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Gravitational radiation reaction in the equations of motion of compact binaries to 3.5 post-Newtonian order

Samaya Nissanke and Luc Blanchet

$\mathcal{G}\mathbb{R}\varepsilon\mathbb{C}\mathcal{O}$—Gravitation et Cosmologie, Institut d'Astrophysique de Paris, CNRS, 98[bis] boulevard Arago, 75014 Paris, France

E-mail: nissanke@iap.fr and blanchet@iap.fr

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Abstract
We compute the radiation-reaction force on the orbital motion of compact binaries to the 3.5 post-Newtonian (3.5PN) approximation, i.e. one PN order beyond the dominant effect. The method is based on a direct PN iteration of the near-zone metric and equations of motion of an extended isolated system, using appropriate 'asymptotically matched' flat-spacetime retarded potentials. The formalism is subsequently applied to binary systems of point particles, with the help of the Hadamard self-field regularization. Our result is the 3.5PN acceleration term in a general harmonic coordinate frame. Restricting the expression to the centre-of-mass frame, we find perfect agreement with the result derived in a class of coordinate systems by Iyer and Will using the energy and angular-momentum balance equations.

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1. Introduction

Since the discovery of the Einstein field equations, gravitational radiation has remained a matter of major theoretical interest, and has led to extensive theoretical studies on the nature and origin of gravitational wave emission from isolated sources. Approximation methods in general relativity, such as the post-Newtonian (PN) expansion, result in both accurate and measurable details of the emission. The prospect of the detection of gravitational waves bathing the Earth by ground and space based interferometers LIGO, VIRGO and LISA, provides a further impetus to such theoretical investigations. Not only would such detectors enable an important comparison between astrophysical observations and theoretical predictions, but they would also ultimately provide strong tests for general relativity.

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A favourable potential source for the detectors is the radiation-reaction dominated inspiral and eventual coalescence of two compact objects (neutron stars or black holes). For such systems, which will undergo hundreds to thousands (depending on the masses) orbital cycles in the frequency bandwidth of LIGO and VIRGO, relativistic corrections to the Newtonian order in the orbital phasing and wave form play a crucial role in preparing the theoretical templates. Indeed, the detection and analysis of these waves in the detectors require at least a third post-Newtonian (3PN) correction to both the energy flux radiated at infinity and in the binary's equations of motion.

The objective of this paper is to compute explicitly the radiation-reaction force in the equations of motion of a compact binary system at the 3.5PN order $\sim\mathcal{O}(c^{-7})$ in harmonic coordinates (in both a general frame and the centre-of-mass frame). It is well known that the leading order radiation-reaction effect occurs at the 2.5PN order $\sim\mathcal{O}(c^{-5})$ [1–8] (see, e.g., [9] for a review). We shall, therefore, compute the radiation reaction at the 1PN relative order, which corresponds, via appropriate balance equations, to the 1PN corrections in the energy and angular-momentum radiated by the system at future null infinity (relative to the standard quadrupole formulae). The 1PN radiation-reaction force is also responsible for the dominant effect in the loss of linear momentum, widely referred to as the gravitational radiation 'recoil'.

Up to the 3.5PN order, the conservative terms in the equations of motion are clearly distinct from the non-conservative, radiation reaction, terms. This clean separation manifests itself as 'even' (Newtonian, 1PN, 2PN and 3PN) and 'odd' (2.5PN and 3.5PN) orders respectively.[1] The 4PN approximation, however, contains both some conservative terms, and also, a contribution from the radiation reaction. The former terms are given by some 'instantaneous' functionals of the source, whilst the latter is associated with the gravitational wave tails [10], and is given by a 'hereditary'-type integral, extending over the past history of the source.

The PN assumption of slow motion limits the validity of the PN expansion to the so-called near zone of the source ($r \ll cT$, where T is a typical period of variation of the source). An important consequence of such a near-zone limitation is that one cannot incorporate directly the radiation reaction into the local PN expansion, since the radiation effects depend on the boundary conditions imposed on the radiation field at infinity ($r \gg a$, where a is the size of the source), notably the famous no-incoming radiation condition imposed at past null infinity, $r \to +\infty$ with $t + r/c =$ const. At present, two different approaches have been proposed and implemented for treating the problem.

The first method is based on an asymptotic matching between the PN expansion valid in the near zone, and the multipolar expansion for the field outside the source. The matching occurs in the so-called exterior near zone of the source, defined by $a < r \ll cT$. The asymptotic matching was introduced in this field by Burke and Thorne [4, 11]. The radiation-reaction force has been derived by matching up to the 3.5PN order for general matter systems [12], and even at the 4PN order, which, as we mentioned above, consists of the contribution of tails [10, 13]. The most developed treatment for the exterior multipolar expansion is the so-called MPM expansion, which combines the multipolar (M) expansion with a post-Minkowskian (PM) scheme [14]. The general solution of the matching equation between the MPM exterior and PN inner fields has recently been obtained [15, 16]. In this calculation, we shall parametrize the PN metric by some appropriate 'asymptotically matched' retarded potentials, which incorporate the 3.5PN radiation-reaction effects, and are introduced in [17, 18]. These potentials at 3.5PN order result from some direct integration of the field equations by means of retarded integrals in the same manner as in [19]. The end result, which

[1] In this paper we adopt the terminology that an even (respectively odd) term is one having an even (odd) power of $1/c$ in front.

we shall obtain, has already been determined by Königsdörffer, Faye and Schäfer [20] (based on the previous works [21, 22]) within the framework of the ADM Hamiltonian formalism, and by Pati and Will [23] (see also [24, 25]) using their variant iteration of the relaxed Einstein field equations in harmonic coordinates.

The second method is exclusively applicable to compact binary systems (modelled by point particles). It consists of using the known PN expressions for the energy and angular momentum radiated at infinity, and assuming that these fluxes are balanced by the corresponding losses of energy and angular momentum in the binary's local equations of motion. This method, based on the energy and angular-momentum balance equations, has been developed by Iyer and Will [26, 27]. As shown in [26, 27], the requirement of energy and momentum balance determines uniquely the radiation-reaction force at 3.5PN order in a class of coordinate systems. The residual coordinate freedom is entirely specified by two arbitrary gauge parameters at 2.5PN order and by six further ones at 3.5PN. The 2.5PN parameters assume some specific values in the case of the scalar radiation-reaction potential of Burke and Thorne at 2.5PN order. They are specified by other values in the case of the 2.5PN radiation reaction in harmonic coordinates as calculated by Damour and Deruelle [28–30]. At 3.5PN order, there is also complete agreement, in the sense that a unique set of 3.5PN gauge parameters can be determined each time, with the scalar and vectorial radiation-reaction potentials of Blanchet [12], which are valid in some extended Burke–Thorne-type gauge, with the end result of Pati and Will [23], who work in harmonic coordinates, and with Königsdörffer *et al* [20], who use ADM coordinates. The method of balance equations has also been extended up to 4.5PN order (excluding the tails at 4PN) in [31].

The two previous approaches have, therefore, enabled the successful determination of the 2.5 and 3.5PN radiation-reaction terms of the compact binaries' orbital dynamics. Our new derivation at 3.5PN order is in complete agreement with the latter works, and, in particular, we confirm the earlier result of [23]. The principles of the method followed in [23] are similar to ours, since both methods are based on the PN iteration of the Einstein field equations relaxed by the harmonic coordinate condition. There are, however, important differences in the implementation of the asymptotic matching procedure, as well as several more minor technical differences. In fact, our own method is justified by the end result of the particular matching procedure we use, as given in [16]. Furthermore, though of less relevance in the present problem, the external field in our approach is described by a choice of multipole moments which is different from the choice adopted in [23–25]. In addition, our treatment of the compact objects model the particles by delta-function singularities, with the help of the Hadamard self-field regularization, whilst Pati and Will [23] do not implement a regularization scheme and instead, model each of the bodies by some spherical, non-rotating, extended fluid balls.

The theoretical framework of this paper is the 3.5PN equations of motion of a general matter system, derived in [18] by a direct PN iteration of the metric in harmonic coordinates. As mentioned previously, the metric is expressed as a functional of a particular set of nonlinear retarded potentials. Such a metric is then specialized to the model of two delta-function singularities by using the standard prescription for a distributional stress-energy tensor in general relativity. To cure the divergences associated with each particle's infinite self-field, we systematically apply the Hadamard 'partie-finie' regularization [32–34], or, more precisely, a specific variant of it defined in [35, 36]. In addition to the divergences due to the point-particle singularities, which are dealt with by Hadamard's regularization, Poisson-like integrals arise at high PN orders and are typically divergent at 'spatial infinity'. However, this problem is technically overcome by the introduction of alternative, general solutions to the Poisson equation, in the form of some regularized versions of the usual Poisson integral, constructed from a specific finite part procedure called FP. This conforms with our definition of the multipole moments in the external field ([15] describes the general formalism which allows one to use such a finite part). Furthermore, the recent work [16] shows that the same finite part FP is also to be applied when computing the 3.5PN radiation-reaction effects.

When investigating the equations of motion at the 3.5PN level, we will not encounter any logarithmic divergences of integrals similar to those found at the 3PN conservative level. Such logarithmic divergences are responsible for an 'incompleteness' of Hadamard's regularization in treating the sources' singular nature. Indeed, this incompleteness results in the appearance of one physical undetermined parameter, called λ, in the 3PN equations of motion in harmonic coordinates [18, 37], or alternatively, the equivalent parameter known as the static ambiguity ω_s in the 3PN Hamiltonian of the particles in ADM-type coordinates [38, 39]. A recent application of dimensional regularization, in the framework of which the logarithmic divergences correspond to poles when the spatial dimension d approaches 3, yielded the numerical value of these parameters and showed that they are indeed equivalent [40, 41].[2] A complete calculation of the 3PN equations of motion has also been performed using an independent method by Itoh *et al* [43, 44]. Here, for the computation of the terms at 3.5PN order, we can use Hadamard's standard regularization or any of the proposed variants of it without encountering such problems. In fact, since there are no logarithmic divergences at the 3.5PN order, nor any associated poles, the result obtained from Hadamard's regularization is identical to that arising from dimensional regularization.

The plan of this paper is as follows. In section 2, we present the expressions for the metric and equations of motion up to 3.5PN order for general isolated matter systems. Section 3 considers the specific mathematical model of two compact objects, which will be described by point-particle singularities. All the elementary potentials needed for the 3.5PN radiation-reaction terms are computed in section 4. Finally, we present our final result for the binary's 3.5PN acceleration in section 5, where we also compare it with the existing literature.

2. Formalism for general matter systems

This section presents the 3.5PN equations of motion, which are expressed as a function of a particular set of elementary nonlinear potentials, and are valid for a general smooth hydrodynamical 'fluid' system in harmonic coordinates. Thus, we assume (initially) that the matter system possesses neither singularities nor black holes, and can be described by some Eulerian-type equations involving some high relativistic corrections.

2.1. Definition of a set of retarded nonlinear potentials

We begin by stating the result of the direct PN iterative method for the near-zone metric, valid for $r \ll cT$, which is parametrized by the retarded potentials (given by some retarded integral) introduced in [17, 18]. Convenience and convention dictate the specific form of these retarded potentials. The near-zone metric asymptotically matches to an exterior far-zone radiative-type metric in the overlapping exterior near zone. The exterior metric is known from a multipolar post-Minkowskian (MPM) formalism [14]. The matching to the PN inner metric yields a solution which is globally defined over all spacetime (in a formal sense of PN expansions) in the harmonic coordinate system [15, 16]. The 3.5PN iterated metric is given by [18]

[2] Other ambiguity parameters, ξ, κ and ζ, present in the radiation field of point-particle binaries at 3PN order, have also been resolved by means of dimensional regularization [42].

$$g_{00} = -1 + \frac{2}{c^2}V - \frac{2}{c^4}V^2 + \frac{8}{c^6}\left(\hat{X} + V_iV_i + \frac{V^3}{6}\right) + \frac{32}{c^8}\left(\hat{T} - \frac{1}{2}V\hat{X} + \hat{R}_iV_i - \frac{1}{2}VV_iV_i - \frac{V^4}{48}\right) + \mathcal{O}\left(\frac{1}{c^{10}}\right), \tag{2.1a}$$

$$g_{0i} = -\frac{4}{c^3}V_i - \frac{8}{c^5}\hat{R}_i - \frac{16}{c^7}\left(\hat{Y}_i + \frac{1}{2}\hat{W}_{ij}V_j + \frac{1}{2}V^2V_i\right) + \mathcal{O}\left(\frac{1}{c^9}\right), \tag{2.1b}$$

$$g_{ij} = \delta_{ij}\left[1 + \frac{2}{c^2}V + \frac{2}{c^4}V^2 + \frac{8}{c^6}\left(\hat{X} + V_kV_k + \frac{V^3}{6}\right)\right] + \frac{4}{c^4}\hat{W}_{ij} + \frac{16}{c^6}\left(\hat{Z}_{ij} + \frac{1}{2}V\hat{W}_{ij} - V_iV_j\right) + \mathcal{O}\left(\frac{1}{c^8}\right). \tag{2.1c}$$

This metric explicitly involves only some *even* powers of $1/c$. Indeed, the *odd* terms, notably the 3.5PN terms we are looking for, are implicitly contained in the definitions of the elementary potentials V, V_i, $\hat{W}_{ij}, \ldots$, which parametrize the metric, as shown below when we perform a PN expansion of the retardation of these potentials.

The matter stress-energy tensor, $T^{\mu\nu}$, is conventionally expressed in terms of certain mass, current and stress densities, given respectively as

$$\sigma \equiv \frac{T^{00} + T^{ii}}{c^2}, \tag{2.2a}$$

$$\sigma_i \equiv \frac{T^{0i}}{c}, \tag{2.2b}$$

$$\sigma_{ij} \equiv T^{ij}, \tag{2.2c}$$

(where $T^{ii} \equiv \delta_{ij}T^{ij}$). The potentials may be grouped depending on the PN order at which they initially appear. For the Newtonian and 1PN orders, they are given by

$$V = \Box^{-1}_{\mathcal{R}}[-4\pi G\sigma], \tag{2.3a}$$

$$V_i = \Box^{-1}_{\mathcal{R}}[-4\pi G\sigma_i], \tag{2.3b}$$

where $\Box^{-1}_{\mathcal{R}}$ denotes the usual flat-spacetime d'Alembertian retarded ($\mathcal{R}$) integral. Next, the potentials which appear at the 2PN order are defined by

$$\hat{X} = \Box^{-1}_{\mathcal{R}}\left[-4\pi GV\sigma_{ii} + \hat{W}_{ij}\partial^2_{ij}V + 2V_i\partial_t\partial_iV + V\partial^2_tV + \tfrac{3}{2}(\partial_tV)^2 - 2\partial_iV_j\partial_jV_i\right], \tag{2.4a}$$

$$\hat{R}_i = \Box^{-1}_{\mathcal{R}}\left[-4\pi G(V\sigma_i - V_i\sigma) - 2\partial_kV\partial_iV_k - \tfrac{3}{2}\partial_tV\partial_iV\right], \tag{2.4b}$$

$$\hat{W}_{ij} = \Box^{-1}_{\mathcal{R}}[-4\pi G(\sigma_{ij} - \delta_{ij}\sigma_{kk}) - \partial_iV\partial_jV]. \tag{2.4c}$$

Finally, the relevant potentials for the highest established order of 3PN are[3],

$$\hat{T} = \Box^{-1}_{\mathcal{R}}\left[-4\pi G\left(\tfrac{1}{4}\sigma_{ij}\hat{W}_{ij} + \tfrac{1}{2}V^2\sigma_{ii} + \sigma V_iV_i\right) + \hat{Z}_{ij}\partial^2_{ij}V + \hat{R}_i\partial_t\partial_iV - 2\partial_iV_j\partial_j\hat{R}_i - \partial_iV_j\partial_t\hat{W}_{ij} + VV_i\partial_t\partial_iV + 2V_i\partial_jV_i\partial_jV + \tfrac{3}{2}V_i\partial_tV\partial_iV + \tfrac{1}{2}V^2\partial^2_tV + \tfrac{3}{2}V(\partial_tV)^2 - \tfrac{1}{2}(\partial_tV_i)^2\right], \tag{2.5a}$$

[3] When performing the PN iteration for point particles in the context of the extended Hadamard regularization [35, 36], there are some extra contributions to be added to the 3PN potentials, which are due to the violation of the Leibniz rule for the derivative of a product by the distributional derivatives, see equations (3.27) in [18]. These so-called 'Leibniz' terms arise, however, only at 3PN order and do not contribute to the present computation at 3.5PN order, where, as previously discussed, the Hadamard regularization can be applied without the problems encountered at the previous 3PN order.

$$\hat{Y}_i = \Box^{-1}_{\mathcal{R}}\Big[-4\pi G\left(-\sigma\hat{R}_i - \sigma V V_i + \tfrac{1}{2}\sigma_k\hat{W}_{ik} + \tfrac{1}{2}\sigma_{ik}V_k + \tfrac{1}{2}\sigma_{kk}V_i\right) + \hat{W}_{kl}\partial_{kl}V_i$$
$$- \partial_t\hat{W}_{ik}\partial_k V + \partial_i\hat{W}_{kl}\partial_k V_l - \partial_k\hat{W}_{il}\partial_l V_k - 2\partial_k V\partial_i\hat{R}_k - \tfrac{3}{2}V_k\partial_i V\partial_k V$$
$$- \tfrac{3}{2}V\partial_t V\partial_i V - 2V\partial_k V\partial_k V_i + V\partial_t^2 V_i + 2V_k\partial_k\partial_t V_i\Big], \qquad (2.5b)$$

$$\hat{Z}_{ij} = \Box^{-1}_{\mathcal{R}}\Big[-4\pi G V(\sigma_{ij} - \delta_{ij}\sigma_{kk}) - 2\partial_{(i}V\partial_t V_{j)} + \partial_i V_k\partial_j V_k + \partial_k V_i\partial_k V_j$$
$$- 2\partial_{(i}V_k\partial_k V_{j)} - \delta_{ij}\partial_k V_m(\partial_k V_m - \partial_m V_k) - \tfrac{3}{4}\delta_{ij}(\partial_t V)^2\Big]. \qquad (2.5c)$$

The spatial traces of the potentials will be denoted by $\hat{W} \equiv \hat{W}_{ii}$ and $\hat{Z} \equiv \hat{Z}_{ii}$. The harmonic gauge condition of the near-zone PN expansion results in four independent PN differential identities to be satisfied by the latter potentials,

$$\partial_t\left\{V + \frac{1}{c^2}\left[\frac{1}{2}\hat{W} + 2V^2\right] + \frac{4}{c^4}\left[\hat{X} + \frac{1}{2}\hat{Z} + \frac{1}{2}V\hat{W} + \frac{2}{3}V^3\right]\right\} + \partial_i\left\{V_i + \frac{2}{c^2}[\hat{R}_i + VV_i]\right.$$
$$\left. + \frac{4}{c^4}\left[\hat{Y}_i - \frac{1}{2}\hat{W}_{ij}V_j + \frac{1}{2}\hat{W}V_i + V\hat{R}_i + V^2V_i\right]\right\} = \mathcal{O}\left(\frac{1}{c^6}\right), \qquad (2.6a)$$

$$\partial_t\left\{V_i + \frac{2}{c^2}[\hat{R}_i + VV_i]\right\} + \partial_j\left\{\hat{W}_{ij} - \frac{1}{2}\hat{W}\delta_{ij} + \frac{4}{c^2}\left[\hat{Z}_{ij} - \frac{1}{2}\hat{Z}\delta_{ij}\right]\right\} = \mathcal{O}\left(\frac{1}{c^4}\right). \qquad (2.6b)$$

The harmonicity conditions, equations (2.6), thus provide a verification of certain computed potentials which are required at the 3.5PN order. We note, however, that these conditions are unable to provide a direct check to the potentials with the most challenging form which are necessary for our purposes.

As suggested by their definitions (2.3)–(2.5), several recurring structures may be identified in our retarded potentials. The advantages of decomposing the potentials are substantial; general schemata are developed to solve integrals of a particular form, which not only simplify considerably the computational aspect but also reveal interesting analytical solutions. The potentials comprise essentially three types of hierarchical terms of increasing complexity [17, 18]:

(a) *Compact* (C) potentials involve spatially compact source terms, proportional to the mass, current and stress densities, σ, σ_i and σ_{ij}. The support of the source of these potentials is limited to the domain of the matter system, which will be given in this instance (the black hole 'particle-model'), by δ-function singularities (see section 3). We have, e.g.,

$$V^{(\mathrm{C})} \equiv V = \Box^{-1}_{\mathcal{R}}[-4\pi G\sigma], \qquad (2.7a)$$

$$\hat{X}^{(\mathrm{C})} = \Box^{-1}_{\mathcal{R}}[-4\pi G V\sigma_{ii}], \qquad (2.7b)$$

$$\hat{W}^{(\mathrm{C})}_{ij} = \Box^{-1}_{\mathcal{R}}[-4\pi G(\sigma_{ij} - \delta_{ij}\sigma_{kk})]. \qquad (2.7c)$$

The C potentials are relatively simple, but, for instance, $V^{(\mathrm{C})}$ must be calculated with the full 3.5PN precision.

(b) *Quadratic non-compact* (QNC) potentials are generated by spatially non-compact supported distribution of the (source-induced) gravitational field. Specifically, such potentials include terms of the symbolic type $\sim\Box^{-1}_{\mathcal{R}}\,\partial V\partial V$, which denote the quadratic product of two compact potentials V and/or V_i and their spacetime derivatives. Examples of QNC potential terms are

$$\hat{W}^{(\mathrm{QNC})}_{ij} = \Box^{-1}_{\mathcal{R}}[-\partial_i V\partial_j V], \qquad (2.8a)$$

$$\hat{Z}^{(\mathrm{QNC})}_{ij} = \Box^{-1}_{\mathcal{R}}\big[-2\partial_{(i}V\partial_t V_{j)} + \partial_i V_k \partial_j V_k + \partial_k V_i \partial_k V_j - 2\partial_{(i}V_k\partial_k V_{j)} - \delta_{ij}\partial_k V_m(\partial_k V_m - \partial_m V_k) - \tfrac{3}{4}\delta_{ij}(\partial_t V)^2\big], \tag{2.8b}$$

$$\hat{X}^{(\mathrm{QNC})} = \Box^{-1}_{\mathcal{R}}\big[\hat{W}^{(\mathrm{C})}_{ij}\partial^2_{ij}V + 2V_i\partial_t\partial_i V + V\partial_t^2 V + \tfrac{3}{2}(\partial_t V)^2 - 2\partial_i V_j \partial_j V_i\big]. \tag{2.8c}$$

(c) *Cubic non-compact* (CNC) potentials include more complicated integral expressions of the form of the product of a quadratic QNC potential and a compact C potential. The symbolic form reads $\sim \Box^{-1}_{\mathcal{R}}\big[\Box^{-1}_{\mathcal{R}}(\partial V\partial V)\partial V\big]$. The paradigm of such terms is,

$$\hat{X}^{(\mathrm{CNC})} = \Box^{-1}_{\mathcal{R}}\big[\hat{W}^{(\mathrm{QNC})}_{ij}\partial^2_{ij}V\big], \tag{2.9}$$

which must be evaluated at relative 1.5PN order. The other CNC terms, present in the potentials $\hat{T}$ and $\hat{Y}_i$, will only need to be controlled at the 0.5PN order.

2.2. *The equations of motion in terms of the potentials*

The 3.5PN equations of motion in the case of a smooth hydrodynamical fluid are obtained by replacing the expression of metric (2.1) into the law of covariant conservation of the matter stress-energy tensor, $\nabla_\nu T^{\mu\nu} = 0$, which is equivalent to the equation of geodesics in the case of point-particle sources. The resulting equation can be expressed as

$$\frac{\mathrm{d}P^i}{\mathrm{d}t} = F^i, \tag{2.10}$$

where P^i and F^i, introduced here for convenience, can be thought of as some effective *linear momentum density* and *force density* of the matter system respectively, and are defined by

$$P^i = \frac{g_{i\mu}v^\mu}{\sqrt{-g_{\rho\sigma}\frac{v^\rho v^\sigma}{c^2}}}, \tag{2.11a}$$

$$F^i = \frac{1}{2}\frac{\partial_i g_{\mu\nu}v^\mu v^\nu}{\sqrt{-g_{\rho\sigma}\frac{v^\rho v^\sigma}{c^2}}}. \tag{2.11b}$$

Here, $v^i = \mathrm{d}x^i/\mathrm{d}t$ denotes the *coordinate* velocity field ($t = x^0/c$), and we set $v^\mu = (c, v^i)$. Substituting metric (2.1) into the above expressions (2.11), and performing the PN re-expansion, gives [18]

$$\begin{aligned}
P^i = {} & v^i + \frac{1}{c^2}\left(\frac{1}{2}v^2v^i + 3Vv^i - 4V_i\right) + \frac{1}{c^4}\left(\frac{3}{8}v^4v^i + \frac{7}{2}Vv^2v^i - 4V_jv^iv^j - 2V_iv^2\right.\\
& \left. + \frac{9}{2}V^2v^i - 4VV_i + 4\hat{W}_{ij}v^j - 8\hat{R}_i\right) + \frac{1}{c^6}\left(\frac{5}{16}v^6v^i + \frac{33}{8}Vv^4v^i - \frac{3}{2}V_iv^4\right.\\
& - 6V_jv^iv^jv^2 + \frac{49}{4}V^2v^2v^i + 2\hat{W}_{ij}v^jv^2 + 2\hat{W}_{jk}v^iv^jv^k - 10VV_iv^2\\
& - 20VV_jv^iv^j - 4\hat{R}_iv^2 - 8\hat{R}_jv^iv^j + \frac{9}{2}V^3v^i + 12V_jV_jv^i + 12\hat{W}_{ij}Vv^j\\
& \left. + 12\hat{X}v^i + 16\hat{Z}_{ij}v^j - 10V^2V_i - 8\hat{W}_{ij}V_j - 8V\hat{R}_i - 16\hat{Y}_i\right) + \mathcal{O}\left(\frac{1}{c^8}\right),
\end{aligned} \tag{2.12a}$$

$$\begin{aligned}F^i = \partial_i V &+ \frac{1}{c^2}\left(-V\partial_i V + \frac{3}{2}\partial_i V v^2 - 4\partial_i V_j v^j\right)\\
&+ \frac{1}{c^4}\left(\frac{7}{8}\partial_i V v^4 - 2\partial_i V_j v^j v^2 + \frac{9}{2}V\partial_i V v^2 + 2\partial_i \hat{W}_{jk} v^j v^k - 4V_j\partial_i V v^j\right.\\
&\left. - 4V\partial_i V_j v^j - 8\partial_i \hat{R}_j v^j + \frac{1}{2}V^2\partial_i V + 8V_j\partial_i V_j + 4\partial_i \hat{X}\right)\\
&+ \frac{1}{c^6}\left(\frac{11}{16}v^6\partial_i V - \frac{3}{2}\partial_i V_j v^j v^4 + \frac{49}{8}V\partial_i V v^4 + \partial_i \hat{W}_{jk} v^2 v^j v^k\right.\\
&- 10V_j\partial_i V v^2 v^j - 10V\partial_i V_j v^2 v^j - 4\partial_i \hat{R}_j v^2 v^j + \frac{27}{4}V^2\partial_i V v^2\\
&+ 12V_j\partial_i V_j v^2 + 6\hat{W}_{jk}\partial_i V v^j v^k + 6V\partial_i \hat{W}_{jk} v^j v^k + 6\partial_i \hat{X} v^2\\
&+ 8\partial_i \hat{Z}_{jk} v^j v^k - 20V_j V\partial_i V v^j - 10V^2\partial_i V_j v^j - 8V_k\partial_i \hat{W}_{jk} v^j\\
&- 8\hat{W}_{jk}\partial_i V_k v^j - 8\hat{R}_j\partial_i V v^j - 8V\partial_i \hat{R}_j v^j - 16\partial_i \hat{Y}_j v^j\\
&- \frac{1}{6}V^3\partial_i V - 4V_j V_j\partial_i V + 16\hat{R}_j\partial_i V_j + 16V_j\partial_i \hat{R}_j\\
&\left. - 8V V_j\partial_i V_j - 4\hat{X}\partial_i V - 4V\partial_i \hat{X} + 16\partial_i \hat{T}\right) + \mathcal{O}\left(\frac{1}{c^8}\right). \qquad (2.12b)\end{aligned}$$

From this, we deduce the coordinate acceleration as

$$a^i = F^i - \frac{\mathrm{d}}{\mathrm{d}t}(P^i - v^i). \qquad (2.13)$$

The radiation-reaction terms in the equations of motion, (2.12), (2.13), appear explicitly when performing the PN expansion of our elementary nonlinear potentials. They are obtained by the careful consideration of all possible contributions at the 2.5PN or 3.5PN orders in each of these potentials. More precisely, the 2.5PN and 3.5PN contributions arise both from the expansion of the retardation of the inverse d'Alembertian retarded integrals, and from the PN corrections already present in the sources of the potentials. For instance, the sources, equations (2.3)–(2.5), involve the mass, current and stress densities, σ, σ_i and σ_{ij}, given by (2.2), which depend themselves on the potentials in a manner consistent with the iterative PN formalism. In addition, some required contributions also occur from the systematic *order reduction* of the accelerations, which is applied when calculating the time derivatives associated with the retardations of the potentials, or for instance, when performing the total time derivative of the linear momentum density function, P^i in equation (2.13). By order reduction, we refer to the replacement of the acceleration by its explicit expression given by the PN equations of motion in terms of the bodies' positions and velocities, and the subsequent PN re-expansion, which is performed in a consistent manner at the PN order in question.

From equations (2.12), (2.13), it is apparent that the expressions for all the potentials or their spatial derivatives are required at a relative 0.5PN order above the existing explicit 3PN expansions computed in [18]. They must be expressed solely as a function of the masses and velocities (after the order reduction of the accelerations). For the moment, equations (2.12), (2.13) were derived in the case of general matter systems, and the precise mathematical description of the source, i.e. the matter stress-energy tensor $T^{\mu\nu}$, is required next. Section 3 introduces the δ-function model of the compact binary and the associated regularization.

The PN precision in each of the potentials and their gradients, which we require in order to control the 3.5PN acceleration, is given in the following table. It is convenient to distinguish between the computation of a potential and that of its gradient, because in the case of the odd terms, the gradient is often easier to compute. This is due to the simplification that the 0.5PN relative term in the expansion of the retardation of a potential (we refer later to such an odd-parity term as 'retardation-like') is always a mere function of time, which thus vanishes when taking the gradient. On the other hand, the gradient of a potential is often required at a higher PN order than the potential itself, so it is generally good practice to perform a separate computation for the gradient. At 3.5PN order, it is necessary to develop the odd terms in,

$$\begin{array}{llll@{\qquad}lll}
\partial_i V & \text{to order} & \mathcal{O}(c^{-7}), & \partial_i \hat{X} & \text{to order} & \mathcal{O}(c^{-3}), \\
V & \text{to order} & \mathcal{O}(c^{-5}), & \hat{X} & \text{to order} & \mathcal{O}(c^{-1}), \\
V_i \text{ and } \partial_j V_i & \text{to order} & \mathcal{O}(c^{-5}), & \hat{Z}_{ij} \text{ and } \partial_k \hat{Z}_{ij} & \text{to order} & \mathcal{O}(c^{-1}), \\
\hat{W}_{ij} \text{ and } \partial_k \hat{W}_{ij} & \text{to order} & \mathcal{O}(c^{-3}), & \hat{Y}_i \text{ and } \partial_j \hat{Y}_i & \text{to order} & \mathcal{O}(c^{-1}), \\
\hat{R}_i \text{ and } \partial_j \hat{R}_i & \text{to order} & \mathcal{O}(c^{-3}), & \partial_i \hat{T} & \text{to order} & \mathcal{O}(c^{-1}).
\end{array}$$

Apart from the purely compact-support potentials, V and V_i, the above potentials consist of both compact (C) and non-compact (QNC and/or CNC) support distributed sources. The following sections 3, 4 systematically treat how to evaluate each of these different types of contributions. We shall find that the evaluation of the required integrals for the QNC and CNC terms yield explicit *closed-form* expressions, valid at any field point over all space, for all the odd parts of potentials in the previous table. This is in contrast with the computation of the equations of motion at the previous 3PN order [18], where closed-form solutions to certain nonlinear Poisson-like integrals could not be given at any field point but existed only at the location of each particle (in a regularized sense). As section 4 illustrates, the alternative 'direct' evaluation method of Poisson-like integrals at the location of the particles, using the same method as for the 3PN equations of motion [18, 35], provides a further verification of our analytic closed-form Poisson-like solutions.

3. Application to point particles

The compact binary system is modelled as two structureless point particles with masses m_1 and m_2, which are described by δ-function singularities, and move on a general, not necessarily circular, orbit. We neglect the intrinsic rotations (spins) of the particles[4]. As part of the formalism of [18], we assume that although the equations of motion (2.10)–(2.13) were derived under the assumption of a general smooth stress-energy tensor, i.e. $C^\infty(\mathbb{R}^3)$, they remain valid in the case of point particles, provided that we supplement the calculation by a consistent use of a self-field regularization. As discussed in the introduction, the Hadamard self-field regularization is appropriate for the present purpose.

The use of the point-particle model is physically justified in the case of *compact* objects by the fact that, using a Newtonian argument, the tidal effects are formally equivalent to a correction of the order 5PN $\sim \mathcal{O}(c^{-10})$ compared to the Newtonian force law (see, e.g., [48]). An *a posteriori* justification is also that the 2.5PN equations of motion of self-gravitating, extended compact bodies, as derived in [23, 49–51], are in complete agreement with those derived for the model of point particles in [17, 28–30]. The same is true at 3PN order: there is agreement between the 3PN equations of motion of extended compact bodies [43, 44] and those for point particles [18, 41]. A general way to justify the use of structureless point particles in order to describe gravitationally condensed objects is to invoke the 'effacing property' of general relativity (a consequence of the strong version of the equivalence principle). According to this property, the internal structure is effaced when considering the motion and the radiation of the compact bodies, so that one can describe them only by their masses [30]. Once the use

[4] If necessary, the spins may be added to the formalism along the lines of [45–47].

of delta functions is physically justified, the advantage, of course, is that they considerably simplify the calculations.

Following the standard prescription in general relativity, we write the distributional stress-energy tensor of point particles as

$$T^{\mu\nu}(\mathbf{x},t) = \mu_1(t) v_1^\mu(t) v_1^\nu(t) \delta(\mathbf{x}-\mathbf{y}_1(t)) + 1 \leftrightharpoons 2, \tag{3.1}$$

where $\mathbf{x}$ is the field point, $\mathbf{y}_1$ and $\mathbf{y}_2$ are the positions of the particles, $\mathbf{v}_1(t) = \mathrm{d}\mathbf{y}_1(t)/\mathrm{d}t$ is the coordinate velocity in harmonic coordinates and $v_1^\mu \equiv (c, \mathbf{v}_1)$. The symbol $1 \leftrightharpoons 2$ means the same terms but with the particle labels 1 and 2 exchanged, and δ denotes the usual Dirac three-dimensional delta-function. Alternatively, for the mass, current and stress densities defined by equations (2.2), we find,

$$\sigma = \widetilde{\mu}_1 \delta(\mathbf{x}-\mathbf{y}_1) + 1 \leftrightharpoons 2, \tag{3.2a}$$

$$\sigma_i = \mu_1 v_1^i \delta(\mathbf{x}-\mathbf{y}_1) + 1 \leftrightharpoons 2, \tag{3.2b}$$

$$\sigma_{ij} = \mu_1 v_1^i v_1^j \delta(\mathbf{x}-\mathbf{y}_1) + 1 \leftrightharpoons 2, \tag{3.2c}$$

where the quantities μ_1 and $\widetilde{\mu}_1$ are some explicit functions of coordinate time t, through the source trajectories, $\mathbf{y}_1(t)$ and $\mathbf{y}_2(t)$, and velocities, $\mathbf{v}_1(t)$ and $\mathbf{v}_2(t)$, given by

$$\mu_1(t) = \frac{m_1}{\sqrt{(g g_{\rho\sigma})_1 \frac{v_1^\rho v_1^\sigma}{c^2}}}, \tag{3.3a}$$

$$\widetilde{\mu}_1(t) = \mu_1(t)\left[1 + \frac{\mathbf{v}_1^2}{c^2}\right]. \tag{3.3b}$$

Here, $(g_{\rho\sigma})_1$ and $(g)_1$ denote the values of the metric and its determinant computed at the position of particle 1 following the prescription of the Hadamard *partie finie* introduced in equation (3.5). As we emphasize later, we do not encounter any problems associated with the 'non-distributivity' of Hadamard's regularization at the present 3.5PN order; so, for instance, $(g g_{\rho\sigma})_1$ in equation (3.3) may be replaced by the product of regularizations, $(g)_1 (g_{\rho\sigma})_1$. Thus, it is unnecessary to take into account the subtleties associated with several possible choices for the stress-energy tensor in the context of Hadamard's regularization, which depend on whether the factors of the delta function $\delta(\mathbf{x}-\mathbf{y}_1)$ are supposed to be evaluated at any field point $\mathbf{x}$ or at the particle's position $\mathbf{y}_1$ as assumed in equations (3.3).[5] The latter problems resulted in the appearance of some ambiguities in the application of Hadamard's regularization at the 3PN order. The ambiguities have since then been resolved by means of dimensional regularization [40–42], but do not concern us here as we are interested in the 3.5PN approximation which is unambiguous. We are thus following here the straightforward prescription for the stress-energy tensor of point particles in general relativity.

We must now be more specific on the method chosen for computing the metric coefficients at point 1 in equations (3.3). Let $F(\mathbf{x})$ be a typical function we encounter in the problem, where for convenience, we indicate only the relevant dependence of the function on the field point $\mathbf{x}$ (it depends also on coordinate time t through the source positions and velocities).

[5] For instance, a different prescription, valid in the context of the extended Hadamard regularization, was given in [36]. In this prescription, the factor in equation (3.3a) involving the determinant of the metric is calculated at $\mathbf{x}$ whilst the other factor is evaluated at $\mathbf{y}_1$; in addition, a special version of the Dirac delta function, designed in such a way that it permits to keep track of the Lorentz invariance of the formalism, is assumed in [36].

The function $F(\mathbf{x})$ is smooth on $\mathbb{R}^3$, except at the singular points $\mathbf{y}_1$ and $\mathbf{y}_2$, around which it admits a power-like singular expansion of the type (for any $N \in \mathbb{N}$),

$$F(\mathbf{x}) = \sum_{a_0 \leqslant a \leqslant N} r_1^a f_1^{[a]}(\mathbf{n}_1) + \mathcal{O}\left(r_1^N\right), \tag{3.4}$$

(and similarly when $1 \leftrightharpoons 2$), where $r_1 \equiv |\mathbf{x} - \mathbf{y}_1| \to 0$ and the coefficients $f_1^{[a]}$ of the various powers of r_1 depend on the unit direction of approach to the singularity, $\mathbf{n}_1 \equiv (\mathbf{x} - \mathbf{y}_1)/r_1$. The powers of r_1 are relative integers, $a \in \mathbb{Z}$, and bounded from below by some typically negative integer a_0, depending on the F in question. The coefficients $f_1^{[a]}$ for which $a < 0$ are called the *singular* coefficients of F. The class of functions such as F is called $\mathcal{F}$. The Hadamard *partie finie* of $F \in \mathcal{F}$ at the singular point 1 is then defined by the angular average

$$(F)_1 = \int \frac{\mathrm{d}\Omega_1}{4\pi} f_1^{[0]}(\mathbf{n}_1), \tag{3.5}$$

where $\mathrm{d}\Omega_1 \equiv \mathrm{d}\Omega\,(\mathbf{n}_1)$ is the solid angle which is centred on $\mathbf{y}_1$ and in the direction $\mathbf{n}_1$. In principle, the Hadamard *partie finie* is non-distributive (with respect to multiplication) in the sense that $(FG)_1 \neq (F)_1(G)_1$ in general for F and G belonging to $\mathcal{F}$. However, for the terms occurring at 3.5PN order, it is unnecessary to account for this feature, as all the functions encountered at this order will in fact be such that $(FG)_1 = (F)_1(G)_1$.

The Hadamard *partie finie* of an integral, in short Pf $\int \mathrm{d}^3\mathbf{x}\, F$, which has divergences due to the singular expansion (3.4) of the function around the singular points $\mathbf{y}_1$ and $\mathbf{y}_2$, is defined by the always existing limit,

$$\begin{aligned}\mathrm{Pf} \int \mathrm{d}^3\mathbf{x}\, F = \lim_{s \to 0} \Bigg\{ &\int_{\mathcal{S}(s)} \mathrm{d}^3\mathbf{x}\, F(\mathbf{x}) \\ &+ \sum_{a+3<0} \frac{s^{a+3}}{a+3} \int d\Omega_1 f_1^{[a]}(\mathbf{n}_1) + \ln\left(\frac{s}{s_1}\right) \int \mathrm{d}\Omega_1\, f_1^{[-3]}(\mathbf{n}_1) + 1 \leftrightharpoons 2 \Bigg\},\end{aligned} \tag{3.6}$$

where in the RHS the integration extends on the domain $\mathcal{S}(s) \equiv \mathbb{R}^3 \setminus \mathcal{B}_1(s) \cup \mathcal{B}_2(s)$, i.e. defined by the whole space from which one has excised two coordinate balls $\mathcal{B}_1(s)$ and $\mathcal{B}_2(s)$ centred on the two particles and having the (same) radius s. The other terms are defined with the help of the singular coefficients in the expansion of the function given by (3.4). Regularization (3.6) depends in principle on two constants, s_1 and s_2, appearing in the logarithmic terms. These constants play an important role at 3PN order, but will never appear in this work, since there are no logarithmic divergences at the 3.5PN order. In equation (3.6), we suppose that the integral converges at infinity, when $|\mathbf{x}| \to +\infty$. We shall explain below how one treats the integral in the case where it diverges at infinity.

In the extended version of Hadamard's regularization [35, 36], one associates with any $F \in \mathcal{F}$ a *partie-finie* 'pseudo-function', i.e. a linear form defined on the set $\mathcal{F}$, which permits us to give a precise meaning to the notions of Dirac delta functions, and derivatives of singular functions in a distributional sense, when they are multiplied by or act on other singular functions in the class $\mathcal{F}$. The detailed construction of [35] was useful at 3PN order, but is not needed in this work. Nevertheless, it is convenient, because of the availability of the computer programs used in [18], to adopt all the rules of the extended Hadamard regularization (we know anyway that the different variants of Hadamard's regularization give the same answer at 3.5PN order). In particular, the Dirac delta function is defined in the extended Hadamard regularization by (for any $F \in \mathcal{F}$),

$$\mathrm{Pf} \int \mathrm{d}^3\mathbf{x}\, \delta(\mathbf{x} - \mathbf{y}_1) F(\mathbf{x}) = (F)_1, \tag{3.7}$$

where $(F)_1$ is the *partie finie* of the function given by (3.5), and where the indication Pf reminds us that the equality is true in the sense of the *partie-finie* integral (3.6).[6]

In addition, the derivatives of singular functions, say $\partial_i F$, are to be performed in a distributional sense, and for this work, we use the explicit formula of the extended Hadamard regularization,

$$\partial_i F = (\partial_i F)_{\text{ordinary}} + \mathrm{D}_i[F], \tag{3.8}$$

where the first term represents the derivative in the ordinary sense (as algebraic computer programs would compute it), and where the second term is the purely distributional part of the derivative, given explicitly by,

$$\mathrm{D}_i[F] = 4\pi n_1^i \left[\frac{1}{2} r_1 f_1^{[-1]}(\mathbf{n}_1) + \sum_{k\geqslant 0} \frac{1}{r_1^k} f_1^{[-2-k]}(\mathbf{n}_1) \right] \delta(\mathbf{x}-\mathbf{y}_1) + 1 \leftrightarrows 2. \tag{3.9}$$

The distributional terms depend only on the singular coefficients of the expansion of F. It was shown in [35] that this derivative generalizes the usual distributional derivative of distribution theory in the context of the class of singular functions $\mathcal{F}$. It is such that one can integrate by parts any integrals; in particular, the integral of the gradient of any $F \in \mathcal{F}$, considered in the previous distributional sense, is always zero. Multiple derivatives, as well as time derivatives, are treated in a similar way, and the reader is referred to [35] for details. Note, however, that the distributional derivative (3.8), (3.9), like the usual distributional derivative of distribution theory [33], is seen not to satisfy the Leibniz rule for the derivation of a product. This poses a problem at the 3PN order (by the presence of certain ambiguity parameters, later resolved by means of dimensional regularization), but not at the next order of 3.5PN.

In our investigations, we shall always consider singular functions $F \in \mathcal{F}$ in the form of a PN expansion. In order to be systematic, we introduce a special notation for the PN coefficients (with odd- or even-type parity) of the function F, say,

$$F(\mathbf{x}, t, c) = \sum_n \frac{1}{c^n} \underset{(n)}{F}(\mathbf{x}, t). \tag{3.10}$$

In this paper, we neglect the dependence of the PN coefficients on the logarithm of c, since such $\ln c$ terms do not occur at the 3.5PN order. The coefficients $F_{(n)}$ (and their gradients) will be computed at the points 1 or 2 by means of the *partie finie* (3.5), leading to the evaluation of such objects like

$$\underset{(n)}{(F)}_1 = \int \frac{\mathrm{d}\Omega_1}{4\pi} \underset{(n)}{f}{}_1^{[0]}(\mathbf{n}_1), \tag{3.11}$$

where F stands for any of the PN iterated potentials V, V_i, $\hat{W}_{ij}, \ldots$ defined in section 2.

4. Computation of the nonlinear potentials

4.1. Compact-support potentials

Following the nomenclature convention introduced in section 2, compact (C) support potentials refer to inverse d'Alembertian retarded integrals of some source terms, which possess as a factor, the source mass, current or stress densities defined by equations (2.2); the latter, therefore, for our model of point particles, are of the form $F_1\delta_1$ and $1 \leftrightarrows 2$, where $F_1(\mathbf{x}) \in \mathcal{F}$ and $\delta_1(\mathbf{x}) \equiv \delta(\mathbf{x}-\mathbf{y}_1)$. We shall illustrate the scheme by the derivation of the 3.5PN term in

[6] The '*partie-finie* delta function', satisfying (3.7), has been developed in [35] by the limiting case of a particular class of pseudo-functions defined from the notion of the Riesz delta function [52].

the purely compact-support potential $V \equiv V^{(\mathrm{C})}$ defined by equation (2.3a). Performing the expansion of retardations inside the d'Alembertian integral, we obtain,

$$V(\mathbf{x}',t) = G\sum_{n=0}^{7}\frac{(-)^n}{n!c^n}\left(\frac{\partial}{\partial t}\right)^n\int \mathrm{d}^3\mathbf{x}\,|\mathbf{x}'-\mathbf{x}|^{n-1}\sigma(\mathbf{x},t) + \mathcal{O}\left(\frac{1}{c^8}\right). \tag{4.1}$$

The source density, σ, is substituted by its expression valid for two point masses, equation (3.2a), where we recall that $\widetilde{\mu}_1(t)$ is a function of time defined by (3.3). This results in,

$$\begin{aligned} V = G\Bigg\{&\frac{\widetilde{\mu}_1}{r_1} - \frac{1}{c}\frac{\mathrm{d}}{\mathrm{d}t}(\widetilde{\mu}_1) + \frac{1}{2c^2}\frac{\partial^2}{\partial t^2}(\widetilde{\mu}_1 r_1) - \frac{1}{6c^3}\frac{\partial^3}{\partial t^3}\left(\widetilde{\mu}_1 r_1^2\right) + \frac{1}{24c^4}\frac{\partial^4}{\partial t^4}\left(\widetilde{\mu}_1 r_1^3\right)\\ &- \frac{1}{120c^5}\frac{\partial^5}{\partial t^5}\left(\widetilde{\mu}_1 r_1^4\right) + \frac{1}{720c^6}\frac{\partial^6}{\partial t^6}\left(\widetilde{\mu}_1 r_1^5\right) - \frac{1}{5040c^7}\frac{\partial^7}{\partial t^7}\left(\widetilde{\mu}_1 r_1^6\right)\Bigg\}\\ &+ 1 \leftrightharpoons 2 + \mathcal{O}\left(\frac{1}{c^8}\right), \end{aligned} \tag{4.2}$$

where $r_1 = |\mathbf{x} - \mathbf{y}_1|$ (for convenience we call $\mathbf{x}$ the field point in this formula). Note that the $1/c$ term is a mere function of time, and thus vanishes when taking the spatial gradient (hence we employ for this term the notation for a total time derivative $\mathrm{d}/\mathrm{d}t$ instead of the partial derivative $\partial/\partial t$). Furthermore, one can see that this term is actually of order $1/c^3$ since the Newtonian approximation to $\widetilde{\mu}_1$, namely m_1, is constant. Performing repeatedly the time derivatives of r_1 introduces some accelerations which must be consistently order reduced by means of the PN equations of motion. In equation (4.2), we require the expression of $\widetilde{\mu}_1(t)$ up to 3.5PN order, which we easily find by inserting the PN metric (2.1) into definition (3.3), and results in,

$$\begin{aligned} \frac{\widetilde{\mu}_1}{m_1} = 1 &+ \frac{1}{c^2}\left[-(V)_1 + \frac{3}{2}v_1^2\right] + \frac{1}{c^4}\left[-2(\hat{W})_1 + \frac{1}{2}(V^2)_1 + \frac{1}{2}(V)_1 v_1^2 - 4(V_i)_1 v_1^i + \frac{7}{8}v_1^4\right]\\ &+ \frac{1}{c^6}\Bigg[-8(\hat{Z})_1 - 4(\hat{X})_1 + 2(\hat{W})_1(V)_1 - 4(V_i)_1(V_i)_1 - \frac{1}{6}(V^3)_1\\ &+ \frac{11}{4}(V^2)_1 v_1^2 - 8(\hat{R}_i)_1 v_1^i + 2(\hat{W}_{ij})_1 v_1^i v_1^j - 3(\hat{W})_1 v_1^2\\ &- 4(V)_1(V_i)_1 v_1^i - 10(V_i)_1 v_1^i v_1^2 + \frac{33}{8}(V)_1 v_1^4 + \frac{11}{16}v_1^6\Bigg] + \mathcal{O}\left(\frac{1}{c^8}\right). \end{aligned} \tag{4.3}$$

Here, the value of each of the elementary potentials is taken at the singularity 1 following the Hadamard *partie finie* (3.5). For all the terms in equation (4.3), the distributivity of Hadamard's *partie finie* at this order is verified, e.g. $(\hat{W}V)_1 = (\hat{W})_1(V)_1$.

Evidently, the computation of $\widetilde{\mu}_1$ and V proceeds using the PN iteration, where one begins with V at Newtonian order, given by (using the notation (3.10), (3.11)),

$$\underset{(0)}{V} = \frac{Gm_1}{r_1} + 1 \leftrightharpoons 2, \tag{4.4a}$$

$$\underset{(0)}{(V)_1} = \frac{Gm_2}{r_{12}}, \tag{4.4b}$$

where $r_{12} \equiv |\mathbf{y}_1 - \mathbf{y}_2|$. This is then inserted into the 1PN term of (4.3) in order to obtain $\widetilde{\mu}_1$ at 1PN order, which hence enables one to deduce V itself at 1PN order, and so on. By taking into account the fact that there is no odd term in $\widetilde{\mu}_1$ at order $1/c^3$, we find that the first odd term in V at the level $1/c^3$ is given by

$$\underset{(3)}{V} = G\left\{-\frac{\mathrm{d}}{\mathrm{d}t}\underset{(2)}{(\widetilde{\mu}_1)} - \frac{1}{6}\frac{\partial^3}{\partial t^3}\left(\underset{(0)}{\widetilde{\mu}_1} r_1^2\right)\right\} + 1 \leftrightharpoons 2, \tag{4.5}$$

(where $\widetilde{\mu}_{(0)1} = m_1$). However, the dominant odd term in the gradient $\partial_i V$ is only at order $1/c^5$. Note that there is an odd term $\sim 1/c$ in the case of $\hat{W}_{ij}$ and for all the potentials besides V and V_i. The computation of both $(\hat{W})_1$ at 0.5PN order and $(V)_1$ at 1.5PN order is required to get $\widetilde{\mu}_1$ at 2.5PN order, and we have,

$$\underset{(5)}{\widetilde{\mu}_1} = m_1 \left\{ -(\underset{(3)}{V})_1 - 2(\underset{(1)}{\hat{W}})_1 \right\}. \tag{4.6}$$

A useful feature of the potential $\hat{W} \equiv \hat{W}_{ii}$ is that it can be expressed at the 0.5PN order in a simple way using the following compact-support form (easily deduced from (2.4*c*));

$$\hat{W} = \Box_{\mathcal{R}}^{-1} \left[8\pi G \left(\sigma_{ii} - \frac{1}{2}\sigma V \right) \right] - \frac{1}{2} V^2 + \mathcal{O}\left(\frac{1}{c^2} \right), \tag{4.7}$$

hence we obtain,

$$\underset{(1)}{\hat{W}} = 2G \frac{\mathrm{d}}{\mathrm{d}t} \int \mathrm{d}^3\mathbf{x} \left(\underset{(0)}{\sigma}_{ii} - \frac{1}{2} \underset{(0)}{\sigma} \underset{(0)}{V} \right) = 2Gm_1 \frac{\mathrm{d}}{\mathrm{d}t} \left(v_1^2 - \frac{1}{2} (\underset{(0)}{V})_1 \right) + 1 \leftrightarrows 2. \tag{4.8}$$

The 0.5PN term is a purely spatial integral with compact support; it is only a function of time, hence $(\hat{W}_{(1)})_1 = \hat{W}_{(1)}$. However, for some more involved potentials, the 0.5PN term will also depend on space, because of the contribution from the Poisson integral of a corresponding 0.5PN term in the source of the potential.

Continuing in this manner, and using the explicit computations of the NC potentials as explained in the following sections, we then obtain $\widetilde{\mu}_1$ at the required 3.5PN order,

$$\begin{aligned} \underset{(7)}{\widetilde{\mu}_1} = m_1 \Big\{ & -(\underset{(5)}{V})_1 - 2(\underset{(3)}{\hat{W}})_1 + (\underset{(0)}{V})_1 (\underset{(3)}{V})_1 + \tfrac{1}{2} (\underset{(3)}{V})_1 v_1^2 - 4 (\underset{(3)}{V_i})_1 v_1^i - 8 (\underset{(1)}{\hat{Z}})_1 \\ & - 4 (\underset{(1)}{\hat{X}})_1 + 2 (\underset{(0)}{V})_1 (\underset{(1)}{\hat{W}})_1 - 8 (\underset{(1)}{\hat{R}_i})_1 v_1^i + 2 (\underset{(1)}{\hat{W}_{ij}})_1 v_1^i v_1^j - 3 (\underset{(1)}{\hat{W}})_1 v_1^2 \Big\}, \end{aligned} \tag{4.9}$$

which allows the computation of V at this order in a straightforward way,

$$\begin{aligned} \underset{(7)}{V} = G \Big\{ & \frac{1}{r_1} \underset{(7)}{\widetilde{\mu}_1} - \frac{\mathrm{d}}{\mathrm{d}t} (\underset{(6)}{\widetilde{\mu}_1}) + \frac{1}{2} \frac{\partial^2}{\partial t^2} (\underset{(5)}{\widetilde{\mu}_1} r_1) - \frac{1}{6} \frac{\partial^3}{\partial t^3} (\underset{(4)}{\widetilde{\mu}_1} r_1^2) \\ & - \frac{1}{120} \frac{\partial^5}{\partial t^5} (\underset{(2)}{\widetilde{\mu}_1} r_1^4) - \frac{1}{5040} \frac{\partial^7}{\partial t^7} (\underset{(0)}{\widetilde{\mu}_1} r_1^6) \Big\} + 1 \leftrightarrows 2. \end{aligned} \tag{4.10}$$

In fact, we only require the gradient of V at 3.5PN order, and so it is unnecessary to compute the second term in the RHS of (4.10), which vanishes when taking the gradient. All these calculations are systematically performed using algebraic computer programs.

4.2. *Quadratic non-compact-support potentials*

The structure of non compact (NC) support potentials is that of a d'Alembertian retarded integral whose 'source' term has a spatially non-compact-support distribution. The integral is perfectly well defined provided that some sensitive boundary conditions are given for the decay of the field at past null infinity (the no incoming radiation condition). However, when we expand the retardations of the d'Alembertian integral, some Poisson-like integrals will appear at high PN order, which typically become divergent due to the boundary of the integral at (spatial) infinity. This is the well-known problem of divergences of the PN expansion, which is related to the near-zone limitation of the validity of the PN expansion.

A solution of the latter problem has recently been proposed in [15]. Essentially, the work [15] showed that the PN expansion can in fact be iterated *ad infinitum* by using a particular solution of the Poisson equation at each step, which constitutes an appropriate generalization of the usual Poisson integral with a non-compact-support source. In this specific approach, the source term of the Poisson integral is multiplied by a factor $|\mathbf{x}|^B$, where B is a complex parameter, and the solution is defined by the 'finite part' in the Laurent expansion of the Poisson integral when the parameter B tends to zero. In a more recent work [16], we have written the end result of [15] in an alternative form, which shows that one proceeds with the expansion of retardations in the PN algorithm by inserting the factor $|\mathbf{x}|^B$ inside the integrand and taking the finite part in the above sense. This procedure will give the correct result for the radiation-reaction odd terms up to the 3.5PN order (we are of course still within our specific approach). Starting from the 4PN level, this procedure will also have to take into account the appearance of tail contributions in the radiation reaction (see [15, 16] for details).

Hence, we compute the PN expansion of any elementary potential by using such a finite part prescription (denoted $\mathrm{FP}_{B=0}$ in the following) to cure the problem of divergences of the integrals at the boundary at infinity. Let us consider as an example the computation of the non-compact-support part of the potential $\hat{W}_{ij}$, say,

$$\hat{W}_{ij}^{(\mathrm{QNC})} \equiv \Box_{\mathcal{R}}^{-1}[-\partial_i V \partial_j V], \tag{4.11}$$

which is required, as we have already seen, up to the relative 1.5PN level. By expanding the retardations up to this level, we obtain,

$$\hat{W}_{ij}^{(\mathrm{QNC})}(\mathbf{x}', t) = \mathop{\mathrm{FP}}_{B=0} \left\{ \Delta^{-1}[-r^B \partial_i V \partial_j V] + \frac{1}{c^2}\frac{\partial^2}{\partial t^2}\Delta^{-2}[-r^B \partial_i V \partial_j V] - \frac{1}{4\pi c}\frac{\mathrm{d}}{\mathrm{d}t} \times \int \mathrm{d}^3\mathbf{x}\, |\mathbf{x}|^B \partial_i V \partial_j V - \frac{1}{24\pi c^3}\frac{\partial^3}{\partial t^3}\int \mathrm{d}^3\mathbf{x}\, |\mathbf{x}|^B |\mathbf{x}' - \mathbf{x}|^2 \partial_i V \partial_j V \right\} + \mathcal{O}\left(\frac{1}{c^4}\right). \tag{4.12}$$

Once again, this particular way of performing the PN expansion, using the regularization $\mathrm{FP}_{B=0}$ 'at infinity', is justified by our previous works [15, 16]. From this expression, one sees that the odd terms at the levels $1/c$ and $1/c^3$ will come either from the Poisson or Poisson-like integrals (which are always even) applied to the odd terms already present in the corresponding source, or from the odd terms coming directly from the expansion of the retardations (as applied to the even part of the source). We shall henceforth refer to the first type of odd-parity terms as *Poisson-like*, and to the second type as *retardation-like*; 'Poisson-like' terms correspond to an even-parity operator applied to an odd source[7], whilst 'retardation-like' terms consist of an odd integral (containing explicitly an odd power of $1/c$ in front) with an even integrand.

To begin with, we see that up to 1.5PN order in equation (4.12), the odd terms are only retardation-like, i.e. given by the two terms with explicit powers of $1/c$ and $1/c^3$ (indeed the first odd term in the gradient $\partial_i V$ occurs only at 2.5PN order). We now compute these retardation-like terms; later, on the occasion of more complicated potentials, we shall also see how to compute the Poisson-like terms. We insert into equation (4.12) the precise form for the non-compact-support source, $\partial_i V \partial_j V$, using the PN expansion (4.2) up to order 1.5PN,

$$\partial_i V \partial_j V = G^2 \widetilde{\mu}_1^2 \partial_i \left(\frac{1}{r_1}\right) \partial_j \left(\frac{1}{r_1}\right) + G^2 \widetilde{\mu}_1 \widetilde{\mu}_2 \partial_i \left(\frac{1}{r_1}\right) \partial_j \left(\frac{1}{r_2}\right) - \frac{G^2 m_1^2}{c^2}\left[a_1^k \partial_{(i}\left(\frac{1}{r_1}\right)\partial_{j)k}(r_1) - v_1^k v_1^l \partial_{(i}\left(\frac{1}{r_1}\right)\partial_{j)kl}(r_1)\right]$$

[7] We call it the operator of the instantaneous potentials in [15, 16], denoted by

$$\mathcal{I}^{-1} = \sum_{k=0}^{+\infty}\left(\frac{\partial}{c\partial t}\right)^{2k}\Delta^{-1-k}.$$

$$- \frac{G^2 m_1 m_2}{c^2} \left[a_1^k \partial_{(i} \left(\frac{1}{r_2} \right) \partial_{j)k} (r_1) - v_1^k v_1^l \partial_{(i} \left(\frac{1}{r_2} \right) \partial_{j)kl} (r_1) \right] + 1 \leftrightarrows 2$$
$$+ \mathcal{O}\left(\frac{1}{c^4}\right). \tag{4.13}$$

Since $\widetilde{\mu}_1$ is a mere function of time, it will not affect the subsequent reasoning which deals with the spatial integrations. Simply, in the end, we replace $\widetilde{\mu}_1$ by its explicit expression at 1.5PN order as deduced from equation (4.3). Note again that (4.13) is purely 'even' up to 2PN order. We have replaced, where appropriate, the masses $\widetilde{\mu}_1$ and $\widetilde{\mu}_2$ by their Newtonian values m_1 and m_2.

We can distinguish two types of terms in (4.13); as suggested by their names, 'self' terms refer solely to a single particle and are proportional to e.g. $\widetilde{\mu}_1^2$ or m_1^2, whereas 'interaction' terms are functions of both particles and involve for instance the product $\widetilde{\mu}_1\widetilde{\mu}_2$. Furthermore, once the interaction terms are known, the 'self' terms can easily be deduced by taking the limit $\mathbf{y}_2 \to \mathbf{y}_1$ (and $1 \leftrightarrows 2$). We will, therefore, now focus on how to solve the interaction terms (we shall find that the self-terms are in fact zero in this calculation).

In order to compute firstly the term of order $1/c$ in (4.12) to the 1.5PN order, we note that each of the partial derivatives in (4.13), which acts at the field point $\mathbf{x}$, can be transformed into a derivative acting on the source point, either $\mathbf{y}_1$ or $\mathbf{y}_2$, and thus, one can merge the factors together and factor the differential operator outside the integral. Hence, for instance,

$$\underset{B=0}{\mathrm{FP}} \int \mathrm{d}^3\mathbf{x}\, |\mathbf{x}|^B \partial_i \left(\frac{1}{r_1}\right) \partial_j \left(\frac{1}{r_2}\right) = \frac{\partial}{\partial y_1^i} \frac{\partial}{\partial y_2^j} \underset{B=0}{\mathrm{FP}} \int \mathrm{d}^3\mathbf{x}\, \frac{|\mathbf{x}|^B}{r_1 r_2}. \tag{4.14}$$

Now, the remaining integral in the RHS of (4.14) can be operationally computed as a particular case of the known elementary integral called Y_L. In fact, we shall need two different types of such integrals, defined by

$$Y_L(\mathbf{y}_1, \mathbf{y}_2) = -\frac{1}{2\pi} \underset{B=0}{\mathrm{FP}} \int \mathrm{d}^3\mathbf{x}\, |\mathbf{x}|^B \frac{\hat{x}_L}{r_1 r_2}, \tag{4.15a}$$

$$T_L(\mathbf{y}_1, \mathbf{y}_2) = -\frac{1}{2\pi} \underset{B=0}{\mathrm{FP}} \int \mathrm{d}^3\mathbf{x}\, |\mathbf{x}|^B \hat{x}_L \frac{r_1}{r_2}, \tag{4.15b}$$

where $L \equiv i_1 \ldots i_\ell$ is a multi-spatial index of order ℓ, $\hat{x}_L$ denotes the STF product of spatial vectors: $\hat{x}_L \equiv \mathrm{STF}(x^{i_1} \ldots x^{i_\ell})$ also denoted $x_{\langle L\rangle} \equiv \hat{x}_L$ and the factor $-1/2\pi$ has been installed for later convenience. These integrals were introduced and computed in [53–55]. The general expressions are given in their simplest form by (see [55] for a detailed derivation),

$$Y_L = \frac{r_{12}}{\ell+1} \sum_{p=0}^{\ell} y_1^{\langle L-P} y_2^{P\rangle}, \tag{4.16a}$$

$$T_L = \frac{r_{12}^3}{3(\ell+1)(\ell+2)} \sum_{p=0}^{\ell} (p+1) y_1^{\langle L-P} y_2^{P\rangle}. \tag{4.16b}$$

Note that the finite part operator $\mathrm{FP}_{B=0}$ plays a crucial role in such calculations; it removes any divergence at infinity and makes our computation perfectly clean and well controlled. From the particular case $\ell = 0$, we find $Y = r_{12}$ and $T = r_{12}^3/6$, and thus obtain the integrals needed to compute the $1/c$ term in equation (4.12).

Consider next the retardation-like $1/c^3$ term in equation (4.12); as shown by (4.12) and (4.13), this consists in essence of finding the solution to the integral of the type

$$\int \mathrm{d}^3\mathbf{x}\, |\mathbf{x}|^B \frac{|\mathbf{x}' - \mathbf{x}|^2}{r_1 r_2} = \int \mathrm{d}^3\mathbf{x}\, |\mathbf{x}|^B \frac{|\mathbf{x}|^2}{r_1 r_2} - 2x'^k \int \mathrm{d}^3\mathbf{x}\, |\mathbf{x}|^B \frac{x^k}{r_1 r_2} + |\mathbf{x}'|^2 \int \mathrm{d}^3\mathbf{x}\, \frac{|\mathbf{x}|^B}{r_1 r_2}. \tag{4.17}$$

As we see, the second and last terms in the RHS can be computed from Y_L with $\ell = 1$ and $\ell = 0$, respectively. From (4.16*a*), one obtains $Y_k = r_{12}\big(y_1^k + y_2^k\big)\big/2$. The first term, on the other hand, involves *a priori* a new structure, and one defines an elementary function for this term, which reads[8],

$$S_L(\mathbf{y}_1, \mathbf{y}_2) = -\frac{1}{2\pi} \mathop{\mathrm{FP}}_{B=0} \int \mathrm{d}^3\mathbf{x}\, |\mathbf{x}|^{B+2} \frac{\hat{x}_L}{r_1 r_2}. \tag{4.18}$$

An explicit expression for this elementary function is (see [55]),

$$S_L(\mathbf{y}_1, \mathbf{y}_2) = \frac{r_{12}}{(\ell+1)(\ell+2)} \sum_{p=0}^{\ell} y_1^{\langle L-P} y_2^{P\rangle} \times \left[(\ell+1-p)\mathbf{y}_1^2 - \frac{2}{3}(p+1)(\ell+1-p)r_{12}^2 + (p+1)\mathbf{y}_2^2\right], \tag{4.19}$$

which thus permits us to close our computation of the above retardation-like odd terms. (The self terms, obtained from the interaction terms by the limit $\mathbf{y}_2 \to \mathbf{y}_1$, are zero.)

Summarizing, we find that our potential $\hat{W}_{ij}^{(\mathrm{QNC})}$ admits the following odd parts. At the 0.5PN level, it is given by a mere function of time t (through the time dependence of the source points $\mathbf{y}_1, \mathbf{y}_2$ and the velocities $\mathbf{v}_1, \mathbf{v}_2$). For later convenience, we introduce a special notation for it, and set,

$$\underset{(1)}{\hat{W}}{}_{ij}^{(\mathrm{QNC})} = \underset{(1)}{\alpha}{}_{ij}(t), \tag{4.20a}$$

$$\underset{(1)}{\alpha}{}_{ij} \equiv G^2 m_1 m_2 \frac{\mathrm{d}}{\mathrm{d}t}\left[\frac{\partial^2 r_{12}}{\partial y_1^{(i} \partial y_2^{j)}}\right]. \tag{4.20b}$$

At the next 1.5PN order, the potential is given by a quadratic function of the position, which we express in the form

$$\underset{(3)}{\hat{W}}{}_{ij}^{(\mathrm{QNC})} = \underset{(3)}{\alpha}{}_{ij}(t) + x_k\, \underset{(3)}{\beta}{}_{ij}^{k}(t) + \mathbf{x}^2\, \underset{(3)}{\gamma}{}_{ij}(t), \tag{4.21}$$

where the functions of time, as introduced, are respectively given by

$$\underset{(3)}{\alpha}{}_{ij} = \frac{G^2}{2}\frac{\mathrm{d}}{\mathrm{d}t}\left[(m_1 \underset{(2)}{\widetilde{\mu}}{}_2 + m_2 \underset{(2)}{\widetilde{\mu}}{}_1)\frac{\partial^2 r_{12}}{\partial y_1^i \partial y_2^j}\right] + \frac{G^2 m_1 m_2}{12}\frac{\mathrm{d}}{\mathrm{d}t}\left[a_1^k \frac{\partial^3 r_{12}^3}{\partial y_2^{(i}\partial y_1^{j)k}} + v_1^k v_1^l \frac{\partial^4 r_{12}^3}{\partial y_2^{(i}\partial y_1^{j)kl}} + \frac{1}{2}\frac{\mathrm{d}^2}{\mathrm{d}t^2}\frac{\partial^2}{\partial y_1^i \partial y_2^j}\left(r_{12}\mathbf{y}_1^2 + r_{12}\mathbf{y}_2^2 - \frac{2}{3}r_{12}^3\right)\right] + 1 \leftrightharpoons 2, \tag{4.22a}$$

$$\underset{(3)}{\beta}{}_{ij}^{k} = -\frac{G^2 m_1 m_2}{6}\frac{\mathrm{d}^3}{\mathrm{d}t^3}\frac{\partial^2}{\partial y_1^{(i}\partial y_2^{j)}}\left(r_{12} y_1^k + r_{12} y_2^k\right), \tag{4.22b}$$

$$\underset{(3)}{\gamma}{}_{ij} = \frac{G^2 m_1 m_2}{6}\frac{\mathrm{d}^3}{\mathrm{d}t^3}\frac{\partial^2 r_{12}}{\partial y_1^{(i}\partial y_2^{j)}}. \tag{4.22c}$$

As we shall see subsequently, the form of (4.21) is of particular use when one derives the expression of the crucial cubic-order potential.

[8] Actually, one easily checks that this function is related to the previous ones by

$$S_L = \left(1 - 2y_1^k \frac{\partial}{\partial y_1^k}\right) T_L + \mathbf{y}_1^2 Y_L.$$

4.3. Cubic non-compact-support potentials

In order to illustrate the more difficult types of computation which involve both Poisson-like and retardation-like spatial integrals, we consider the example of the part of the non-compact-support potential $\hat{X}$, referred to as cubic non-compact (CNC) in equation (2.9), and given by

$$\hat{X}^{(\mathrm{CNC})} \equiv \Box^{-1}_{\mathcal{R}}\left[\hat{W}^{(\mathrm{QNC})}_{ij}\partial^2_{ij}V\right], \tag{4.23}$$

in which we recall that $\hat{W}^{(\mathrm{QNC})}_{ij}$ is defined by (4.11). It is necessary to compute the odd term in $\hat{X}^{(\mathrm{CNC})}$ up to the required order 1.5PN. By expanding the retardations up to this level, we obtain,

$$\begin{aligned}\hat{X}^{(\mathrm{CNC})} = \underset{B=0}{\mathrm{FP}}\Bigg\{&\Delta^{-1}\left[r^B\hat{W}^{(\mathrm{QNC})}_{ij}\partial^2_{ij}V\right] + \frac{1}{c^2}\frac{\partial^2}{\partial t^2}\Delta^{-2}\left[r^B\hat{W}^{(\mathrm{QNC})}_{ij}\partial^2_{ij}V\right]\\ &+\frac{1}{4\pi c}\frac{\mathrm{d}}{\mathrm{d}t}\int \mathrm{d}^3\mathbf{x}\,|\mathbf{x}|^B\hat{W}^{(\mathrm{QNC})}_{ij}\partial^2_{ij}V\\ &+\frac{1}{24\pi c^3}\frac{\partial^3}{\partial t^3}\int \mathrm{d}^3\mathbf{x}\,|\mathbf{x}|^B|\mathbf{x}'-\mathbf{x}|^2\hat{W}^{(\mathrm{QNC})}_{ij}\partial^2_{ij}V\Bigg\} + \mathcal{O}\left(\frac{1}{c^4}\right).\end{aligned} \tag{4.24}$$

The last two explicit terms in the latter expression represent the retardation-like odd terms and are computed using the same techniques as in the case of the quadratic non-compact-support potential, $\hat{W}^{(\mathrm{QNC})}_{ij}$ (see the previous section).

In contrast, however, to the case of the QNC potential, we see from equation (4.24) that since the source of the Poisson integral will contain some odd terms at the $1/c$ and $1/c^3$ levels, there are, in addition to the retardation-like terms, some 'Poisson-like' contributions to the odd terms in the CNC potential (the first two terms in equation (4.24)). Since there are no terms at orders 0.5PN and 1.5PN in the gradient of V, one finds that the 0.5PN and 1.5PN pieces in the source term, $\hat{W}^{(\mathrm{QNC})}_{ij}\partial^2_{ij}V$, come only from the odd terms in the potential $\hat{W}^{(\mathrm{QNC})}_{ij}$ itself, which has already been computed in equations (4.20)–(4.22). We show only the more difficult case of the 1.5PN order. Hence, the source term for the Poisson integral is in the form

$$\left[\hat{W}^{(\mathrm{QNC})}_{ij}\partial^2_{ij}V\right]_{(3)} = \underset{(3)}{\alpha}_{ij}\partial^2_{ij}\underset{(0)}{V} + \underset{(1)}{\alpha}_{ij}\partial^2_{ij}\underset{(2)}{V} + \underset{(3)}{\beta}{}^k_{ij}x^k\partial^2_{ij}\underset{(0)}{V} + \underset{(3)}{\gamma}_{ij}\mathbf{x}^2\partial^2_{ij}\underset{(0)}{V}\,. \tag{4.25}$$

We work out this expression using the known even part of the V potential. By thus transforming the derivatives with respect to the field point into derivatives with respect to the source points (using $\partial_i = -\partial/\partial y_1^i$ when acting on a function of r_1), one arrives at a new expression which can be immediately integrated, with the result

$$\begin{aligned}\underset{B=0}{\mathrm{FP}}\,\Delta^{-1}\left[r^B\hat{W}^{(\mathrm{QNC})}_{ij}\partial^2_{ij}V\right]_{(3)} &= \underset{(3)}{\alpha}_{ij}\frac{\partial^2}{\partial y_1^{ij}}\left[Gm_1\frac{r_1}{2}\right] + \underset{(1)}{\alpha}_{ij}\frac{\partial^2}{\partial y_1^{ij}}\left[G\underset{(2)}{\tilde{\mu}}_1\frac{r_1}{2} + \frac{Gm_1}{24}\frac{\partial^2 r_1^3}{\partial t^2}\right]\\ &+\underset{(3)}{\beta}{}^k_{ij}\frac{\partial^2}{\partial y_1^{ij}}\left[\frac{Gm_1}{4}\left(x^k+y_1^k\right)r_1\right]\\ &+\underset{(3)}{\gamma}_{ij}\frac{\partial^2}{\partial y_1^{ij}}\left[Gm_1\left(\frac{r_1^3}{12}+\frac{1}{2}y_1^kx^kr_1\right)\right] + 1 \leftrightharpoons 2,\end{aligned} \tag{4.26}$$

where we recall that the time-dependent coefficients have been given in equations (4.20*b*) and (4.22). Concerning the other Poisson-like integral, the computation is easier because it is required only at 0.5PN order, and we simply obtain,

$$\underset{B=0}{\mathrm{FP}}\,\Delta^{-2}\left[r^B\hat{W}^{(\mathrm{QNC})}_{ij}\partial^2_{ij}V\right]_{(1)} = \underset{(1)}{\alpha}_{ij}\frac{\partial^2}{\partial y_1^{ij}}\left[Gm_1\frac{r_1^3}{24}\right] + 1 \leftrightharpoons 2. \tag{4.27}$$

Having in hand all the required terms at any field point $\mathbf{x}$, we can now compute the appropriate gradients of these potentials which enter the equations of motion (2.12), (2.13), and then obtain their value at the location of particle 1 using Hadamard's *partie finie* (3.5). For instance, the gradient of term (4.27), taken at the location of particle 1, reads

$$\left(\partial_k \mathop{\mathrm{FP}}_{B=0} \Delta^{-2}\left[r^B \hat{W}^{(\mathrm{QNC})}_{ij}\partial^2_{ij}V\right]_{(1)}\right)_1 = \mathop{\alpha}_{(1)}{}_{ij}\frac{\partial^3}{\partial y_1^{ijk}}\left[Gm_2\frac{r_{12}^3}{24}\right], \tag{4.28}$$

where $r_{12} = |\mathbf{y}_1 - \mathbf{y}_2|$ represents the particles' separation.

4.4. Alternative derivation of the contribution of NC potentials

As sections 4.2 and 4.3 discuss, the previous method consisted of first computing all the required NC potentials *at any field point* $\mathbf{x}$, and then accounting for their contributions to the equations of motion of particle 1 (say), by applying the Hadamard *partie finie* (3.5). One can, however, also compute *directly* the value at point 1 of any potential given in the form of a Poisson integral. This alternative technique is indeed used in the computation of the equations of motion at 3PN order in [18]. In the problem of the 3PN equations of motion, it was impossible to obtain the value of all the potentials at an arbitrary field point $\mathbf{x}$, and thus, the complete result could only be achieved by using the latter technique. In this case of the 3.5PN term, as seen previously, all the potentials at any field point could be computed by analytical methods. We are, therefore, able to perform an important verification of the result by directly evaluating the potentials 'on the line', i.e. at the source point $\mathbf{y}_1$ (say).

Such a direct evaluation on the line applies equally well to the computation of retardation-like odd terms or the Poisson and Poisson-like integrals. It proceeds from the Hadamard *partie-finie* formalism reviewed in section 3, and, in particular, from the expression of the *partie-finie* integral (3.6). We check that all the retardation-like odd terms can be cast in the form of some sum of *partie-finie* integrals of type (3.6), which are eventually multiplied by some spatial vector positions; this is due to the factor, say $|\mathbf{x} - \mathbf{x}'|^{2k}$, which enters such integrals, and can always be expanded as in equation (4.17). The computation using this technique of the Poisson-like terms is more complicated but the full proofs are given in section 5 of [35]. Here, we provide only a summary of this computation, which has been systematically performed to obtain full confirmation of our result. Typically, one is dealing with the Poisson integral of some $F \in \mathcal{F}$, say,

$$P(\mathbf{x}') = -\frac{1}{4\pi}\mathrm{Pf}\int \frac{\mathrm{d}^3\mathbf{x}}{|\mathbf{x} - \mathbf{x}'|}F(\mathbf{x}), \tag{4.29}$$

where the symbol Pf refers to Hadamard's *partie finie* in the sense of equation (3.6). We are interested in the value of this Poisson integral when $\mathbf{x}'$ tends to the particle's position $\mathbf{y}_1$, and this value takes the meaning of equation (3.5). In other words, the computation of the quantity $(P)_1$, and also of the corresponding gradient, $(\partial_i P)_1$, are required. They have already been given in equations (5.5) and (5.17*a*) of [35], and read as

$$(P)_1 = -\frac{1}{4\pi}\mathrm{Pf}\int \mathrm{d}^3\mathbf{x}\,\frac{F}{r_1} + \left[\ln\left(\frac{r_1'}{s_1}\right) - 1\right]\left(r_1^2 F\right)_1, \tag{4.30a}$$

$$(\partial_i P)_1 = -\frac{1}{4\pi}\mathrm{Pf}\int \mathrm{d}^3\mathbf{x}\,\frac{n_1^i}{r_1^2}F + \ln\left(\frac{r_1'}{s_1}\right)\left(n_1^i r_1 F\right)_1. \tag{4.30b}$$

Recall that the *partie-finie* integral (3.6) depends on two constants s_1 and s_2. In addition, $r_1' = |\mathbf{x}' - \mathbf{y}_1|$ in (4.30) is a 'constant' which tends toward zero when evaluating the *partie* *finie*. Although these constants played an important role in the computation of the equations of motion at 3PN order, none of them appear in this work. For this calculation, we also require the formulae concerning the twice-iterated Poisson integral, and the results were provided by equations (5.16) and (5.17*b*) in [35].

We have found by using the above formulae that the direct method is in complete agreement for all the terms with the analytical methods reviewed in sections 4.2 and 4.3. However, note that for the agreement to work, one must crucially take into account, in addition to the formulae such as (4.30), the contribution of the distributional part of derivatives. For this calculation, we use formula (3.9), which gives the distributional derivative in the extended Hadamard regularization.

5. The 3.5PN compact binary acceleration

We finally present our result, which gives the complete radiation-reaction force in the equations of motion of the compact binary at the 3.5PN order, for general orbits and in a general harmonic coordinate frame. We write the 3.5PN acceleration of particle 1, say, in the form,

$$\mathbf{a}_1 = \mathbf{A}_1^{\mathrm{N}} + \frac{1}{c^2}\mathbf{A}_1^{1\mathrm{PN}} + \frac{1}{c^4}\mathbf{A}_1^{2\mathrm{PN}} + \frac{1}{c^5}\mathbf{A}_1^{2.5\mathrm{PN}} + \frac{1}{c^6}\mathbf{A}_1^{3\mathrm{PN}} + \frac{1}{c^7}\mathbf{A}_1^{3.5\mathrm{PN}} + \mathcal{O}\left(\frac{1}{c^8}\right), \tag{5.1}$$

where the first term is given by the famous Newtonian law

$$\mathbf{A}_1^{\mathrm{N}} = -\frac{Gm_2}{r_{12}^2}\mathbf{n}_{12}. \tag{5.2}$$

The acceleration of the second particle is obtained by exchanging all the labels $1 \leftrightarrows 2$. The conservative (with even-parity) approximations 1PN, 2PN and 3PN have been computed elsewhere; they are thoroughly given by equation (7.16) in [18] or by equation (131) in [48], together with the value of the ambiguity parameter $\lambda = -1987/3080$ computed in [41].

The result central to our study concerns the radiation-reaction (odd-order) acceleration coefficients $\mathbf{A}_1^{2.5\mathrm{PN}}$ and $\mathbf{A}_1^{3.5\mathrm{PN}}$. We find, for the 2.5PN term[9],

$$\begin{aligned}\mathbf{A}_1^{2.5\mathrm{PN}} = {} & \frac{4G^2 m_1 m_2}{5 r_{12}^3}\left((n_{12}v_{12})\left[-6\frac{Gm_1}{r_{12}} + \frac{52}{3}\frac{Gm_2}{r_{12}} + 3v_{12}^2\right]\mathbf{n}_{12}\right.\\ & \left. + \left[2\frac{Gm_1}{r_{12}} - 8\frac{Gm_2}{r_{12}} - v_{12}^2\right]\mathbf{v}_{12}\right).\end{aligned} \tag{5.3}$$

This result is in perfect agreement with previous calculations in a general harmonic coordinate frame [17, 18, 28–30, 43, 44, 51], as well as in the centre-of-mass frame [23]. The new result is the complete expression of the 3.5PN coefficient in an arbitrary frame which we find to be given by

$$\begin{aligned}\mathbf{A}_1^{3.5\mathrm{PN}} = {} & \frac{G^2 m_1 m_2}{r_{12}^3}\left\{\frac{G^2 m_1^2}{r_{12}^2}\left[\left(\frac{3992}{105}(n_{12}v_1) - \frac{4328}{105}(n_{12}v_2)\right)\mathbf{n}_{12} - \frac{184}{21}\mathbf{v}_{12}\right]\right.\\ & + \frac{G^2 m_1 m_2}{r_{12}^3}\left[\left(-\frac{13\,576}{105}(n_{12}v_1) + \frac{2872}{21}(n_{12}v_2)\right)\mathbf{n}_{12} + \frac{6224}{105}\mathbf{v}_{12}\right]\\ & + \frac{G^2 m_2^2}{r_{12}^3}\left[-\frac{3172}{21}(n_{12}v_{12})\mathbf{n}_{12} + \frac{6388}{105}\mathbf{v}_{12}\right]\end{aligned}$$

[9] Our notation is $r_{12} = |\mathbf{y}_1 - \mathbf{y}_2|$, $\mathbf{n}_{12} = (\mathbf{y}_1 - \mathbf{y}_2)/r_{12}$ and $\mathbf{v}_1 = \mathrm{d}\mathbf{y}_1/\mathrm{d}t$, $\mathbf{a}_1 = \mathrm{d}\mathbf{v}_1/\mathrm{d}t$ for the harmonic-coordinates velocity and acceleration (together with $1 \leftrightarrows 2$). We set $\mathbf{v}_{12} = \mathbf{v}_1 - \mathbf{v}_2$ for the relative velocity. The parentheses indicate the usual Euclidean scalar product, e.g. $(n_{12}v_{12}) = \mathbf{n}_{12} \cdot \mathbf{v}_{12}$.

$$
\begin{aligned}
&+\frac{Gm_1}{r_{12}}\left[\left(48(n_{12}v_1)^3-\frac{696}{5}(n_{12}v_1)^2(n_{12}v_2)+\frac{744}{5}(n_{12}v_1)(n_{12}v_2)^2\right.\right.\\
&-\frac{288}{5}(n_{12}v_2)^3-\frac{4888}{105}(n_{12}v_1)v_1^2+\frac{5056}{105}(n_{12}v_2)v_1^2+\frac{2056}{21}(n_{12}v_1)(v_1v_2)\\
&\left.-\frac{2224}{21}(n_{12}v_2)(v_1v_2)-\frac{1028}{21}(n_{12}v_1)v_2^2+\frac{5812}{105}(n_{12}v_2)v_2^2\right)\mathbf{n}_{12}\\
&+\left(\frac{52}{15}(n_{12}v_1)^2-\frac{56}{15}(n_{12}v_1)(n_{12}v_2)-\frac{44}{15}(n_{12}v_2)^2-\frac{132}{35}v_1^2\right.\\
&\left.\left.+\frac{152}{35}(v_1v_2)-\frac{48}{35}v_2^2\right)\mathbf{v}_{12}\right]+\frac{Gm_2}{r_{12}}\left[\left(-\frac{582}{5}(n_{12}v_1)^3\right.\right.\\
&+\frac{1746}{5}(n_{12}v_1)^2(n_{12}v_2)-\frac{1954}{5}(n_{12}v_1)(n_{12}v_2)^2\\
&+158(n_{12}v_2)^3+\frac{3568}{105}(n_{12}v_{12})(v_1v_1)-\frac{2864}{35}(n_{12}v_1)(v_1v_2)\\
&\left.+\frac{10\,048}{105}(n_{12}v_2)(v_1v_2)+\frac{1432}{35}(n_{12}v_1)v_2^2-\frac{5752}{105}(n_{12}v_2)v_2^2\right)\mathbf{n}_{12}\\
&+\left(\frac{454}{15}(n_{12}v_1)^2-\frac{372}{5}(n_{12}v_1)(n_{12}v_2)+\frac{854}{15}(n_{12}v_2)^2-\frac{152}{21}v_1^2\right.\\
&\left.\left.+\frac{2864}{105}(v_1v_2)-\frac{1768}{105}v_2^2\right)\mathbf{v}_{12}\right]+\left(-56(n_{12}v_{12})^5+60(n_{12}v_1)^3v_{12}^2\right.\\
&-180(n_{12}v_1)^2(n_{12}v_2)v_{12}^2+174(n_{12}v_1)(n_{12}v_2)^2v_{12}^2\\
&-54(n_{12}v_2)^3v_{12}^2-\frac{246}{35}(n_{12}v_{12})v_1^4+\frac{1068}{35}(n_{12}v_1)v_1^2(v_1v_2)\\
&-\frac{984}{35}(n_{12}v_2)v_1^2(v_1v_2)-\frac{1068}{35}(n_{12}v_1)(v_1v_2)^2+\frac{180}{7}(n_{12}v_2)(v_1v_2)^2\\
&-\frac{534}{35}(n_{12}v_1)v_1^2v_2^2+\frac{90}{7}(n_{12}v_2)v_1^2v_2^2+\frac{984}{35}(n_{12}v_1)(v_1v_2)v_2^2\\
&\left.-\frac{732}{35}(n_{12}v_2)(v_1v_2)v_2^2-\frac{204}{35}(n_{12}v_1)v_2^4+\frac{24}{7}(n_{12}v_2)v_2^4\right)\mathbf{n}_{12}\\
&+\left(60(n_{12}v_{12})^4-\frac{348}{5}(n_{12}v_1)^2v_{12}^2+\frac{684}{5}(n_{12}v_1)(n_{12}v_2)v_{12}^2\right.\\
&-66(n_{12}v_2)^2v_{12}^2+\frac{334}{35}v_1^4-\frac{1336}{35}v_1^2(v_1v_2)+\frac{1308}{35}(v_1v_2)^2\\
&\left.\left.+\frac{654}{35}v_1^2v_2^2-\frac{1252}{35}(v_1v_2)v_2^2+\frac{292}{35}v_2^4\right)\mathbf{v}_{12}\right\}.
\end{aligned}
\tag{5.4}
$$

Recall that we obtain results (5.3) and (5.4) directly from (2.12) and (2.13) by summing up all the contributions of the regularized values of the potentials. The latter were computed following the two (rather independent) methods proposed in section 4.

We next give the result for the 2.5PN and 3.5PN relative accelerations in the centre-of-mass frame. For this calculation, we require the transformation equations for converting from the positions and velocities of the particles in the general frame to those in the centre-of-mass frame. Naturally, the centre-of-mass is defined by the nullity of the binary's mass dipole moment at the required PN order. All these relations have been worked out at the 3PN order in [56]. Specifically, the transformation equations take the form[10],

$$\mathbf{y}_1 = [X_2 + \nu(X_1 - X_2)\mathcal{P}]\,\mathbf{x} + \nu(X_1 - X_2)\mathcal{Q}\mathbf{v}, \tag{5.5a}$$

$$\mathbf{y}_2 = [-X_1 + \nu(X_1 - X_2)\mathcal{P}]\,\mathbf{x} + \nu(X_1 - X_2)\mathcal{Q}\mathbf{v}. \tag{5.5b}$$

The coefficients $\mathcal{P}$ and $\mathcal{Q}$ are given at 3PN order by equations (3.11), (3.12) in [56]. For our purposes, we require the relative 1PN even correction term in $\mathcal{P}$ and $\mathcal{Q}$, since we are computing the 3.5PN radiation-reaction force, which is at relative 1PN order. In addition, we have found that the 2.5PN odd correction term in the latter transformation equations, gives a crucial contribution at the 3.5PN order in the radiation-reaction force. Inspection of equations (3.11), (3.12) in [56], reveals that the 1PN correction term exists only in the coefficient $\mathcal{P}$ (since $\mathcal{Q}$ begins at the $1/c^4$ level), whilst the odd contribution at order $1/c^5$ is proportional to the velocity, and hence is contained only in $\mathcal{Q}$. Using our previous notation (3.10) for PN coefficients, we have the required terms

$$\underset{(2)}{\mathcal{P}} = \frac{v^2}{2} - \frac{Gm}{2r}, \tag{5.6a}$$

$$\underset{(5)}{\mathcal{Q}} = \frac{4Gm}{5}\left[v^2 - \frac{2Gm}{r}\right]. \tag{5.6b}$$

We have inserted equations (5.5a), (5.6a) into our general-frame result, in order, therefore, to obtain the 3.5PN relative centre-of-mass acceleration in the form,

$$\begin{aligned}\mathbf{a} &\equiv \mathbf{a}_1 - \mathbf{a}_2\\ &= \mathbf{A}^{\mathrm{N}} + \frac{1}{c^2}\mathbf{A}^{1\mathrm{PN}} + \frac{1}{c^4}\mathbf{A}^{2\mathrm{PN}} + \frac{1}{c^5}\mathbf{A}^{2.5\mathrm{PN}} + \frac{1}{c^6}\mathbf{A}^{3\mathrm{PN}} + \frac{1}{c^7}\mathbf{A}^{3.5\mathrm{PN}} + \mathcal{O}\left(\frac{1}{c^8}\right).\end{aligned} \tag{5.7}$$

All the terms up to 3PN order have already been computed in equations (3.16)–(3.18) of [56]. The radiation-reaction 2.5PN term reads

$$\mathbf{A}^{2.5\mathrm{PN}} = \frac{8G^2m^2\nu}{5r^3}\left\{\left[-3\frac{Gm}{r} - v^2\right]\mathbf{v} + (nv)\left[\frac{17}{3}\frac{Gm}{r} + 3v^2\right]\mathbf{n}\right\}, \tag{5.8}$$

whilst the 3.5PN contribution is given as

$$\begin{aligned}\mathbf{A}^{3.5\mathrm{PN}} = \frac{G^2m^2\nu}{r^3}\Bigg\{&\Bigg[\frac{G^2m^2}{r^2}\left(\frac{1060}{21} + \frac{104}{5}\nu\right) + \frac{Gmv^2}{r}\left(-\frac{164}{21} - \frac{148}{5}\nu\right)\\ &+ v^4\left(\frac{626}{35} + \frac{12}{5}\nu\right) + \frac{Gm(nv)^2}{r}\left(\frac{82}{3} + \frac{848}{15}\nu\right)\\ &+ v^2(nv)^2\left(-\frac{678}{5} - \frac{12}{5}\nu\right) + 120(nv)^4\Bigg]\mathbf{v}\\ &+ (nv)\Bigg[\frac{G^2m^2}{r^2}\left(-\frac{3956}{35} - \frac{184}{5}\nu\right) + \frac{Gmv^2}{r}\left(-\frac{692}{35} + \frac{724}{15}\nu\right)\\ &+ v^4\left(-\frac{366}{35} - 12\nu\right) + \frac{Gm(nv)^2}{r}\left(-\frac{294}{5} - \frac{376}{5}\nu\right)\\ &+ v^2(nv)^2\,(114 + 12\nu) - 112(nv)^4\Bigg]\mathbf{n}\Bigg\}.\end{aligned} \tag{5.9}$$

[10] For centre-of-mass quantities, we denote by $\mathbf{x} = \mathbf{y}_1 - \mathbf{y}_2$, $r = |\mathbf{x}|$, $\mathbf{n} = \mathbf{x}/r$ the relative binary's separation (formerly denoted $r_{12} = r$ and $\mathbf{n}_{12} = \mathbf{n}$), and by $\mathbf{v} = \mathrm{d}\mathbf{x}/\mathrm{d}t = \mathbf{v}_1 - \mathbf{v}_2$ the relative velocity. The mass parameters are given as

$$m = m_1 + m_2, \qquad X_1 = \frac{m_1}{m}, \qquad X_2 = \frac{m_2}{m}, \qquad \nu = X_1X_2.$$

Expressions (5.8) and (5.9) allow for an important and mutual consistency check with the results of Iyer and Will [26, 27]. These authors computed the radiation-reaction force at 3.5PN order for compact binary systems in a class of coordinate systems, by applying the energy and angular-momentum balance equations to relative 1PN order. As a consequence of the gauge freedom, the radiation reaction at 2.5PN order depends on two gauge parameters, denoted α and β, whilst at the 3.5PN order, it depends on six further gauge parameters, denoted δ_A, $A = 1, \ldots, 6$ [26, 27]. The eight parameters, α, β, δ_A, were proved to correspond exactly to the unconstrained degrees of freedom which relate to coordinate transformations. For instance, the 3.5PN radiation-reaction potentials of [12] were shown when evaluated in the case of compact binaries to correspond to a unique, self-consistent choice of all these gauge parameters [26, 27].

In this case, one also finds perfect agreement between our specific expressions, (5.8) and (5.9), derived from 'first principles', with the end result of [26, 27]. This is provided that the eight parameters, α, β, δ_A, assume some constant values reflecting the present choice of harmonic coordinates. By comparing our results (5.8) and (5.9) with equations (2.12) together with (2.18) in [27] (note the change of notation: $\varepsilon_5 \to \delta_6$), we indeed obtain a unique and consistent choice for these parameters, given by

$$\begin{aligned}
&\alpha = -1, && \beta = 0, \\
&\delta_1 = \tfrac{271}{28} + 6\nu, && \delta_2 = -\tfrac{77}{4} - \tfrac{3}{2}\nu, \\
&\delta_3 = \tfrac{79}{14} - \tfrac{92}{7}\nu, && \delta_4 = 10, \\
&\delta_5 = \tfrac{5}{42} + \tfrac{242}{21}\nu, && \delta_6 = -\tfrac{439}{28} + \tfrac{18}{7}\nu.
\end{aligned} \tag{5.10}$$

These values correspond to harmonic coordinates. For these, we find complete agreement with the result obtained (also from first principles) by Pati and Will [23] (see also [57]). On the other hand, some other values correspond to ADM coordinates, as computed by Königsdörffer *et al* [20]. Still other values for these parameters, obtained in [27], correspond to the 3.5PN radiation-reaction potentials [12] valid in extended Burke–Thorne gauge.

In conclusion, we have computed at 3.5PN order the radiation-reaction effect in the local equations of motion of a compact binary system in a general harmonic coordinates frame. The result was derived using a direct PN iterated expansion of the metric in harmonic coordinates at the 3.5PN approximation derived in [17, 18]. The 3.5PN metric is expressed as a function of a particular set of nonlinear retarded potentials, defined for any general smooth 'hydrodynamical' matter distribution. The existence of singular functions and divergent integrals, a consequence of our modelization by two delta-function singularities, required the Hadamard *partie-finie* regularization in order to remove each particle's infinite self-field. Analytical techniques were used to solve the more complicated integrals of the NC supported distribution of the gravitational field, resulting in a final expression in closed analytic form. We found that the 3.5PN term in the equations of motion in the centre-of-mass frame is perfectly consistent with the general expression of the 1PN radiation reaction force, derived by energy and angular-momentum balance equations in a class of coordinate systems [26, 27]. This study thus confirms that all the results to date for the 3.5PN equations of motion, computed either by 'direct' iterative PN expansion or by 'indirect' energy and angular-momentum balance considerations, are fully in agreement with each other. Since the equations of motion up to the 3PN order have already been derived elsewhere ([18, 37, 41, 43, 44] in harmonic coordinates, [38–40] in ADM coordinates), we conclude that the problem of the local equations of motion of compact binaries is solved up to 3.5PN order.

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