Dipolar particles in general relativity

Dipolar particles in general relativity

Luc Blanchet

$\mathcal{GR}\varepsilon\mathbb{C}\mathcal{O}$, Institut d'Astrophysique de Paris, CNRS, Université Pierre et Marie Curie,
98 bis Boulevard Arago, 75014 Paris, France

E-mail: blanchet@iap.fr

Abstract
The dynamics of 'dipolar particles', i.e. particles endowed with a 4-vector mass
dipole moment, is investigated using an action principle in general relativity.
The action is a specific functional of the particle's worldline, and of the dipole
moment vector, considered as a dynamical variable. The first part of the action
is inspired by that of a particle with spin moving on an arbitrary gravitational
background. The second part is intended to describe, at some effective level, the
internal non-gravitational force linking together the 'microscopic' constituents
of the dipole. We find that some solutions of the equations of motion and
evolution of the dipolar particles correspond to an equilibrium state for the
dipole moment in a gravitational field. Under some hypothesis we show that
a fluid of dipolar particles, supposed to constitute the dark matter, reproduces
the modified Newtonian dynamics (MOND) in the non-relativistic limit. We
recover the main characteristics of a recently proposed quasi-Newtonian model
of 'gravitational polarization'.

PACS numbers: 04.20.−q, 95.35.+d, 95.30.Sf

1. Introduction

1.1. Astrophysical motivation

It is recognized (see [1–4] for reviews) that the modified Newtonian dynamics or MOND,
which has been proposed by Milgrom [5–7] as an alternative to dark matter, works extremely
well at predicting the form of rotation curves of galaxies from the observed distribution of
stars and gas. In addition, MOND naturally explains the relation between the luminosity of
galaxies and their asymptotic rotation velocity—the so-called Tully–Fisher law [8]. Though
there might be some examples where MOND does not fully account for the observed kinematics
of galaxies [9], overall the fit achieved by MOND of the rotation curves of most galaxies is
very impressive and calls for a possible physical explanation. On the other hand, we know

that the mass discrepancy of clusters of galaxies is not completely accounted for by MOND [10], and that at the cluster scale there is still empirical evidence for unseen dark matter [11].

In the usual interpretation, MOND is viewed as a modification of the fundamental law of gravity, without the need of dark matter. Several relativistic extensions of MOND, sharing this view of modifying the sector of gravity, postulate the existence of supplementary fields associated with the gravitational force, in addition to the metric tensor of general relativity [12–16]. So far the most developed of the relativistic MOND theories is the tensor–scalar–vector theory of Bekenstein and Sanders [14–16].

At this stage we seem to face two alternatives to the issue of dark matter:

 (1) Either accept the existence of cold dark matter particles, e.g. predicted by super-symmetric extensions of the standard model of particle physics (see [17] for a review). However, these particles are yet to be discovered, and the simplest models of cold dark matter fail to reproduce in a natural way the flat rotation curves of galaxies [2].
 (2) Or postulate an alteration of our fundamental theory of gravity (namely MOND and its relativistic extensions). But the motivation for altering the law of gravity is ad hoc, and one can argue that the relativistic MOND theory [14–16] does not in fact explain the phenomenology of MOND—at least until the extra fields find a well motivated explanation coming from fundamental physics.

In the present paper, following [18][1], we propose a third alternative. Namely we

 (3) Keep the standard law of gravity, i.e. general relativity and its Newtonian limit, but we add to the distribution of ordinary matter some specific form of dark matter in such a way as to naturally explain MOND. The dark matter consists of 'polarization masses' associated with a medium of dipole moments aligned in the gravitational field of ordinary matter. (But, in this paper, we shall leave aside the problem of clusters of galaxies [10, 11].)

Our basic motivation is that MOND can be naturally interpreted as resulting from an effect of gravitational polarization of a dipolar medium. Paper I argued, on the basis of a simple quasi-Newtonian model, that the polarization tends to enhance the magnitude of the gravitational field of ordinary galaxies in a way consistent with MOND. This effect constitutes the gravitational analogue of the electric polarization of a dielectric material (whose atoms can be modelled by electric dipoles) by an applied electric field [19]. Thus, the phenomenology of MOND results from the non-standard influence of the dark matter on ordinary matter.

Arguably proposal (3) does not really provide an explanation for MOND. Indeed the models of paper I and this paper are only *effective*—the fundamental nature of the dipolar particles (i.e. their 'internal' structure) will not be elucidated. However, the proposal is conceptually simple and fits naturally with the MOND phenomenology of the flat rotation curves and the Tully–Fisher empirical relation. In addition, the model cannot be said to be ad hoc because it invokes the (gravitational analogue of the) well known physical mechanism of polarization by an external field.

The quasi-Newtonian model of paper I is based on a microscopic description of the dipole moment using negative gravitational-type masses (or gravitational charges). As a result, the motion of dipolar particles in this model violates the equivalence principle. In the present paper, we elaborate a model of dipolar particles and gravitational polarization in the standard general relativity theory[2], without negative (passive) gravitational masses and consistent with the equivalence principle. Consequently, we shall find that the equations of motion of the

---

[1] Hereafter [18] will be referred to as paper I.
[2] The concept of gravitational polarization at the quadrupolar order in relativistic gravity theories has been investigated in [20].

dipolar particles in the non-relativistic (NR) limit of the present model are different from those of paper I. However, we shall recover the main characteristics of the quasi-Newtonian model of paper I and notably the interesting connection with the phenomenology of MOND.

1.2. Concept of dipole moment in general relativity

In theories satisfying the equivalence principle the mass dipole moment of a mass distribution, say $\pi_g^i = \sum m_g x^i$ (adopting a Newtonian picture), is proportional, by equivalence between the gravitational and inertial masses, $m_g = m_i$, to the position of the centre of mass of the system, $C^i = \sum m_i x^i / M_i$ (where $M_i = \sum m_i$). Similarly, the current dipole moment, $\mu_g^i = \sum m_g \varepsilon^{ijk} x^j v^k$, is equivalent to the spin angular momentum $S^i = \sum m_i \varepsilon^{ijk} x^j v^k$. Defining the mass dipole moment of a particle (supposed to be composed of some sub-particles) is *a priori* delicate because one faces the problem that $\pi_g^i = 0$ in the centre-of-mass frame where $C^i = 0$ by definition; however, the current dipole moment μ_g^i equivalent to the spin S^i is admissible. Thus, while the notion of a particle carrying a mass dipole moment seems to be possible only at the price of violating the equivalence principle (like in the model of paper I), the notion of a particle endowed with a current dipole moment or spin is perfectly legitimate.

Particles with spins have been the subject of many fundamental works in general relativity [21–26], with applications to the problem of dynamics of spinning black holes in binary systems [27–31]. The general formalism has been encapsulated in an action principle for a particle with spin moving on an arbitrary gravitational background. The action, due to Bailey and Israel [26], is derived by expressing the standard action for a non-spinning particle (i.e. the integral of the proper time), as defined with respect to some 'eccentric' worldline located at the edge of some composite particle, in terms of the quantities belonging to a different worldline, which is viewed as the 'reference' or 'central' worldline located at the centre of the particle. Crucial to the formalism is the vector separation between the two worldlines, defined as the gradient of the geodesic distance separating them[3]. The spin of the particle is then given by the anti-symmetric product between the latter separation vector and the particle's linear momentum. Hence, the separation vector is the 'lever arm' associated with the spin angular momentum, as defined with respect to the fiducial central worldline. The point for our purpose is that such a lever arm appears essentially to be the analogue of a mass-type dipole moment. We thus see the possibility of describing dipolar particles in general relativity by starting from (some variant of) the Bailey–Israel action [26].

The previous paradox concerning the dipole moment which should vanish in the frame of the centre of mass will be solved because (roughly speaking) there are two notions of dipole moments. The first moment is the one we mentioned earlier, π_g^i, but let us call it now the 'passive' dipole moment π_p^i to emphasize the fact that the gravitational mass it contains is really the *passive* gravitational mass $m_p = m_g$ (i.e. that mass which enters on the right-hand side of the law of motion, in factor of the gravitational field). Assuming as before that the dipolar particle is composed of sub-particles, its passive dipole moment $\pi_p^i \sim \sum m_p x^i$ will indeed be zero in the particle's centre of mass, by equivalence between the inertial and passive gravitational masses of the sub-particles, $m_p = m_i$. However, the dipole moment we shall consider in this paper is different: this is the moment parametrizing the dipolar part of the stress–energy tensor $T^{\mu\nu}$ of the particles (notably its 00 component or energy density). In general relativity the stress–energy tensor is the source for the gravitational field, so we can

---

[3] The geodesic distance is nothing but the world function in Synge's formalism [32]; it is a bi-scalar in the general theory of bi-tensors [33].

rightly refer to this moment as the *active* dipole moment. In conclusion, we shall have $\pi_{\rm p}^i = 0$ by the equivalence principle, but we still have at our disposal the 'active gravitational' version of the moment, call it $\pi_{\rm a}^i$, which enters the stress–energy tensor of the particles, say through a term of the form $\sim -\partial_i\left(\pi_{\rm a}^i \delta\right)$ in the energy density at the NR approximation (where δ is the Dirac function). In the following we shall investigate a specific model for the relativistic dynamics and evolution of particles endowed with a 4-vector dipolar moment $\pi^\mu \equiv \pi_{\rm a}^\mu$ of the active type[4].

By analogy with the model of paper I we shall denote by $2m$ the total inertial mass of the dipolar particle, equivalent to its total passive gravitational mass, i.e. $M_{\rm i} = M_{\rm p} = 2m$. However, this equivalence does not mean that the motion of the dipolar particle is geodesic. Indeed, motivated by paper I, we shall introduce a force F^μ, which will be considered as 'internal' to the dipolar particle and is aimed at 'stabilizing' the dipole moment embedded in an exterior gravitational field. The force F^μ has a non-gravitational origin and will derive from a scalar potential function V in the action. Because of the presence of this force reflecting its internal structure, the dipolar particle is not a test particle: its motion is not geodesic, its 4-acceleration is non-zero; the particle can be thought of as a 'rocket', self-accelerated by the internal force F^μ. We shall find an approximate solution in which F^μ accelerates the particle in such a way as to compensate for the local gravitational field, so that the dipolar particle stays essentially at rest in a gravitational field (like in the model of paper I).

The monopolar part of the stress–energy tensor defines what can be regarded as the particle's active gravitational mass $M_{\rm a}$. Intuition from paper I would lead us to expect that for a dipolar particle $M_{\rm a}$ is zero, i.e. the particle does not generate any monopolar gravitational field. In the present model, we shall find that this is not possible: $M_{\rm a}$ cannot be zero, and in fact we shall have $M_{\rm a} = V/c^2$, where V is the potential function from which derives the internal force F^μ. However, thanks to the explicit factor $1/c^2$ it contains, $M_{\rm a}$ turns out to be very small since it vanishes in the NR limit, i.e. $M_{\rm a} = \mathcal{O}(c^{-2})$ when $c \to \infty$. Thus, the dipolar particle will indeed be 'purely dipolar' in the NR limit[5].

Applying the standard general relativistic coupling to gravity, we prove that the Einstein field equations reduce to a Poisson equation with a dipolar source term in addition to the density of ordinary matter, having the structure $\sim -\partial_i\left(\pi_\perp^i \delta\right)$ in the limit $c \to \infty$ (where $\pi_\perp^i$ refers to an appropriate orthogonal projection). We are then close to the MOND equation; to recover MOND it suffices that the dipole moment be aligned with the gravitational field and polarized in a certain way. In the present paper we shall find, under some hypothesis, some solutions which correspond to gravitational polarization, i.e. we shall show that the polarization scenario yielding MOND is consistent with our equations. There will be essentially a one-to-one correspondence between the internal potential scalar function V entering the action and the Milgrom [5–7] function μ (which is linked in the interpretation of paper I with the gravitational susceptibility χ of the dipolar medium by $\mu = 1 + \chi$).

The paper is organized as follows. We first deal with the relativistic model based on the action presented in section 2.1. The general equations of motion and evolution, and the stress–energy tensor, are derived in section 2.2. In section 2.3, we restrict ourselves to a particular solution which corresponds to some 'equilibrium' state for the dipole moment. For

---

[4] Hereafter we no longer mention that this moment can be regarded as the active one. Note that the names we are giving here and below for different types of masses and dipoles are useful for the discussion but are not required in the present formalism, which is entirely based on the action (2.1).

[5] Note that the fact that the particle carries some different active and passive gravitational masses, $M_{\rm a}$ and $M_{\rm p}$, does not contradict the principle of conservation of the linear momentum (and the law of action and reaction). Indeed, the particle's stress–energy tensor contains in addition to the monopolar part a dipolar contribution (parametrized by π^μ) which ensures the satisfaction of the usual conservation laws.

that solution we obtain in section 3.1 the dynamics of dipolar particles in the NR approximation where $c \to \infty$. By neglecting the tidal gravitational field, we find in section 3.2 that when the fluid of dipole moments is at rest with respect to the ordinary matter distribution the medium is polarized and the MOND equation follows. Finally, we investigate in section 4 (still in the NR limit) the case of a stationary fluid of dipole moments aligned in the central gravitational field of a point mass. Section 5 summarizes and concludes the paper. In the appendix, we show on general grounds how to vary a dipolar action functional.

2. Relativistic dynamics of dipolar particles

2.1. Action principle

The starting point of our approach is the formalism for spinning particles in general relativity [21–26] and especially the Bailey–Israel action [26]. In this formalism, the equations of motion of the spinning particle (moving on an arbitrary gravitational background) are derived from the extremum of the action with respect to an infinitesimal displacement of the particle's worldline; however, the spin variable itself is not varied. Instead, one postulates some condition for the evolution of the spin, called the spin supplementary condition (SSC), which essentially tells that the spin is parallel-transported along the particle's worldline (see, e.g., [30, 31]). In the present paper, we shall interpret the 'lever arm' associated with the spin angular momentum, in the formalism of spinning particles, as the mass-type dipole moment carried by the particle. However, we shall not impose some analogue of spin supplementary condition. Instead we shall promote the dipole moment as a *dynamical* variable, which will have to be varied independently of the variation of the particle's position.

Let the particle follow a timelike worldline x^μ in a four-dimensional manifold and being endowed with the dipole moment 4-vector π^μ (having the dimension of a mass times a length)[6]. We propose that the particle's dynamics in a prescribed gravitational field $g_{\mu\nu}$ be derived from the following matter action, consisting of three terms,

$$S = \sum \int_{-\infty}^{+\infty} \mathrm{d}\tau \left[-c\sqrt{g_{\mu\nu} \left(2mu^\mu - \dot{\pi}^\mu\right)\left(2mu^\nu - \dot{\pi}^\nu\right)} + \frac{\dot{\pi}_\mu \dot{\pi}^\mu}{4m} - V\left(\frac{\pi_\perp}{m}\right) \right]. \tag{2.1}$$

The sum goes over all the particles of this type. We denote the particle's proper time by $\mathrm{d}\tau = [-g_{\mu\nu}\,\mathrm{d}x^\mu\,\mathrm{d}x^\nu/c^2]^{1/2}$ and its 4-velocity by $u^\mu = \mathrm{d}x^\mu/\mathrm{d}\tau$ (which is normalized to $g_{\mu\nu}u^\mu u^\nu = -c^2$). The dynamical variables are the spacetime position $x^\mu(\tau)$ and the dipole moment $\pi^\mu(\tau)$, both of them are functions of the proper time.

The first term in equation (2.1) is inspired by the known action for the dynamics of spinning particles [26] and describes the inertial properties of the dipolar particle, i.e. its equations of motion in the gravitational field. The second term is a kinetic-type term for the dynamical evolution of the dipole moment itself and serves to tell how this evolution will differ from parallel transport. The third term of (2.1) is made of a scalar potential function V, supposed to describe, at some effective 'macroscopic' level, a non-gravitational force that is internal to the dipolar particle.

We denote by a dot the covariant derivative of the dipole moment with respect to the proper time, namely

$$\dot{\pi}^\mu \equiv \frac{\mathrm{D}\pi^\mu}{\mathrm{d}\tau} = \frac{\mathrm{d}\pi^\mu}{\mathrm{d}\tau} + \Gamma^\mu_{\rho\sigma} u^\rho \pi^\sigma. \tag{2.2}$$

---

[6] Greek indices take the spacetime values $\mu, \nu = 0, 1, 2, 3$; Latin ones range on spatial values $i, j = 1, 2, 3$; the convention for the Riemann curvature tensor is $R^\mu{}_{\nu\rho\sigma} = \partial_\rho \Gamma^\mu_{\nu\sigma} - \partial_\sigma \Gamma^\mu_{\nu\rho} + \cdots$; the Lorentzian signature is $- + ++$; all factors in the speed of light c and Newton's constant G are indicated throughout.

More generally the dot will always refer to the covariant time derivative $D/d\tau$. The potential function V in the third term of the action depends on the norm $\pi_\perp$ of the orthogonal projection of the dipole moment perpendicular to the 4-velocity, defined by

$$\pi_\perp^\mu = \perp_\nu^\mu \pi^\nu, \tag{2.3}$$

where $\perp_\nu^\mu \equiv \delta_\nu^\mu + u^\mu u_\nu/c^2$ is the corresponding projection operator. The vector (2.3) is spacelike and its norm is given by

$$\pi_\perp = \left[g_{\mu\nu} \pi_\perp^\mu \pi_\perp^\nu \right]^{1/2} = \sqrt{\perp_{\mu\nu} \pi^\mu \pi^\nu}. \tag{2.4}$$

The fact that V depends on the orthogonal projection $\pi_\perp^\mu$ rather than on π^μ itself is crucial for the present formalism.

The mass m parametrizing the action (2.1) is a certain mass parameter associated with the dipole moment, and is such that $x \equiv \pi_\perp/m$ represents the typical size of the dipolar particle. The potential function V will typically be (such as in equation (3.27)) a quadratic function of the particle's size $x = \pi_\perp/m$. The mass m can be viewed as the relativistic analogue of the mass in the model of paper I (we have already discussed in the introduction some heuristic interpretation of $2m$).

An important feature of the action (2.1) is that its first term is given by the norm of a *spacelike* vector. It will be useful to keep in mind this vector, so we introduce a special notation for it:

$$p^\mu = \frac{2mu^\mu - \dot\pi^\mu}{\Lambda}, \tag{2.5a}$$

$$\text{where} \quad \Lambda \equiv \sqrt{-1 - \frac{u_\nu \dot\pi^\nu}{mc^2} + \frac{\dot\pi_\nu \dot\pi^\nu}{4m^2 c^2}}. \tag{2.5b}$$

This spacelike vector, satisfying $p^2 = +4m^2c^2$, will *not* represent the linear 4-momentum of the dipolar particle (otherwise the particle would be a tachyon)[7]. As we shall see, since there are other terms in the action, the linear momentum of the particle will differ from p^μ and be normally timelike.

Because of the spacelike character of p^μ we note that the action (2.1) makes no sense in the limit where the dipole moment vanishes, since this limit corresponds to a particle with an imaginary mass or tachyon. In this sense, the dipolar particle described by (2.1) exists only through the existence of the dipole moment, contrary to a particle carrying a spin, which is described both by a spin and a mass, and reduces to an ordinary point mass in the absence of the spin. (We are speaking of classical particles, whose spin is given by the classical notion of angular momentum.) Here, the mass parameter m is not independent of the dipole moment; it is such that $\pi_\perp = mx$ where x is the particle's size.

Unlike the case of a spinning particle [26], no explicit coupling to the Riemann curvature tensor has been introduced at the level of the action (2.1). Recall that the Riemann tensor in the Bailey–Israel action [26] comes from the particular way this action is constructed, by conveying the point particle action from some eccentric worldline of some composite particle to the physical central worldline. As a result, if one varies this action with respect to the spin, one obtains the same equation as with the variation with respect to the position—both equations containing the same coupling to the Riemann curvature (at least at linear order in the spin). In the present approach, by contrast, the dipole moment will be considered as an independent dynamical variable, and we shall demand that the variation with respect to the dipole moment provides an independent evolution equation.

---

[7] This is in contrast with the Bailey–Israel action for a spinning particle [26], which is given by the norm of the particle's linear momentum—a timelike vector as usual.

The introduction of the scalar potential function V is motivated by the quasi-Newtonian model of paper I, consisting of a pair of sub-particles, with opposite gravitational masses $\pm m$ and positive inertial masses $+m$ (in analogy with the electric dipole in electrostatics), and interacting via some non-gravitational force making possible the existence of a stable dipolar configuration. (Indeed, free gravitational masses with opposite signs will accelerate apart from each other, and consequently must be bound by an internal force able to counteract their gravitational repulsion.) In the present relativistic model, we shall also need to invoke such a force—supposed to describe some non-gravitational attractive interaction between the constituents of the dipole. This force will derive from the potential V in the action.

Let us stress however that we shall not use any model for the elementary constituents of the dipole, nor shall we invoke the notion of a negative mass. The formalism follows completely from the action (2.1), which contains simply the moment π^μ and a positive mass parameter m. As already commented in the introduction, the mass $2m$ represents the total inertial mass M_i of the dipolar particle (in agreement with paper I), but here, contrary to paper I, this mass can also be interpreted as the passive gravitational mass M_p, i.e. $M_\mathrm{i} = M_\mathrm{p} = 2m$. Later we shall argue that one can also define an active gravitational mass M_a by the monopole part of the particle's stress–energy tensor and that this mass turns out to be negligible in the NR limit, $M_\mathrm{a} = \mathcal{O}(c^{-2})$.[8]

2.2. Equations of motion and evolution

By the principle of stationary action we require that S be unchanged under the infinitesimal variation $\delta x^\mu(\tau)$ and $\delta \pi^\mu(\tau)$ in the dynamical variables, with the condition that they vanish on the boundary of the region of integration, i.e. $\delta x^\mu(\pm\infty) = 0$ and $\delta \pi^\mu(\pm\infty) = 0$. We provide in the appendix a summary of the way we vary the dipolar action.

Varying first S with respect to the change $\delta \pi^\mu$ in the dipole moment, holding the particle's worldline fixed, $\delta x^\mu = 0$, we obtain an equation taking the form of the force law

$$\dot{P}^\mu = -2F^\mu, \tag{2.6}$$

where we recall that the dot means the covariant derivative with respect to proper time: $\dot{P}^\mu \equiv DP^\mu/\mathrm{d}\tau$. On the left-hand side (LHS) we have defined what will turn out to be the linear momentum P^μ of the dipolar particle; it is given in terms of the spacelike vector defined in equation (2.5) by

$$P^\mu = p^\mu + \dot{\pi}^\mu. \tag{2.7}$$

As we see P^μ differs from p^μ, and we shall check later that P^μ is timelike. On the right-hand side (RHS) of (2.6), F^μ is the quadri-force derived from the potential V present in the action and is given by

$$F^\mu = \frac{\pi_\perp^\mu}{\pi_\perp} \frac{\mathrm{d}V}{\mathrm{d}x}\left(\frac{\pi_\perp}{m}\right). \tag{2.8}$$

Here, $\mathrm{d}V/\mathrm{d}x$ is the derivative of V with respect to its natural argument $x = \pi_\perp/m$. This force is proportional to the orthogonal projection of the dipole moment (2.3), thus it satisfies the constraint

$$u_\mu F^\mu = 0. \tag{2.9}$$

We next perform a variation δx^μ of the particle's position, holding the components of the dipole moment π^μ 'constant' during the displacement of the worldline. For instance, one

[8] Notice the difference with the quasi-Newtonian 'microscopic' model of paper I, in which we have $M_\mathrm{i} = 2m$ and $M_\mathrm{p} = M_\mathrm{a} = 0$. This represents the fundamental difference between the model of paper I and the (NR limit of the) present relativistic model.

can think of the dipole moment vector as being transported parallel along the displacement vector δx^μ. We find that the variation of the covariant time derivative of the dipole moment, $\dot{\pi}^\mu = D\pi^\mu/d\tau$, yields a Riemann curvature term, which can be understood as coming from the non-commutation of the variational derivative δ with the covariant derivative $D/d\tau$. During the variation we must take into account the contribution due to the orthogonal projection operator $\perp_\nu^\mu$ present in the term $V(\pi_\perp/m)$. With all computations done we end up with the equation (where $\dot{\Omega}^\mu \equiv D\Omega^\mu/d\tau$)

$$\dot{\Omega}^\mu = -\frac{1}{2m} R^\mu{}_{\rho\nu\sigma} u^\rho \pi^\nu P^\sigma. \tag{2.10}$$

See the appendix for a general derivation of this equation. The RHS represents the analogue of the famous Papapetrou coupling to the Riemann curvature tensor in the equation of motion of a particle with spin [21][9]. The 'linear momentum' Ω^μ on the LHS of (2.10) is found to be different from P^μ and to be given by

$$\Omega^\mu = \omega^\mu - p^\mu, \tag{2.11}$$

in which we used again the convenient definition of p^μ, equation (2.5), and where ω^μ represents another intermediate quantity given by

$$\omega^\mu = \frac{u^\mu}{c^2} \left(\frac{\dot{\pi}_\nu \dot{\pi}^\nu}{4m} + V \right) - \frac{u_\nu \pi^\nu}{mc^2} F^\mu. \tag{2.12}$$

At this point, we observe that the complete dynamics of the dipolar particle is encoded into two equations (2.6) and (2.10).

Next we obtain the particle's stress–energy tensor by varying the action (2.1) with respect to an infinitesimal change in the background metric, $\delta g_{\mu\nu}$, vanishing at the edges of the spacetime manifold, when $|x^\mu| \to \infty$. Obviously, we must take into account all metric contributions, including crucially those arising from the Christoffel symbols in the covariant time derivative $\dot{\pi}^\mu$ and those coming from $\perp_{\mu\nu} = g_{\mu\nu} + u_\mu u_\nu/c^2$. The conserved number density n of the dipolar particles satisfies the covariant continuity equation

$$\nabla_\nu(nu^\nu) = 0. \tag{2.13}$$

By straightforward calculations—see the appendix—we find that the stress–energy tensor $T^{\mu\nu}$ (with the dimension of an energy density) of the dipolar particles can be expressed in terms of the two basic linear momenta Ω^μ and P^μ as

$$T^{\mu\nu} = n\Omega^{(\mu} u^{\nu)} - \frac{1}{2m} \nabla_\rho(n[\pi^\rho P^{(\mu} - P^\rho \pi^{(\mu}] u^{\nu)}). \tag{2.14}$$

We readily verify that the covariant conservation law,

$$\nabla_\nu T^{\mu\nu} = 0, \tag{2.15}$$

holds as a consequence of the equations of motion (2.6) and (2.10).

Some physical interpretation follows from the expression of $T^{\mu\nu}$. It is clear that the first term in (2.14) takes the form of a *monopolar* contribution, appropriate for a point-like particle having velocity u^μ and linear momentum Ω^μ. We thus see that the particles we are considering are not purely dipolar, since they also involve a monopolar contribution in this sense. The monopolar piece in the stress–energy tensor will generate a monopolar gravitational field via the Einstein field equations. So, the mass associated with the linear momentum Ω^μ can be naturally interpreted—since it represents the point-like source of a monopolar gravitational field—as the active gravitational mass of the particle (see (2.31) for the computation of this

---

[9] In the case of a spinning particle, the anti-symmetric spin tensor is given in terms of the dipole moment variable and the particle's linear momentum by $S^{\mu\nu} = \pi^{[\mu} P^{\nu]}/m$.

mass within a particular solution of the equations). Similarly Ω^μ can be referred to as the 'active linear momentum', while P^μ which enters the law of motion (2.6) should rather be regarded as the 'inertial linear momentum'. (Technically, Ω_μ represents the conjugate momentum of x^μ and P_μ the conjugate momentum of π^μ; see equations (A.11) in the appendix.)

The second term in equation (2.14) is clearly dipolar and represents the relativistic generalization of the quasi-Newtonian density of polarization $\rho_{\text{polar}} = -\partial_i \Pi^i$ in paper I— i.e. minus the divergence of the polarization vector Π^i. To emphasize the dipolar character of this term, we rewrite the stress–energy tensor as

$$T^{\mu\nu} = n\Omega^{(\mu} u^{\nu)} - c^2 \nabla_\rho \Pi^{\rho\mu\nu}, \tag{2.16}$$

where $\Pi^{\rho\mu\nu}$ can be called the polarization tensor and is given by

$$\Pi^{\rho\mu\nu} = \frac{n}{2mc^2} [\pi^\rho P^{(\mu} - P^\rho \pi^{(\mu}] u^{\nu)}. \tag{2.17}$$

The gravitational field generated by the distribution of dipolar particles will be computed in the standard way by adding the matter action (2.1) to the Einstein–Hilbert action for the gravitational field. Equivalently, we shall put the stress–energy tensor (2.14) on the RHS of the Einstein field equations of general relativity. (Obviously, we can also use the gravitational action and field equations of any favourite metric theory of gravity.)

Finally, let us exploit some constraint equations which are satisfied by the general solution of equations (2.6) and (2.10). First we note that a consequence of equation (2.10), implied by the antisymmetry of the Riemann tensor with respect to its first pair of indices, is

$$u_\mu \dot{\Omega}^\mu = 0. \tag{2.18}$$

Let us prove that in fact this relation is identically satisfied, in the sense that it is implied by the other equations we have. We recall that $\Omega^\mu = \omega^\mu - p^\mu$. Using definition (2.11) we first compute the contraction $u_\mu \dot{\omega}^\mu$ and reduce it thanks to the easy-to-check formula

$$\dot{V} = \frac{1}{m} F_\mu \dot{\pi}_\perp^\mu, \tag{2.19}$$

which follows from the fact that V is a function of $\pi_\perp/m$ only. In equation (2.19), we have used the expression of the quadri-force (2.8) and we have defined[10]

$$\dot{\pi}_\perp^\mu \equiv \frac{\mathrm{D}\pi_\perp^\mu}{\mathrm{d}\tau}. \tag{2.20}$$

At this stage we arrive at the intermediate result $u_\mu \dot{\omega}^\mu = -\dot{\pi}_\mu (\ddot{\pi}^\mu + 2F^\mu)/(2m)$. Next we have recourse to the other equation (2.6), and we recall the fact that $p_\mu \dot{p}^\mu = 0$ since the norm of p^μ is a constant, $p^2 = 4m^2 c^2$. This immediately yields $u_\mu \dot{\omega}^\mu = u_\mu \dot{p}^\mu$ which is indeed the constraint (2.18) we wanted to prove. So this constraint is consistent with our basic equations (2.6) and (2.10).

We now show that there is another constraint relation, which is a non-trivial consequence of our equations, and will be used below to find an interesting particular solution of those equations. This relation is obtained by contracting equation (2.6) with the 4-velocity. Because F^μ is orthogonal to the 4-velocity (cf equation (2.9)), we obtain

$$u_\mu \dot{P}^\mu = 0, \tag{2.21}$$

---

[10] Beware of the fact that since the motion of the particle will not be geodesic ($\dot{u}^\mu \neq 0$), the latter definition $\dot{\pi}_\perp^\mu \equiv \mathrm{D}[\perp_\nu^\mu \pi^\nu]/\mathrm{d}\tau$ is in general *different* from $\perp_\nu^\mu \mathrm{D}\pi^\nu/\mathrm{d}\tau$. Thus, one is not allowed to commute the operations of perpendicular projection $\perp_\nu^\mu$ and of taking the covariant time derivative $\mathrm{D}/\mathrm{d}\tau$.

which can easily be transformed, using the definition (2.7) of P^μ, into[11]

$$\dot\Lambda u_\mu p^\mu + (\Lambda - 1)u_\mu \dot p^\mu = 0.$$ (2.22)

The constraint relation (2.22) could be viewed as a differential equation for the quantity Λ which is defined by (2.5b). Since it is a consequence of our main equations (2.6) and (2.10), relation (2.22) is to be satisfied by any solutions of those equations.

2.3. Particular solution of the equations

The general dynamics of dipolar particles in the present approach is given by equations (2.6) and (2.10), whose consequence as we have just seen is equation (2.22). From now on we shall restrict our attention to a particular class of solutions owning some features that are consistent, as we shall see, with the intuitive idea of a dipole moment in 'equilibrium' in the gravitational field. This class of solutions is obtained by solving the constraint equation (2.22) in the simplest way that

$$\Lambda = 1.$$ (2.23)

This choice will greatly simplify the equations of motion (2.6) and (2.10) and make them quite attractive. We shall however leave open the question of how generic the solutions satisfying (2.23) are, and under which conditions (if any) could they be made unique[12]. In more detail, relation (2.23) reads

$$\frac{\dot\pi_\mu \dot\pi^\mu}{4m} - u_\mu \dot\pi^\mu = 2mc^2.$$ (2.24)

This can be regarded as an equation giving the timelike component of $\dot\pi^\mu$, which is parallel to the velocity u^μ (i.e. $u_\mu \dot\pi^\mu$), in terms of the spacelike components, perpendicular to u^μ. The solution of (2.24) is best expressed in terms of the vector p^μ (which is now $p^\mu = 2mu^\mu - \dot\pi^\mu$ because $\Lambda = 1$), and we get

$$u_\mu p^\mu / c = \varepsilon \sqrt{\perp_{\mu\nu} \dot\pi^\mu \dot\pi^\nu - 4m^2 c^2},$$ (2.25)

where $\varepsilon = \pm 1$ tells us whether p^μ is future or past directed.

Using equation (2.23) it will become clear that the linear momentum P^μ represents the flow of inertial or equivalently passive-type gravitational mass, while the other linear momentum Ω^μ is associated with some active-type gravitational mass. Furthermore, for the class of solutions satisfying (2.23), it will happen most remarkably that the 'physical' dipole moment, namely the one which appears in the final equations of motion and stress–energy tensor, is the projection orthogonal $\pi_\perp^\mu$ to the 4-velocity, rather than π^μ itself. Interestingly, we shall find that the longitudinal component, $u_\mu \pi^\mu$, which never appears in the final equations and is therefore unphysical (i.e. unobservable), is actually given by a complex number, see equation (3.10).

---

[11] We can also write the alternative equivalent expression

$$\dot\Lambda(2m + u_\mu \dot\pi^\mu) + \Lambda(\Lambda - 1)u_\mu \ddot\pi^\mu = 0.$$

[12] One could be tempted to replace $\Lambda = 1$ back into the original action (2.1), therefore defining the alternative action

$$\widetilde{S} = \int d\tau \left[-2mc^2 + \frac{\dot\pi_\mu \dot\pi^\mu}{4m} - V \right].$$

However, this action would not describe a dipolar particle in the sense we want. Notably since as we see $\widetilde{S}$ now involves a mass term in the ordinary sense (with mass $2m$), its stress–energy tensor will contain some unwanted 'monopolar' mass contribution $\sim mnu^\mu u^\nu$.

From equations (2.5) and (2.7) we find that when $\Lambda = 1$ the linear momentum P^μ is simplified to the following vector, which is timelike,

$$P^\mu = 2mu^\mu. \tag{2.26}$$

Then (2.21) is satisfied simply because the quadri-norm of u^μ is constant. Thus, the equation of motion (2.6) gives the particle's quadri-acceleration $a^\mu \equiv \dot{u}^\mu$ as

$$2ma^\mu = -2F^\mu. \tag{2.27}$$

Since the mass coefficient $2m$ is in factor of the acceleration (which incorporates both inertial and gravitational effects), it can equivalently be interpreted as the particle's inertial mass and passive gravitational mass, $M_i = M_p = 2m$; in this sense the equivalence principle is satisfied. However, the motion is not geodesic as a result of the force F^μ which is supposed to reflect the internal structure of the dipolar particle.

Let us reduce next the expression of Ω^μ, defined by (2.11) and (2.12). To this end, we make use of equation (2.24) to obtain first the alternative expression

$$\Omega^\mu = \frac{u^\mu}{c^2}[V + u_\nu \dot{\pi}^\nu] + \dot{\pi}^\mu - \frac{u_\nu \pi^\nu}{mc^2}F^\mu. \tag{2.28}$$

The point is that we can express Ω^μ entirely in terms of the orthogonal projection $\pi_\perp^\mu = \perp_\nu^\mu \pi^\nu$. We replace $\dot{\pi}^\mu$ in (2.28) by its equivalent expression in terms of the time derivative of $\pi_\perp^\mu$, namely $\dot{\pi}_\perp^\mu = D\pi_\perp^\mu/d\tau$ already defined in (2.20), and which we recall is different from the alternative object $(\dot{\pi}^\mu)_\perp = \perp_\nu^\mu \dot{\pi}^\mu$. An easy computation, also using equation (2.27), brings then Ω^μ into the simple form

$$\Omega^\mu = \frac{V}{c^2}u^\mu + \perp_\nu^\mu \dot{\pi}_\perp^\nu, \tag{2.29}$$

displaying the longitudinal versus perpendicular decomposition of Ω^μ with respect to the 4-velocity. Note that in the second term of (2.29) the orthogonal projector $\perp_\nu^\mu$ appears two times: one explicitly in front of the term and other contained into $\pi_\perp^\nu$. Expression (2.29) makes it clear that

$$u_\mu \Omega^\mu + V = 0. \tag{2.30}$$

Motivated by the fact that Ω^μ parametrizes the 'mass term' or 'monopole part' of the stress–energy tensor (2.14), we define the particle's *active* gravitational mass M_a—namely the mass which 'actively' generates the gravitational field—as the coefficient of the velocity u^μ in equation (2.29). More precisely, M_a is defined by the longitudinal part of Ω^μ along u^μ as[13]

$$M_a \equiv -\frac{1}{c^2}u_\mu \Omega^\mu = \frac{V}{c^2}. \tag{2.31}$$

This mass is not conserved because of the work done by the internal force F^μ, and we find (consistently with equation (2.19))

$$\dot{M}_a = \frac{1}{mc^2}F_\mu \Omega^\mu. \tag{2.32}$$

With this notation equation (2.29) can be viewed as the classic relation between the particle's linear momentum Ω^μ and the 4-velocity u^μ, namely $\Omega^\mu = M_a u^\mu + $(dipolar effects). A similar relation holds in the case of spinning particles, where the linear momentum and velocity differ by spin effects [21–23].

---

[13] Such a definition is proposed here for heuristic discussion, but is not used directly in the formalism.

The evolution equation (2.10) reads now[14]

$$\dot{\Omega}^{\mu} \equiv \frac{\mathrm{D}}{\mathrm{d}\tau}\left[\frac{V}{c^2}u^{\mu}+ \perp_{\nu}^{\mu} \dot{\pi}_{\perp}^{\nu}\right] = -\pi_{\perp}^{\nu} R^{\mu}_{\ \rho\nu\sigma}u^{\rho}u^{\sigma}, \tag{2.33}$$

where we have taken advantage of the symmetries of the Riemann tensor to replace π^{ν} by $\pi_{\perp}^{\nu}$ on the RHS. This point is not without interest because we discover that the equation depends only on the orthogonal projection $\pi_{\perp}^{\mu}$—indeed recall that V itself is a function of $\pi_{\perp} = \sqrt{g_{\mu\nu}\pi_{\perp}^{\mu}\pi_{\perp}^{\nu}}$. In section 3.1, we shall interpret the non-relativistic limit of equation (2.33) as an equilibrium condition for the dipole moment in a gravitational field.

Finally, the stress–energy tensor for the class of solutions satisfying $\Lambda = 1$ is given by

$$T^{\mu\nu} = n\left[\frac{V}{c^2}u^{\mu}u^{\nu} + u^{(\mu} \perp_{\rho}^{\nu)} \dot{\pi}_{\perp}^{\rho}\right] - c^2\nabla_{\rho}\Pi^{\rho\mu\nu}, \tag{2.34}$$

where the polarization tensor (2.17) reads now

$$\Pi^{\rho\mu\nu} = \frac{n}{c^2}\left[\pi_{\perp}^{\rho}u^{(\mu} - u^{\rho}\pi_{\perp}^{(\mu}\right]u^{\nu)}. \tag{2.35}$$

Here also we have been able to replace π^{μ} by $\pi_{\perp}^{\mu}$, thanks to the antisymmetry of the two terms in (2.35).

To conclude, the choice of solution (2.23) enables one to appreciably simplify the equations and to ease their interpretation. A consequence is that the number of independent components of the dipole moment is reduced from four down to three. The component of the dipole moment that is along the 4-velocity (namely $u_{\mu}\pi^{\mu}$) never appears in the final equations and is unobservable. The physical dipole moment is entirely described by the orthogonal projection variable $\pi_{\perp}^{\mu}$, which is a *spacelike* vector.

3. Dipolar particles in the non-relativistic limit

3.1. Quasi-Newtonian equations

We investigate the non-relativistic (NR) approximation of the dynamics of dipolar particles, as described by the solution found in section 2.3, characterized by the equations of motion (2.27) and (2.33) and by the stress–energy tensors (2.34) and (2.35). To proceed, we explicitly write all factors of c's, and consider the formal limit when $c \to \infty$, which is equivalent to the usual $v/c \to 0$, where v is the typical value of the coordinate velocity of the dipolar particles. In the following, we systematically indicate the neglected remainder terms and write them as some $\mathcal{O}(c^{-n})$.

We suppose that the dipolar particles evolve in the gravitational field of some ordinary matter system with spatially compact support. We introduce a Cartesian coordinate grid $\{t, x^i\}$ (with $t = x^0/c$) and we choose it to be inertial, i.e. without rotation no acceleration with respect to some averaged cosmological matter distribution at large distances from the local matter system. In these coordinates, the metric is asymptotically flat (the local matter system is freely falling in the cosmological background field).

In the NR limit, the gravitational field is described by a single Newtonian-like potential U, whose source will be the sum of the Newtonian densities of the ordinary matter and of the dipolar particles. Such potential will satisfy a Poisson equation coming from the NR limit of

---

[14] In more detail this equation can also be written as (making use of equations (2.19) and (2.27))

$$\left(V + u_{\nu}\dot{\pi}_{\perp}^{\nu}\right)\frac{a^{\mu}}{c^2} + \perp_{\nu}^{\mu} \ddot{\pi}_{\perp}^{\nu} = -\pi_{\perp}^{\nu} R^{\mu}_{\ \rho\nu\sigma}u^{\rho}u^{\sigma}.$$

the Einstein field equations. In the usual notation U has the dimension of a velocity squared and is of order one when $c \to \infty$, which we denote by $U = \mathcal{O}(c^0)$. The metric coefficients are given by

$$g_{00} = -1 + \frac{2U}{c^2} + \mathcal{O}\left(\frac{1}{c^4}\right), \tag{3.1a}$$

$$g_{0i} = \mathcal{O}\left(\frac{1}{c^3}\right), \tag{3.1b}$$

$$g_{ij} = \delta_{ij}\left(1 + \frac{2U}{c^2}\right) + \mathcal{O}\left(\frac{1}{c^4}\right). \tag{3.1c}$$

This metric is accurate enough to obtain the Poisson equation satisfied by U in the NR approximation and to discuss the motion of massive particles (ordinary stars) as well as relativistic particles (ordinary photons) in the gravitational field.

The 4-velocity u^μ of the dipolar particle is written as $u^\mu = (u^0, u^0 v^i/c)$, where $v^i = \mathrm{d}x^i/\mathrm{d}t$ is the coordinate velocity and $u^0 = c\mathrm{d}t/\mathrm{d}\tau$. Since the particle will be non-relativistic, v^i is of order unity when $c \to \infty$, i.e. $v^i = \mathcal{O}(c^0)$. In the NR limit, the proper time reduces to the coordinate time, $\mathrm{d}\tau = \mathrm{d}t + \mathcal{O}(c^{-2})$, hence $u^0 = c + \mathcal{O}(c^{-1})$ and $u^i = v^i + \mathcal{O}(c^{-2})$. The particle's quadri-acceleration, given by $a^\mu = \mathrm{d}u^\mu/\mathrm{d}\tau + \Gamma^\mu_{\rho\sigma} u^\rho u^\sigma$, is then found to reduce in the NR limit to $a^0 = \mathcal{O}(c^{-1})$ and $a^i = \mathrm{d}^2 x^i/\mathrm{d}t^2 + c^2 \Gamma^i_{00} + \mathcal{O}(c^{-2})$, where $\mathrm{d}^2 x^i/\mathrm{d}t^2$ is the ordinary coordinate acceleration. The Christoffel symbol reads $\Gamma^i_{00} = -c^{-2}g^i + \mathcal{O}(c^{-4})$, where $g^i = \partial_i U$ is the Newtonian gravitational field, so we have

$$a^i = \frac{\mathrm{d}^2 x^i}{\mathrm{d}t^2} - g^i + \mathcal{O}\left(\frac{1}{c^2}\right). \tag{3.2}$$

(Of course, geodesic motion simply means that $a^i = 0$ hence $\mathrm{d}^2 x^i/\mathrm{d}t^2 = g^i + \mathcal{O}(c^{-2})$.)

Consider the internal force F^μ, which is defined by equation (2.8) and appears on the RHS of the law of motion (2.27). As a basic (and quite natural) hypothesis, we impose that this force exists in the NR limit, in the sense that its *spatial* components F^i admit a non-zero and finite limit when $c \to \infty$:

$$F^i = \mathcal{O}(c^0). \tag{3.3}$$

Since F^μ is orthogonal to the velocity, equation (2.9), its zeroth component F^0 vanishes in the NR limit, $F^0 = \mathcal{O}(c^{-1})$ (and more precisely we have $F^0 = v^i F^i/c + \mathcal{O}(c^{-3})$). It is now clear that the dipolar particle's law of motion (2.27) reduces in the NR approximation to the non-geodesic equation[15]

$$m\frac{\mathrm{d}^2 x^i}{\mathrm{d}t^2} = mg^i - F^i + \mathcal{O}\left(\frac{1}{c^2}\right). \tag{3.4}$$

We emphasize that $g^i = \partial_i U$ represents here the *total* gravitational field, which is sourced not only by the ordinary matter but also by the dipolar particles themselves.

Consistently with the order of magnitude (3.3) we assume that the potential function V from which derives the force F^i is also of the same order,

$$V = \mathcal{O}(c^0). \tag{3.5}$$

Equation (3.5) means that we preclude any constant term in the expression of V, which would be of the order of $\mathcal{O}(c^2)$ and therefore be of the form $\mathcal{M}c^2$, where $\mathcal{M}$ is some constant mass

---

[15] Since both a^0 and F^0 are of order $\mathcal{O}(c^{-1})$ we see that the zeroth component of the law of motion is also satisfied in the NR limit.

parameter. Such a term corresponds to a mass term in the action (2.1) and would imply that the dipolar particle is endowed not only by the dipole moment π^μ (and its associated mass m), but also by the mass $\mathcal{M}$ in the ordinary sense—dark matter particles would carry some mass in the ordinary sense. This $\mathcal{M}$ yields an unwanted monopolar contribution to the stress–energy tensor (non-vanishing in the NR limit), so we simply pose $\mathcal{M} = 0$. Related to this we note that equation (3.5) implies that the active gravitational mass $M_a = V/c^2$ we defined in (2.31) vanishes in the NR limit:

$$M_a = \mathcal{O}\left(\frac{1}{c^2}\right). \tag{3.6}$$

In the NR limit, the dipolar particle has zero active-type gravitational mass, hence its stress–energy tensor is purely dipolar, in agreement with the elementary intuition from paper I.

Consider next the other equation (2.33). Our basic dipole moment variable $\pi_\perp^\mu$ is assumed to be such that (consistently with (3.3) and (3.5))

$$\pi_\perp^i = \mathcal{O}(c^0), \tag{3.7}$$

hence $\pi_\perp^0 = \mathcal{O}(c^{-1})$ which follows from the orthogonality to the velocity. The fact that $\pi_\perp^0$ vanishes in the NR limit is crucial. It implies that the four norm of $\pi_\perp^\mu$ reduces to the Euclidean norm of $\pi_\perp^i$ in the NR limit: $\pi_\perp = \left[\delta_{ij}\pi_\perp^i\pi_\perp^j\right]^{1/2} + \mathcal{O}(c^{-2})$. Hence, the force F^i is a function only of $\pi_\perp^j$ in this limit[16]. We need the covariant proper time derivative of $\pi_\perp^\mu$ which is given by $\dot\pi_\perp^\mu = \mathrm{d}\pi_\perp^\mu/\mathrm{d}\tau + \Gamma_{\nu\rho}^\mu u^\nu \pi_\perp^\rho$. Because $\pi_\perp^0$ vanishes in the NR limit, we readily find that $\dot\pi_\perp^\mu$ is not coupled to gravity in this limit, in the sense that the Christoffel symbols make a contribution which is of higher order. Hence, we find that $\dot\pi_\perp^0 = \mathcal{O}(c^{-1})$, and most importantly that the spatial components $\dot\pi_\perp^i$ reduce to the *ordinary* time derivative, namely $\dot\pi_\perp^i = \mathrm{d}\pi_\perp^i/\mathrm{d}t + \mathcal{O}(c^{-2})$. The linear momentum (2.29) then becomes $\Omega^0 = \mathcal{O}(c^{-1})$ and $\Omega^i = \mathrm{d}\pi_\perp^i/\mathrm{d}t + \mathcal{O}(c^{-2})$. Applying the same reasoning we find that $\dot\Omega^0 = \mathcal{O}(c^{-1})$ and $\dot\Omega^i = \mathrm{d}^2\pi_\perp^i/\mathrm{d}t^2 + \mathcal{O}(c^{-2})$. Finally, the Riemann curvature tensor on the RHS of (2.33) yields a coupling to the tidal gravitational field $\partial_{ij}U$. Therefore, we have proved that the NR limit of this equation is

$$\frac{\mathrm{d}^2\pi_\perp^i}{\mathrm{d}t^2} = \pi_\perp^j \partial_{ij}U + \mathcal{O}\left(\frac{1}{c^2}\right). \tag{3.8}$$

The law of evolution given by equation (3.8) is interesting for our purpose because it can be regarded as a condition of equilibrium for the dipole moment in a gravitational field. It states that when the tidal gravitational field can be neglected, the components of the dipole moment $\pi_\perp^i$ stay constant (or evolve linearly in time). Note that equation (3.8) is formally identical with the equation of *geodesic deviation* for the (spacelike) 'separation' vector $\pi_\perp^\mu$ in the NR limit. However, because of other terms in equation (2.33), the equation of geodesic deviation will not hold outside this limit.

Thus, our equations are the law of motion (3.4) and the equilibrium condition (3.8). Let us emphasize that these equations have a structure different from those of paper I, which were concocted by analogy with the model of a dipole moment in electrostatics, and in particular violate the equivalence principle. In the present relativistic description of the dipole, we are consistent with the equivalence principle, and as a result the law of motion (3.4) is different from equation (12) in paper I. We also obtain directly an equilibrium condition (3.8) for the dipole moment, instead of the evolution equation (13) of paper I. Amazingly, we find that in

---

[16] More precisely we should write that $F^i[\pi_\perp^\mu] = \tilde{F}^i[\pi_\perp^j] + \mathcal{O}(c^{-2})$, but in the following we shall identify, by a slight abuse of notation, the functional $\tilde{F}^i$ with the original one F^i.

order to reproduce the present equations (3.4) and (3.8) of the (NR limit of the) relativistic model, the two RHS of equations (12) and (13) in paper I should exactly be interchanged.

As we pointed out, because of the constraint relation (2.23), the end equations depend only on the orthogonal projection of the dipole moment $\pi_\perp^\mu$ (or $\pi_\perp^i$ in the NR limit). Still it is interesting to control the original, unprojected dipole moment variable π^μ which entered into the action (2.1). We do it here, by considering equation (2.25) in the NR limit. From the results $\dot{\pi}^0 = \mathcal{O}(c)$ and $\dot{\pi}^i = \mathcal{O}(c^0)$, we find that $\perp_{\mu\nu} \dot{\pi}_\perp^\mu \dot{\pi}_\perp^\nu = \mathcal{O}(c^0)$ (because of the orthogonal projection), hence this term is negligible with respect to the mass term in equation (2.25). This equation thus becomes $u_\mu p^\mu = 2\varepsilon \mathrm{i} m c^2 + \mathcal{O}(c^0)$ where $\mathrm{i} = \sqrt{-1}$ is the imaginary number (and $\varepsilon = \pm 1$). The appearance of a complex quantity is due to the spacelike character of p^μ (which satisfies $p^2 = 4m^2c^2$). Now we further deduce $p^0 = -2\varepsilon \mathrm{i} mc + \mathcal{O}(c^{-1})$ and $p^i = \mathcal{O}(c^0)$ in the NR limit. The interesting finding is that the components of π^μ are seen to be *complex* in the NR approximation for that solution and given by

$$\pi^0 = 2mc\,(1 + \varepsilon \mathrm{i})\,t + \mathcal{O}\left(\frac{1}{c}\right), \tag{3.9a}$$

$$\pi^i = \pi_\perp^i + 2mv^i\,(1 + \varepsilon \mathrm{i})t + \mathcal{O}\left(\frac{1}{c^2}\right). \tag{3.9b}$$

In particular, the longitudinal component along the velocity is

$$u_\mu \pi^\mu = -2mc^2(1 + \varepsilon \mathrm{i})t + \mathcal{O}(c^0). \tag{3.10}$$

However, the components of π^μ itself do not appear into the final equations of motion and stress–energy tensor and are therefore unphysical; only the components of $\pi_\perp^\mu$, which parametrize the final equations, represent the physical variables, and these are (to be considered as) real.

We compute the stress–energy tensors (2.34) and (2.35) in the NR limit. In such a limit the number density n of dipolar particles satisfies the Eulerian continuity equation,

$$\partial_t n + \partial_i(nv^i) = \mathcal{O}\left(\frac{1}{c^2}\right). \tag{3.11}$$

From equation (3.5) (or equivalently (3.6)), together with the fact that $\dot{\pi}_\perp^0 = \mathcal{O}(c^{-1})$, we deduce that the monopolar term in equation (2.34) is zero in the NR limit so that (most satisfactorily) the stress–energy tensor becomes purely dipolar. The covariant derivative of the polarization tensor can be approximated by an ordinary derivative, so the components of $T^{\mu\nu}$ are simply obtained as

$$T^{00} = -c^2 \partial_i \Pi^i + \mathcal{O}(c^0), \tag{3.12a}$$

$$T^{0i} = \mathcal{O}(c), \tag{3.12b}$$

$$T^{ij} = \mathcal{O}(c^0), \tag{3.12c}$$

in which the density of dipole moment or polarization vector is defined by

$$\Pi^i = n\pi_\perp^i. \tag{3.13}$$

It is clear from equations (3.12) that since $T^{\mu\nu}$ of dipolar particles is to be added to the one of the other matter fields, the only effect of the dipolar particles in the NR limit is to add to the Newtonian density of the ordinary matter the density of polarization

$$\rho_{\text{polar}} = -\partial_i \Pi^i. \tag{3.14}$$

The Einstein field equations reduce in the NR limit to the Poisson equation for the potential U defined by the metric coefficients (3.1). The point is that the source of the Poisson equation is the *total* matter density; therefore, we have proved that in the NR limit

$$\Delta U = -4\pi G\left(\rho + \rho_{\text{polar}}\right) + \mathcal{O}\left(\frac{1}{c^2}\right), \tag{3.15}$$

where ρ denotes the Newtonian density of the ordinary (monopolar) matter while ρ_{polar} is the dipolar matter density found in (3.14).

The gravitational field, given by the metric (3.1) where U is solution of (3.15), affects the dynamics of any matter distribution. In the case of the dipolar particles, we have already derived the equation of motion in equation (3.4). For non-relativistic ordinary particles ('ordinary stars' assimilated as point masses) we have the standard acceleration law, coming from the universal coupling to gravity and the geodesic equation, namely

$$\left(\frac{\mathrm{d}^2 x^i}{\mathrm{d}t^2}\right)_{\text{ordinary}} = g^i + \mathcal{O}\left(\frac{1}{c^2}\right). \tag{3.16}$$

In the case of the motion of *relativistic* ordinary particles (photons), we find that the equation of motion is exactly given by what general relativity predicts, i.e.

$$\left(\frac{\mathrm{d}^2 x^i}{\mathrm{d}t^2}\right)_{\text{photon}} = [1 + \beta^j \beta^j] g^i - 4\beta^i \beta^j g^j + \mathcal{O}\left(\frac{1}{c}\right), \tag{3.17}$$

where the velocity of the photon $c\beta^i \equiv (\mathrm{d}x^i/\mathrm{d}t)_{\text{photon}}$ satisfies $\beta^j \beta^j (1 + 2U/c^2) = 1 - 2U/c^2$. Indeed, the result (3.17) is the consequence of the metric (3.1)[17] and the expression of the stress–energy tensor (3.12). However, recall that $g^i = \partial_i U$ where U is solution of equation (3.15), so the light deflection of photons is given by the usual formula (3.17), but in which U is generated by the ordinary stars *and* also by the distribution of dipolar particles themselves.

The latter point on light deflection constitutes an attractive feature of the present model of dipolar dark matter. Indeed, the distribution of dark matter is felt not only by ordinary stars (and gas) moving around galaxies, but also by photons, as is evidenced in experiments probing the large-scale structure using light deflection and amplification (weak lensing experiments [34]). The problem of light deflection turned out to be crucial in the construction of relativistic MOND theories based on extra fields besides the metric tensor of general relativity [14–16]. To account for the observed strong light bending in lensing experiments, one is obliged in these theories to modify the standard conformal coupling of a scalar field to matter by means of some vector field (preferred or dynamical) especially designed for this purpose [14–16]. The resulting tensor–scalar–vector theory is inevitably complicated. In the present model, by contrast, we have derived in a natural way the formula for the light bending (3.17), which takes into account, via the Poisson equation (3.15), the effect due to the distribution of dipolar dark matter.

To summarize this section, the equations giving the dynamics of the fluid of dipolar particles in the NR limit are

(i) their equation of motion (3.4);
(ii) an 'equilibrium' condition for the dipole moments (3.8);
(iii) the conservation of the number of particles or continuity equation (3.11);
(iv) the Poisson equation (3.15) for the gravitational field.

In addition, we have equations (3.16) and (3.17) for the motion of ordinary matter and photons, respectively.

[17] We assumed that the spatial metric g_{ij} is the general relativistic one, with post-Newtonian parameter $\gamma = 1$. Obviously, the calculation could be done for a general value of the parameter γ.

3.2. Link with the modified Newtonian dynamics

Having arrived at the Poisson equation (3.15), whose source contains the density of polarization ρ_{polar} given by equation (3.14), we apply the arguments of paper I and recover the MOND equation [5–7] when the polarization vector Π^i is aligned with the gravitational field. We pose

$$\Pi^i = -\frac{\chi}{4\pi G} g^i, \tag{3.18}$$

and interpret χ, following paper I, as a coefficient of 'gravitational susceptibility' reflecting the properties of the dipolar medium. We assume that χ depends on the norm $g = |g^i|$ of the gravitational field, in analogy with the electric susceptibility of a dielectric material which depends on the norm of the electric field. Next we define

$$\mu = 1 + \chi. \tag{3.19}$$

Without loss of generality, we can scale g by means of some constant acceleration a_0, so that (3.19) defines a certain function $\mu(g/a_0)$. The Poisson equation (3.15) then becomes equivalent to the MOND equation (which has the form of a modified Poisson equation[18])

$$\partial_i(\mu g^i) = -4\pi G\rho. \tag{3.20}$$

To account with astronomical observations we know that the Newtonian law (without dark matter) must be valid for strong enough (though non-relativistic) gravitational fields, much above the constant acceleration scale a_0. Thus, $\mu \sim 1$ when $g \gg a_0$. The fact that the acceleration scale is the relevant one is not trivial but was found early on by Milgrom [5–7]. Indeed, a remarkable fit to many observations of rotation curves of galaxies has been achieved by assuming that $\mu \sim g/a_0$ in the regime of weak gravitational fields, for $g \ll a_0$. The acceleration scale a_0 is empirically found to be of the order of 10^{-10} m s^{-2} (the same numerical value, of course, for all galaxies). Interestingly, the numerical value of a_0 is close to the Hubble scale, $a_0 \approx cH_0$.

Interpolating between the Newtonian and MOND regimes we have $0 \leqslant \mu < 1$, hence the gravitational susceptibility defined by equation (3.19) must be negative:

$$-1 \leqslant \chi < 0. \tag{3.21}$$

This fact was interpreted in paper I within a model consisting of a pair of sub-particles with positive inertial masses but opposite gravitational masses $\pm m$ (in analogy with the electric dipole made of two charges $\pm q$). The negative sign in (3.21) was then seen to reflect the fact that gravity is governed by a *negative* Coulomb law in the NR limit—like masses attract and unlike ones repel [35]. As a result, the polarization masses tend to *increase* the magnitude of the gravitational field, by an effect which can be referred to as gravitational 'anti-screening', and is opposite to the usual screening of electric charges by polarization charges in electrostatics. The negative sign of χ was emphasized in paper I as the main argument for viewing MOND as a mechanism of gravitational polarization.

In principle, our task would be to justify the proportionality relation between the polarization Π^i and the gravitational field g^i, equation (3.18). In the model of paper I, we invoked an equilibrium in which the distance between the sub-particles constituting the dipole remains constant. In the present paper, we have to modify the argument because the equations of the relativistic model (in the NR limit) are different from those of paper I. Essentially, we shall find that when the tidal gravitational field can be neglected there is a solution for which the dipole moments are at rest with respect to the local matter distribution.

---

[18] We are adopting the variant of the MOND equation which is derivable from a non-relativistic Lagrangian [12]. Henceforth we no longer indicate the neglected remainder term $\mathcal{O}(c^{-2})$.

For this solution, the internal force F^i must compensate exactly for the gravitational force. When this is realized we find that the dipole moment is aligned with the gravitational field.

Suppose that the fluid of dipole moments fills an asymptotically flat spacetime generated by the local distribution of ordinary matter. When choosing such an asymptotically flat spacetime we implicitly assume that the isolated matter system is freely falling in the cosmological gravitational field generated by far-distant masses in the universe. The MOND regime will take place in the region far from the isolated system where gravity is weak, so we expect that a good approximation is to neglect the tidal gravitational field:

$$\partial_{ij} U \approx 0. \tag{3.22}$$

The discussion which follows is based on this approximation. More detailed calculations taking into account the tidal gravitational field will be performed in section 4.

Equation (3.8) tells us that when (3.22) holds $\pi_\perp^i$ is constant or varies linearly with time. Discarding any linear variation in time, we assume it to be constant,

$$\pi_\perp^i \approx \text{const}, \tag{3.23}$$

where the approximation sign $\approx$ reminds us that this is true when the tidal gravitational field is neglected. Now we have seen that the force F^i is a function of $\pi_\perp^i$ in the NR limit, so it must also be constant, $F^i \approx \text{const}$. Similarly for the coordinate acceleration of the dipolar particle given by equation (3.4), $d^2 x^i / dt^2 \approx \text{const}$. Our basic motive is that we shall *assume*, without justification if it were not by the existence of the solution, that the coordinate acceleration of the dipolar particles is everywhere approximately *zero*:

$$\frac{d^2 x^i}{dt^2} \approx 0. \tag{3.24}$$

This is consistent with the expectation that the dipole moments stay at rest with respect to some averaged cosmological matter distribution on cosmological scales. In this view, the fluid of dipolar particles appears essentially to be an immobile (static) 'ether', weakly influenced by the ordinary matter distribution, in agreement with the model of paper I. Substituting (3.24) into the law of motion (3.4) we obtain

$$F^i \approx mg^i, \tag{3.25}$$

so in this solution the internal force F^i is equal to the weight mg^i, exactly like in paper I. Note that equations (3.24) and (3.25) actually say that the particle is accelerated in a quadri-dimensional sense, because its motion is non-geodesic (recall $ma^\mu = -F^\mu$). The quadri-force F^μ acts like a rocket to accelerate the dipolar particle and hold it at rest in the gravitational field.

Because F^i is proportional to $\pi_\perp^i$ (see equation (2.8)), the result (3.25) implies that $\pi_\perp^i$ and hence the polarization vector $\Pi^i = n\pi_\perp^i$ are aligned with the gravitational field. The proportionality relation (3.18) is therefore justified and we have verified the MOND equation (3.20). Following paper I, we then look for the expression of F^i that corresponds to the specific MOND regime where $\mu \sim g/a_0$ and obtain it in the form of an expansion when $\pi_\perp \to 0$:

$$F^i \approx km\pi_\perp^i \left[1 + \frac{k}{a_0} \pi_\perp + \mathcal{O}\left(\pi_\perp^2\right) \right]. \tag{3.26}$$

Such expansion can be viewed as a short-distance expansion in the separation $x = \pi_\perp/m$ between the dipole's constituents. We have posed $k = 4\pi Gn$ which is assumed here to be

constant and uniform[19]. The potential function V from which derives the latter force reads (in the approximation where k is independent of $\pi_\perp^i$)

$$V \approx \frac{k}{2}\pi_\perp^2 \left[1 + \frac{2}{3}\frac{k}{a_0}\pi_\perp + \mathcal{O}\left(\pi_\perp^2\right) \right]. \qquad (3.27)$$

It is noteworthy that this potential takes the form of a harmonic oscillator, i.e. is quadratic at dominant order when $\pi_\perp \to 0$ (see paper I for discussion). In the present formalism, the MOND acceleration scale a_0 appears to be related via equations (3.26) and (3.27) to the properties of the internal dipolar interaction at short distances.

To summarize, the present discussion confirms the relation between MOND and a specific form of dipolar dark matter. Admittedly this relation is striking—cf the electrostatic analogy for the MOND equation and the polarization density (3.14), the correct sign of the susceptibility coefficient (3.21), the harmonic form (3.27) of the potential V. Dark matter could consist of a fluid of dipole moments, polarized in the gravitational field of ordinary galaxies in the MOND regime where $g \ll a_0$ (but inactive in the regime of Newtonian gravitational fields where $g \gg a_0$). This fluid of dark matter would essentially be static with respect to the averaged cosmological matter distribution at large scales.

4. Dipolar particles in a central gravitational field

The previous section was restricted to the case where the tidal gravitational field is negligible, equation (3.22). Though probably a good approximation, this restriction hides the way the complete equations could be integrated in more realistic situations (possibly using numerical methods). Recall from section 3.1 that the dynamics of dipolar particles in the NR limit is given by the equation of motion (3.4), the equation of evolution (3.8), the continuity equation (3.11) and the Poisson equation (3.15). In the present section, we integrate those equations in the particular case where the gravitational field is generated by a point mass M (made of ordinary matter).

We shall assume that (1) the dipole moments are aligned with the gravitational field and (2) the fluid of dipolar particles is *stationary* with a purely radial flow. With those hypotheses, we shall basically show that there is a one-to-one correspondence between the MOND function $\mu(g/a_0)$ and the potential function $V(\pi_\perp/m)$ describing the 'internal physics' of the dipole moment. Note that by (1) we *assume* that the dipolar medium is polarized in the gravitational field but do not *prove* it[20]. However, we shall find a consistent solution which describes the polarization state of this medium—this partly justifies the assumption.

The gravitational field generated by M is central. We denote by $r = |x^i|$ the radial distance to the mass, by $U(r)$ the gravitational potential and by $g^i = \partial_i U = U'(r)n^i$ the gravitational field, where $n^i = x^i/r$ is the direction to the observer and the prime denotes the derivative with respect to r. We have $U'(r) < 0$ and $g = |g^i| = -U'(r)$. By assumption (1) the dipole moment is aligned with g^i so we pose $\pi_\perp^i = -\pi_\perp n^i$, where the minus sign reflects the fact that the moment should, like g^i, be directed towards M, in conformity with the sign of equation (3.21). Here, $\pi_\perp$ is the Euclidean norm of $\pi_\perp^i$; for simplicity we keep the same name as for the four norm (2.4) because both agree in the NR limit. Similarly, the internal force F^i,

---

[19] Thus, equation (3.26) depends on the space density n of particles. Note however that (3.26) represents the value of the force at equilibrium, when the equilibrium condition (3.25) is satisfied. This force is computed from the internal force F^μ defined in general by equation (2.8), which constitutes a definition *intrinsic* to the dipole, i.e. valid for the single dipole moment independently of n. Thus, F^μ is derived from the part of the action (2.1) corresponding to a single particle, without reference to n.

[20] This could be viewed as an indication that the model is more 'effective' than fundamental (like some models of dielectrics where the electric polarization is parallel to the electric field [19]).

which is parallel to the dipole moment and in the same direction, is written as $F^i = -Fn^i$. Thus, equation (2.8) becomes

$$F = m \frac{dV}{d\pi_\perp}.$$
(4.1)

By assumption (2) the velocity of the dipolar particles is radial, so we write it as $v^i = dx^i/dt = n^i dr/dt$ (such a hypothesis will partly be justified by the consistency of the solution). With these notations, equation (3.8) becomes

$$\frac{d^2\pi_\perp}{dt^2} = \pi_\perp U'',$$
(4.2)

where U'' is the second derivative with respect to r, while equation (3.4) reads

$$\frac{d^2r}{dt^2} = \frac{dV}{d\pi_\perp} - g.$$
(4.3)

Let us now adopt an Eulerian description of the dipolar fluid. For a stationary fluid there is no dependence on time, and in the central potential the norms of the velocity and dipole moment depend only on the radial distance: $v(r)$ and $\pi_\perp(r)$. Their total time derivatives are then given by the usual Eulerian derivative as $dr/dt = v$, then $dv/dt = vv'$ and $d\pi_\perp/dt = v\pi_\perp'$. Equation (4.2) is thus reduced to

$$\gamma\pi_\perp' + v^2\pi_\perp'' = \pi_\perp U'',$$
(4.4)

where $\gamma = vv'$ denotes the acceleration, which is itself given by equation (4.3) as

$$\gamma = \frac{dV}{d\pi_\perp} - g.$$
(4.5)

A consequence of the latter equation is

$$V' = \pi_\perp' \left(\frac{v^2}{2} - U \right)'.$$
(4.6)

We now eliminate γ between (4.4) and (4.5) to obtain

$$V' + U'\pi_\perp' + v^2\pi_\perp'' = \pi_\perp U'',$$
(4.7)

which can be transformed by further manipulations (using equation (4.6)) into an equation having the form of a 'conservation' law:

$$(V - v^2\pi_\perp' + \pi_\perp U')' = 0.$$
(4.8)

Upon integration of this conservation law we shall obtain an arbitrary constant (independent of r), i.e. $V - v^2\pi_\perp' + \pi_\perp U' = $ const. However, this constant can be absorbed into the definition of the potential V (which is anyway defined up to a constant), and we arrive at the simple result

$$V = \pi_\perp g + v^2\pi_\perp'.$$
(4.9)

Moreover, we have the independent equation (4.5) which we rewrite using (4.9) as

$$\frac{dV}{d\pi_\perp} = g + \frac{d}{dr}\left[\frac{V - \pi_\perp g}{2\pi_\perp'} \right].$$
(4.10)

The previous relations are valid for any central potential $U(r)$, and we want now to specify that $U(r)$ is generated by the point mass M. The density of ordinary matter on the RHS of the Poisson equation (3.15) is therefore $\rho = M\delta(x^i)$, where δ is the three-dimensional Dirac

function. Using also the density of polarization, (3.13) and (3.14), we find that the Poisson equation becomes

$$\partial_i \left(g^i - k\pi_\perp^i \right) = -4\pi G M \delta, \tag{4.11}$$

in which we employ the notation $k = 4\pi G n$ already used in section 3.2. In the present case of spherical symmetry, the latter equation is equivalent to

$$g - k\pi_\perp = \frac{GM}{r^2}, \tag{4.12}$$

where the RHS is made of the Newtonian potential for the mass M. From equation (3.18) we have $k\pi_\perp = -\chi g$, where the (negative) susceptibility coefficient χ is related to the MOND function by $\mu = 1 + \chi$. Thus,

$$g\mu = \frac{GM}{r^2}, \tag{4.13}$$

which is nothing but the MOND equation in the case of a point-mass source, as originally postulated in [5–7][21]. There is no surprise in recovering (4.13) because we have already shown on general grounds that the Poisson equation with a dipolar source term satisfying (3.18) is equivalent to the MOND equation (3.20). Equation (4.13) could also be directly deduced from (3.20). On the other hand, we still have the relation between $\pi_\perp$ and g which we prefer to write in terms of μ as

$$k\pi_\perp = (1 - \mu)g. \tag{4.14}$$

We deal next with the equation of conservation of the number of particles (3.11). This equation is easy to solve because the fluid is stationary, the velocity field is purely radial and we are in spherical symmetry. It suffices to say that the flow of dipolar particles crossing the surface of the sphere $S = 4\pi r^2$ is constant, which means that $nvS = \widetilde{C}$ where $\widetilde{C}$ represents the number of particles passing through S per unit of time—a constant. Combining this with equation (4.13) we get

$$\frac{kv}{g\mu} = C, \tag{4.15}$$

where C is a constant giving the flow through S per unit of central mass M (i.e. $C = \widetilde{C}/M$). Both C and v are negative for an inward flow directed towards the central mass and positive for an outward flow. We shall find that for the same configuration for $\pi_\perp, n$ and V, the two solutions are possible and can be deduced from each other by a time reversal. For definiteness we shall choose $C > 0$.

Our equations are the first integral (4.9), the equation of motion (4.10), the gravitational field equation (4.13), the polarization relation (4.14) and the conservation of particles equivalent to (4.15). Thus, five equations in all for the five unknowns $g, \pi_\perp, V, v$ and k (where k is equivalent to the number density n). Note that in this counting we have considered that the MOND function μ is known, but we have included the potential function V in the list of unknowns. Indeed we shall show that once μ is specified (for instance $\mu = 1 - e^{-g/a_0}$ as chosen below), the potential function V is determined. It would seem *a priori* more natural to proceed the other way around, namely to specify first V because it is part of the more fundamental action (2.1), and only then to deduce μ which would tell which kind of MOND phenomenology this action corresponds to. Since there will basically be a bijective correspondence between V and μ, it is clear that the two approaches are equivalent, and we can

---

[21] In the extreme MOND regime we have $\mu \sim g/a_0$ and relation (4.13) yields the famous logarithmic potential $U \sim -\sqrt{GMa_0} \ln r$ explaining the flat rotation curves of galaxies and the Tully–Fisher relation $v_{\mathrm{flat}} \sim (GMa_0)^{1/4}$.

say that a choice for the action (2.1) determines μ as expected (at least under the hypothesis of the present calculation).

From equations (4.14) and (4.15) we obtain v as

$$v = C\pi_\perp \frac{\mu}{1 - \mu}. \tag{4.16}$$

Replacing it into (4.9) yields the differential equation

$$\pi_\perp' = \frac{1}{C^2 \pi_\perp} \left[\frac{V}{\pi_\perp} - g \right] \left(\frac{1 - \mu}{\mu} \right)^2. \tag{4.17}$$

Furthermore we can write (4.10) into the equivalent form

$$\frac{V'}{\pi_\perp'} = \frac{V}{\pi_\perp} + C^2 \pi_\perp^2 \frac{\mu \mu'}{(1 - \mu)^3}. \tag{4.18}$$

Since μ is given, g is known by equation (4.13), and the equations (4.17)–(4.18) form a system of coupled differential equations for $\pi_\perp$ and V. In a first stage, the ratio between V and $\pi_\perp$ is integrated with the result

$$\frac{V}{\pi_\perp} = g + \frac{\mu}{1 - \mu} \left[g - \int \frac{\mathrm{d}g}{\mu} \right]. \tag{4.19}$$

(The standard notation is used for the indefinite integral, defined modulo an integration constant.) Then, the solution for $\pi_\perp$ is found to be

$$\pi_\perp = \left(\frac{2}{C^2} \int \mathrm{d}r \frac{1 - \mu}{\mu} \left[g - \int \frac{\mathrm{d}g}{\mu} \right] \right)^{1/2}. \tag{4.20}$$

The velocity field is then given by (4.16), the density of dipole moments $n = k/(4\pi G)$ follows from (4.15), and V is deduced from (4.16). As we said, for any choice of function μ we can determine V and vice versa. In this sense we have related the phenomenology of MOND to some more 'basic' physics associated with the description of dark matter.

Let us exemplify the previous resolution by showing the case of the MOND function that corresponds to the simple susceptibility coefficient $\chi = -\mathrm{e}^{-g/a_0}$, namely

$$\mu = 1 - \mathrm{e}^{-g/a_0}. \tag{4.21}$$

This function is obviously interesting (though maybe very special) because it is exponentially close to its Newtonian limit when $g \to \infty$. In this case we readily integrate equation (4.16) and find

$$\frac{V}{\pi_\perp} = g \, \mathrm{e}^{g/a_0} - a_0 (\mathrm{e}^{g/a_0} - 1) \ln(\mathrm{e}^{g/a_0} - 1). \tag{4.22}$$

Here, the integration constant has been chosen in such a way that the ratio $V/\pi_\perp$ becomes equivalent to g in the Newtonian regime $g \to \infty$ (any other choice would make it growing exponentially). The dipole moment is then given by

$$\pi_\perp = \left(-\frac{2a_0}{C^2} \int_0^r \mathrm{d}r' \frac{\ln(1 - \mathrm{e}^{-g'/a_0})}{\mathrm{e}^{g'/a_0} - 1} \right)^{1/2}, \tag{4.23}$$

where $g' \equiv g(r')$. We have fixed the integration constant (in the lower bound of the integral) in order to ensure that $\pi_\perp \to 0$ in the Newtonian limit corresponding to $r \to 0$. Using the fact that $g(r')$ is implicitly given by equation (4.13), we obtain

$$\pi_\perp = \frac{(GMa_0)^{1/4}}{C} \left(-\int_{g/a_0}^\infty \mathrm{d}y \, y^{-3/2} \, \mathrm{e}^{-y} \frac{1 + (y - 1)\,\mathrm{e}^{-y}}{(1 - \mathrm{e}^{-y})^{5/2}} \ln(1 - \mathrm{e}^{-y}) \right)^{1/2}. \tag{4.24}$$

DIPOLE MOMENT VARIABLE
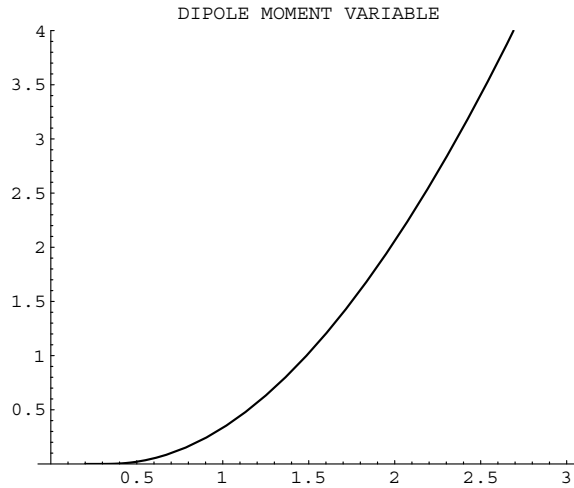
Figure 1. Dipole moment $\pi_\perp$ as a function of the rescaled distance $\rho = r/r_0$ defined by (4.25). On the horizontal axis $\rho = 1$ corresponds to the transition between the Newtonian and MOND regimes. The unit of dipole moment on the vertical axis is $(GMa_0)^{1/4}/C$.
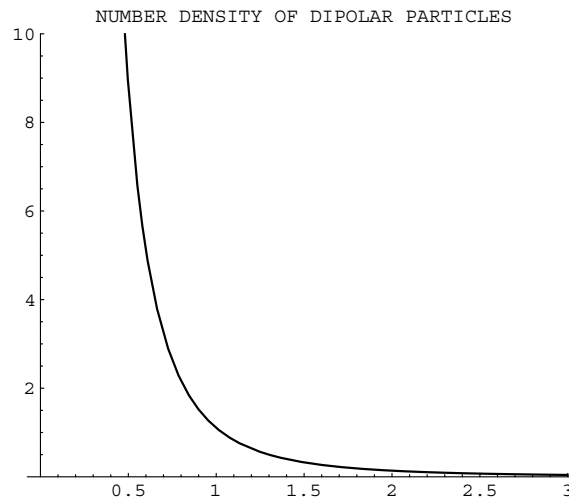
VELOCITY OF DIPOLAR PARTICLES
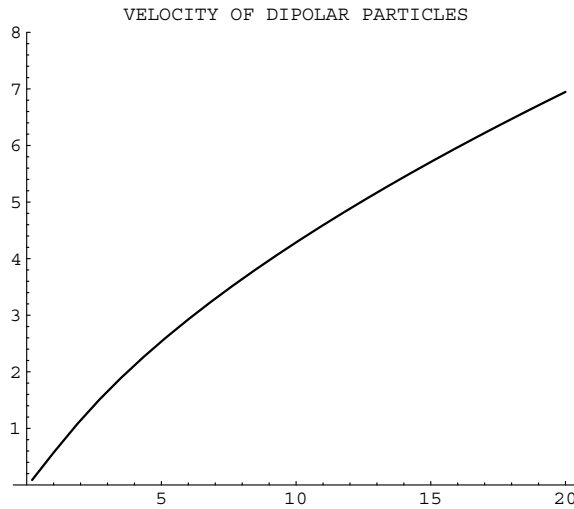
Figure 2. Velocity of the dipolar particles v as a function of the distance ρ. The unit of velocity on the vertical axis is $(GMa_0)^{1/4}$.

In figure 1, we plot the dipole moment $\pi_\perp$ as a function of the rescaled distance defined by $\rho \equiv r/r_0$, where r_0 is the characteristic length at which the transition between the Newtonian and MOND regimes occurs, i.e. which is such that $g(r_0) = a_0$ and is given by

$$r_0 = \left(\frac{GM}{a_0(1 - \mathrm{e}^{-1})} \right)^{1/2}. \tag{4.25}$$

In figures 2 and 3, we plot the velocity v and the number density n of dipolar particles as functions of $\rho = r/r_0$. We give the polarization of the medium in figure 4, the density of polarization masses in figure 5 and the potential function V in figure 6.

NUMBER DENSITY OF DIPOLAR PARTICLES

Figure 3. Number density of the dipolar particles n as a function of the distance ρ. The unit of number density on the vertical axis is $(a_0^3/GM)^{1/4}C/(4\pi G)$.
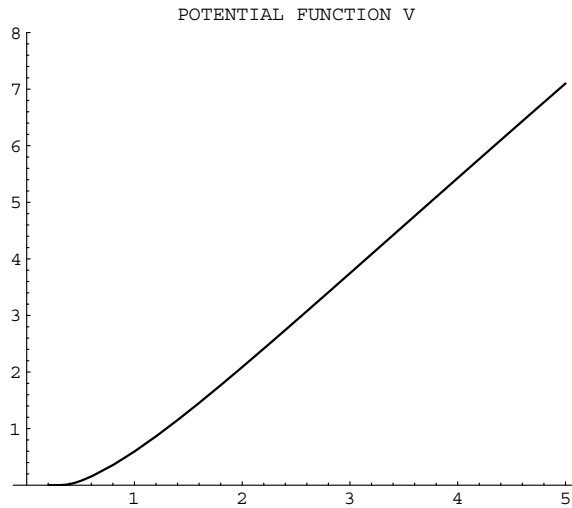
POLARIZATION OF THE MEDIUM
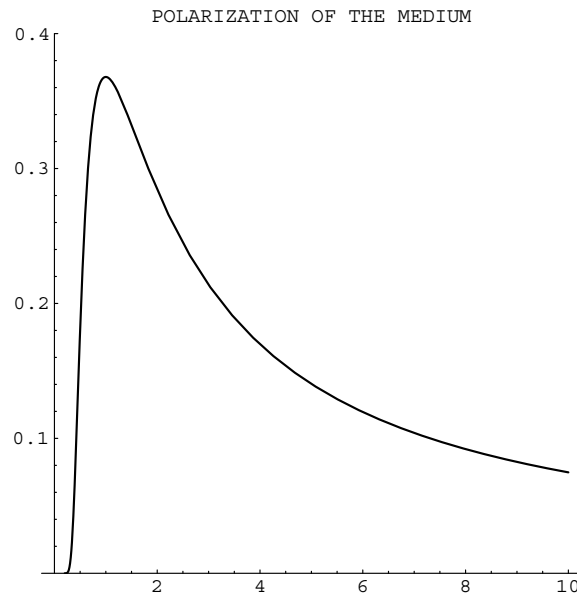
Figure 4. Polarization of the medium Π as a function of the distance ρ. The unit of polarization on the vertical axis is $a_0/(4\pi G)$.

As we can see from figure 1 the moment $\pi_\perp$ tends (exponentially) to zero when $r \to 0$, so the dipole moments actually do not exist in the vicinity of the ordinary mass. However, $\pi_\perp$ diverges in the MOND domain where $r \to \infty$; more precisely, we can check that $\pi_\perp$ behaves like $\sim r(\ln r)^{1/2}$ when $r \to \infty$. Moreover, we see from figure 3 that the number density n of dipole moments decreases at infinity, at the rate $\sim r^{-2}(\ln r)^{-1/2}$. To ensure the conservation of the mass flow the velocity v must then increase like $\sim(\ln r)^{1/2}$, which is indeed what we observe in figure 2 (recall that the particles do not obey the geodesic equation).

The fact that the dipole moment $\pi_\perp$ opens up more and more and diverges when $r \gg r_0$ (deep in the MOND regime, far from the ordinary matter) may appear to be surprising.

DENSITY OF POLARIZATION MASSES
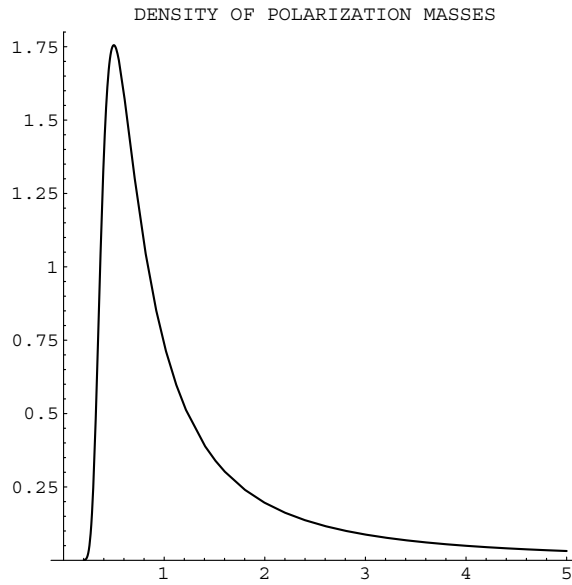
Figure 5. Density of polarization masses ρ_{polar} as a function of the distance ρ. The unit of mass density on the vertical axis is $a_0/(4\pi G r_0)$.

POTENTIAL FUNCTION V

Figure 6. Potential function V as a function of the distance ρ. The unit of V on the vertical axis is $(GMa_0^5)^{1/4}/C$.

However, what is important is the *density* of dipole moments or polarization $\Pi = n\pi_\perp$. As we have seen the number density of dipole moments tends to zero in the MOND domain $r \to \infty$, and as a result the polarization (plotted in figure 4) does decrease at large distances as expected (at the rate $\sim r^{-1}$).[22] On the other hand, we see that with the present choice of MOND function (4.21) the polarization decays exponentially in the Newtonian regime at short distances.

---

[22] Figure 4 is the analogue of figure 1 in the Newtonian-like model of paper I (indeed the density n of dipole moments is considered to be constant in paper I, so the dipole moment varies like the polarization).

In the present model, the polarization reflects the distribution of the gravitational field that is induced by the fluid of dipolar particles. The polarization shown in figure 4 is in fact given by the analytic formula

$$\Pi = \frac{g \, e^{-g/a_0}}{4\pi G}. \tag{4.26}$$

(Note that this formula, equivalent to (3.18), could be written directly from the standard MOND equation [5–7]; only the interpretation we propose here as the polarization of the dipolar medium is new.)

Even more relevant than the polarization to illustrate the effect of dark matter is the density of the polarization masses ρ_{polar} defined by equation (3.14). In the present case (cf assumptions (1) and (2)), we find that

$$\rho_{\text{polar}} = \Pi' + \frac{2\Pi}{r}. \tag{4.27}$$

Figure 5 displays ρ_{polar} as a function of the distance $\rho = r/r_0$. In the present model, ρ_{polar} really represents the mass density of dark matter (namely, that density which is added to the density of ordinary matter in the source of the Poisson equation for the Newtonian potential U). We can check from figure 5 that ρ_{polar} has the correct behaviour $\sim r^{-2}$ at infinity which reproduces the flat rotation curves of galaxies; in more detail we have

$$\rho_{\text{polar}} \sim \frac{1}{4\pi r^2}\sqrt{\frac{Ma_0}{G}} \qquad \text{when} \quad r \to \infty, \tag{4.28}$$

which corresponds to the polarization mass

$$M_{\text{polar}} \sim \sqrt{\frac{Ma_0}{G}} r \qquad \text{when} \quad r \to \infty. \tag{4.29}$$

Note also that a prominent feature of figure 5 is that it predicts no accumulation of dark matter in the close vicinity of ordinary masses, when $r \ll r_0$. Of course, we meet here the natural explanation by MOND (and therefore also by the present model) for the absence of observed cusps of dark matter in the central regions of galaxies.

We have thus shown how the equations of motion and evolution of the dipolar particles can be integrated in a specific example. Although the solution we have considered is rather idealized—the fluid of dipolar particles is stationary and the dipole moments are exactly aligned in the central gravitational field of a point mass[23]—it has the merit of exhibiting in detail the link between the phenomenology of MOND and the physics of dipolar particles, specified by the potential function $V(\pi_\perp/m)$ which enters the action (2.1) (the function V is plotted in figure 6 for the particular case of MOND function (4.21)).

5. Summary and conclusions

Motivated by the quasi-Newtonian model of paper I [18], we interpret the phenomenology of MOND as resulting from an effect of gravitational polarization of a medium made of dipole moments aligned in the gravitational field of ordinary masses. We propose an action principle, based on the matter action (2.1), to describe the dynamics of dipolar particles and the evolution of the dipole moments in standard general relativity. The action involves a kinetic-like term for the evolution of the dipole moment and a scalar function V which is supposed to describe (at some effective level) the non-gravitational interaction between the constituents of the dipole.

---

[23] Note that the question of the stability of this solution against gravitational perturbations has not been investigated.

The dynamical variables are the dipolar particle's spacetime position x^μ, and the dipole moment 4-vector π^μ, which are varied independently yielding the two basic equations (2.6) and (2.10). Variation with respect to the metric yields the stress–energy tensor (2.14). We find a particular class of solutions, defined by the constraint (2.23), corresponding to the intuitive idea of a dipole moment in equilibrium. For this class of solutions, the physical dipole moment variable is the projection $\pi_\perp^\mu$ orthogonal to the particle's 4-velocity.

The non-relativistic (NR) limit of the model is investigated next. The basic equations in this limit are (i) the equation of motion (3.4) of the dipolar particle, (ii) an equilibrium condition (3.8) for the dipole moment $\pi_\perp^i$, (iii) the conservation law (3.11) of the number of particles n and (iv) the Poisson equation (3.15) for the gravitational potential U. Equations (i) and (ii) are different from those of the quasi-Newtonian model of paper I (indeed the model of paper I violates the equivalence principle and cannot result from the NR limit of a general relativistic model). In addition, we have the equations of motion of ordinary massive particles (3.16) and of photons (3.17).

Assuming that the tidal gravitational field $\partial_{ij}U$ can be neglected, we find that there is a solution for which the dipole moments are aligned with the gravitational field. For that solution the Poisson equation for the gravitational field reduces to the MOND equation like in paper I, confirming the close relation between the phenomenology of MOND and the dipolar dark matter.

Finally, the equations in the NR limit have been integrated in the idealized case where the fluid of dipolar particles is stationary and the dipole moments are exactly aligned in the gravitational field generated by a point mass M. The polarization of the medium is plotted in figure 4 and the density of polarization masses (which represents in this model the density of dark matter accumulated around M) is shown in figure 5. There is a correspondence between the potential function V in the action and the MOND function μ.

To conclude, we usually face two alternatives to the issue of dark matter: either accept the existence of cold dark matter particles but which fail to reproduce in a natural way the rotation curves of galaxies, or postulate an ad hoc alteration of the fundamental theory of gravity (MOND and its relativistic extensions). In the present paper (following paper I), we proposed a third alternative: keep the standard law of gravity but add to the ordinary matter some non-standard dark matter in order to 'explain' MOND. More precisely we invoke a mechanism of gravitational polarization, in which the ordinary masses (galaxies) are 'anti-screened' by polarization masses associated with gravitational dipoles playing the role of dark matter. The dipolar dark matter particles are in equilibrium in the gravitational field because of their internal structure driven by some postulated non-gravitational force (whose fundamental origin is unknown).

Acknowledgments

Appendix. Variation of the action functional

In this appendix, we show how to vary the dipolar action functional (2.1), which is of the general form

$$S = \sum \int_{-\infty}^{+\infty} d\tau\, L[g_{\mu\nu}, u^\mu, \pi^\mu, \dot{\pi}^\mu]. \tag{A.1}$$

The dynamical variables are the particle's spacetime position $x^\mu(\tau)$ and the dipole moment carried by the particle $\pi^\mu(\tau)$, both depending on the proper time parametrizing the worldline and denoted by $d\tau = \sqrt{-g_{\mu\nu}dx^\mu dx^\nu/c^2}$. The Lagrangian L is a function of the dynamical variables through the metric $g_{\mu\nu}(x)$ evaluated at $x^\mu(\tau)$, the particle's 4-velocity $u^\mu = dx^\mu/d\tau$, the dipole moment itself $\pi^\mu(\tau)$ and the covariant time derivative of the moment $\dot\pi^\mu = d\pi^\mu/d\tau + \Gamma^\mu_{\rho\sigma}u^\rho\pi^\sigma$. The four variables $g_{\mu\nu}$, u^μ, π^μ and $\dot\pi^\mu$ are considered to be independent in equation (A.1).

We first establish a relation stating that the action (A.1) is a scalar vis-à-vis the arbitrary infinitesimal coordinate transformation $x'^\mu = x^\mu + \varepsilon^\mu(x)$. The requested linear transformations of the independent variables are

$$\left.\begin{array}{l} g'_{\mu\nu} = g_{\mu\nu} - 2g_{\rho(\mu}\partial_{\nu)}\varepsilon^\rho \\ u'^\mu = u^\mu + u^\rho\partial_\rho\varepsilon^\mu \\ \pi'^\mu = \pi^\mu + \pi^\rho\partial_\rho\varepsilon^\mu \\ \dot\pi'^\mu = \dot\pi^\mu + \dot\pi^\rho\partial_\rho\varepsilon^\mu \end{array}\right\} + \mathcal{O}(\varepsilon^2), \tag{A.2}$$

together with $d\tau' = d\tau$ which is a scalar. Here, the prime refers to the new values of the components of the tensor when evaluated at the same spacetime event. In the last line, we have used the fact that $\dot\pi^\mu$ is a 4-vector. The 'scalarity' of the action means that, under the previous coordinate transformation,

$$\int_{-\infty}^{+\infty} d\tau\, L[g_{\mu\nu}, u^\mu, \pi^\mu, \dot\pi^\mu] = \int_{-\infty}^{+\infty} d\tau'\, L[g'_{\mu\nu}, u'^\mu, \pi'^\mu, \dot\pi'^\mu], \tag{A.3}$$

in which the *same* functional L appears on both sides of the equation. Using equations (A.2), we immediately deduce the scalarity condition

$$2g_{\mu\rho}\frac{\partial L}{\partial g_{\nu\rho}} = u^\nu\frac{\partial L}{\partial u^\mu} + \pi^\nu\frac{\partial L}{\partial \pi^\mu} + \dot\pi^\nu\frac{\partial L}{\partial \dot\pi^\mu}. \tag{A.4}$$

We first vary the action with respect to the moment $\pi^\mu(\tau)$ subject as usual to the condition that $\pi^\mu(\pm\infty) = 0$. The variations are $\delta\pi^\mu$ and $\delta\dot\pi^\mu = d\delta\pi^\mu/d\tau + \Gamma^\mu_{\rho\sigma}u^\rho\delta\pi^\sigma$. We integrate the ordinary time derivative $d\delta\pi^\mu/d\tau$ by part, and find that the role of the Christoffel symbol is to make the corresponding field equation covariant,

$$\frac{D}{d\tau}\left(\frac{\partial L}{\partial \dot\pi^\mu}\right) = \frac{\partial L}{\partial \pi^\mu}, \tag{A.5}$$

where $D/d\tau$ is the covariant time derivative. The variation with respect to the position $x^\mu(\tau)$ (satisfying $x^\mu(\pm\infty) = 0$) is more involved, due to the dependence of $d\tau$, u^μ and $g_{\mu\nu}$ on the position. Many Christoffel symbols and their derivatives are generated in the calculation. The partial derivative of the Lagrangian with respect to the metric is simplified with the help of the condition (A.4). We finally obtain a manifestly covariant equation, given by

$$\frac{D}{d\tau}\left(\frac{\partial L}{\partial u^\mu} + u_\mu\left[u^\rho\frac{\partial L}{\partial u^\rho} + \dot\pi^\rho\frac{\partial L}{\partial \dot\pi^\rho} - L\right]\right) = R^\nu{}_{\rho\mu\sigma}\pi^\rho u^\sigma\frac{\partial L}{\partial \dot\pi^\nu}, \tag{A.6}$$

where $R^\nu{}_{\rho\mu\sigma}$ is the Riemann curvature tensor.

Finally, we obtain the stress–energy tensor $T^{\mu\nu}$ of the fluid of dipolar particles with number density n satisfying the continuity equation

$$\nabla_\mu(nu^\mu) = 0. \tag{A.7}$$

We thus vary the action with respect to the metric, which enters explicitly in the first slot of L in equation (A.1), and implicitly through the proper time $d\tau$ and the covariant time derivative

$\dot{\pi}^{\mu}$ of the dipole moment[24]. In this way, we obtain

$$T^{\mu\nu} = n\left(2\frac{\partial L}{\partial g_{\mu\nu}} + u^{\mu}u^{\nu}\left[u^{\rho}\frac{\partial L}{\partial u^{\rho}} + \dot{\pi}^{\rho}\frac{\partial L}{\partial \dot{\pi}^{\rho}} - L\right]\right)$$
$$+ \nabla_{\rho}\left\{n\left[u^{(\mu}\pi^{\nu)}\frac{\partial L}{\partial \dot{\pi}_{\rho}} - u^{\rho}\pi^{(\mu}\frac{\partial L}{\partial \dot{\pi}_{\nu)}} - \pi^{\rho}u^{(\mu}\frac{\partial L}{\partial \dot{\pi}_{\nu)}}\right]\right\}, \tag{A.8}$$

in which we denote $\partial L/\partial\dot{\pi}_{\rho} \equiv g^{\rho\sigma}\partial L/\partial\dot{\pi}^{\sigma}$. The second, dipolar-type term in this expression takes the form of the divergence of a tensor and has a structure similar to some related term in the Belinfante–Rosenfeld [36, 37] symmetric stress–energy tensor of integer spin fields. An alternative expression of (A.8) is readily derived with the help of the equation of motion (A.5) and the scalarity condition (A.4); we have (with $\partial L/\partial u_{\rho} \equiv g^{\rho\sigma}\partial L/\partial u^{\sigma}$)

$$T^{\mu\nu} = n\left(u^{(\mu}\frac{\partial L}{\partial u_{\nu)}} + u^{\mu}u^{\nu}\left[u^{\rho}\frac{\partial L}{\partial u^{\rho}} + \dot{\pi}^{\rho}\frac{\partial L}{\partial \dot{\pi}^{\rho}} - L\right]\right) - \nabla_{\rho}\left\{n\left[\pi^{\rho}\frac{\partial L}{\partial \dot{\pi}_{(\mu}} - \pi^{(\mu}\frac{\partial L}{\partial \dot{\pi}_{\rho}}\right]u^{\nu)}\right\}. \tag{A.9}$$

We check that the stress–energy tensor is conserved 'on shell', i.e. when the equations of motion (A.5) and (A.6) are satisfied:

$$\nabla_{\nu}T^{\mu\nu} = 0. \tag{A.10}$$

Finally, let us make the link with the linear momenta P_{μ} and Ω_{μ} introduced in equations (2.7) and (2.11), and which we have seen characterize entirely the dipolar particle. These momenta appear in fact to be the conjugate momenta associated with the dynamical variables π^{μ} and x^{μ}, respectively, in the sense that

$$P_{\mu} \equiv 2m\frac{\partial L}{\partial \dot{\pi}^{\mu}}, \tag{A.11a}$$

$$\Omega_{\mu} \equiv \frac{\partial L}{\partial u^{\mu}} + u_{\mu}\left[u^{\rho}\frac{\partial L}{\partial u^{\rho}} + \dot{\pi}^{\rho}\frac{\partial L}{\partial \dot{\pi}^{\rho}} - L\right]. \tag{A.11b}$$

The relation for Ω_{μ} seems to be more complicated than a simple conjugation relation, but this is due to our use of the parametrization by the proper time $d\tau$ in the action, instead of a parametrization which is independent of the dynamical variable x^{μ}. With such definitions the equations of motion (A.5) and (A.6) become

$$\dot{P}_{\mu} = 2m\frac{\partial L}{\partial \pi^{\mu}}, \tag{A.12a}$$

$$\dot{\Omega}_{\mu} = \frac{1}{2m}R_{\mu\sigma\nu\rho}u^{\sigma}P^{\nu}\pi^{\rho}, \tag{A.12b}$$

and are seen to be equivalent to the expressions derived in equations (2.6) and (2.10) (indeed $F_{\mu} = -m\partial L/\partial\pi^{\mu}$ with the explicit action (2.1)). Furthermore, using the alternative expression (A.9), the stress–energy tensor is obtained as

$$T^{\mu\nu} = n\Omega^{(\mu}u^{\nu)} - \frac{1}{2m}\nabla_{\rho}(n[\pi^{\rho}P^{(\mu} - P^{\rho}\pi^{(\mu}]u^{\nu)}), \tag{A.13}$$

which is precisely the result given by equation (2.14).

---

[24] The variation of the Christoffel symbol present in $\dot{\pi}^{\mu}$ is a tensor given by the Palatini formula as $\delta\Gamma^{\mu}_{\rho\sigma} = \frac{1}{2}g^{\mu\nu}[\nabla_{\rho}\delta g_{\sigma\nu} + \nabla_{\sigma}\delta g_{\rho\nu} - \nabla_{\nu}\delta g_{\rho\sigma}]$.

References

[1] Milgrom M 2002 *New Astron. Rev.* **46** 741
[2] Sanders R and McGaugh S 2002 *Ann. Rev. Astron. Astrophys.* **40** 263 (*Preprint* astro-ph/0204521)
[3] Bekenstein J 2005 *Proc. 28th John Hopkins Workshop: Current Problems in Particle Theory* (*Preprint* astro-ph/0412652)
[4] Sanders R 2006 *Proc. 3rd Aegian Summer School. The Invisible Universe: Dark Matter and Dark Energy* (*Preprint* astro-ph/0601431)
[5] Milgrom M 1983a *Astrophys. J.* **270** 365
[6] Milgrom M 1983b *Astrophys. J.* **270** 371
[7] Milgrom M 1983c *Astrophys. J.* **270** 384
[8] Tully R and Fisher J 1977 *Astron. Astrophys.* **54** 661
[9] Gentile P, Salucci P, Klein U, Vergani D and Kalberta P 2004 *Mon. Not. R. Astron. Soc.* **351** 903
[10] Gerbal D, Durret F, Lachièze-Rey M and Lima-Neto G 1992 *Astron. Astrophys.* **262** 395
[11] Clowe D, Bradac M, Gonzalez A H, Markevitch M, Randall S W, Jones C and Zaritsky D 2006 *Astrophys. J. Lett.* **648** L109 (*Preprint* astro-ph/0608407)
[12] Bekenstein J and Milgrom M 1984 *Astrophys. J.* **286** 7
[13] Bekenstein J and Sanders R 1994 *Astrophys. J.* **429** 480 (*Preprint* astro-ph/9311063)
[14] Sanders R 1997 *Astrophys. J.* **480** 492 (*Preprint* astro-ph/9612099)
[15] Bekenstein J 2004 *Phys. Rev.* D **70** 083509 (*Preprint* astro-ph/0403694)
[16] Sanders R 2005 *Mon. Not. R. Astron. Soc.* **363** 459 (*Preprint* astro-ph/0502222)
[17] Bertone G, Hooper D and Silk J 2005 *Phys. Rep.* **405** 279 (*Preprint* hep-th/0404175)
[18] Blanchet L 2007 *Class. Quantum Grav.* **24** 3529 (*Preprint* astro-ph/0605637)
[19] Jackson J 1999 *Classical Electromagnetism* 3rd edn (New York: Wiley)
[20] Bel L and Zia H Sirousse 1985 *Phys. Rev.* D **32** 3128
[21] Papapetrou A 1951 *Proc. R. Soc.* A **209** 248
[22] Trautman A 2002 *Gen. Relat. Grav.* **34** 721 (reprinted from lectures delivered in 1958)
[23] Dixon W 1979 *Isolated Systems in General Relativity* ed J Ehlers (Amsterdam: North-Holland) p 156
[24] Barker B and O'Connell R 1975 *Phys. Rev.* D **12** 329
[25] Barker B and O'Connell R 1979 *Gen. Relat. Grav.* **11** 149
[26] Bailey I and Israel W 1980 *Ann. Phys.* **130** 188
[27] Kidder L, Will C and Wiseman A 1993 *Phys. Rev.* D **47** R4183
[28] Kidder L 1995 *Phys. Rev.* D **52** 821 (*Preprint* gr-qc/9506022)
[29] Tagoshi H, Ohashi A and Owen B 2001 *Phys. Rev.* D **63** 044006
[30] Faye G, Blanchet L and Buonanno A 2006 *Phys. Rev.* D **74** 104033 (*Preprint* gr-qc/0605139)
[31] Blanchet L, Buonanno A and Faye G 2006 *Phys. Rev.* D **74** 104034 (*Preprint* gr-qc/0605140)
[32] Synge J 1964 *Relativity: The General Theory* (Amsterdam: North-Holland)
[33] De Witt B and Brehme R 1960 *Ann. Phys., NY* **9** 220
[34] Fort B and Mellier Y 1994 *Astron. Astrophys. Rev.* **5** 239
[35] Bondi H 1957 *Rev. Mod. Phys.* **29** 423
[36] Belinfante F 1939 *Physica* **6** 887
[37] Rosenfeld L 1940 *Mem. Acad. R. Belg.* **18** 3