + \frac{5}{16} c_i^2 - \frac{1}{192} c_i^4 + \nu \left(-\frac{49}{96} + \frac{1}{8} c_i^2 + \frac{1}{96} c_i^4 \right) \right] \\
& + \cos 2\psi \left[-2\pi \left(1 + c_i^2\right) \right] + s_i \Delta \cos 3\psi \left[-\frac{657}{128} - \frac{45}{16} c_i^2 + \frac{81}{128} c_i^4 \right. \\
& \left. + \nu \left(\frac{225}{64} - \frac{9}{8} c_i^2 - \frac{81}{64} c_i^4 \right) \right] + s_i \Delta \cos 5\psi \left[\frac{625}{384} s_i^2 \left(1 + c_i^2\right)(1 - 2\nu) \right], \qquad (8.9d) \\
H_+^{(2)} =\; & \pi s_i \Delta \cos\psi \left[-\frac{5}{8} - \frac{1}{8} c_i^2 \right] + \cos 2\psi \left[\frac{11}{60} + \frac{33}{10} c_i^2 + \frac{29}{24} c_i^4 - \frac{1}{24} c_i^6 \right. \\
& \left. + \nu \left(\frac{353}{36} - 3 c_i^2 - \frac{251}{72} c_i^4 + \frac{5}{24} c_i^6 \right) + \nu^2 \left(-\frac{49}{12} + \frac{9}{2} c_i^2 - \frac{7}{24} c_i^4 - \frac{5}{24} c_i^6 \right) \right] \\
& + \pi s_i \Delta \cos 3\psi \left[\frac{27}{8} \left(1 + c_i^2\right) \right] + \frac{2}{15} s_i^2 \cos 4\psi \left[59 + 35 c_i^2 - 8 c_i^4 \right. \\
& \left. - \frac{5}{3} \nu \left(131 + 59 c_i^2 - 24 c_i^4\right) + 5\nu^2 \left(21 - 3 c_i^2 - 8 c_i^4\right) \right] \\
& + \cos 6\psi \left[-\frac{81}{40} s_i^4 \left(1 + c_i^2\right)(1 - 5\nu + 5\nu^2) \right] \\
& + s_i \Delta \sin\psi \left[\frac{11}{40} + \frac{5 \ln 2}{4} + c_i^2 \left(\frac{7}{40} + \frac{\ln 2}{4} \right) \right] \\
& + s_i \Delta \sin 3\psi \left[\left(-\frac{189}{40} + \frac{27}{4} \ln(3/2) \right) \left(1 + c_i^2\right) \right], \qquad (8.9e) \\
H_+^{(2.5)} =\; & s_i \Delta \cos\psi \left[\frac{1771}{5120} - \frac{1667}{5120} c_i^2 + \frac{217}{9216} c_i^4 - \frac{1}{9216} c_i^6 \right. \\
& + \nu \left(\frac{681}{256} + \frac{13}{768} c_i^2 - \frac{35}{768} c_i^4 + \frac{1}{2304} c_i^6 \right) \\
& \left. + \nu^2 \left(-\frac{3451}{9216} + \frac{673}{3072} c_i^2 - \frac{5}{9216} c_i^4 - \frac{1}{3072} c_i^6 \right) \right] \\
& + \pi \cos 2\psi \left[\frac{19}{3} + 3 c_i^2 - \frac{2}{3} c_i^4 + \nu \left(-\frac{16}{3} + \frac{14}{3} c_i^2 + 2 c_i^4 \right) \right] \\
& + s_i \Delta \cos 3\psi \left[\frac{3537}{1024} - \frac{22\,977}{5120} c_i^2 - \frac{15\,309}{5120} c_i^4 + \frac{729}{5120} c_i^6 \right. \\
& + \nu \left(-\frac{23\,829}{1280} + \frac{5529}{1280} c_i^2 + \frac{7749}{1280} c_i^4 - \frac{729}{1280} c_i^6 \right) \\
& \left. + \nu^2 \left(\frac{29\,127}{5120} - \frac{27\,267}{5120} c_i^2 - \frac{1647}{5120} c_i^4 + \frac{2187}{5120} c_i^6 \right) \right] \\
& + \cos 4\psi \left[-\frac{16\pi}{3} \left(1 + c_i^2\right) s_i^2 (1 - 3\nu) \right] \\
& + s_i \Delta \cos 5\psi \left[-\frac{108\,125}{9216} + \frac{40\,625}{9216} c_i^2 + \frac{83\,125}{9216} c_i^4 - \frac{15\,625}{9216} c_i^6 \right. \\
& + \nu \left(\frac{8125}{256} - \frac{40\,625}{2304} c_i^2 - \frac{48\,125}{2304} c_i^4 + \frac{15\,625}{2304} c_i^6 \right) \\
& \left. + \nu^2 \left(-\frac{119\,375}{9216} + \frac{40\,625}{3072} c_i^2 + \frac{44\,375}{9216} c_i^4 - \frac{15\,625}{3072} c_i^6 \right) \right]
\end{aligned}
$$

$$
\begin{aligned}
&+\Delta\cos 7\psi\left[\frac{117\,649}{46\,080}s_i^5\left(1+c_i^2\right)(1-4\nu+3\nu^2)\right]\\
&+\sin 2\psi\left[-\frac{9}{5}+\frac{14}{5}c_i^2+\frac{7}{5}c_i^4+\nu\left(32+\frac{56}{5}c_i^2-\frac{28}{5}c_i^4\right)\right]\\
&+s_i^2\left(1+c_i^2\right)\sin 4\psi\left[\frac{56}{5}-\frac{32\ln 2}{3}+\nu\left(-\frac{1193}{30}+32\ln 2\right)\right],
\end{aligned}
\qquad (8.9f)
$$

$$
\begin{aligned}
H_+^{(3)} = \pi\,\Delta s_i\cos\psi&\left[\frac{19}{64}+\frac{5}{16}c_i^2-\frac{1}{192}c_i^4+\nu\left(-\frac{19}{96}+\frac{3}{16}c_i^2+\frac{1}{96}c_i^4\right)\right]\\
&+\cos 2\psi\left[-\frac{465\,497}{11\,025}+\left(\frac{856C}{105}-\frac{2\pi^2}{3}+\frac{428}{105}\ln(16x)\right)\left(1+c_i^2\right)\right.\\
&-\frac{3561\,541}{88\,200}c_i^2-\frac{943}{720}c_i^4+\frac{169}{720}c_i^6-\frac{1}{360}c_i^8\\
&+\nu\left(\frac{2209}{360}-\frac{41\pi^2}{96}\left(1+c_i^2\right)+\frac{2039}{180}c_i^2+\frac{3311}{720}c_i^4-\frac{853}{720}c_i^6+\frac{7}{360}c_i^8\right)\\
&+\nu^2\left(\frac{12\,871}{540}-\frac{1583}{60}c_i^2-\frac{145}{108}c_i^4+\frac{56}{45}c_i^6-\frac{7}{180}c_i^8\right)\\
&\left.+\nu^3\left(-\frac{3277}{810}+\frac{19\,661}{3240}c_i^2-\frac{281}{144}c_i^4-\frac{73}{720}c_i^6+\frac{7}{360}c_i^8\right)\right]\\
&+\pi\,\Delta s_i\cos 3\psi\left[-\frac{1971}{128}-\frac{135}{16}c_i^2+\frac{243}{128}c_i^4+\nu\left(\frac{567}{64}-\frac{81}{16}c_i^2-\frac{243}{64}c_i^4\right)\right]\\
&+s_i^2\cos 4\psi\left[-\frac{2189}{210}+\frac{1123}{210}c_i^2+\frac{56}{9}c_i^4-\frac{16}{45}c_i^6\right.\\
&+\nu\left(\frac{6271}{90}-\frac{1969}{90}c_i^2-\frac{1432}{45}c_i^4+\frac{112}{45}c_i^6\right)\\
&+\nu^2\left(-\frac{3007}{27}+\frac{3493}{135}c_i^2+\frac{1568}{45}c_i^4-\frac{224}{45}c_i^6\right)\\
&\left.+\nu^3\left(\frac{161}{6}-\frac{1921}{90}c_i^2-\frac{184}{45}c_i^4+\frac{112}{45}c_i^6\right)\right]\\
&+\Delta\cos 5\psi\left[\frac{3125\pi}{384}s_i^3\left(1+c_i^2\right)(1-2\nu)\right]+s_i^4\cos 6\psi\left[\frac{1377}{80}+\frac{891}{80}c_i^2-\frac{729}{280}c_i^4\right.\\
&+\nu\left(-\frac{7857}{80}-\frac{891}{16}c_i^2+\frac{729}{40}c_i^4\right)+\nu^2\left(\frac{567}{4}+\frac{567}{10}c_i^2-\frac{729}{20}c_i^4\right)\\
&\left.+\nu^3\left(-\frac{729}{16}-\frac{243}{80}c_i^2+\frac{729}{40}c_i^4\right)\right]\\
&+\cos 8\psi\left[-\frac{1024}{315}s_i^6\left(1+c_i^2\right)(1-7\nu+14\nu^2-7\nu^3)\right]\\
&+\Delta s_i\sin\psi\left[-\frac{2159}{40\,320}-\frac{19\ln 2}{32}+\left(-\frac{95}{224}-\frac{5\ln 2}{8}\right)c_i^2+\left(\frac{181}{13\,440}+\frac{\ln 2}{96}\right)c_i^4\right.\\
&\left.+\nu\left(\frac{81\,127}{10\,080}+\frac{19\ln 2}{48}+\left(-\frac{41}{48}-\frac{3\ln 2}{8}\right)c_i^2+\left(-\frac{313}{480}-\frac{\ln 2}{48}\right)c_i^4\right)\right]\\
&+\sin 2\psi\left[-\frac{428\pi}{105}\left(1+c_i^2\right)\right]
\end{aligned}
$$

$$
\begin{aligned}
&+ \Delta s_i \sin 3\psi \left[\frac{205\,119}{8960} - \frac{1971}{64}\ln(3/2) + \left(\frac{1917}{224} - \frac{135}{8}\ln(3/2)\right) c_i^2\right.\\
&+ \left(-\frac{43\,983}{8960} + \frac{243}{64}\ln(3/2)\right) c_i^4\\
&+ \nu\left(-\frac{54\,869}{960} + \frac{567}{32}\ln(3/2) + \left(-\frac{923}{80} - \frac{81}{8}\ln(3/2)\right) c_i^2\right.\\
&\left.\left.+ \left(\frac{41\,851}{2880} - \frac{243}{32}\ln(3/2)\right) c_i^4\right)\right]\\
&+ \Delta s_i^3\left(1 + c_i^2\right)\sin 5\psi\left[-\frac{113\,125}{5376} + \frac{3125}{192}\ln(5/2)\right.\\
&\left.+ \nu\left(\frac{17\,639}{320} - \frac{3125}{96}\ln(5/2)\right)\right].
\end{aligned}
\tag{8.9g}
$$

For the cross polarizations we obtain

$$H_\times^{(0)} = -2c_i \sin 2\psi, \tag{8.10a}$$

$$H_\times^{(0.5)} = s_i c_i \Delta\left[-\frac{3}{4}\sin\psi + \frac{9}{4}\sin 3\psi\right], \tag{8.10b}$$

$$H_\times^{(1)} = c_i \sin 2\psi\left[\frac{17}{3} - \frac{4}{3}c_i^2 + \nu\left(-\frac{13}{3} + 4c_i^2\right)\right] + c_i s_i^2 \sin 4\psi\left[-\frac{8}{3}(1 - 3\nu)\right], \tag{8.10c}$$

$$
\begin{aligned}
H_\times^{(1.5)} = {}& s_i c_i \Delta \sin\psi\left[\frac{21}{32} - \frac{5}{96}c_i^2 + \nu\left(-\frac{23}{48} + \frac{5}{48}c_i^2\right)\right] - 4\pi c_i \sin 2\psi\\
&+ s_i c_i \Delta \sin 3\psi\left[-\frac{603}{64} + \frac{135}{64}c_i^2 + \nu\left(\frac{171}{32} - \frac{135}{32}c_i^2\right)\right]\\
&+ s_i c_i \Delta \sin 5\psi\left[\frac{625}{192}(1 - 2\nu)s_i^2\right],
\end{aligned}
\tag{8.10d}
$$

$$
\begin{aligned}
H_\times^{(2)} = {}& s_i c_i \Delta \cos\psi\left[-\frac{9}{20} - \frac{3}{2}\ln 2\right] + s_i c_i \Delta \cos 3\psi\left[\frac{189}{20} - \frac{27}{2}\ln(3/2)\right]\\
&- s_i c_i \Delta\left[\frac{3\pi}{4}\right]\sin\psi + c_i \sin 2\psi\left[\frac{17}{15} + \frac{113}{30}c_i^2 - \frac{1}{4}c_i^4\right.\\
&\left.+ \nu\left(\frac{143}{9} - \frac{245}{18}c_i^2 + \frac{5}{4}c_i^4\right) + \nu^2\left(-\frac{14}{3} + \frac{35}{6}c_i^2 - \frac{5}{4}c_i^4\right)\right]\\
&+ s_i c_i \Delta \sin 3\psi\left[\frac{27\pi}{4}\right] + \frac{4}{15}c_i s_i^2 \sin 4\psi\left[55 - 12c_i^2 - \frac{5}{3}\nu\left(119 - 36c_i^2\right)\right.\\
&\left.+ 5\nu^2\left(17 - 12c_i^2\right)\right] + c_i \sin 6\psi\left[-\frac{81}{20}s_i^4\left(1 - 5\nu + 5\nu^2\right)\right],
\end{aligned}
\tag{8.10e}
$$

$$
\begin{aligned}
H_\times^{(2.5)} = {}& \frac{6}{5}s_i^2 c_i \nu + c_i \cos 2\psi\left[2 - \frac{22}{5}c_i^2 + \nu\left(-\frac{282}{5} + \frac{94}{5}c_i^2\right)\right]\\
&+ c_i s_i^2 \cos 4\psi\left[-\frac{112}{5} + \frac{64}{3}\ln 2 + \nu\left(\frac{1193}{15} - 64\ln 2\right)\right]\\
&+ s_i c_i \Delta \sin\psi\left[-\frac{913}{7680} + \frac{1891}{11\,520}c_i^2 - \frac{7}{4608}c_i^4\right.\\
&\left.+ \nu\left(\frac{1165}{384} - \frac{235}{576}c_i^2 + \frac{7}{1152}c_i^4\right) + \nu^2\left(-\frac{1301}{4608} + \frac{301}{2304}c_i^2 - \frac{7}{1536}c_i^4\right)\right]
\end{aligned}
$$

$$
\begin{aligned}
&+\pi c_i \sin 2\psi\left[\frac{34}{3}-\frac{8}{3}c_i^2+\nu\left(-\frac{20}{3}+8c_i^2\right)\right]\\
&+s_i c_i\Delta \sin 3\psi\left[\frac{12\,501}{2560}-\frac{12\,069}{1280}c_i^2+\frac{1701}{2560}c_i^4\right.\\
&+\nu\left(-\frac{19\,581}{640}+\frac{7821}{320}c_i^2-\frac{1701}{640}c_i^4\right)\\
&\left.+\nu^2\left(\frac{18\,903}{2560}-\frac{11\,403}{1280}c_i^2+\frac{5103}{2560}c_i^4\right)\right]+s_i^2 c_i\sin 4\psi\left[-\frac{32\pi}{3}(1-3\nu)\right]\\
&+\Delta s_i c_i \sin 5\psi\left[-\frac{101\,875}{4608}+\frac{6875}{256}c_i^2-\frac{21\,875}{4608}c_i^4\right.\\
&+\nu\left(\frac{66\,875}{1152}-\frac{44\,375}{576}c_i^2+\frac{21\,875}{1152}c_i^4\right)\\
&\left.+\nu^2\left(-\frac{100\,625}{4608}+\frac{83\,125}{2304}c_i^2-\frac{21\,875}{1536}c_i^4\right)\right]\\
&+\Delta s_i^5 c_i \sin 7\psi\left[\frac{117\,649}{23\,040}(1-4\nu+3\nu^2)\right],
\end{aligned}
\tag{8.10f}
$$

$$
\begin{aligned}
H_\times^{(3)} = {}&\Delta s_i c_i\cos\psi\left[\frac{11\,617}{20\,160}+\frac{21}{16}\ln 2+\left(-\frac{251}{2240}-\frac{5}{48}\ln 2\right)c_i^2\right.\\
&\left.+\nu\left(-\frac{48\,239}{5040}-\frac{5}{24}\ln 2+\left(\frac{727}{240}+\frac{5}{24}\ln 2\right)c_i^2\right)\right]\\
&+c_i\cos 2\psi\left[\frac{856\pi}{105}\right]\\
&+\Delta s_i c_i\cos 3\psi\left[-\frac{36\,801}{896}+\frac{1809}{32}\ln(3/2)+\left(\frac{65\,097}{4480}-\frac{405}{32}\ln(3/2)\right)c_i^2\right.\\
&\left.+\nu\left(\frac{28\,445}{288}-\frac{405}{16}\ln(3/2)+\left(-\frac{7137}{160}+\frac{405}{16}\ln(3/2)\right)c_i^2\right)\right]\\
&+\Delta s_i^3 c_i\cos 5\psi\left[\frac{113\,125}{2688}-\frac{3125}{96}\ln(5/2)+\nu\left(-\frac{17\,639}{160}+\frac{3125}{48}\ln(5/2)\right)\right]\\
&+\pi\,\Delta s_i c_i\sin\psi\left[\frac{21}{32}-\frac{5}{96}c_i^2+\nu\left(-\frac{5}{48}+\frac{5}{48}c_i^2\right)\right]\\
&+c_i\sin 2\psi\left[-\frac{3\,620\,761}{44\,100}+\frac{1712C}{105}-\frac{4\pi^2}{3}+\frac{856}{105}\ln(16x)\right.\\
&-\frac{3413}{1260}c_i^2+\frac{2909}{2520}c_i^4-\frac{1}{45}c_i^6\\
&+\nu\left(\frac{743}{90}-\frac{41\pi^2}{48}+\frac{3391}{180}c_i^2-\frac{2287}{360}c_i^4+\frac{7}{45}c_i^6\right)\\
&+\nu^2\left(\frac{7919}{270}-\frac{5426}{135}c_i^2+\frac{382}{45}c_i^4-\frac{14}{45}c_i^6\right)\\
&\left.+\nu^3\left(-\frac{6457}{1620}+\frac{1109}{180}c_i^2-\frac{281}{120}c_i^4+\frac{7}{45}c_i^6\right)\right]\\
&+\pi\,\Delta s_i c_i\sin 3\psi\left[-\frac{1809}{64}+\frac{405}{64}c_i^2+\nu\left(\frac{405}{32}-\frac{405}{32}c_i^2\right)\right]
\end{aligned}
$$

$$
\begin{aligned}
&+s_i^2 c_i \sin 4\psi \left[-\frac{1781}{105}+\frac{1208}{63}c_i^2-\frac{64}{45}c_i^4\right.\\
&+\nu\left(\frac{5207}{45}-\frac{536}{5}c_i^2+\frac{448}{45}c_i^4\right)\\
&\left.+\nu^2\left(-\frac{24838}{135}+\frac{2224}{15}c_i^2-\frac{896}{45}c_i^4\right)+\nu^3\left(\frac{1703}{45}-\frac{1976}{45}c_i^2+\frac{448}{45}c_i^4\right)\right]\\
&+\Delta \sin 5\psi\left[\frac{3125\pi}{192}s_i^3 c_i(1-2\nu)\right]\\
&+s_i^4 c_i \sin 6\psi\left[\frac{9153}{280}-\frac{243}{35}c_i^2+\nu\left(-\frac{7371}{40}+\frac{243}{5}c_i^2\right)\right.\\
&\left.+\nu^2\left(\frac{1296}{5}-\frac{486}{5}c_i^2\right)+\nu^3\left(-\frac{3159}{40}+\frac{243}{5}c_i^2\right)\right]\\
&+\sin 8\psi\left[-\frac{2048}{315}s_i^6 c_i(1-7\nu+14\nu^2-7\nu^3)\right].
\end{aligned}
\tag{8.10g}
$$

Note the obvious fact that the polarization waveforms remain invariant when we rotate by π the separation direction between the particles and simultaneously exchange the labels of the two particles, i.e. when we apply the transformation $(\psi, \Delta) \to (\psi+\pi, -\Delta)$. Moreover, due to the parity invariance, H_+ is unchanged after the replacement $i \to \pi - i$, while $H_\times$ being the projection of h_{ij}^{TT} on a tensorial product of two vectors of inverse parity types, is changed into its opposite.

We have performed two important tests on these expressions. First of all, we have verified that the perturbative limit $\nu \to 0$ of the polarization waveforms (8.9) and (8.10) is in full agreement up to 3PN order with the result of black-hole perturbation theory as reported in the appendix B of [69][33]. Our second test is the verification that the wave polarizations (8.9) and (8.10) give back the correct energy flux at 3PN order. The asymptotic flux is given in terms of the polarizations by

$$
\mathcal{F}^{\mathrm{GW}} = \lim_{R\to+\infty}\frac{R^2c^3}{4G}\int\frac{\mathrm{d}\Omega}{4\pi}[(\dot{h}_+)^2+(\dot{h}_\times)^2],
\tag{8.11}
$$

where $\mathrm{d}\Omega$ is the solid angle element associated with the direction of propagation $\mathbf{N}$. We have $\mathrm{d}\Omega = \sin\Theta\,\mathrm{d}\Theta\,\mathrm{d}\Phi$ where (Θ, Φ) are the angles defining $\mathbf{N}$, following the notation of section 2. To obtain the polarizations corresponding to this general convention for $\mathbf{N}$ we have to make some simple replacements in (8.9) and (8.10) for i and ψ. As is clear from the geometry of the problem we must replace $(i, \psi) \to (\Theta, \psi + \pi/2 - \Phi)$. The time derivative of the polarizations is computed in the adiabatic approximation, using $\dot{\phi} = \omega$ and $\dot{\omega}$ given by (6.2b). Of course one must take into account the difference between ϕ and the variable ψ used in (8.9) and (8.10). Finally, the angular integration in (8.11) is readily performed and the result is in perfect agreement with the 3PN energy flux given by (12.9) of [13][34].

As already mentioned there are some 'true' logarithms which remain in the FWF at the 3PN order—i.e. after it has been expressed with the help of the PN parameter x and the phase variable ψ. Inspection of (8.9) and (8.10) shows that these logarithms have the effect of correcting the Newtonian polarizations in the following way:

$$
\begin{pmatrix} H_+ \\ H_\times \end{pmatrix} = \begin{pmatrix} -\left(1+c_i^2\right)\cos 2\psi \\ -2c_i \sin 2\psi \end{pmatrix}\left(1-\frac{428}{105}x^3\ln(16x)\right)+\cdots+\mathcal{O}\left(\frac{1}{c^7}\right),
\tag{8.12}
$$

[33] In [17] a misprint was spotted in appendix B of [69]: the sign of the harmonic coefficient $\zeta^\times_{7,3}$ should be changed, so that one should read $\zeta^\times_{7,3} = +\frac{729}{10\,250\,240}\cos(\theta)(167+\cdots)\sin(\theta)(v^5\cos(3\psi)-\cdots)$.

[34] The ambiguity parameters therein are now known to be $\lambda = -\frac{1987}{3080}$ [65, 58] and $\theta = \xi + 2\kappa + \zeta = -\frac{11\,831}{9240}$ [15].

where the dots represent the terms independent of logarithms. In our previous computation of the 3PN flux using (8.11) we have already checked that these logarithms are consistent with similar logarithms occurring at 3PN order in the flux. Indeed we easily see that they correspond in the 3PN flux to the terms

$$\mathcal{F}^{\mathrm{GW}} = \frac{32c^5}{5G}\nu^2 x^5\left[1 - \frac{856}{105}x^3\ln(16x) + \cdots + \mathcal{O}\left(\frac{1}{c^7}\right)\right], \tag{8.13}$$

already known from (12.9) in [13]. Technically the logarithm in (8.12) or (8.13) is due to the tails-of-tails at 3PN order. Note that this logarithm survives in the test-mass limit $\nu \to 0$ and is therefore also seen to appear in linear black-hole perturbations [68–70].

9. 3PN spherical harmonic modes for numerical relativity

The spin-weighted spherical harmonic modes of the polarization waveforms at 3PN order can now be obtained from using the angular integration (2.5). An alternative route would be to use the relations (2.6) and (2.7) giving the modes directly in terms of separate contributions of the radiative moments U_L and V_L. In the present paper, the two routes are equivalent because all the radiative moments are 'uniformly' given with the approximation that is necessary and sufficient to control the 3PN waveform.

In this respect, one should be careful about what we mean by controlling the modes up to 3PN order. We mean—having in mind the standard PN practice—that the accuracy of the modes is exactly the one which is needed to obtain the 3PN waveform. Thus the dominant mode h^{22} will have full 3PN accuracy, but higher-order modes, which start at some higher PN order, will have a lower *relative* PN accuracy. For instance we shall see that the mode h^{44} starts at 1PN order thus it will be given only with 2PN relative accuracy.

The angular integration in (2.5) is over the angles (Θ, Φ). Like in our previous computation of the flux (8.11), it should be performed after substituting $(i, \psi) \to (\Theta, \psi + \pi/2 - \Phi)$ in the wave polarizations. Denoting $h = h_+ - \mathrm{i}h_\times$ the integral we consider is thus

$$h^{\ell m} = \int \mathrm{d}\Omega\, h(\Theta, \psi + \pi/2 - \Phi)\overline{Y}^{\ell m}_{-2}(\Theta, \Phi). \tag{9.1}$$

Changing Φ into $\psi + \pi/2 - \psi'$ and Θ into $i' = \arccos c'_i$, and using the known dependence of the spherical harmonics on the azimuthal angle Φ (see (2.4)), we obtain

$$h^{\ell m} = (-\mathrm{i})^m\,\mathrm{e}^{-\mathrm{i}m\psi}\int_0^{2\pi}\mathrm{d}\psi'\int_{-1}^{1}\mathrm{d}c'_i h(i', \psi')Y^{\ell m}_{-2}(i', \psi'), \tag{9.2}$$

exhibiting the azimuthal factor $\mathrm{e}^{-\mathrm{i}m\psi}$ appropriate for each mode. Let us factorize out in all the modes an overall coefficient including $\mathrm{e}^{-\mathrm{i}m\psi}$, and such that the dominant mode with $(\ell, m) = (2, 2)$ starts with one (by pure convention) at the Newtonian order. Remembering also our previous factorization in (8.4) we pose

$$h^{\ell m} = \frac{2Gm\nu x}{Rc^2}H^{\ell m}, \tag{9.3a}$$

$$H^{\ell m} = \sqrt{\frac{16\pi}{5}}\hat{H}^{\ell m}\,\mathrm{e}^{-\mathrm{i}m\psi}, \tag{9.3b}$$

and list all the results in terms of $\hat{H}^{\ell m}$,[35]

[35] The modes having $m < 0$ are easily deduced using $\hat{H}^{\ell,-m} = (-)^\ell\overline{\hat{H}}^{\ell m}$.

$$\hat{H}^{22} = 1 + x\left(-\frac{107}{42} + \frac{55\nu}{42}\right) + 2\pi x^{3/2} + x^2\left(-\frac{2173}{1512} - \frac{1069\nu}{216} + \frac{2047\nu^2}{1512}\right)$$
$$+ x^{5/2}\left(-\frac{107\pi}{21} - 24\mathrm{i}\nu + \frac{34\pi\nu}{21}\right) + x^3\left(\frac{27\,027\,409}{646\,800} - \frac{856C}{105} + \frac{428\mathrm{i}\pi}{105} + \frac{2\pi^2}{3}\right.$$
$$\left. + \left(-\frac{278\,185}{33\,264} + \frac{41\pi^2}{96}\right)\nu - \frac{20\,261\nu^2}{2772} + \frac{114\,635\nu^3}{99\,792} - \frac{428}{105}\ln(16x)\right)$$
$$+ \mathcal{O}\left(\frac{1}{c^7}\right), \tag{9.4a}$$

$$\hat{H}^{21} = \frac{1}{3}\mathrm{i}\Delta\left[x^{1/2} + x^{3/2}\left(-\frac{17}{28} + \frac{5\nu}{7}\right) + x^2\left(\pi + \mathrm{i}\left(-\frac{1}{2} - 2\ln 2\right)\right)\right.$$
$$+ x^{5/2}\left(-\frac{43}{126} - \frac{509\nu}{126} + \frac{79\nu^2}{168}\right) + x^3\left(-\frac{17\pi}{28} + \frac{3\pi\nu}{14}\right.$$
$$\left.\left. + \mathrm{i}\left(\frac{17}{56} + \nu\left(-\frac{995}{84} - \frac{3\ln 2}{7}\right) + \frac{17\ln 2}{14}\right)\right)\right] + \mathcal{O}\left(\frac{1}{c^7}\right), \tag{9.4b}$$

$$\hat{H}^{20} = -\frac{5}{14\sqrt{6}} + \mathcal{O}\left(\frac{1}{c^7}\right), \tag{9.4c}$$

$$\hat{H}^{33} = -\frac{3}{4}\mathrm{i}\sqrt{\frac{15}{14}}\Delta\left[x^{1/2} + x^{3/2}(-4 + 2\nu) + x^2\left(3\pi + \mathrm{i}\left(-\frac{21}{5} + 6\ln(3/2)\right)\right)\right.$$
$$+ x^{5/2}\left(\frac{123}{110} - \frac{1838\nu}{165} + \frac{887\nu^2}{330}\right) + x^3\left(-12\pi + \frac{9\pi\nu}{2}\right.$$
$$\left.\left. + \mathrm{i}\left(\frac{84}{5} - 24\ln(3/2) + \nu\left(-\frac{48\,103}{1215} + 9\ln(3/2)\right)\right)\right)\right] + \mathcal{O}\left(\frac{1}{c^7}\right), \tag{9.4d}$$

$$\hat{H}^{32} = \frac{1}{3}\sqrt{\frac{5}{7}}\left[x(1 - 3\nu) + x^2\left(-\frac{193}{90} + \frac{145\nu}{18} - \frac{73\nu^2}{18}\right) + x^{5/2}\left(2\pi - 6\pi\nu + \mathrm{i}\left(-3 + \frac{66\nu}{5}\right)\right)\right.$$
$$\left. + x^3\left(-\frac{1451}{3960} - \frac{17\,387\nu}{3960} + \frac{5557\nu^2}{220} - \frac{5341\nu^3}{1320}\right)\right] + \mathcal{O}\left(\frac{1}{c^7}\right), \tag{9.4e}$$

$$\hat{H}^{31} = \frac{\mathrm{i}\Delta}{12\sqrt{14}}\left[x^{1/2} + x^{3/2}\left(-\frac{8}{3} - \frac{2\nu}{3}\right) + x^2\left(\pi + \mathrm{i}\left(-\frac{7}{5} - 2\ln 2\right)\right)\right.$$
$$+ x^{5/2}\left(\frac{607}{198} - \frac{136\nu}{99} - \frac{247\nu^2}{198}\right) + x^3\left(-\frac{8\pi}{3} - \frac{7\pi\nu}{6}\right.$$
$$\left.\left. + \mathrm{i}\left(\frac{56}{15} + \frac{16\ln 2}{3} + \nu\left(-\frac{1}{15} + \frac{7\ln 2}{3}\right)\right)\right)\right] + \mathcal{O}\left(\frac{1}{c^7}\right), \tag{9.4f}$$

$$\hat{H}^{30} = -\frac{2}{5}\mathrm{i}\sqrt{\frac{6}{7}}x^{5/2}\nu + \mathcal{O}\left(\frac{1}{c^7}\right), \tag{9.4g}$$

$$\hat{H}^{44} = -\frac{8}{9}\sqrt{\frac{5}{7}}\left[x(1 - 3\nu) + x^2\left(-\frac{593}{110} + \frac{1273\nu}{66} - \frac{175\nu^2}{22}\right)\right.$$
$$+ x^{5/2}\left(4\pi - 12\pi\nu + \mathrm{i}\left(-\frac{42}{5} + \nu\left(\frac{1193}{40} - 24\ln 2\right) + 8\ln 2\right)\right)$$
$$\left. + x^3\left(\frac{1\,068\,671}{200\,200} - \frac{1\,088\,119\nu}{28\,600} + \frac{146\,879\nu^2}{2340} - \frac{226\,097\nu^3}{17\,160}\right)\right] + \mathcal{O}\left(\frac{1}{c^7}\right), \tag{9.4h}$$

$$\hat{H}^{43} = -\frac{9\mathrm{i}\Delta}{4\sqrt{70}}\left[x^{3/2}(1-2\nu) + x^{5/2}\left(-\frac{39}{11} + \frac{1267\nu}{132} - \frac{131\nu^2}{33}\right)\right.$$
$$\left. + x^3\left(3\pi - 6\pi\nu + \mathrm{i}\left(-\frac{32}{5} + \nu\left(\frac{16\,301}{810} - 12\ln(3/2)\right) + 6\ln(3/2)\right)\right)\right] + \mathcal{O}\left(\frac{1}{c^7}\right), \tag{9.4i}$$

$$\hat{H}^{42} = \frac{1}{63}\sqrt{5}\left[x(1-3\nu) + x^2\left(-\frac{437}{110} + \frac{805\nu}{66} - \frac{19\nu^2}{22}\right) + x^{5/2}\left(2\pi - 6\pi\nu\right.\right.$$
$$\left.\left. + \mathrm{i}\left(-\frac{21}{5} + \frac{84\nu}{5}\right)\right) + x^3\left(\frac{1\,038\,039}{200\,200} - \frac{606\,751\nu}{28\,600} + \frac{400\,453\nu^2}{25\,740} + \frac{25\,783\nu^3}{17\,160}\right)\right]$$
$$+ \mathcal{O}\left(\frac{1}{c^7}\right), \tag{9.4j}$$

$$\hat{H}^{41} = \frac{\mathrm{i}\Delta}{84\sqrt{10}}\left[x^{3/2}(1-2\nu) + x^{5/2}\left(-\frac{101}{33} + \frac{337\nu}{44} - \frac{83\nu^2}{33}\right)\right.$$
$$\left. + x^3\left(\pi - 2\pi\nu + \mathrm{i}\left(-\frac{32}{15} - 2\ln 2 + \nu\left(\frac{1661}{30} + 4\ln 2\right)\right)\right)\right] + \mathcal{O}\left(\frac{1}{c^7}\right), \tag{9.4k}$$

$$\hat{H}^{40} = -\frac{1}{504\sqrt{2}} + \mathcal{O}\left(\frac{1}{c^7}\right), \tag{9.4l}$$

$$\hat{H}^{55} = \frac{625\mathrm{i}\Delta}{96\sqrt{66}}\left[x^{3/2}(1-2\nu) + x^{5/2}\left(-\frac{263}{39} + \frac{688\nu}{39} - \frac{256\nu^2}{39}\right)\right.$$
$$\left. + x^3\left(5\pi - 10\pi\nu + \mathrm{i}\left(-\frac{181}{14} + \nu\left(\frac{105\,834}{3125} - 20\ln(5/2)\right) + 10\ln(5/2)\right)\right)\right]$$
$$+ \mathcal{O}\left(\frac{1}{c^7}\right), \tag{9.4m}$$

$$\hat{H}^{54} = -\frac{32}{9\sqrt{165}}\left[x^2(1-5\nu+5\nu^2) + x^3\left(-\frac{4451}{910} + \frac{3619\nu}{130} - \frac{521\nu^2}{13} + \frac{339\nu^3}{26}\right)\right] + \mathcal{O}\left(\frac{1}{c^7}\right), \tag{9.4n}$$

$$\hat{H}^{53} = -\frac{9}{32}\mathrm{i}\sqrt{\frac{3}{110}}\Delta\left[x^{3/2}(1-2\nu) + x^{5/2}\left(-\frac{69}{13} + \frac{464\nu}{39} - \frac{88\nu^2}{39}\right)\right.$$
$$\left. + x^3\left(3\pi - 6\pi\nu + \mathrm{i}\left(-\frac{543}{70} + \nu\left(\frac{83\,702}{3645} - 12\ln(3/2)\right) + 6\ln(3/2)\right)\right)\right]$$
$$+ \mathcal{O}\left(\frac{1}{c^7}\right), \tag{9.4o}$$

$$\hat{H}^{52} = \frac{2}{27\sqrt{55}}\left[x^2(1-5\nu+5\nu^2) + x^3\left(-\frac{3911}{910} + \frac{3079\nu}{130} - \frac{413\nu^2}{13} + \frac{231\nu^3}{26}\right)\right] + \mathcal{O}\left(\frac{1}{c^7}\right), \tag{9.4p}$$

$$\hat{H}^{51} = \frac{\mathrm{i}\Delta}{288\sqrt{385}}\left[x^{3/2}(1-2\nu) + x^{5/2}\left(-\frac{179}{39} + \frac{352\nu}{39} - \frac{4\nu^2}{39}\right)\right.$$
$$\left. + x^3\left(\pi - 2\pi\nu + \mathrm{i}\left(-\frac{181}{70} - 2\ln 2 + \nu\left(\frac{626}{5} + 4\ln 2\right)\right)\right)\right] + \mathcal{O}\left(\frac{1}{c^7}\right), \tag{9.4q}$$

$$\hat{H}^{50} = \mathcal{O}\left(\frac{1}{c^7}\right), \tag{9.4r}$$

$$\hat{H}^{66} = \frac{54}{5\sqrt{143}}\left[x^2(1-5\nu+5\nu^2) + x^3\left(-\frac{113}{14} + \frac{91\nu}{2} - 64\nu^2 + \frac{39\nu^3}{2}\right)\right] + \mathcal{O}\left(\frac{1}{c^7}\right), \tag{9.4s}$$

$$\hat{H}^{65} = \frac{3125\mathrm{i}x^{5/2}\Delta}{504\sqrt{429}}[1-4\nu+3\nu^2] + \mathcal{O}\left(\frac{1}{c^7}\right), \tag{9.4t}$$

$$\hat{H}^{64} = -\frac{128}{495}\sqrt{\frac{2}{39}}\left[x^2(1-5\nu+5\nu^2) + x^3\left(-\frac{93}{14} + \frac{71\nu}{2} - 44\nu^2 + \frac{19\nu^3}{2}\right)\right] + \mathcal{O}\left(\frac{1}{c^7}\right), \tag{9.4u}$$

$$\hat{H}^{63} = -\frac{81\mathrm{i}x^{5/2}\Delta}{616\sqrt{65}}[1-4\nu+3\nu^2] + \mathcal{O}\left(\frac{1}{c^7}\right), \tag{9.4v}$$

$$\hat{H}^{62} = \frac{2}{297\sqrt{65}}\left[x^2(1-5\nu+5\nu^2) + x^3\left(-\frac{81}{14} + \frac{59\nu}{2} - 32\nu^2 + \frac{7\nu^3}{2}\right)\right] + \mathcal{O}\left(\frac{1}{c^7}\right), \tag{9.4w}$$

$$\hat{H}^{61} = \frac{\mathrm{i}x^{5/2}\Delta}{8316\sqrt{26}}[1-4\nu+3\nu^2] + \mathcal{O}\left(\frac{1}{c^7}\right), \tag{9.4x}$$

$$\hat{H}^{60} = \mathcal{O}\left(\frac{1}{c^7}\right), \tag{9.4y}$$

$$\hat{H}^{77} = -\frac{16\,807\mathrm{i}x^{5/2}\Delta}{1440}\sqrt{\frac{7}{858}}[1-4\nu+3\nu^2] + \mathcal{O}\left(\frac{1}{c^7}\right), \tag{9.4z}$$

$$\hat{H}^{76} = \frac{81}{35}\sqrt{\frac{3}{143}}x^3(1-7\nu+14\nu^2-7\nu^3) + \mathcal{O}\left(\frac{1}{c^7}\right), \tag{9.4aa}$$

$$\hat{H}^{75} = \frac{15\,625\mathrm{i}x^{5/2}\Delta}{26\,208\sqrt{66}}[1-4\nu+3\nu^2] + \mathcal{O}\left(\frac{1}{c^7}\right), \tag{9.4bb}$$

$$\hat{H}^{74} = -\frac{128x^3}{1365}\sqrt{\frac{2}{33}}(1-7\nu+14\nu^2-7\nu^3) + \mathcal{O}\left(\frac{1}{c^7}\right), \tag{9.4cc}$$

$$\hat{H}^{73} = -\frac{243\mathrm{i}x^{5/2}\Delta}{160\,160}\sqrt{\frac{3}{2}}[1-4\nu+3\nu^2] + \mathcal{O}\left(\frac{1}{c^7}\right), \tag{9.4dd}$$

$$\hat{H}^{72} = \frac{x^3(1-7\nu+14\nu^2-7\nu^3)}{3003\sqrt{3}} + \mathcal{O}\left(\frac{1}{c^7}\right), \tag{9.4ee}$$

$$\hat{H}^{71} = \frac{\mathrm{i}x^{5/2}\Delta}{864\,864\sqrt{2}}[1-4\nu+3\nu^2] + \mathcal{O}\left(\frac{1}{c^7}\right), \tag{9.4ff}$$

$$\hat{H}^{70} = \mathcal{O}\left(\frac{1}{c^7}\right), \tag{9.4gg}$$

$$\hat{H}^{88} = -\frac{16\,384}{63}\sqrt{\frac{2}{85\,085}}x^3(1-7\nu+14\nu^2-7\nu^3) + \mathcal{O}\left(\frac{1}{c^7}\right), \tag{9.4hh}$$

$$\hat{H}^{87} = \mathcal{O}\left(\frac{1}{c^7}\right), \tag{9.4ii}$$

$$\hat{H}^{86} = \frac{243}{35}\sqrt{\frac{3}{17\,017}}x^3(1 - 7\nu + 14\nu^2 - 7\nu^3) + \mathcal{O}\left(\frac{1}{c^7}\right), \tag{9.4jj}$$

$$\hat{H}^{85} = \mathcal{O}\left(\frac{1}{c^7}\right), \tag{9.4kk}$$

$$\hat{H}^{84} = -\frac{128}{4095}\sqrt{\frac{2}{187}}x^3(1 - 7\nu + 14\nu^2 - 7\nu^3) + \mathcal{O}\left(\frac{1}{c^7}\right), \tag{9.4ll}$$

$$\hat{H}^{83} = \mathcal{O}\left(\frac{1}{c^7}\right), \tag{9.4mm}$$

$$\hat{H}^{82} = \frac{x^3}{9009\sqrt{85}}(1 - 7\nu + 14\nu^2 - 7\nu^3) + \mathcal{O}\left(\frac{1}{c^7}\right), \tag{9.4nn}$$

$$\hat{H}^{81} = \mathcal{O}\left(\frac{1}{c^7}\right), \tag{9.4oo}$$

$$\hat{H}^{80} = \mathcal{O}\left(\frac{1}{c^7}\right), \tag{9.4pp}$$

while all the higher-order modes fall into the PN remainder and are negligible. However, we shall give here for the reader's convenience their leading order expressions for nonzero m (see the derivation in [19]). For $\ell + m$ even we find

$$\hat{H}^{\ell m} = \frac{(-)^{(\ell-m+2)/2}}{2^{\ell+1}\left(\frac{\ell+m}{2}\right)!\left(\frac{\ell-m}{2}\right)!(2\ell-1)!!}\left(\frac{5(\ell+1)(\ell+2)(\ell+m)!(\ell-m)!}{\ell(\ell-1)(2\ell+1)}\right)^{1/2} s_\ell(\nu)(\mathrm{i}m)^\ell x^{\ell/2-1} + \mathcal{O}\left(\frac{1}{c^{\ell-2}}\right), \tag{9.5}$$

where we recall that the function $s_\ell(\nu)$ is defined in (5.22). For $\ell + m$ odd we have

$$\hat{H}^{\ell m} = \frac{(-)^{(\ell-m-1)/2}}{2^{\ell-1}\left(\frac{\ell+m-1}{2}\right)!\left(\frac{\ell-m-1}{2}\right)!(2\ell+1)!!}\left(\frac{5(\ell+2)(2\ell+1)(\ell+m)!(\ell-m)!}{\ell(\ell-1)(\ell+1)}\right)^{1/2} \times s_{\ell+1}(\nu)\mathrm{i}(\mathrm{i}m)^\ell x^{(\ell-1)/2} + \mathcal{O}\left(\frac{1}{c^{\ell-2}}\right). \tag{9.6}$$

When $m = 0$, $\hat{H}^{\ell m}$ may not vanish due to DC contributions of the memory integrals. We already know that such an effect arises at Newtonian order (see (8.9*a*), hence the nonzero values of $\hat{H}^{20}$ and $\hat{H}^{40}$.

We find that the result for $\hat{H}^{22}$ at 3PN order given by (9.4*a*) is in complete agreement with the result of Kidder [19]. The only difference is our use of the particular phase variable (8.8) which permits us to remove most of the logarithmic terms, showing that they are actually negligible modulations of the orbital phase. For the other harmonics we find agreement with the results of [19] up to 2.5PN order, but the results have here been completed by all the 3PN contributions.

Acknowledgments

A part of the results of this paper was obtained by means of the tensor package xTensor developed for Mathematica by J Martín-García [55]. We thank him for the technical support he provided as we were elaborating the code. LB, GF and BRI are grateful to the Indo-French collaboration IFCPAR for its support during the initial stages of this work.

Appendix. List of symbols

$\mathbf{a}$	relative acceleration of binary masses in harmonic coordinates
$\alpha_L^{\ell m}$	STF tensor connecting the usual spherical harmonics basis $Y^{\ell m}$ to the STF tensors basis $N_{\langle L\rangle}$
c_i	$\cos i$
Δ	mass difference ratio; $\Delta = (m_1 - m_2)/m$
$\mathcal{F}^{\mathrm{GW}}$	total gravitational wave energy flux
FWF	full gravitational waveform at 3PN order
γ	PN parameter; $\gamma = \frac{Gm}{rc^2}$
h_{ij}^{TT}	transverse traceless (TT) projection of metric deviation; equation (2.1)
$h_{+,\times}$	'plus' and 'cross' polarization states of the FWF; equations (2.2)
$H_{+,\times}$	same as $h_{+,\times}$ modulo an overall factor; equations (8.5), (8.9) and (8.10)
$h^{\ell m}$	spin-weighted spherical harmonic modes of the FWF; equation (2.5)
$H^{\ell m}$, $\hat{H}^{\ell m}$	same as $h^{\ell m}$ modulo overall factors; equations (9.3)
I_L	mass-type source multipole moment STF with ℓ multipolar spatial indices; is given for $\ell = 2, 3, 4, 5, 6, 7, 8$ by equations (5.12), (5.15*a*), (5.16*a*), (5.17*a*), (5.18*a*), (5.19*a*), (5.20*a*) respectively
i	inclination angle of the binary orbit
J_L	current-type source multipole moment STF with ℓ multipolar spatial indices; is given for $\ell = 2, 3, 4, 5, 6, 7$ by equations (5.15*b*), (5.16*b*), (5.17*b*), (5.18*b*), (5.19*b*), (5.20*b*) respectively
ℓ	multipolar order
$\boldsymbol{\lambda}$	unit vector in the orbital plane; $\boldsymbol{\lambda} = \hat{\mathbf{z}} \times \mathbf{n}$
m_1, m_2	individual masses of binary components
m	total mass of the binary; $m = m_1 + m_2$
M	source mass-type monopole moment; equation (5.23)
M_L	canonical mass-type STF moment with ℓ multipolar spatial indices, related to source moment I_L by equations (4.25*a*) and (5.9), (5.10*a*)
$\mathbf{n}$	binary's separation direction, from m_2 to m_1
ν	symmetric mass-ratio; $\nu = \frac{m_1 m_2}{m^2}$
$\mathbf{N}$	direction of propagation of gravitational wave
$\mathbf{P}$	unit vector along the direction of the ascending node $\mathcal{N}$
$\mathcal{P}_{ijkl}^{\mathrm{TT}}$	TT projection operator; equation (2.1)
Φ	azimuthal angle of $\mathbf{N}$ in spherical polar coordinates
$\phi(t)$	orbital phase of the binary's relative orbit, the angle between $\mathbf{n}$ and $\mathbf{P}$, increasing in the direction of $\boldsymbol{\lambda}$
$\psi(t)$	effective orbital phase of the binary's relative orbit, as modified by 4PN log term; equation (8.8)
$\mathbf{Q}$	unit polarization vector in the plane of the sky; $\mathbf{Q} = \mathbf{N} \times \mathbf{P}$
RWF	restricted post-Newtonian gravitational waveform
r	relative separation of binary masses in harmonic coordinates
R	distance to the source in radiative coordinates
STF	symmetric-trace-free projection
s_i	$\sin i$
S_L	canonical current-type STF moment with ℓ multipolar spatial indices, related to source moment J_L by equations (4.25*b*) and (5.10*b*)
Θ	polar angle of $\mathbf{N}$ in spherical polar coordinates
TT	transverse traceless projection

U_L	mass-type radiative multipole moment STF with ℓ multipolar spatial indices; is given for $\ell = 2, 3, 4, 5$ by equations (5.4), (5.5a), (5.6a), (5.7a) respectively
$U^{\ell m}$	radiative mass moment (non-STF form) corresponding to $h^{\ell m}$; equation (2.7)
$\mathbf{v}$	relative velocity of binary masses in harmonic coordinates
V_L	mass-type radiative multipole moment STF with ℓ multipolar spatial indices; is given for $\ell = 2, 3, 4$ by equations (5.5b), (5.6b), (5.7b) respectively
$V^{\ell m}$	radiative current moment (non-STF form) corresponding to $h^{\ell m}$; equation (2.7)
ω	angular velocity of the relative orbit in harmonic coordinates; equation (6.4)
W_L	gauge moment, entering the relation between canonical and source moments; equations (5.9), (5.10)
X_L	gauge moment
x	gauge invariant PN expansion parameter; equation (6.5)
x^μ	harmonic coordinate system in the near zone
X^μ	radiative-type coordinate system in the far zone
$Y_{-2}^{\ell m}(\Theta, \Phi)$	spin-weighted spherical harmonics of weight -2; equation (2.4)
$Y^{\ell m}(\Theta, \Phi)$	standard spherical harmonics
Y_L	gauge moment
Z_L	gauge moment
$\hat{\mathbf{z}}$	unit vector normal to the binary orbital plane

References

[1] http://www.ligo.caltech.edu/

[2] http://www.virgo.infn.it/

[3] http://lisa.jpl.nasa.gov/

[4] Blanchet L 2006 *Living Rev. Rel.* **9** 4 (*Preprint* gr-qc/0202016)

[5] Pretorius F 2005 *Phys. Rev. Lett.* **95** 121101 (*Preprint* gr-qc/0507014)

[6] Baker J G, Centrella J, Choi D-I, Koppitz M and van Meter J 2006 *Phys. Rev. Lett.* **96** 111102 (*Preprint* gr-qc/0511103)

[7] Campanelli M, Lousto C O, Marronetti P and Zlochower Y 2006 *Phys. Rev. Lett.* **96** 111101 (*Preprint* gr-qc/0511048)

[8] Baker J, Campanelli M, Pretorius F and Zlochower Y 2007 *Class. Quantum Grav.* **24** S25

[9] Buonanno A, Cook G B and Pretorius F 2007 *Phys. Rev.* D **75** 124018

[10] Berti E, Cardoso V, Gonzalez J A, Sperhake U, Hannam M, Husa S and Bruegmann B 2007 *Phys. Rev.* D **76** 064034

[11] Hannam M, Husa S, Gonzalez J A, Sperhake U and Bruegmann B 2008 *Phys. Rev.* D **77** 044020 (*Preprint* arXiv:0706.1305)

[12] Boyle M, Brown D A, Kidder L E, Mroue A H, Pfeiffer H P, Scheel M A, Cook G B and Teukolsky S A 2007 *Preprint* arXiv:0710.0158

[13] Blanchet L, Iyer B R and Joguet B 2002 *Phys. Rev.* D **65** 064005
Blanchet L, Iyer B R and Joguet B 2005 *Phys. Rev.* D **71** 129903(E) (*Preprint* gr-qc/0105098) (erratum)

[14] Blanchet L, Faye G, Iyer B R and Joguet B 2002 *Phys. Rev.* D **65** 061501(R)
Blanchet L, Faye G, Iyer B R and Joguet B 2005 *Phys. Rev.* D **71** 129902(E) (*Preprint* gr-qc/0105099) (erratum)

[15] Blanchet L, Damour T, Esposito-Farèse G and Iyer B R 2004 *Phys. Rev. Lett.* **93** 091101 (*Preprint* gr-qc/0406012)

[16] Blanchet L, Iyer B R, Will C M and Wiseman A G 1996 *Class. Quantum Grav.* **13** 575 (*Preprint* gr-qc/9602024)

[17] Arun K G, Blanchet L, Iyer B R and Qusailah M S 2004 *Class. Quantum Grav.* **21** 3771
Arun K G, Blanchet L, Iyer B R and Qusailah M S 2005 *Class. Quantum Grav.* **22** 3115 (*Preprint* gr-qc/0404185) (erratum)

[18] Kidder L, Blanchet L and Iyer B R 2007 *Class. Quantum Grav.* **24** 5307 (*Preprint* arXiv:0706.0726)

[19] Kidder L 2008 *Phys. Rev.* D **77** 044016 (*Preprint* arXiv:0710.0614)

[20] Cutler C *et al* 1993 *Phys. Rev. Lett.* **70** 2984

[21] Cutler C and Flanagan E 1994 *Phys. Rev.* D **49** 2658

[22] Sathyaprakash B 1994 *Phys. Rev.* D **50** 7111
[23] Poisson E and Will C 1995 *Phys. Rev.* D **52** 848
[24] Balasubramanian R, Sathyaprakash B and Dhurandhar S 1996 *Phys. Rev.* D **53** 3033
[25] Damour T, Iyer B R and Sathyaprakash B 1998 *Phys. Rev.* D **57** 885
[26] Buonanno A, Chen Y and Vallisneri M 2003 *Phys. Rev.* D **67** 024016 (*Preprint* gr-qc/0205122)
[27] Hellings R W and Moore T A 2003 *Class. Quantum Grav.* **20** S181
[28] Moore T A and Hellings R W 2003 *Phys. Rev.* D **65** 062001
[29] Sintes A M and Vecchio A 2000 *Rencontres de Moriond: Gravitational Waves and Experimental Gravity* ed J Dumarchez (France: Editions Frontières) (*Preprint* gr-qc/0005058)
[30] Sintes A M and Vecchio A 2000 *Third Amaldi Conference on Gravitational Waves* ed S Meshkov (New York: AIP) (*Preprint* gr-qc/0005059)
[31] Van Den Broeck C 2006 *Class. Quantum Grav.* **23** L51
[32] Van Den Broeck C and Sengupta A S 2007 *Class. Quantum Grav.* **24** 155
[33] Arun K G, Iyer B R, Sathyaprakash B S and Sinha S 2007 *Phys. Rev.* D **75** 124002
[34] Holz D E and Hughes S A 2005 *Astrophys. J.* **629** 15
[35] Dalal N, Holz D E, Hughes S A and Jain B 2006 *Phys. Rev.* D **74** 063006
[36] Arun K G, Iyer B R, Sathyaprakash B S, Sinha S and Van Den Broeck C 2007 *Phys. Rev.* D **76** 104016
[37] Trias M and Sintes A 2008 *Phys. Rev.* D **77** 024030 (*Preprint* arXiv:0707.4434)
[38] Blanchet L and Damour T 1986 *Phil. Trans. R. Soc. Lond.* A **320** 379
[39] Blanchet L 1987 *Proc. R. Soc. Lond.* A **409** 383
[40] Blanchet L and Damour T 1992 *Phys. Rev.* D **46** 4304
[41] Blanchet L 1995 *Phys. Rev.* D **51** 2559 (*Preprint* gr-qc/9501030)
[42] Blanchet L 1996 *Phys. Rev.* D **54** 1417
Blanchet L 2005 *Phys. Rev.* D **71** 129904(E) (*Preprint* gr-qc/9603048) (erratum)
[43] Blanchet L 1998 *Class. Quantum Grav.* **15** 1971 (*Preprint* gr-qc/9801101)
[44] Will C and Wiseman A 1996 *Phys. Rev.* D **54** 4813 (*Preprint* gr-qc/9608012)
[45] Pati M and Will C 2000 *Phys. Rev.* D **62** 124015
[46] Pati M and Will C 2002 *Phys. Rev.* D **65** 104008 (*Preprint* gr-qc/0201001)
[47] Thorne K 1980 *Rev. Mod. Phys.* **52** 299
[48] Blanchet L, Damour T and Iyer B R 1995 *Phys. Rev.* D **51** 5360 (*Preprint* gr-qc/9501029)
[49] Blanchet L and Iyer B R 2004 *Phys. Rev.* D **71** 024004 (*Preprint* gr-qc/0409094)
[50] Blanchet L 1998 *Class. Quantum Grav.* **15** 113
Blanchet L 2005 *Class. Quantum Grav.* **22** 3381 (*Preprint* gr-qc/9710038) (erratum)
[51] Blanchet L 1998 *Class. Quantum Grav.* **15** 89 (*Preprint* gr-qc/9710037)
[52] Christodoulou D 1991 *Phys. Rev. Lett.* **67** 1486
[53] Wiseman A and Will C 1991 *Phys. Rev.* D **44** R2945
[54] Thorne K 1992 *Phys. Rev.* D **45** 520
[55] Martín-García J 2002 http://metric.iem.csic.es/Martin-Garcia/xAct/
[56] Blanchet L and Faye G 2000 *Phys. Lett.* A **271** 58 (*Preprint* gr-qc/0004009)
[57] Blanchet L and Faye G 2001 *Phys. Rev.* D **63** 062005 (*Preprint* gr-qc/0007051)
[58] Blanchet L, Damour T and Esposito-Farèse G 2004 *Phys. Rev.* D **69** 124007 (*Preprint* gr-qc/0311052)
[59] MacMahon P A 1960 *Combinatory Analysis* vols I, II (Cambridge: Cambridge University Press) (reprinted Chelsea, New York)
[60] Itoh Y, Futamase T and Asada H 2001 *Phys. Rev.* D **63** 064038
[61] Itoh Y and Futamase T 2003 *Phys. Rev.* D **68** 121501(R)
[62] Itoh Y 2004 *Phys. Rev.* D **69** 064018
[63] Jaranowski P and Schäfer G 1998 *Phys. Rev.* D **57** 7274
[64] Jaranowski P and Schäfer G 1999 *Phys. Rev.* D **60** 124003
[65] Damour T, Jaranowski P and Schäfer G 2001 *Phys. Lett.* B **513** 147
[66] Gradshteyn I and Ryzhik I 1980 *Table of Integrals, Series and Products* (New York: Academic)
[67] Blanchet L and Schäfer G 1993 *Class. Quantum Grav.* **10** 2699
[68] Sasaki M 1994 *Prog. Theor. Phys.* **92** 17
[69] Tagoshi H and Sasaki M 1994 *Prog. Theor. Phys.* **92** 745
[70] Tanaka T, Tagoshi H and Sasaki M 1996 *Prog. Theor. Phys.* **96** 1087