A note on the radiation reaction in the 2.5PN waveform from inspiralling binaries in quasi-circular orbits

2007 Class. Quantum Grav. 24 5307

(http://iopscience.iop.org/0264-9381/24/20/N01)

Class. Quantum Grav. **24** (2007) 5307–5312 doi:10.1088/0264-9381/24/20/N01

COMMENTS, REPLIES AND NOTES

A note on the radiation reaction in the 2.5PN waveform from inspiralling binaries in quasi-circular orbits

Lawrence E Kidder[1], Luc Blanchet[2] and Bala R Iyer[3]

[1] Center for Radiophysics and Space Research, Cornell University, Ithaca, NY 14853, USA
[2] $\mathcal{G}\mathbb{R}\varepsilon\mathbb{C}\mathcal{O}$ Institut d'Astrophysique de Paris, C.N.R.S., 98[bis] boulevard Arago, 75014 Paris, France
[3] Raman Research Institute, Bangalore 560 080, India

E-mail: kidder@astro.cornell.edu, blanchet@iap.fr and bri@rri.res.in

Received 5 June 2007, in final form 6 June 2007
Published 2 October 2007
Online at stacks.iop.org/CQG/24/5307

Abstract
In this note we compute the contributions of the radiation reaction force in the 2.5 post-Newtonian (PN) gravitational wave polarizations for compact binaries in quasi-circular orbits. (i) We point out and correct an inconsistency in the derivation of [1]. (ii) We prove that all contributions from the radiation reaction in the 2.5PN waveform are actually negligible since they can be absorbed into a modification of the orbital phase at the 5PN order.

PACS numbers: 04.25.Nx, 0.4.30.−w, 97.60.Jd, 97.60.Lf

1. Introduction

The second and a half post-Newtonian (2.5PN $\sim c^{-5}$) waveform for inspiralling compact binaries moving in quasi-circular orbits was computed by Arun *et al* [1]. Starting from the expressions of the radiative multipole moments given by equations (3.4)–(3.6) of [1], and of the source moments given by (3.16)–(3.18) of [1], the waveform is made of the instantaneous terms (equations (5.1)–(5.4) of [1]), the hereditary memory-type contributions (equations (4.24) of [1]) and the tail contributions (equations (4.38) corrected by the published erratum [1]). In this note we are concerned with the piece of the instantaneous waveform that is the contribution from radiation reaction (RR) at the 2.5PN order in the dynamics of the inspiralling compact binary. We point out and correct an inconsistency in the derivation of RR terms in [1], but argue that in fact the RR terms are negligible in the 2.5PN waveform, since they can be absorbed into a 5PN order contribution to the orbital phase evolution.

2. Quasi-circular inspiral at the 2.5PN order

Adopting the conventions of [1], $r = |\mathbf{x}|$ is the binary separation with $\mathbf{x} = \mathbf{y}_1 - \mathbf{y}_2$ being the vectorial separation between the particles, $\mathbf{v} = \mathbf{v}_1 - \mathbf{v}_2$ is the relative velocity, $m = m_1 + m_2$ is the total mass of the binary system (distinct from the mass monopole moment M given by the ADM mass of the system), and $\nu = m_1 m_2/m^2$ is the reduced mass divided by the total mass.

In modeling the orbital motion of the binary at the 2.5PN order, [1] stressed the importance of including the radiation-reaction force in the 2.5PN expression of the binary acceleration. Consider an arbitrary orbit confined to the x–y plane. The relative position, velocity and acceleration are given by

$$\mathbf{x} = r\mathbf{n}, \tag{1}$$

$$\mathbf{v} = \dot{r}\mathbf{n} + r\omega\boldsymbol{\lambda}, \tag{2}$$

$$\mathbf{a} = (\ddot{r} - r\omega^2)\mathbf{n} + (r\dot{\omega} + 2\dot{r}\omega)\boldsymbol{\lambda}. \tag{3}$$

Here $\boldsymbol{\lambda} = \hat{\mathbf{z}} \times \mathbf{n}$ where $\hat{\mathbf{z}}$ is the unit vector along the z-direction. The orbital frequency ω is related as usual to the orbital phase ϕ via $\omega = \dot{\phi}$.

Through 2PN order, it is possible to model the motion of the binary as a circular orbit with the solution $\ddot{r} = \dot{r} = \dot{\omega} = 0$ and $r\omega^2 = -\mathbf{n}\cdot\mathbf{a}$. Detailed calculations of the 2PN equations of motion for a circular orbit (in harmonic coordinates) yield

$$\omega^2 = \frac{Gm}{r^3}\left\{1 + (-3+\nu)\frac{Gm}{rc^2} + \left(6 + \frac{41}{4}\nu + \nu^2\right)\left(\frac{Gm}{rc^2}\right)^2\right\}. \tag{4}$$

At the 2.5PN order, however, the effect of the inspiral motion must be taken into account. The leading order contribution to the inspiral can be obtained by examining the Newtonian orbital energy,

$$E = -\frac{1}{2}\frac{Gm^2\nu}{r}, \tag{5}$$

and the leading order gravitational-wave luminosity,

$$-\frac{\mathrm{d}E}{\mathrm{d}t} = \frac{32}{5}\frac{G^4 m^5 \nu^2}{r^5 c^5}. \tag{6}$$

Here we assume that the energy radiated by the gravitational waves is balanced by the change in the orbital energy. This yields

$$\dot{r} = \frac{\mathrm{d}E/\mathrm{d}t}{\mathrm{d}E/\mathrm{d}r} = -\frac{64}{5}\frac{G^3 m^3 \nu}{r^3 c^5}. \tag{7}$$

Similarly the orbital frequency changes by (using $Gm = r^3\omega^2$ at leading order)

$$\dot{\omega} = \frac{\mathrm{d}E/\mathrm{d}t}{\mathrm{d}E/\mathrm{d}\omega} = \frac{96}{5}\frac{Gm\nu}{r^3}\left(\frac{Gm}{rc^2}\right)^{5/2}, \tag{8}$$

while the orbital phase $\phi = \int \omega\, \mathrm{d}t$, found by integrating (8), is

$$\phi = \phi_0 - \frac{1}{32\nu}\left(\frac{Gm}{rc^2}\right)^{-5/2}. \tag{9}$$

Substituting (7) and (8) into (2)–(3), and noting that $\ddot{r} \sim \mathcal{O}(c^{-10})$ is of the order of the *square* of RR effects, the expressions for the 2.5PN inspiral relative velocity and acceleration in harmonic coordinates are obtained,

$$\mathbf{v} = r\omega\boldsymbol{\lambda} - \frac{64}{5}\frac{G^3m^3\nu}{r^3c^5}\mathbf{n}, \tag{10}$$

$$\mathbf{a} = -\omega^2\mathbf{x} - \frac{32}{5}\frac{G^3m^3\nu}{r^4c^5}\mathbf{v}, \tag{11}$$

where ω is given by (4). As we shall detail, the 2.5PN RR terms in both the inspiral velocity (10) and acceleration (11) should be substituted into the gravitational waveform at the 2.5PN order.

3. Computation of polarization waveforms

In the leading quadrupole approximation the gravitational waveform is given by

$$h_{ij}^{\mathrm{TT}} = \frac{2G}{c^4 R}\mathcal{P}_{ijkl}(\mathbf{N})[\ddot{I}_{kl} + \mathcal{O}(c^{-1})], \tag{12}$$

where R is the distance to the observer and $\mathcal{P}_{ijkl}$ is the transverse-traceless (TT) projection operator $\mathcal{P}_{ijkl} = \mathcal{P}_{ik}\mathcal{P}_{jl} - \frac{1}{2}\mathcal{P}_{ij}\mathcal{P}_{kl}$, with $\mathcal{P}_{ij} = \delta_{ij} - N_iN_j$ and $\mathbf{N} = (N_i)$ the radial direction from the source to the observer. Here $\ddot{I}_{kl}$ is the second time derivative of some appropriate source quadrupole moment defined from a general post-Newtonian multipole moment formalism. The remainder $\mathcal{O}(c^{-1})$ indicates higher PN corrections coming notably from all the higher multipolar orders.

Defining an orthonormal triad $(\mathbf{N}, \mathbf{P}, \mathbf{Q})$ where $\mathbf{P}$ and $\mathbf{Q}$ are unit polarization vectors transverse to the direction of propagation, the polarization waveforms h_+ and $h_\times$ are computed from the waveform (12) by

$$h_+ = \tfrac{1}{2}(P_iP_j - Q_iQ_j)h_{ij}^{\mathrm{TT}}, \tag{13}$$

$$h_\times = \tfrac{1}{2}(P_iQ_j + P_jQ_i)h_{ij}^{\mathrm{TT}}. \tag{14}$$

If the orbital plane is chosen to be the x–y plane with the orbital phase ϕ measuring the direction of the unit vector $\mathbf{n} = \mathbf{x}/r$ along the relative separation vector, then

$$\mathbf{n} = \cos\phi\,\hat{\mathbf{x}} + \sin\phi\,\hat{\mathbf{y}}. \tag{15}$$

[1] has chosen the polarization vector $\mathbf{P}$ to lie along the x-axis and the observer to be in the y–z plane with

$$\mathbf{N} = \sin i\,\hat{\mathbf{y}} + \cos i\,\hat{\mathbf{z}}, \tag{16}$$

where i is the orbit's inclination angle ($0 \leqslant i \leqslant \pi$). With these definitions $\mathbf{P}$ lies along the intersection of the orbital plane with the plane of the sky in the direction of the ascending node, and the orbital phase ϕ is the angle between the ascending node and the direction of body one (say). The rotating orthonormal triad $(\mathbf{n}, \boldsymbol{\lambda}, \hat{\mathbf{z}})$ describing the motion of the binary is then related to the polarization triad $(\mathbf{N}, \mathbf{P}, \mathbf{Q})$ by

$$\mathbf{n} = \cos\phi\,\mathbf{P} + \sin\phi\,(c_i\mathbf{Q} + s_i\mathbf{N}), \tag{17}$$

$$\boldsymbol{\lambda} = -\sin\phi\,\mathbf{P} + \cos\phi\,(c_i\mathbf{Q} + s_i\mathbf{N}), \tag{18}$$

$$\hat{\mathbf{z}} = -s_i\mathbf{Q} + c_i\mathbf{N}, \tag{19}$$

where we pose $c_i = \cos i$ and $s_i = \sin i$.

4. Radiation reaction contributions to the waveform

All contributions arising from the gravitational RR are contained in the leading order quadrupolar waveform given by (12). They have three different origins. The first type of

RR term is issued directly from the expression of the (post-Newtonian) quadrupole moment I_{ij} at the order 2.5PN, and reads (see equation (3.16a) of [1])

$$I_{ij} = \nu m x^{\langle i} x^{j\rangle} + \cdots + \frac{48}{7} \frac{G^2 m^3 \nu^2}{r c^5} x^{\langle i} v^{j\rangle}. \tag{20}$$

The first term is the usual expression of the Newtonian quadrupole moment (the brackets surrounding indices refer to the symmetric-trace-free projection). The dots indicate the 1PN and 2PN conservative terms, and terms strictly higher than 2.5PN. Taking two time derivatives of (20) we get

$$\ddot{I}_{ij} = 2\nu m (v^{\langle i} v^{j\rangle} + x^{\langle i} a^{j\rangle}) + \cdots - \frac{192}{7} \frac{G^3 m^4 \nu^2}{r^4 c^5} x^{\langle i} v^{j\rangle}. \tag{21}$$

The second type of RR term comes from inserting the expression of the acceleration **a** given by (11), which produces another RR term which modifies the term in (21) as

$$\ddot{I}_{ij} = 2\nu m \left(v^{\langle i} v^{j\rangle} - \frac{Gm}{r^3} x^{\langle i} x^{j\rangle} \right) + \cdots - \frac{1408}{35} \frac{G^3 m^4 \nu^2}{r^4 c^5} x^{\langle i} v^{j\rangle}. \tag{22}$$

The sum of these RR contributions is exactly what has been computed in [1], where it was denoted by $\rho_{ij}^{(5/2)}$ and given by equation (5.2). We now replace (22) into the waveform (12) and compute the two GW polarizations according to (13)–(14), hence

$$h_{+,\times} = \frac{2G}{c^4 R} \begin{pmatrix} \frac{P_i P_j - Q_i Q_j}{2} \\ \frac{P_i Q_j + P_j Q_i}{2} \end{pmatrix} \left[2\nu m \left(v^i v^j - \frac{Gm}{r^3} x^i x^j \right) + \cdots - \frac{1408}{35} \frac{G^3 m^4 \nu^2}{r^4 c^5} x^i v^j \right]. \tag{23}$$

For ease of notation, in equation (23) above and similar equations later the first row (line) corresponds to + and the second row (line) to ×. The RR term in (22) has already been included in the final result of [1]. Its contribution to the polarization waveforms is given by

$$\delta_{12} h_{+,\times} = \frac{2G\nu m x}{c^2 R} \begin{cases} \frac{352}{35} (1 + c_i^2) \nu x^{5/2} \sin 2\phi, \\ -\frac{352}{35} (2c_i) \nu x^{5/2} \cos 2\phi, \end{cases} \tag{24}$$

where we pose $x = (Gm\omega/c^3)^{2/3}$. With the notation δ_{12} we recall that this term was made of two distinct pieces.

However, let us show there is also another contribution to RR that has been overlooked in [1]. Indeed the instantaneous terms in the waveform h_{ij}^{TT} (equations (5.1)–(5.4) of [1]) are given in terms of the relative position **n** and velocity **v** of the binary, as well as the PN parameter $\gamma = Gm/(rc^2)$. In computing the polarization waveforms, i.e., projecting out h_{ij}^{TT} to get $h_{+,\times}$, [1] has substituted, at the last stage of the computation, $\mathbf{v} = r\omega\boldsymbol{\lambda}$ in the waveform to obtain equations (5.9)–(5.10) of [1]. This is correct for all the terms *except* the two leading order 'Newtonian' terms in (23) for which one must use the true expression of the velocity (10) *including* RR inspiral. We find that substituting the inspiral velocity (10) into the leading terms of the waveform (23) yields the following additional contribution to the polarization waveforms,

$$\delta_3 h_{+,\times} = \frac{2G\nu m x}{c^2 R} \begin{cases} \frac{64}{5} \left(1 + c_i^2\right) \nu x^{5/2} \sin 2\phi, \\ -\frac{64}{5} (2c_i) \nu x^{5/2} \cos 2\phi. \end{cases} \tag{25}$$

Note that both results (24) and (25) have the same structure. Since the contribution (25) has not been taken into account in [1] we find that the following terms in the 2.5PN polarization waveforms of [1] (equations (5)–(6) of the erratum) are to be changed from

$$H^{(2.5)}_{+,\times}\Big|_{\text{old}} = \begin{cases} \cdots + \sin 2\psi \left[-\frac{9}{5} + \frac{14}{5}c_i^2 + \frac{7}{5}c_i^4 + \nu\left(\frac{96}{5} - \frac{8}{5}c_i^2 - \frac{28}{5}c_i^4\right)\right] + \cdots, \\ \cdots + c_i \cos 2\psi \left[2 - \frac{22}{5}c_i^2 + \nu\left(-\frac{154}{5} + \frac{94}{5}c_i^2\right)\right] + \cdots, \end{cases} \tag{26}$$

to

$$H^{(2.5)}_{+,\times} = \begin{cases} \cdots + \sin 2\psi \left[-\frac{9}{5} + \frac{14}{5}c_i^2 + \frac{7}{5}c_i^4 + \nu\left(32 + \frac{56}{5}c_i^2 - \frac{28}{5}c_i^4\right)\right] + \cdots, \\ \cdots + c_i \cos 2\psi \left[2 - \frac{22}{5}c_i^2 + \nu\left(-\frac{282}{5} + \frac{94}{5}c_i^2\right)\right] + \cdots. \end{cases} \tag{27}$$

We recall that the phase variable ψ differs from ϕ and is given by equation (3) of erratum [1]. The difference between ψ and ϕ is at the order 4PN at least (see [1] for discussion). Note that the new correction (25) is in terms that vanish in the limit $\nu \to 0$, and so is still consistent with the results of black hole perturbation theory [2].

5. Phase modulation due to radiation reaction

We are now going to show that *all* the RR contributions are in fact negligible in the 2.5PN waveform. From the sum of the previous results (24) and (25) we end up with the following total contribution due to RR in the GW polarizations:

$$\delta^{\text{RR}} h_{+,\times} = \frac{2G\nu m x}{c^2 R} \begin{cases} \frac{160}{7}\left(1 + c_i^2\right)\nu x^{5/2} \sin 2\phi, \\ -\frac{160}{7}(2c_i)\nu x^{5/2} \cos 2\phi. \end{cases} \tag{28}$$

Comparing this result with the GW polarizations at the Newtonian order,

$$h^{(\text{N})}_{+,\times} = \frac{2G\nu m x}{c^2 R} \begin{cases} -\left(1 + c_i^2\right) \cos 2\phi, \\ -(2c_i) \sin 2\phi, \end{cases} \tag{29}$$

we see that the RR terms can be absorbed into a redefinition of the phase variable as

$$\Phi = \phi + \tfrac{80}{7}\nu x^{5/2}. \tag{30}$$

This means that the 2.5PN waveform will take exactly the same expression as if we had neglected the RR contributions (28), but with phase variable Φ in place of ϕ. Now the point is that the added term in (30) represents an extremely small modification of the phasing. Indeed, we take into account the expression for the phase as a function of frequency at the leading order (i.e., due to leading order RR effects), which has been computed in (9) and whose order $\sim c^5$ is the *inverse* of the order $\sim c^{-5}$ of RR effects. Thus we find that the redefined phase is equivalent to

$$\Phi = \phi_0 - \frac{x^{-5/2}}{32\nu}\left[1 + \cdots - \frac{2560}{7}\nu^2 x^5\right]. \tag{31}$$

This means that the RR terms seen as modulations of the phase evolution, contribute to the phase much beyond the 2.5PN order, namely at order 5PN $\sim x^5$ beyond the leading phase evolution. Indeed there is in principle no point in including these terms because they are comparable to 5PN terms in the phase that are unknown—only the phase evolution up to the 3.5PN order is known.

We conclude therefore that the RR terms can in fact be neglected in the 2.5PN waveform for circular orbits. This is similar to what happens in the energy flux for circular orbits where we know that the 2.5PN radiation reaction gives finally a contribution only at the 5PN order [3] (the 2.5PN terms in the energy flux are only due to tails). Hence we can by the redefinition of the phase variable (30) completely remove the RR contribution given by (28). However we can also decide to include such terms in the 2.5PN waveform (as usual there are many different ways of presenting PN results at a given order of approximation). Here we simply propose to include the RR terms as they are in the templates of binary inspiral, i.e. by correcting the inconsistency in [1] (which has by the above argument no physical effect at 2.5PN) and keeping the corrected waveform in the form given by equations (27). For completeness, we note that if we choose to include all the RR terms into the phase redefinition Φ, equation (27) will then be modified into

$$H'^{(2.5)}_{+,\times} = \begin{cases} \cdots + \sin 2\Psi \left[-\frac{9}{5} + \frac{14}{5}c_i^2 + \frac{7}{5}c_i^4 + \nu\left(\frac{64}{7} - \frac{408}{35}c_i^2 - \frac{28}{5}c_i^4\right)\right] + \cdots, \\ \cdots + c_i \cos 2\Psi \left[2 - \frac{22}{5}c_i^2 + \nu\left(-\frac{374}{35} + \frac{94}{5}c_i^2\right)\right] + \cdots, \end{cases} \tag{32}$$

where Ψ is equal to ψ plus the added RR contribution given by the second term in (30), and where ψ itself differs from ϕ by equation (3) in erratum [1]. (All the other terms in the waveform are then to be expressed with the same phase variable Ψ.)

Acknowledgments

This work was supported in part by grants from the Sherman Fairchild Foundation, NSF grants PHY-0354631, DMS-0553677 and NASA grant NNG05GG51G at Cornell. LB and BRI thank the Indo–French Collaboration (IFCPAR) for its support.

References

[1] Arun K G, Blanchet L, Iyer B R and Qusailah M S S 2004 *Class. Quantum Grav.* **21** 3771–801
Arun K G, Blanchet L, Iyer B R and Qusailah M S S 2005 *Class. Quantum Grav.* **22** 3115–7 (erratum)
[2] Tagoshi H and Sasaki M 1994 *Prog. Theor. Phys.* **92** 745–71
[3] Blanchet L 1996 *Phys. Rev.* D **54** 1417–1438
Blanchet L 2005 *Phys. Rev.* D **71** 129904 (erratum)