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Surface-integral expressions for the multipole moments of post-Newtonian sources and the boosted Schwarzschild solution

Luc Blanchet[1,2], **Thibault Damour**[3] **and Bala R Iyer**[4]

[1] $\mathcal{G}\mathbb{R}\varepsilon\mathbb{C}\mathcal{O}$, Institut d'Astrophysique de Paris, CNRS, 98[bis] boulevard Arago, 75014 Paris, France
[2] Yukawa Institute for Theoretical Physics, Kyoto University, Kyoto 606-8502, Japan
[3] Institut des Hautes Études Scientifiques, 35 route de Chartres, 91440 Bures-sur-Yvette, France
[4] Raman Research Institute, Bangalore 560 080, India

E-mail: blanchet@iap.fr, damour@ihes.fr and bri@rri.res.in

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Abstract

New expressions for the multipole moments of an isolated post-Newtonian source, in the form of surface integrals in the outer near-zone, are derived. As an application we compute the 'source' quadrupole moment of a Schwarzschild solution boosted to uniform velocity, at the third post-Newtonian (3PN) order. We show that the consideration of this boosted Schwarzschild solution (BSS) is enough to uniquely determine one of the ambiguity parameters in the recent computation of the gravitational wave generation by compact binaries at 3PN order: $\zeta = -7/33$. We argue that this value is the only one for which the Poincaré invariance of the 3PN wave generation formalism is realized. As a check, we confirm the value of ζ by a different method, based on the far-zone expansion of the BSS at fixed retarded time, and a calculation of the relevant nonlinear multipole interactions in the external metric at the 3PN order.

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1. Introduction

The first aim of the present paper is to obtain, in the continuation of previous work [1, 2], some new expressions for the multipole moments of an isolated gravitating source within the post-Newtonian (PN) approximation scheme of general relativity. The 'source' multipole moments, as they were derived in [2], are given, for any multipolar order ℓ, by certain functionals of the PN expansion of the pseudo-tensor of the matter and gravitational fields, formally valid up to any PN order. The moments have been obtained by solving the matching equation between the inner PN field and the outer multipolar expansion. They represent the generalization of the 1PN multipole moments obtained earlier in [3–5][5]. The source moments are those which parametrize the 'multipolar-post-Minkowskian' (MPM) expansion of the external field of the source, as it was defined and investigated in [6].

In [2] the PN moments have been given in the form of some *volume integrals*, whose support is non-compact because of the contribution of the gravitational field, and extending therefore over the full three-dimensional space. However, as we recall below, their integrand is made from the PN expansion of the stress–energy pseudo tensor $\tau^{\mu\nu}$, whose physical validity is restricted to the *near-zone* of the source. The formal extension of the integrals to $r \equiv |\mathbf{x}| \to +\infty$ was (uniquely) defined by introducing a suitable procedure of analytic continuation.

In the present paper, we derive some equivalent expressions for the PN source moments, in the form of *surface integrals*, formally performed at $r \to +\infty$. However let us emphasize that the physical meaning of the limit $r \to +\infty$ (which is taken after the PN limit $c \to +\infty$), corresponds to considering what one can call the *outer near-zone*, namely the region which is at once far from the source, $r \gg$ (source radius), and still within the near-zone, i.e. $r \ll$ (wavelength). For some physical problems, such alternative surface-integral expressions of the PN moments can be quite useful, as the next result of our paper will illustrate.

Indeed, our second aim is to make a contribution to the problem of theoretical templates of inspiralling compact binaries for gravitational-wave experiments such as LIGO and VIRGO. Current calculations of the gravitational waves are based on the expressions of the PN source multipole moments [1, 2], as well as on high-order post-Minkowskian iteration of the external field [6–8]. The mass-type quadrupole moment has been computed in the case of compact binary systems up to the 2PN order [9, 10] and more recently at 3PN order [11, 12]. The physical motivation for such high-order PN calculations of compact binary inspiral can be found in [13–19].

It has been shown in [11] that at 3PN order the radiation field of compact binaries, modelled as systems of point particles, contains *three ambiguity parameters*, ξ, κ and ζ, due to an incompleteness of the Hadamard self-field regularization, used in the computation of the quadrupole moment of point particles at the 3PN order. In the present paper we shall show how one can determine the value of one of these parameters, ζ, without any use of a self-field regularization scheme. The idea is to consider the situation where one of the two masses is exactly *zero*. Such a limiting case corresponds to a single-particle moving with some uniform velocity. As it turns out, the ambiguity parameter ζ, but only this one, survives in this limit. We can therefore compute it from the particular case of the 3PN quadrupole moment generated by a single object, specifically a spherically symmetric extended matter distribution, moving with uniform velocity in the preferred reference frame with respect to which the multipole moments are defined. The *external* gravitational field of such an object is evidently physically equivalent to a *boosted Schwarzschild solution* (BSS), i.e. a Schwarzschild solution viewed in a frame which is obtained from the usual 'Schwarzschild rest frame' by a Lorentz boost.

To compute the multipole moments of the BSS we propose and implement two methods. The first one is to insert the metric of the BSS into our new expressions for the PN multipole moments in terms of surface integrals in the outer near-zone. The second method, substantially more involved, consists of computing the far-zone or 'radiative' moments[6] of the BSS by expanding the metric in the far zone at fixed retarded time, and comparing the result with an

[5] As usual the nPN approximation refers to all the terms up to the relative order $\sim 1/c^{2n}$, where c is the speed of light. Powers of c and the gravitational constant G will generally be explicitly displayed here.

[6] Although the BSS does not radiate gravitational waves, it is sometimes convenient to employ a language for the different types of multipole moments which is similar to the one used in the more general case of non-stationary matter systems.

analytic calculation of the various nonlinear multipole interactions occurring in the far-zone moments up to 3PN order. The source-type moment will thereby be determined in an indirect way. We find that the results of the two methods agree, and uniquely determine

$$\zeta = -\frac{7}{33}. \tag{1.1}$$

We argue below that (1.1) is the *unique* value for which the 3PN wave generation formalism, as applied to binary systems in [11, 12], incorporates the global Lorentz–Poincaré invariance of general relativity. We have already reported elsewhere [20] that dimensional regularization leads to the same value for ζ (as well as the determination of the values of the two other ambiguity parameters ξ and κ), and therefore has the important feature of 'automatically' preserving the Poincaré invariance of the formalism. The calculation of ζ is similar to the one of its analogue in the 3PN equations of motion of point particle binaries, namely the so-called 'kinetic' ambiguity constant ω_k [21, 22], which has also been fixed from the requirement of Poincaré invariance, either by using an appropriate Lorentz-invariant version of Hadamard's regularization at the level of the equations of motion [23, 24], or by direct imposition that the ADM Hamiltonian be compatible with the existence of phase-space generators satisfying the Poincaré algebra [25]. (The second and last ambiguity in the 3PN equations of motion is the 'static' ambiguity constant ω_s [21, 22] equivalent to the ambiguity parameter λ [23, 24]. Both ω_s and λ (as well as ω_k in fact) have been obtained by means of dimensional regularization [26, 27]. The same results have also been achieved in [28, 29] by expressing the equations of motion at 3PN order in terms of surface integrals surrounding the compact objects.)

The paper is organized as follows. Section 2 is devoted to a general investigation of the PN source multipole moments of extended matter sources. In section 2.1 we present some useful reminders of the formulation of the PN moments, and in section 2.2 we obtain our new expressions of these moments in terms of surface integrals in the outer (or far) near-zone. Section 3 is devoted to the investigation of the multipole moments of the BSS. In section 3.1 we apply the new expressions of the PN source quadrupole moment to the case of the BSS and obtain the crucial coefficient which yields equation (1.1) by comparison with the BSS limit of the quadrupole moment of compact binaries. In section 3.2 we expand the field of the BSS at retarded infinity and obtain the radiative-type quadrupole moment, which we then relate to the source-type moment by computing the nonlinear multipole interactions therein up to 3PN order (the technical details of the nonlinear iteration are relegated to the appendix). In this way we are able to confirm the value (1.1).

2. Multipole moments as surface integrals

2.1. Reminders of the PN multipole moments

As gravitational field variable we employ the standard 'Gothic metric deviation' from flat spacetime, that we shall subject all over this paper to the condition of harmonic coordinates, meaning that

$$h^{\mu\nu} \equiv \sqrt{-g}g^{\mu\nu} - \eta^{\mu\nu}, \tag{2.1a}$$

$$\partial_\nu h^{\mu\nu} = 0, \tag{2.1b}$$

where g denotes the determinant and $g^{\mu\nu}$ the inverse of the covariant metric: $g = \det(g_{\rho\sigma})$ and $g^{\mu\nu}g_{\nu\sigma} = \delta^\mu_\sigma$, and where $\eta^{\mu\nu}$ is the Minkowski metric written in Cartesian coordinates: $\eta^{\mu\nu} = \mathrm{diag}(-1, 1, 1, 1)$. The Einstein field equations, relaxed by the harmonic-coordinates condition, read as

$$\Box h^{\mu\nu} = \frac{16\pi G}{c^4}\tau^{\mu\nu}, \tag{2.2}$$

where $\Box \equiv \Box_\eta$ is the *flat spacetime* d'Alembertian or wave operator, and $\tau^{\mu\nu}$ denotes the total stress–energy pseudo tensor of the matter and gravitational fields in harmonic coordinates, given by

$$\tau^{\mu\nu} \equiv |g|T^{\mu\nu} + \frac{c^4}{16\pi G}\Lambda^{\mu\nu}[h]. \tag{2.3}$$

Here $T^{\mu\nu}$ is the matter stress–energy tensor, and $\Lambda^{\mu\nu}$ is proportional to the stress–energy distribution of the gravitational field—$\Lambda^{\mu\nu}$ is a functional of the field strength h or, more precisely, a function of h and its first and second spacetime derivatives ($\sim\partial h$ and $\partial^2 h$), at least quadratic in $h, \partial h$ or $\partial^2 h$. The pseudo tensor $\tau^{\mu\nu}$ is conserved in a Minkowskian sense,

$$\partial_\nu \tau^{\mu\nu} = 0, \tag{2.4}$$

which is equivalent to the equation of motion of the matter source: $\nabla_\nu T^{\mu\nu} = 0$.

In principle one can solve equation (2.2) by successive iterations in the order of nonlinearity, i.e. by a formal expansion in powers of Newton's constant G. This would constitute what one calls the *post-Minkowskian* expansion (with explicit consideration of the matter source terms). However, such a straightforward post-Minkowskian expansion does not lead to easily implementable iterations (see, e.g., [30, 31]). Another scheme, technically more useful, consists of splitting the problem of solving (2.2) into three sub-problems. First, one solves the *vacuum* equations, obtained by setting the matter stress–energy tensor $T^{\mu\nu}$ to zero in equation (2.3), by a particular post-Minkowskian expansion. Second, one solves the full inhomogeneous Einstein equations (2.2) by a formal PN expansion, in inverse powers of the velocity of light c (with G fixed). And third, one combines together these two expansions by means of an appropriate variant of the method of matched asymptotic expansions.

The particular post-Minkowskian scheme we use to solve the vacuum Einstein equations is the so-called 'multipolar-post-Minkowskian' (MPM) expansion of the metric exterior to the source [6]. This expansion combines a nonlinearity expansion in powers of G, with a multipolar expansion of the successive nonlinear iterations of $h^{\mu\nu}$, say $h_n^{\mu\nu}$, i.e. essentially a decomposition of each function $h_n^{\mu\nu}(t, r, \theta, \varphi)$ in tensorial spherical harmonics on the unit sphere parametrized by θ and φ. It will be convenient, following [2], to denote the MPM expansion of some quantity like $h^{\mu\nu}$, as $\mathcal{M}(h^{\mu\nu})$, where the calligraphic letter $\mathcal{M}$ serves mainly the purpose of reminding the *multipolar* aspect of the expansion. (The post-Minkowskian aspect, though playing a crucial role in the definition and especially the construction of MPM metrics in [6], is less important to keep in mind in the reasoning that we shall follow below.) The MPM expansion of the metric is thus written as

$$\mathcal{M}(h^{\mu\nu}) = \sum_{n=1}^{+\infty} G^n h_n^{\mu\nu}, \tag{2.5}$$

where one must always have in mind that each term $h_n^{\mu\nu}(t, r, \theta, \varphi)$ is decomposed in tensorial spherical harmonics (for instance STF tensorial harmonics like in [6]). The successive post-Minkowskian coefficients $h_n^{\mu\nu}$ are constructed iteratively from the linearized approximation $h_1^{\mu\nu}$ by solving the harmonic-coordinates vacuum field equations. These consist of the harmonicity condition $\partial_\nu \mathcal{M}(h^{\mu\nu}) = 0$ and

$$\Box \mathcal{M}(h^{\mu\nu}) = \mathcal{M}(\Lambda^{\mu\nu}), \tag{2.6}$$

where $\mathcal{M}(\Lambda^{\mu\nu})$ means $\Lambda^{\mu\nu}$ in which we substitute for $h, \partial h, \partial^2 h$ their multipolar expansion: $\mathcal{M}(\Lambda^{\mu\nu}) \equiv \Lambda^{\mu\nu}[\mathcal{M}(h)]$. Again we stress that $\mathcal{M}$(some field) refers to a quantity which is supposed to be decomposed in a spherical-harmonics expansion (with coefficients being some functions of t and r).

The algorithm to explicitly construct the successive post-Minkowskian coefficients $h_n^{\mu\nu}$, from knowledge of the linearized approximation $h_1^{\mu\nu}$, was given in [6]. As everything depends on $h_1^{\mu\nu}$, the parametrization of the linearized approximation will determine the full MPM metric (2.5). In our approach the *source* multipole moments are defined as the 'seed moments' that one introduces at the start of the MPM scheme to parametrize the linearized approximation $h_1^{\mu\nu}$.

With full generality one can write the linearized metric $h_1^{\mu\nu}$, which satisfies $\Box h_1^{\mu\nu} = 0$ together with $\partial_\nu h_1^{\mu\nu} = 0$, in the form

$$h_1^{\mu\nu} = h_{\mathrm{can}1}^{\mu\nu} + \partial^\mu \varphi_1^\nu + \partial^\nu \varphi_1^\mu - \eta^{\mu\nu}\partial_\lambda \varphi_1^\lambda, \tag{2.7}$$

where φ_1^μ is a linearized gauge transformation vector (satisfying $\Box \varphi_1^\mu = 0$, so the harmonic gauge condition is preserved), and where $h_{\mathrm{can}1}^{\mu\nu}$ represents a useful form of the linearized multipolar metric, called 'canonical' and introduced in [32]. The mass-type and current-type source multipole moments, $I_L(t)$ and $J_L(t)$, respectively, are symmetric and trace-free (STF) tensors with respect to their ℓ indices (ℓ is the multipolar order). They parametrize the canonical metric via the following definition:

$$h_{\mathrm{can}\,1}^{00} = -\frac{4}{c^2}\sum_{\ell=0}^{+\infty}\frac{(-)^\ell}{\ell!}\partial_L\left[\frac{1}{r}I_L(u)\right], \tag{2.8a}$$

$$h_{\mathrm{can}\,1}^{0i} = \frac{4}{c^3}\sum_{\ell=1}^{+\infty}\frac{(-)^\ell}{\ell!}\left\{\partial_{L-1}\left[\frac{1}{r}\dot{I}_{iL-1}(u)\right] + \frac{\ell}{\ell+1}\varepsilon_{iab}\partial_{aL-1}\left[\frac{1}{r}J_{bL-1}(u)\right]\right\}, \tag{2.8b}$$

$$h_{\mathrm{can}\,1}^{ij} = -\frac{4}{c^4}\sum_{\ell=2}^{+\infty}\frac{(-)^\ell}{\ell!}\left\{\partial_{L-2}\left[\frac{1}{r}\ddot{I}_{ijL-2}(u)\right] + \frac{2\ell}{\ell+1}\partial_{aL-2}\left[\frac{1}{r}\varepsilon_{ab(i}\dot{J}_{j)bL-2}(u)\right]\right\}. \tag{2.8c}$$

Our notation is standard[7]. In addition, the components of the gauge transformation vector φ_1^μ can be parametrized by four other sequences of multipole moments, called W_L, X_L, Y_L and Z_L, also being STF in their indices L, in the way specified by equation (4.13) in [2], i.e.

$$\varphi_1^0 = \frac{4}{c^3}\sum_{\ell=0}^{+\infty}\frac{(-)^\ell}{\ell!}\partial_L\left[\frac{1}{r}W_L(u)\right], \tag{2.9a}$$

$$\begin{aligned}\varphi_1^i = &-\frac{4}{c^4}\sum_{\ell=0}^{+\infty}\frac{(-)^\ell}{\ell!}\partial_{iL}\left[\frac{1}{r}X_L(u)\right]\\ &-\frac{4}{c^4}\sum_{\ell=1}^{+\infty}\frac{(-)^\ell}{\ell!}\left\{\partial_{L-1}\left[\frac{1}{r}Y_{iL-1}(u)\right] + \frac{\ell}{\ell+1}\varepsilon_{iab}\partial_{aL-1}\left[\frac{1}{r}Z_{bL-1}(u)\right]\right\}.\end{aligned} \tag{2.9b}$$

The complete set of moments parametrizing the linearized approximation (2.7) with (2.8) and (2.9), i.e. $\{I_L, J_L, W_L, X_L, Y_L, Z_L\}$, is collectively referred to as the *source* multipole moments. All these moments are defined in such a way that they admit a nonzero finite

[7] $L \equiv i_1\cdots i_\ell$ denotes a multi-index composed of ℓ multipolar indices $i_1,\ldots,i_\ell$; $\partial_L \equiv \partial_{i_1}\cdots\partial_{i_\ell}$ means a product of ℓ partial derivatives $\partial_i \equiv \partial/\partial x^i$; similarly $x_L \equiv x_{i_1}\cdots x_{i_\ell}$ is a product of ℓ spatial vectors $x_i \equiv x^i$; symmetric-trace-free products are denoted with hats so that $\hat{x}_L \equiv \mathrm{STF}(x_L)$; sometimes we shall also use some brackets surrounding the STF indices: $x_{\langle L\rangle} \equiv \hat{x}_L$; the dots refer to the partial time derivation; ε_{abi} is the Levi–Civita totally antisymmetric symbol (such that $\varepsilon_{123} = 1$); index symmetrization means $(ij) \equiv \frac{ij+ji}{2}$; and $r \equiv |\mathbf{x}|$ and $u \equiv t - r/c$.

Newtonian limit, when $c \to +\infty$. It is clear that the most important of these moments are the mass-type moment I_L and the current-type one J_L. Indeed the other moments, $W_L, \ldots, Z_L$, parametrize a gauge transformation and thus do not play any physical role at the linearized order (though they do play a role at the nonlinear level).

In section 3.2 we shall also recall the definition of two and only two sets of moments, named the 'canonical' moments, denoted by $\{M_L, S_L\}$, which are physically equivalent to the complete set of *six* source moments $\{I_L, J_L, \ldots, Z_L\}$, in the sense that they describe the *same* external gravitational field. However, following [2], we prefer to reserve the name of source moments to the set $I_L, J_L, \ldots, Z_L$ because they are connected via some analytic closed form expressions to the stress–energy tensor of a PN source.

The MPM algorithm computes sequentially any of the nonlinear coefficients in equation (2.5) as follows [6, 33][8]. Suppose that the first $n-1$ coefficients $h_1, \ldots, h_{n-1}$, where h_1 is given by equations (2.7)–(2.9), have been constructed. We then have to solve, at the nth order, the inhomogeneous wave equation,

$$\Box h_n^{\mu\nu} = \Lambda_n^{\mu\nu}, \tag{2.10}$$

whose source term is known from the previous iterations, i.e. $\Lambda_n^{\mu\nu} = \Lambda_n^{\mu\nu}[h_1, \ldots, h_{n-1}]$, and where as always we also have to satisfy the coordinate condition $\partial_\nu h_n^{\mu\nu} = 0$. The solution, satisfying a condition of 'stationarity in the past' ensuring that the correct boundary conditions at Minkowskian past null-infinity are satisfied, reads

$$h_n^{\mu\nu} = u_n^{\mu\nu} + v_n^{\mu\nu}. \tag{2.11}$$

The first term represents the standard retarded integral operator (denoted by $\Box_{\mathrm{R}}^{-1}$ below) acting on the nonlinear source, but augmented by a specific regularization scheme to deal with the divergencies of the retarded integral introduced by the fact that the multipolar expansion diverges at the origin of the spatial coordinates, when $r \equiv |\mathbf{x}| \to 0$ (see, e.g., equation (2.8)). Posing

$$u_n^{\mu\nu} = \underset{B=0}{\mathrm{FP}}\, \Box_{\mathrm{R}}^{-1} \left[\left(\frac{r}{r_0} \right)^B \Lambda_n^{\mu\nu} \right], \tag{2.12}$$

we do solve the required wave equation, i.e. $\Box u_n^{\mu\nu} = \Lambda_n^{\mu\nu}$, provided that the finite part (FP) takes the meaning specified below. The second term in equation (2.11) represents a particular homogeneous solution, i.e. $\Box v_n^{\mu\nu} = 0$, defined in such a way that the harmonic gauge condition $\partial_\nu h_n^{\mu\nu} = 0$ is satisfied at this order (see [6, 33] for the details).

To define the FP process we multiply the source term in equation (2.10) by a factor $(r/r_0)^B$, where $B \in \mathbb{C}$ and r_0 denotes some arbitrary length scale, we compute the B-dependent retarded integral in the domain of the complex B plane in which it converges, i.e. for which $\Re(B)$ is initially a large enough positive number, and we define it in a neighbourhood of the value of interest $B = 0$ by analytic continuation. The finite part in (2.12) means the coefficient a_0 of the zero-th power of B in the Laurent expansion $\sum a_p B^p$ (where $p \in \mathbb{Z}$) of the retarded integral when $B \to 0$. We emphasize that the divergencies cured by the FP regularization in (2.12) are ultraviolet (UV) type divergencies ($r \to 0$). As we shall see the same FP regularization will be used in the multipole moments to deal with their infra-red (IR) divergencies (when $r \to +\infty$).

In order to prevent any confusion, let us clarify the meaning of the various expansions that we shall use, and of the limits $r \to 0$ and $r \to +\infty$ that will arise in the present paper. First, we recall that any MPM-expanded quantity is always thought of as written

[8] In this paper we adopt a slightly modified version of the MPM algorithm, defined in section 2 of [33], which is more convenient in practical computations.

as a double expansion: one expansion in powers of G and one in spherical harmonics, say $h_n = \sum_\ell \hat{n}_L F_L$, where L denotes a multi-index which carries an irreducible representation of the rotation group (we do not write spacetime indices). Then, each of the coefficients of this double expansion is given by some explicit function of t and r, say $F_L(t, r, c)$, where we have also indicated a dependence on the velocity of light c (cf for instance the simple case of the linearized approximation above). An important technical aspect of our formalism is that we shall consider these functions $F_L(t, r, c)$ in the *whole range* of the radial coordinate r, even if this range does not correspond to a spacetime region where the corresponding expansion is physically valid. Moreover, we shall sometimes consider instead of the original MPM coefficients $F_L(t, r, c)$ their PN expansion (or 'near-zone expansion'), which means technically a formal expansion in powers of $1/c$ (with possibly some powers of $\ln c$). We shall denote the PN expansion of any quantity by an overline. For instance, $\overline{F}_L(t, r, c)$ denotes the expansion in powers of $1/c$ of $F_L(t, r, c)$, when keeping fixed the variables r and t.

We deal sometimes (as in equation (2.12) above) with functions $F_L(t, r, c)$ in the region $r \to 0$, which is mathematically well defined (by real analytic continuation in r) but which physically corresponds to a region where the *vacuum* MPM metric should be replaced by a solution of the *inhomogeneous* Einstein equations, and therefore where the actual physical function would be a different function of r (and t) than $F_L(t, r, c)$. On the other hand, the limit $r \to +\infty$ is acceptable both mathematically and physically for $F_L(t, r, c)$ because it comes from a post-Minkowskian expansion which is valid all over the exterior of the source. However, we shall also mathematically deal with the PN re-expansion $\overline{F}_L(t, r, c)$ of the function $F_L(t, r, c)$, and then formally consider the function $\overline{F}_L(t, r, c)$ in the limit $r \to +\infty$. Mathematically, the behaviour of $\overline{F}_L(t, r, c)$ when $r \to +\infty$ is again well defined (by real analytic continuation in r) for each term $F_L(t, r, c)$ in the MPM expansion. Though the limit $r \to +\infty$ seems physically incorrect for a PN expansion, it is here technically (or mathematically) well defined. We stress that the PN limit $c \to +\infty$ is taken *before* considering the $r \to +\infty$ behaviour. If we remember that the PN expansion is physically valid only in the *near-zone* of the source, defined as $r \ll cT$, where T is a characteristic time of variation of the source, we see that the physical domain of validity of the $r \to +\infty$ expansion of the PN expanded functions $\overline{F}_L(t, r, c)$ actually corresponds to the *outer* part of the near-zone, i.e. when r is much larger than the size, say a, of the source (multipole expansion), but still significantly smaller than a gravitational wavelength $\lambda \sim cT$. We shall often refer to this domain as the *far near-zone*.

As explained in [6], in order to be able to define the behaviour for $r \to 0$ and $r \to +\infty$ of all the coefficients $F_L(t, r, c)$ appearing in the successive iterations of the MPM scheme one needs to make some formal technical assumptions: one must start with only a finite number of 'seed' multipole moments, assume that they are infinitely differentiable functions of time, and that they tend to some constants in the infinite past. In our approach we assume that these requirements are initially satisfied, and we formally take, at the end of the calculation, a limit where these requirements are relaxed (so that we extend our results to an infinite number of moments, which are not necessarily past-stationary).

The MPM expansion, sketched above, must be completed by an expansion scheme which covers the source. This is done by considering also a PN approximation for solving the inhomogeneous Einstein equations. The PN scheme is *a priori* valid in the near-zone ($r \ll cT$), while the MPM one is valid in the exterior of the source ($r > a$). The two domains of validity overlap in the exterior near-zone ($a < r \ll cT$). One then imposes a 'matching condition' which will enable us to determine the values of the multipole moments as functionals of the PN source. If we denote as above by $\overline{h}^{\mu\nu}$ the PN expansion the matching condition can be expressed as

$$\mathcal{M}(\overline{h}^{\mu\nu}) \equiv \overline{\mathcal{M}(h^{\mu\nu})}, \tag{2.13}$$

which says that the multipolar re-expansion of the PN metric $\overline{h}^{\mu\nu}$ agrees, in the sense of formal series, with the *near-zone* re-expansion (also denoted with an overbar) of the MPM metric $\mathcal{M}(h^{\mu\nu})$. If we consider that a multipolar expansion is essentially an expansion in inverse powers of r when r gets far away from the source, we can roughly summarize the matching equation (2.13) as saying that the far expansion ($r \to +\infty, t = \text{const}$) of the near-zone metric—the LHS of equation (2.13)—coincides with the near expansion ($r/cT \to 0, t = \text{const}$) of the far (multipolar-expanded) metric—the RHS of (2.13). The common general structure of both sides of (2.13) will be given in equations (2.21) and (2.22).

In [2] the PN source moments were obtained as functionals of the PN expansion of the pseudo-stress energy tensor defined by equation (2.3), namely $\overline{\tau}^{\mu\nu}$. For the main source moments I_L and J_L (with any $\ell \geqslant 2$) we get

$$\begin{aligned} I_L(t) = \frac{1}{c^2} \mathop{\mathrm{FP}}_{B=0} \int \mathrm{d}^3\mathbf{x} \left(\frac{r}{r_0}\right)^B \int_{-1}^{1} \mathrm{d}z \Bigg\{ & \delta_\ell(z) \hat{x}_L (\overline{\tau}^{00} + \overline{\tau}^{ii})(\mathbf{x}, t + zr/c) \\ & - \frac{4(2\ell+1)}{(\ell+1)(2\ell+3)} \delta_{\ell+1}(z) \hat{x}_{iL} \frac{\partial \overline{\tau}^{i0}}{c\partial t}(\mathbf{x}, t + zr/c) \\ & + \frac{2(2\ell+1)}{(\ell+1)(\ell+2)(2\ell+5)} \delta_{\ell+2}(z) \hat{x}_{ijL} \frac{\partial^2 \overline{\tau}^{ij}}{c^2 \partial t^2}(\mathbf{x}, t + zr/c) \Bigg\}, \end{aligned} \tag{2.14a}$$

$$\begin{aligned} J_L(t) = \frac{1}{c} \mathop{\mathrm{FP}}_{B=0} \varepsilon_{ab\langle i_\ell} \int \mathrm{d}^3\mathbf{x} \left(\frac{r}{r_0}\right)^B \int_{-1}^{1} \mathrm{d}z \Bigg\{ & \delta_\ell(z) \hat{x}_{L-1\rangle a} \overline{\tau}^{b0}(\mathbf{x}, t + zr/c) \\ & - \frac{2\ell+1}{(\ell+2)(2\ell+3)} \delta_{\ell+1}(z) \hat{x}_{L-1\rangle ac} \frac{\partial \overline{\tau}^{bc}}{c\partial t}(\mathbf{x}, t + zr/c) \Bigg\}, \end{aligned} \tag{2.14b}$$

where we recall that $\hat{x}_L$ means the STF product of ℓ spatial vectors, $\hat{x}_L \equiv \mathrm{STF}\big(x_{i_1} \cdots x_{i_\ell}\big)$. The other source moments, W_L, X_L, Y_L, Z_L, are given by equations (5.17)–(5.20) in [2]. We shall give below their new expressions in terms of surface integrals. See also [2] for a discussion of the conserved monopole and dipole moments (having $\ell = 0, 1$).

A basic feature of these expressions is that the integral formally extends over the whole support of the PN expansion of the stress–energy pseudo-tensor, i.e. from $r = 0$ up to infinity. As already emphasized, the formal series $\overline{\tau}^{\mu\nu}$ is physically meaningful only within the near-zone. Therefore the integrals (2.14) physically refer to a result obtained from near-zone quantities only (in the formal limit where $c \to +\infty$). However, it was found convenient in [2] to mathematically extend the integrals up to $r \to +\infty$. This was made possible by the use of the prefactor $(r/r_0)^B$, together with a process of analytic continuation in the complex B plane. This shows up in equations (2.14) as the crucial finite part operation, when $B \to 0$, which technically allows one to uniquely define integrals which would otherwise be IR divergent, i.e. divergent at their upper boundary, $|\mathbf{x}| \to +\infty$. See [1, 2] for the proof and details.

Since equations (2.14) are valid only in the sense of PN expansions, the operational meaning of the auxiliary integrals in (2.14), with respect to the variable z, is actually that of an infinite PN series, given by

$$\int_{-1}^{1} \mathrm{d}z \delta_\ell(z) \overline{\tau}^{\mu\nu}(\mathbf{x}, t + zr/c) = \sum_{k=0}^{+\infty} \alpha_{k,\ell} \left(\frac{r}{c}\frac{\partial}{\partial t}\right)^{2k} \overline{\tau}^{\mu\nu}(\mathbf{x}, t), \tag{2.15a}$$

where

$$\alpha_{k,\ell} \equiv \frac{(2\ell+1)!!}{(2k)!!(2\ell+2k+1)!!}. \tag{2.15b}$$

The expression of the function $\delta_\ell(z)$, given in appendix B of [3] as

$$\delta_\ell(z) \equiv \frac{(2\ell+1)!!}{2^{\ell+1}\ell!}(1-z^2)^\ell \qquad \text{such that} \qquad \int_{-1}^{1} \mathrm{d}z\, \delta_\ell(z) = 1, \tag{2.16}$$

is useful when manipulating formal PN expansions such as (2.15), but will not be used explicitly in the present investigation.

2.2. *The multipole moments as surface integrals*

In this section we derive an alternative form of the PN source moments (2.14) in terms of two-dimensional surface integrals. Such a possibility of expressing the moments, for general ℓ and at any PN order, as some surface integrals is quite useful for practical purposes, as we shall show below when considering the application to the BSS case. In keeping with the fact just explained that the 'volume integrals' equations (2.14) physically involve only near-zone quantities, the 'surface integrals' into which we shall transform equations (2.14) physically refer to an operation which extracts some coefficients in the 'far near-zone' expansion of the gravitational field, i.e. in the expansion in increasing powers of $1/r$ of the PN-expanded near-zone metric. Technically, as our starting point (2.14) is made of integrals extended up to $r \to +\infty$, our mathematical manipulations below will involve 'surface terms' on arbitrary large spheres $r = \mathcal{R}$. All our manipulations will be mathematically well defined because of the properties of complex analytic continuation in B.

The basic idea is to go from the 'source term', $\overline{\tau}^{\mu\nu}$, to the corresponding 'solution', $\overline{h}^{\mu\nu}$, via integrating by parts the Laplace operator present in $\overline{\tau}^{\mu\nu} = \frac{c^4}{16\pi G}\Box\overline{h}^{\mu\nu}$. From equation (2.15) we have

$$\int \mathrm{d}^3\mathbf{x}\, r^B \hat{x}_L \int_{-1}^{1} \mathrm{d}z\, \delta_\ell(z)\overline{\tau}^{\mu\nu} = \frac{c^4}{16\pi G}\sum_{k=0}^{+\infty}\alpha_{k,\ell}\left(\frac{\mathrm{d}}{c\,\mathrm{d}t}\right)^{2k}\int \mathrm{d}^3\mathbf{x}\, r^{B+2k}\hat{x}_L\,\Box\overline{h}^{\mu\nu}, \tag{2.17}$$

in which we insert $\Box = \Delta - \left(\frac{\partial}{c\partial t}\right)^2$ on the RHS, and operate the Laplacian by parts using $\Delta(r^{B+2k}\hat{x}_L) = (B+2k)(B+2\ell+2k+1)r^{B+2k-2}\hat{x}_L$. In the process we can ignore the all-integrated surface terms because they are identically zero by complex analytic continuation, from the case where the real part of B is chosen to be a large enough *negative* number. (The complete justification of this is as follows. In the present formalism we are actually dealing with the MPM metric given by the nonlinearity expansion (2.5), and we are working with the calculation of some given *finite* post-Minkowskian approximation n. Then the PN expansion of the post-Minkowskian metric coefficient $h_n^{\mu\nu}$, namely $\overline{h}_n^{\mu\nu}$, will typically diverge at infinity, but not more than a certain finite power of r, say $N(n)$, depending on n and such that $\lim_{n\to\infty} N(n) = \infty$. Using $\overline{h}_n^{\mu\nu} = \mathcal{O}(r^{N(n)})$ it is then clear that the all-integrated terms in question are zero when we choose initially $\Re(B) + 2k + \ell + N(n) + 1 < 0$, hence they are zero by analytic continuation in B.) Using the expression of the coefficients (2.15*b*), we are next led to the alternative expression

$$\begin{aligned}\int \mathrm{d}^3\mathbf{x}\, r^B \hat{x}_L \int_{-1}^{1} \mathrm{d}z\delta_\ell(z)\overline{\tau}^{\mu\nu} \\ = \frac{c^4}{16\pi G}\sum_{k=0}^{+\infty} B(B+2\ell+4k+1)\alpha_{k,\ell}\left(\frac{\mathrm{d}}{c\,\mathrm{d}t}\right)^{2k}\int \mathrm{d}^3\mathbf{x}\, r^{B+2k-2}\hat{x}_L\overline{h}^{\mu\nu}.\end{aligned} \tag{2.18}$$

A remarkable feature of this result, which is the basis of our new expressions, is the presence of an *explicit factor B* in front of the integral. The factor means that the result depends only on the occurrence of *poles*, $\propto 1/B^p$, in the boundary of the integral at infinity:

$r \to +\infty$ with $t =$ const. At this stage it is useful to write down the expressions of the moments I_L and J_L we obtain by substituting (2.18) back into (2.14). These are

$$I_L = \frac{c^2}{16\pi G} \mathop{\mathrm{FP}}_{B=0} B r_0^{-B} \sum_{k=0}^{+\infty} \left\{ (B+2\ell+4k+1)\alpha_{k,\ell} \left(\frac{\mathrm{d}}{c\,\mathrm{d}t}\right)^{2k} \int \mathrm{d}^3\mathbf{x}\, r^{B+2k-2} \hat{x}_L (\overline{h}^{00} + \overline{h}^{ii}) \right.$$
$$- \frac{4(2\ell+1)(B+2\ell+4k+3)}{(\ell+1)(2\ell+3)} \alpha_{k,\ell+1} \left(\frac{\mathrm{d}}{c\,\mathrm{d}t}\right)^{2k+1} \int \mathrm{d}^3\mathbf{x}\, r^{B+2k-2} \hat{x}_{iL} \overline{h}^{i0}$$
$$\left. + \frac{2(2\ell+1)(B+2\ell+4k+5)}{(\ell+1)(\ell+2)(2\ell+5)} \alpha_{k,\ell+2} \left(\frac{\mathrm{d}}{c\,\mathrm{d}t}\right)^{2k+2} \int \mathrm{d}^3\mathbf{x}\, r^{B+2k-2} \hat{x}_{ijL} \overline{h}^{ij} \right\}, \tag{2.19a}$$

$$J_L = \frac{c^3}{16\pi G} \mathop{\mathrm{FP}}_{B=0} B r_0^{-B} \varepsilon_{ab\langle i_\ell} \sum_{k=0}^{+\infty} \left\{ (B+2\ell+4k+1)\alpha_{k,\ell} \left(\frac{\mathrm{d}}{c\,\mathrm{d}t}\right)^{2k} \int \mathrm{d}^3\mathbf{x}\, r^{B+2k-2} \hat{x}_{L-1\rangle a} \overline{h}^{b0} \right.$$
$$\left. - \frac{(2\ell+1)(B+2\ell+4k+3)}{(\ell+2)(2\ell+3)} \alpha_{k,\ell+1} \left(\frac{\mathrm{d}}{c\,\mathrm{d}t}\right)^{2k+1} \int \mathrm{d}^3\mathbf{x}\, r^{B+2k-2} \hat{x}_{L-1\rangle ac} \overline{h}^{bc} \right\}. \tag{2.19b}$$

Let us proceed further. Thanks to the factor B we can replace the integration domain of equation (2.18) by some outer domain of the type $r > \mathcal{R}$, where $\mathcal{R}$ denotes some large arbitrary constant radius. The integral over the inner domain $r < \mathcal{R}$ is always zero in the limit $B \to 0$ because the integrand is constructed from $\overline{\tau}^{\mu\nu}$, and we are considering extended regular PN sources, without singularities. Now, in the outer (but still near-zone) domain we can replace the PN metric coefficients $\overline{h}^{\mu\nu}$ by the expansion in increasing powers of $1/r$ of the PN-expanded metric, which is identical to the multipolar expansion of the PN-expanded metric. This is precisely the quantity which was already introduced in equation (2.13) and denoted there by $\mathcal{M}(\overline{h}^{\mu\nu})$. Hence we have

$$\int \mathrm{d}^3\mathbf{x}\, r^B \hat{x}_L \int_{-1}^{1} \mathrm{d}z \delta_\ell(z) \overline{\tau}^{\mu\nu} = \frac{c^4}{16\pi G} \sum_{k=0}^{+\infty} B(B+2\ell+4k+1)\alpha_{k,\ell} \left(\frac{\mathrm{d}}{c\,\mathrm{d}t}\right)^{2k}$$
$$\times \int_{r>\mathcal{R}} \mathrm{d}^3\mathbf{x}\, r^{B+2k-2} \hat{x}_L \mathcal{M}(\overline{h}^{\mu\nu}). \tag{2.20}$$

We want now to make use of a more explicit form of the far near-zone expansion $\mathcal{M}(\overline{h}^{\mu\nu})$, whose general structure is known. It consists of terms proportional to arbitrary powers of $1/r$, and multiplied by powers of the *logarithm* of r. More precisely,

$$\mathcal{M}(\overline{h}^{\mu\nu})(\mathbf{x}, t) = \sum_{a,b} \frac{(\ln r)^b}{r^a} \varphi^{\mu\nu}_{a,b}(\mathbf{n}, t), \tag{2.21}$$

where a can take any positive or negative integer values, and b can be any positive integer: $a \in \mathbb{Z}, b \in \mathbb{N}$. The coefficients $\varphi^{\mu\nu}_{a,b}$ depend on the unit direction $\mathbf{n} \equiv \mathbf{x}/r$ and on the coordinate time t (in the harmonic coordinate system). The structure (2.21) for the multipolar expansion of the near-zone (PN-expanded) metric is a consequence, via the matching equation (2.13), of the corresponding result concerning the structure of the *near-zone* expansion of the MPM metric (2.5), which has been proved in [6] by using the properties of the MPM algorithm for the iteration of the metric. (Again we stress the fact that the result has been proved at some *arbitrary but finite* post-Minkowskian order n; see equation (5.4) in [6]. In the present paper we assume, following [2], that we are always entitled to sum up formally the post-Minkowskian series, and to view a result like (2.21) as true in the sense of formal power series.)

As a side remark (which is not essential for the following), note that in [6] the near-zone expansion of the MPM metric was viewed as an expansion in 'ascending' powers of r/cT, namely, it was written in the form

$$\overline{\mathcal{M}\,(h^{\mu\nu})}(\mathbf{x},t)=\sum_{p,q} r^p(\ln r)^q f^{\mu\nu}_{p,q}(\mathbf{n},t), \tag{2.22}$$

where $p\in\mathbb{Z}$ and $q\in\mathbb{N}$. Evidently, expansions (2.22) and (2.21) are equivalent. The only difference is that (2.21) was ordered in ascending powers of $1/r$, i.e. in 'descending' powers of r. Actually, both expansions are formal Laurent-type expansions, valid in some intermediate range of radii: $a<r<cT$. Their coefficients are related by a simple re-ordering of the exponents of r,

$$f^{\mu\nu}_{p,q}(\mathbf{n},t)=\varphi^{\mu\nu}_{-p,q}(\mathbf{n},t). \tag{2.23}$$

In the following, we shall use the notation $\varphi^{\mu\nu}_{a,b}(\mathbf{n},t)$, corresponding to equation (2.21), for the coefficients of these equivalent expansions.

Inserting (2.21) into (2.20), we are therefore led to the computation of the integral

$$\int_{r>\mathcal{R}} \mathrm{d}^3\mathbf{x}\, r^{B+2k-2}\hat{x}_L\mathcal{M}(\overline{h}^{\mu\nu})=\sum_{a,b}\int_{\mathcal{R}}^{+\infty}\mathrm{d}r\, r^{B+2k+\ell-a}(\ln r)^b\int \mathrm{d}\Omega\,\hat{n}_L\varphi^{\mu\nu}_{a,b}(\mathbf{n},t), \tag{2.24}$$

where $\mathrm{d}\Omega$ is the solid angle element associated with the unit direction $\mathbf{n}$ (and $\hat{n}_L\equiv\hat{x}_L/r^\ell$). The radial integral can be trivially integrated by analytic continuation in B, with result

$$\int_{\mathcal{R}}^{+\infty}\mathrm{d}r\, r^{B+2k+\ell-a}(\ln r)^b=-\left(\frac{\mathrm{d}}{\mathrm{d}B}\right)^b\left[\frac{\mathcal{R}^{B+2k+\ell-a+1}}{B+2k+\ell-a+1}\right]. \tag{2.25}$$

Remember that we are ultimately interested only in the analytic continuation of such integrals down to $B=0$. And as an integral such as (2.25) is multiplied by a coefficient which is proportional to B, we must control the poles of equation (2.25) at $B=0$. Those poles are in general multiple because of the presence of powers of $\ln r$ in the expansion, and the consecutive multiple differentiation with respect to B in equation (2.25). The poles at $B=0$ clearly come from a single value of a, namely $a=2k+\ell+1$. For that value, the 'multiplicity' of the pole takes the value $b+1$. Here a useful simplification comes from the fact that the factor in front of the integrals in (2.20) is of the form $\sim B(B+K)$. In other words, this factor contains only the first and second powers of B. Therefore, only the simple and double poles $1/B$ and $1/B^2$ in (2.25) can contribute to the final result. Hence, we conclude that it is enough to consider the values $b=0,1$ for the exponent b of $\ln r$ in expansion (2.21).

To express the result in the most convenient manner let us introduce special notation for some relevant combination of far-near-zone coefficients, $\varphi^{\mu\nu}_{a,b}(\mathbf{n},t)$, which, as we just said, correspond exclusively to the values $a=\ell+2k+1$ and $b=0$ or 1. Namely,

$$\Psi^{\mu\nu}_{k,\ell}(\mathbf{n},t)\equiv\alpha_{k,\ell}[-(2\ell+4k+1)\varphi^{\mu\nu}_{2k+\ell+1,0}(\mathbf{n},t)+(1-(2\ell+4k+1)\ln r_0)\varphi^{\mu\nu}_{2k+\ell+1,1}(\mathbf{n},t)], \tag{2.26}$$

in which we have absorbed the numerical coefficient $\alpha_{k,\ell}$ defined by (2.15*b*). (Note that the coefficients $\varphi^{\mu\nu}_{a,b}$ depend *a priori* on the scale r_0.) With this notation we then obtain

$$\underset{B=0}{\mathrm{FP}}\, Br_0^{-B}(B+2\ell+4k+1)\alpha_{k,\ell}\int_{r>\mathcal{R}}\mathrm{d}^3\mathbf{x}\, r^{B+2k-2}\hat{x}_L\mathcal{M}(\overline{h}^{\mu\nu})=4\pi\langle\hat{n}_L\Psi^{\mu\nu}_{k,\ell}\rangle, \tag{2.27}$$

where the brackets refer to the spherical or angular average (at coordinate time t), i.e.

$$\left\langle \hat{n}_L \Psi^{\mu\nu}_{k,\ell} \right\rangle(t) \equiv \int \frac{\mathrm{d}\Omega}{4\pi} \hat{n}_L \Psi^{\mu\nu}_{k,\ell}(\mathbf{n}, t). \tag{2.28}$$

As we can see, any reference to the intermediate scale $\mathcal{R}$ has completely disappeared. The quantities (2.28) are integrals over a unit sphere, and can rightly be referred to as 'surface integrals'. These surface integrals will be the basic blocks entering our new expressions for the multipole moments. If we wish to physically think of them as integrals over some two-surface surrounding the source, we can roughly consider that this two-surface is located at a radius $\mathcal{R}$, with $a \ll \mathcal{R} \ll cT$. Anyway, the important point is that, as we have just remarked, the surface integrals (2.28), and therefore the multipole moments, are strictly independent of the choice of the intermediate scale $\mathcal{R}$ which entered our reasoning.

Finally, we are in a position to write down the following final results for the source multipole moments (2.19*a*) and (2.19*b*), expressed solely in terms of the surface integrals of the type (2.28),

$$I_L = \frac{c^2}{4G} \sum_{k=0}^{+\infty} \left\{ \left(\frac{\mathrm{d}}{c\,\mathrm{d}t}\right)^{2k} \left\langle \hat{n}_L \left(\Psi^{00}_{k,\ell} + \Psi^{ii}_{k,\ell}\right)\right\rangle - \frac{4(2\ell+1)}{(\ell+1)(2\ell+3)} \left(\frac{\mathrm{d}}{c\,\mathrm{d}t}\right)^{2k+1} \left\langle \hat{n}_{iL} \Psi^{i0}_{k,\ell+1}\right\rangle \right.$$
$$\left. + \frac{2(2\ell+1)}{(\ell+1)(\ell+2)(2\ell+5)} \left(\frac{\mathrm{d}}{c\,\mathrm{d}t}\right)^{2k+2} \left\langle \hat{n}_{ijL} \Psi^{ij}_{k,\ell+2}\right\rangle \right\}, \tag{2.29a}$$

$$J_L = \frac{c^3}{4G} \varepsilon_{ab\langle i_\ell} \sum_{k=0}^{+\infty} \left\{ \left(\frac{\mathrm{d}}{c\,\mathrm{d}t}\right)^{2k} \left\langle \hat{n}_{L-1\rangle a} \Psi^{b0}_{k,\ell}\right\rangle - \frac{2\ell+1}{(\ell+2)(2\ell+3)} \left(\frac{\mathrm{d}}{c\,\mathrm{d}t}\right)^{2k+1} \left\langle \hat{n}_{L-1\rangle ac} \Psi^{bc}_{k,\ell+1}\right\rangle \right\}. \tag{2.29b}$$

The other source moments, W_L, X_L, Y_L and Z_L, which parametrize the gauge vector given by equation (2.9), admit similar expressions, which can be derived by the same method. For these we give only the results:

$$W_L = \frac{c^3}{4G} \sum_{k=0}^{+\infty} \left\{ \frac{2\ell+1}{(\ell+1)(2\ell+3)} \left(\frac{\mathrm{d}}{c\,\mathrm{d}t}\right)^{2k} \left\langle \hat{n}_{iL} \Psi^{i0}_{k,\ell+1}\right\rangle \right.$$
$$\left. - \frac{2\ell+1}{2(\ell+1)(\ell+2)(2\ell+5)} \left(\frac{\mathrm{d}}{c\,\mathrm{d}t}\right)^{2k+1} \left\langle \hat{n}_{ijL} \Psi^{ij}_{k,\ell+2}\right\rangle \right\}, \tag{2.30a}$$

$$X_L = \frac{c^4}{4G} \sum_{k=0}^{+\infty} \left\{ \frac{2\ell+1}{2(\ell+1)(\ell+2)(2\ell+5)} \left(\frac{\mathrm{d}}{c\,\mathrm{d}t}\right)^{2k} \left\langle \hat{n}_{ijL} \Psi^{ij}_{k,\ell+2}\right\rangle \right\}, \tag{2.30b}$$

$$Y_L = \frac{c^4}{4G} \sum_{k=0}^{+\infty} \left\{ - \left(\frac{\mathrm{d}}{c\,\mathrm{d}t}\right)^{2k} \left\langle \hat{n}_L \Psi^{ii}_{k,\ell}\right\rangle + \frac{3(2\ell+1)}{(\ell+1)(2\ell+3)} \left(\frac{\mathrm{d}}{c\,\mathrm{d}t}\right)^{2k+1} \left\langle \hat{n}_{iL} \Psi^{i0}_{k,\ell+1}\right\rangle \right.$$
$$\left. - \frac{2(2\ell+1)}{(\ell+1)(\ell+2)(2\ell+5)} \left(\frac{\mathrm{d}}{c\,\mathrm{d}t}\right)^{2k+2} \left\langle \hat{n}_{ijL} \Psi^{ij}_{k,\ell+2}\right\rangle \right\}, \tag{2.30c}$$

$$Z_L = \frac{c^4}{4G} \varepsilon_{ab\langle i_\ell} \sum_{k=0}^{+\infty} \left\{ \frac{2\ell+1}{(\ell+2)(2\ell+3)} \left(\frac{\mathrm{d}}{c\,\mathrm{d}t}\right)^{2k} \left\langle \hat{n}_{L-1\rangle ac} \Psi^{bc}_{k,\ell+1}\right\rangle \right\}. \tag{2.30d}$$

3. Application to a boosted Schwarzschild solution

3.1. Source quadrupole moment of the BSS at 3PN order

As an application of our explicit surface-integral formulae (2.29), we wish to compute the source-type multipole moments of a spherically symmetric extended body moving with *uniform* velocity. Remember that our formalism assumes, in principle, that we are dealing with regular, weakly self-gravitating bodies. We expect, because of the nice 'effacing properties' of Einstein's theory [34], that our final physical results, especially when they are expressed as surface integrals as in (2.29), can be applied to more general sources, such as neutron stars or black holes. Indeed, we are going to confirm this expectation in the simplest possible case, that of an isolated spherically symmetric body which is known, by Birkhoff's theorem, to generate a universal exterior gravitational field, given by the Schwarzschild solution. We shall therefore apply our formulae to a *boosted Schwarzschild solution* (BSS). Actually, in order to justify our use of the BSS in standard harmonic coordinates, we must dispose of a small technicality.

This technicality concerns the non-uniqueness of harmonic coordinates for the Schwarzschild solution, even under the assumption of stationarity (in the rest frame) and spherical symmetry. Indeed, under these assumptions, and starting from the usual Schwarzschild–Droste radial coordinate, say r_S, the (rest frame) radial coordinate of the most general harmonic coordinate system, say $r = k(r_S)$, must satisfy the differential equation (see, e.g., Weinberg [35], p 181)

$$\frac{\mathrm{d}}{\mathrm{d}r_S}\left[\left(r_S^2 - \frac{2GM}{c^2}r_S\right)\frac{\mathrm{d}k}{\mathrm{d}r_S}\right] = 2k. \tag{3.1}$$

The 'standard' solution of equation (3.1), which is considered in all textbooks such as [35], reads simply

$$r = k^{\text{standard}}(r_S) = r_S - \frac{GM}{c^2}. \tag{3.2}$$

In the black-hole case, the solution (3.2) is the only one which is regular on the horizon, i.e. when $r_S = 2GM/c^2$ (as will be clear from equations (3.2) below). However, in the case of the external metric of an extended spherically symmetric body, the regularity on the horizon is not a relevant issue. What is relevant is that the solution of the *external* problem (3.1) be smoothly matched to a *regular* solution of the corresponding *internal* problem. As usual, this matching determines a unique solution everywhere. In general, this unique, everywhere regular, solution will correspond, in the exterior of the body, to a particular case of the general, two-parameter solution of the second-order differential equation (3.1). The latter is of the form

$$r = k^{\text{general}}(r_S) = c_1\left(r_S - \frac{GM}{c^2}\right) + c_2 k_2(r_S), \tag{3.3}$$

where $k_2(r_S)$ denotes the (uniquely defined) 'radially decaying solution' of equation (3.1), and where c_1 and c_2 are two integration constants. Indeed, when considering the flat-space limit of equation (3.1), it is easily seen that there are two independent solutions which behave, when $r_S \to +\infty$, as r_S and r_S^{-2} respectively. An explicit expression for the decaying solution is[9]

[9] Here

$$F(\alpha,\beta,\gamma,z) = 1 + \frac{\alpha\beta}{\gamma}\frac{z}{1!} + \frac{\alpha(\alpha+1)\beta(\beta+1)}{\gamma(\gamma+1)}\frac{z^2}{2!} + \cdots,$$

denotes Gauss's hypergeometric function.

$$k_2(r_S) = \frac{1}{r_S^2} F\left(2, 2, 4, \frac{2GM}{c^2 r_S}\right), \tag{3.4a}$$

$$F(2, 2, 4, z) = -\frac{6}{z^2}\left[2 + \frac{1}{z}(2 - z)\ln(1 - z)\right]. \tag{3.4b}$$

We can always normalize c_1 to the value $c_1 = 1$. Then, with the above definitions, c_2 has the dimension of a length cubed. By considering in more detail the matching of the general solution of the harmonically relaxed Einstein equations at the 2PN level (see, e.g., the book by Fock [36], p 322), one easily finds that the second integration constant is of the order of $c_2 \sim (GM/c^2)^2 a$, where a denotes the radius of the extended body under consideration. It is also easily checked that the constant c_2 parametrizes, at the linearized order, a *gauge vector* φ_1^i of the form $\varphi_1^i \propto c_2 \partial_i(1/r)$, and can thus be referred to as a 'gauge parameter'. Comparing to the general multipole decomposition (2.9), we see that this gauge parameter c_2 corresponds to the monopole ($\ell = 0$) in the gauge multipole sequence X_L.

In contrast to the multipole moments of *stationary* sources, which are geometric invariants (and can be expressed as surface integrals on a sphere at spatial infinity), the 'source multipole moments' defined in [2] (and re-expressed above as surface integrals over spheres in some intermediate region, $a \ll r \ll cT$) are probably not geometric invariants. They are useful intermediate constructs, which allow one to compute physically invariant information, but their definition is linked to the choice of harmonic coordinates covering the source. This means that the various 'gauge multipoles' W_L, X_L, Y_L, Z_L will influence, at some nonlinear order of the MPM iteration, the values of the two sequences of 'physical multipoles': I_L, J_L. Therefore, one should expect that, at some nonlinear order, the physical multipoles I_L, J_L of a boosted general, harmonic-coordinate spherically symmetric metric will start to depend on the value of the gauge parameter c_2.

Here, we are only interested in computing the quadrupole moment I_{ij} of a boosted general spherically symmetric metric. We shall see below that the index structure of I_{ij} will be provided by the STF tensor product of the boost velocity V^i with itself, denoted by $V^{\langle i} V^{j\rangle}$ (assuming that the origin of the coordinates is at the initial position of the centre of symmetry of the BSS). Therefore, any contribution to I_{ij} coming from the gauge parameter c_2 must contain, at least, the factors c_2 and $V^{\langle i} V^{j\rangle}$, and also the total mass M. Taking into account the dimensionality of $c_2 \sim (GM/c^2)^2 a$, which is that of a length cubed, it is easily seen that there is no way to generate such a contribution to I_{ij}. Therefore, we conclude that the source quadrupole moment of a boosted general, harmonic-coordinate spherically symmetric metric is strictly equal to the source quadrupole moment of a boosted *standard harmonic-coordinate* Schwarzschild solution, obtained by setting $c_2 = 0$ (and $c_1 = 1$) in (3.3), i.e. by choosing the standard harmonic radial coordinate (3.2).

In the following, we shall therefore consider only such a boosted Schwarzschild solution in standard form. We shall sometimes refer to the source of this solution as a black hole (though, strictly speaking, one should always have in mind some extended spherical star). For simplicity, we shall translate the origin of the coordinate system so that it is located at the initial position of the black hole at coordinate time $t = 0$. With this choice of origin of the coordinates all the current-type moments J_L of the BSS are zero. We shall concentrate our attention on the mass-type quadrupole moment, I_{ij}, that we shall compute at the 3PN order.

Let us denote by $x^\mu = (ct, \mathbf{x})$ the global reference frame, in which the black hole is moving, and by $X^\mu = (cT, \mathbf{X})$ the rest frame of the black hole—both x^μ and X^μ are assumed to be harmonic coordinates. Let $x^i(t)$ be the rectilinear and uniform trajectory of the (centre of symmetry of the) BSS in the global coordinates x^μ, and $\mathbf{V} = (V^i)$ be the constant coordinate velocity of the BSS,

$$V^i \equiv \frac{\mathrm{d}x^i(t)}{\mathrm{d}t}. \tag{3.5}$$

The rest frame X^μ is transformed from the global one x^μ by the Lorentz boost (for simplicity we consider a pure Lorentz boost without rotation of the spatial coordinates)

$$x^\mu = \Lambda^\mu_\nu(\mathbf{V})X^\nu, \tag{3.6}$$

whose components are explicitly given by

$$\Lambda^0{}_0(\mathbf{V}) = \gamma, \tag{3.7a}$$

$$\Lambda^i{}_0(\mathbf{V}) = \Lambda^0{}_i(\mathbf{V}) = \gamma\frac{V^i}{c}, \tag{3.7b}$$

$$\Lambda^i{}_j(\mathbf{V}) = \delta^i_j + \frac{\gamma^2}{\gamma+1}\frac{V^iV_j}{c^2}, \tag{3.7c}$$

with

$$\gamma \equiv \left(1 - \frac{V^2}{c^2}\right)^{-1/2}. \tag{3.7d}$$

As explained above, we can assume that the metric of the BSS in the rest frame X^μ takes the standard harmonic-coordinate Schwarzschild expression, which we write in terms of the Gothic metric deviation $H^{\mu\nu}$, satisfying $\partial_\nu H^{\mu\nu} = 0$. (We use the same conventions as in equation (2.1), with upper case letters referring to quantities associated with the BSS rest frame.) Hence,

$$H^{00} = 1 - \frac{\left(1+\frac{GM}{c^2R}\right)^3}{1-\frac{GM}{c^2R}}, \tag{3.8a}$$

$$H^{i0} = 0, \tag{3.8b}$$

$$H^{ij} = -\frac{G^2M^2}{c^4R^2}N^iN^j, \tag{3.8c}$$

where M is the total mass, $R \equiv |\mathbf{X}|$ and $N^i \equiv X^i/R$. A well-known feature of the Schwarzschild metric in harmonic coordinates is that the spatial Gothic metric H^{ij} is made of a single quadratic-order term $\propto G^2$ as shown in equation (3.8*c*). The Gothic metric deviation transforms like a Lorentz tensor so the metric of the BSS in the global frame x^μ reads as

$$h^{\mu\nu}(x) = \Lambda^\mu{}_\rho\Lambda^\nu{}_\sigma H^{\rho\sigma}(\Lambda^{-1}x), \tag{3.9}$$

in which the rest-frame coordinates have been expressed by means of the global ones, i.e. $X^\mu(x) = (\Lambda^{-1})^\mu{}_\nu x^\nu$, where the inverse Lorentz transformation is given by $(\Lambda^{-1})^\mu{}_\nu(\mathbf{V}) \equiv \Lambda_\nu{}^\mu(\mathbf{V}) = \Lambda^\mu{}_\nu(-\mathbf{V})$. In our explicit calculations (done with the software Mathematica) we employ the BSS metric in exactly the form given by equation (3.9). The only problem is to derive the explicit relations giving the rest-frame radial coordinate R and the unit direction N^i as functions of their global-frame counterparts r and n^i, of the global coordinate time t, and of the boost velocity V^i. For these relations we find

$$R = r\left[1 + c^2(\gamma^2-1)\left(\frac{t}{r}\right)^2 - 2\gamma^2(Vn)\left(\frac{t}{r}\right) + \gamma^2\frac{(Vn)^2}{c^2}\right]^{1/2}, \tag{3.10a}$$

$$N^i = \frac{n^i - \gamma V^i\left(\frac{t}{r}\right) + \frac{\gamma^2}{\gamma+1}\frac{V^i}{c^2}(Vn)}{\left[1 + c^2(\gamma^2-1)\left(\frac{t}{r}\right)^2 - 2\gamma^2(Vn)\left(\frac{t}{r}\right) + \gamma^2\frac{(Vn)^2}{c^2}\right]^{1/2}}, \tag{3.10b}$$

where $(Vn) \equiv \mathbf{V} \cdot \mathbf{n} = V^j n^j$ is the usual Euclidean scalar product. The latter formulation of the BSS metric, equations (3.8)–(3.10), is well adapted to our calculations because we have to perform, when computing the source multipole moments, an integration over the *coordinate* three-dimensional spatial slice $\mathbf{x} \in \mathbb{R}^3$, with *coordinate* time $t =$ const, and this is easily done using the explicit relations (3.10).

However, let us note that the BSS metric (in standard harmonic coordinates) is best formulated in a manifestly Lorentz covariant way as follows:

$$h^{\mu\nu} = \left(1 - \frac{\left(1 + \frac{GM}{c^2 r_\perp}\right)^3}{1 - \frac{GM}{c^2 r_\perp}}\right) u^\mu u^\nu - \frac{G^2 M^2}{c^4 r_\perp^2} n^\mu n^\nu, \tag{3.11}$$

where u^μ is the timelike unit four-velocity of the centre of symmetry of the BSS, where n^μ is the spacelike unit vector pointing from the BSS to the field point along the direction *orthogonal* (in a Minkowskian sense) to the world line of the BSS, and where $r_\perp$ denotes the orthogonal distance to the world line (square root of the interval). Expression (3.11) is completely equivalent to (and more elegant than) the more 'coordinate-rooted' formulation (3.8)–(3.10). We shall employ it in a future investigation [37].

We compute the quadrupole moment I_{ij} of the BSS, following the prescriptions defined by equation (2.29*a*). To this end we first expand $h^{\mu\nu}$ when $c \to +\infty$, taking into account all the c present both in the expression of the rest frame metric (3.8) as well as those coming from the Lorentz transformation (3.9) and (3.10). In this process the boost velocity $\mathbf{V}$ is to be considered as a constant, 'spectator', vector. Note in passing that, in the present problem, the characteristic size a of the source at time t is given by the displacement from the origin, $a \sim \mathbf{V}t$, where $V \equiv |\mathbf{V}|$, while the near-zone corresponds to $r \ll ct$. Therefore, the far near-zone, where we read off the multipole moments as some combination of expansion coefficients $\varphi^{\mu\nu}_{a,b}(\mathbf{n}, t)$, is the domain $Vt \ll r \ll ct$. We have evidently to assume that $V \ll c$ for this region to exist.

We then first get the near-zone (or PN) expansion of the BSS metric, $\overline{h}^{\mu\nu}$, by expanding in inverse powers of c up to 3PN order. Next we compute the multipolar (or far) re-expansion of each of the PN coefficients when $r \to +\infty$ with $t =$ const. In this way we obtain what we have denoted by $\mathcal{M}(\overline{h}^{\mu\nu})$ in equation (2.21). In the BBS case it is evident that the far-zone expansion (2.21) involves simply some powers of $1/r$, without any logarithm of r (indeed, see, e.g., equation (3.10)).

With $\mathcal{M}(\overline{h}^{\mu\nu})$ in hand we have the coefficients of the various powers of $1/r$, and we obtain thereby the various quantities $\Psi^{\mu\nu}_{k,\ell}$ defined by equation (2.26). It is then a simple matter to compute all the required angular averages present in our formula (2.29*a*) and to obtain the following 3PN mass quadrupole moment of the BSS (the angular brackets surrounding indices referring to the STF projection),

$$I^{\mathrm{BSS}}_{ij} = mt^2 V^{\langle i} V^{j\rangle} \left[1 + \frac{9}{14}\frac{V^2}{c^2} + \frac{83}{168}\frac{V^4}{c^4} + \frac{507}{1232}\frac{V^6}{c^6}\right] + \frac{4}{7}\frac{G^2 M^3}{c^6} V^{\langle i} V^{j\rangle} + \mathcal{O}\left(\frac{1}{c^8}\right). \tag{3.12}$$

The first term represents the standard Newtonian expression, multiplied here by a bunch of relativistic corrections. (Recall that we have chosen the origin of the coordinate system at the initial location of the BSS at $t = 0$.)

The last term in equation (3.12), with coefficient $\mathcal{C} = 4/7$, is the most interesting for our purpose. It is purely of 3PN order, and it contains the first occurrence of the gravitational constant G, which therefore arises in the quadrupole of the BSS only at 3PN order. This term is interesting because it corresponds to one of the regularization 'ambiguities', due to an incompleteness of Hadamard's self-field regularization, which recently appeared in the calculation of the mass-type quadrupole moment of inspiralling point particle binaries at the

3PN order [11, 12]. The associated ambiguity parameter was called ζ, and was introduced as a factor of 3PN terms in the quadrupole moment having the form $\sim m_1^3 v_1^{\langle i} v_1^{j\rangle}$ or $m_2^3 v_2^{\langle i} v_2^{j\rangle}$, where m_1 and m_2 are the two point masses, and v_1^i, v_2^i are their coordinate velocities[10]. The parameter ζ represents the analogue of the 'kinetic ambiguity' parameter ω_k in the 3PN Hamiltonian of compact binaries [21, 22]. It is now clear that ζ can be determined from what we shall now call the *BSS limit* of a binary system, which consists of setting one of the masses of the binary to be *exactly zero*, say $m_2 = 0$.

We have computed the BSS limit of the 3PN mass-type quadrupole moment of compact binaries computed for general binary's orbits in [11, 12]. We have also inserted for the position of the first body $y_1^i = v_1^i t$ in order to conform to our choice for the origin of the coordinates. In this way we obtain

$$I_{ij}^{\text{BSSlimit}} = m_1 t^2 v_1^{\langle i} v_1^{j\rangle} \left[1 + \frac{9}{14}\frac{v_1^2}{c^2} + \frac{83}{168}\frac{v_1^4}{c^4} + \frac{507}{1232}\frac{v_1^6}{c^6}\right] + \left(\frac{232}{63} + \frac{44}{3}\zeta\right)\frac{G^2 m_1^3}{c^6} v_1^{\langle i} v_1^{j\rangle} + \mathcal{O}\left(\frac{1}{c^8}\right). \tag{3.13}$$

The comparison of equations (3.13) and (3.12) reveals a complete match between the two results if and only if we have the expected agreement between the masses, i.e. $M = m_1$, and the velocities, $v_1^i = V^i$ (since the velocity of the body remaining after taking the BSS limit should exactly take the boost velocity), and the ambiguity constant ζ takes the *unique* value

$$\zeta = -\frac{7}{33}. \tag{3.14}$$

Our conclusion, therefore, is that the ambiguity parameter ζ is uniquely determined by the BSS limit. Because of the close relation between the BSS limit with Lorentz boosts, it is clear that ζ is linked to the Lorentz–Poincaré invariance of the multipole moment formalism of [2] as applied to compact binary systems in [11, 12]. This link strongly suggests that the specific value (3.14) represents the only one for which the expression of the 3PN quadrupole moment is compatible with the Poincaré symmetry. In other words the present calculation indicates that the Poincaré invariance should correctly be incorporated into the laws of transformation of the source-type multipole moments for general extended PN sources as given by equations (2.14) or (2.29), though we have not verified this directly.

Note that the situation regarding ζ is the same as for the kinetic ambiguity parameter ω_k in the 3PN equations of motion, whose value has also been uniquely fixed by imposing the Lorentz–Poincaré invariance of the formalism [23–25]. (The other ambiguities ξ and κ in the binary's 3PN mass quadrupole moment [11, 12] parametrize some Galilean invariant terms which cannot be derived from a Lorentz transformation. This can also be seen from the fact that ξ and κ are in factor of some 'interacting' terms, depending on both masses m_1 and m_2, and are thus out of the scope of the BSS limit.)

Let us finally emphasize that we have obtained equation (3.14) here without using any regularization scheme for curing the divergencies associated with the self-field of point particles. However, it has been shown in recent related works [20, 37] that this value is precisely the one derived in the problem of point particles binaries at 3PN order by means of the dimensional self-field regularization, instead of Hadamard's regularization. (This shows that dimensional regularization is able to correctly keep track of the global Poincaré invariance of the general relativistic description of isolated systems.)

[10] See section X of [11] for the discussion on why and how to introduce the ambiguity parameters in the 3PN quadrupole of point particle binaries.

3.2. Retarded far-zone field of the BSS at 3PN order

In this section we present, as a further confirmation of our result, an alternative derivation of the crucial coefficient $\mathcal{C} = 4/7$ in equation (3.12), and hence of the kinetic-type ambiguity parameter $\zeta = -7/33$. This new derivation will be based on the expansion of the BSS metric (3.8) and (3.9) at Minkowskian *future null infinity*, $r \to +\infty$ with $u \equiv t - r/c = \text{const}$. It relies on the complete identification of the metric (3.8) and (3.9), considered in the limit where $r \to +\infty$ with $u = \text{const}$, with the multipolar-post-Minkowskian (MPM) external metric of an isolated system, as given by the quantity $\mathcal{M}(h^{\mu\nu})$ reviewed in section 2.1 above. This identification is justified by the fact that the general harmonic-coordinate MPM exterior metric defined by equation (2.5) smoothly matches, via the matching equation (2.13), the inner PN solution of the inhomogeneous field equations inside the matter source (see [2]). Indeed, so does the BSS metric (3.8) and (3.9), as we have seen during the discussion on the general solution of the harmonic-coordinates condition (3.1) for the Schwarzschild metric, and our proof that the parameter c_2 does not contribute to the quadrupole moment.

In this section we thus have to consider the BSS metric as a functional of the *six* sets of multipole moments $\{I_L, J_L, W_L, X_L, Y_L, Z_L\}$ parametrizing the linearized approximation to the MPM metric, namely $h_1^{\mu\nu}$ defined by equations (2.7)–(2.9), and then take into account the subsequent nonlinear iterations, $h_2^{\mu\nu}, h_3^{\mu\nu}, \ldots$, whose computation is defined by the MPM algorithm of [6], as reviewed in equations (2.11) and (2.12) above. Actually we shall work mostly with the so-called 'canonical' MPM metric, $h_{\text{can}}^{\mu\nu}$, a functional of *two and only two* types of multipole moments called 'canonical', mass-type moment M_L and current-type S_L, instead of the six source moments $I_L, J_L, \ldots, Z_L$. Working with the metric $h_{\text{can}}^{\mu\nu}$ simplifies drastically the computation of the nonlinear interactions. We shall justify below why, for our purpose, we can indeed use just this special case of a particular harmonic-gauge-fixed external metric, rather than the more general harmonic-coordinate external metric. Those metrics are geometrically equivalent, but we shall have to check (as in the case of the c_2 parameter in section 3.1) that gauge effects do not modify the result we are interested in.

In the present section we shall have to compute the quadratic and cubic metric corrections, $h_{\text{can}2}^{\mu\nu}$ and $h_{\text{can}3}^{\mu\nu}$ in the notation of equation (2.5), for some specific multipole interactions. The canonical MPM metric is defined by

$$h_{\text{can}}^{\mu\nu}[M_L, S_L] = \sum_{n=1}^{+\infty} G^n h_{\text{can}\,n}^{\mu\nu}, \tag{3.15}$$

where the linearized approximation is given by the same formulae as equations (2.8) but with the canonical moments M_L and S_L in place of the source moments I_L and J_L, and with all the gauge multipoles $\{W_L, X_L, Y_L, Z_L\}$ set to zero. In other words,

$$h_{\text{can}\,1}^{00}[M_L, S_L] = -\frac{4}{c^2}\sum_{\ell=0}^{+\infty}\frac{(-)^\ell}{\ell!}\partial_L\left[\frac{1}{r}M_L\right], \tag{3.16a}$$

$$h_{\text{can}\,1}^{0i}[M_L, S_L] = \frac{4}{c^3}\sum_{\ell=1}^{+\infty}\frac{(-)^\ell}{\ell!}\left\{\partial_{L-1}\left[\frac{1}{r}\dot{M}_{iL-1}\right] + \frac{\ell}{\ell+1}\varepsilon_{iab}\partial_{aL-1}\left[\frac{1}{r}S_{bL-1}\right]\right\}, \tag{3.16b}$$

$$h_{\text{can}\,1}^{ij}[M_L, S_L] = -\frac{4}{c^4}\sum_{\ell=2}^{+\infty}\frac{(-)^\ell}{\ell!}\left\{\partial_{L-2}\left[\frac{1}{r}\ddot{M}_{ijL-2}\right] + \frac{2\ell}{\ell+1}\partial_{aL-2}\left[\frac{1}{r}\varepsilon_{ab(i}\dot{S}_{j)bL-2}\right]\right\}. \tag{3.16c}$$

Then the nonlinear metrics $h_{\text{can}\,n}^{\mu\nu}[M_L, S_L]$, for $n \geqslant 2$, are obtained from (3.16) by means of the same algorithm as before, explained in equations (2.11) and (2.12), in the case where the gauge vector $\varphi_1^\mu = 0$, together with the replacement $\{I_L, J_L\} \to \{M_L, S_L\}$.

We have already alluded to the important point, proved in [6], that although the canonical metric (3.15) and (3.16) is simpler than our previous construction (2.5)–(2.9), it is physically or geometrically equivalent to it, i.e. it describes the same physical matter system, provided that M_L and S_L are related to $I_L, J_L, \ldots, Z_L$ by some specific relations of the type

$$M_L = I_L + \mathcal{F}_L[I, J, W, X, Y, Z], \tag{3.17a}$$

$$S_L = J_L + \mathcal{G}_L[I, J, W, X, Y, Z], \tag{3.17b}$$

where $\mathcal{F}_L$ and $\mathcal{G}_L$ denote two nonlinear functionals (at least quadratic in the moments) of the original set of source moments $I_L, J_L, \ldots, Z_L$. However, if the use of only two sets of moments, M_L and S_L, is very useful when computing the nonlinear multipole interactions, it remains that these moments have still to be related to the more 'fundamental' source moments by equations (3.17). Indeed we know the analytic closed-form expressions of the source moments $I_L, J_L, \ldots, Z_L$ (see section 2.1) but similar formulae for M_L, S_L, valid to all PN orders, are not known to exist. Equations (3.17) need to be investigated anew for each specific cases. Fortunately, M_L and S_L are 'almost' equal to their counterparts I_L and J_L. Indeed we know that in the case of the mass-type moments for instance, equation (3.17*a*) when further PN expanded reads as

$$M_L = I_L + \frac{1}{c^5}\delta I_L + \mathcal{O}\left(\frac{1}{c^7}\right), \tag{3.18}$$

where δI_L denotes some correction term arising at order 2.5PN only (if necessary this term is given by equation (4.24) in [38]). The remainder in equation (3.18) is of order 3.5PN. Equation (3.18) shows that the 3PN term in the quadrupole moment of the BSS we are looking for—last term in equation (3.12)—will be *the same* for I_{ij} as for M_{ij}. In addition to the relations (3.17) and (3.18) we must also take into account the possible effect of the coordinate transformation between the canonical metric (3.15) and the metric (2.5), since as we mentioned, it is the latter which must be identified with the retarded far-zone expansion of the BSS metric (3.9).

Let us consider the general structure of the mass-type moments M_L (or I_L) in the case of the BSS. In the present problem there is only one vector which can be used to build the moment: namely the boost velocity V^i, so the index structure of M_L must necessarily be made of the STF product $V_{\langle L\rangle} \equiv V_{\langle i_1} \cdots V_{i_\ell\rangle}$. (Indeed we recall that we choose the origin of the coordinate system to lie on the trajectory of the BSS, so we do not have at our disposal the vectorial separation between the origin and the BSS world line. It is clear that such a restriction is not physically crucial, however it simplifies the presentation and several arguments in this section very much.) In addition, we readily see on dimensional grounds that $V_{\langle L\rangle}$ must be multiplied by either u^ℓ, where $u \equiv t - r/c$, or by the product $(GM/c^3)u^{\ell-1}$, or by $(GM/c^3)^2 u^{\ell-2}$, and so on, and that each of the latter terms can be multiplied by some relativistic corrections of the type $(V^2/c^2)^n$ up to any PN order. Here M denotes the BSS constant mass monopole, $\ell = 0$, or ADM mass. The general structure of the mass-type moment M_L (and also of I_L as well) at the 3PN order therefore reads

$$M_L(u) = M V_{\langle L\rangle}\left[u^\ell\left(1 \,\&\, \frac{V^2}{c^2} \,\&\, \frac{V^4}{c^4} \,\&\, \frac{V^6}{c^6}\right) \,\&\, u^{\ell-1}\frac{GM}{c^3}\left(1 \,\&\, \frac{V^2}{c^2}\right)\right.$$
$$\left.\,\&\, u^{\ell-2}\left(\frac{GM}{c^3}\right)^2 + \mathcal{O}\left(\frac{1}{c^7}\right)\right], \tag{3.19}$$

where the notation & means that we have to add a term having the structure that is indicated next. In the quadrupole case, $\ell = 2$, we recognize in the last (explicit) term of equation (3.19) the interesting form of the contribution to the ambiguity ζ in the BSS limit.

The mass multipole moment of the BSS varies with time typically like some u^ℓ. This fact seems to be incompatible with the construction of MPM metrics in [6], since it was assumed there, in order to implement this construction, that the matter source is stationary before some fixed finite instant $-\mathcal{T}$ in the remote past (and also that the coordinates are mass-centred in the sense that the dipole moment M_i is always zero). These assumptions, made in [6] for purely technical reasons, imply in principle that all the multipole moments M_L are constant before the date $-\mathcal{T}$ (i.e. for $u < -\mathcal{T}$), and thus cannot *a priori* be applied to the physical situation of the BSS. Nevertheless, we shall admit in the present paper that we are allowed to use the construction and the results of [6] even in the case of the BSS. Indeed there has been several indications in our previous works using the MPM formalism, notably when the formalism was used for the computation of gravitational wave tails and 'tails-of-tails' [7, 8], that the MPM expansion can in fact be applied to more general sources which have always been non-stationary, for instance an inspiralling compact binary formed by capture of two particles initially moving on some hyperbolic-like orbits. On the other hand, as we shall see our application of the MPM formalism to the external field of the BSS will yield some consistent result, which is independent of any initial instant $-\mathcal{T}$ and is in agreement with the result of section 3.1. This justifies *a posteriori* (to some extent) our expectation that the MPM formalism is still valid in the case of the BSS. Furthermore, we shall give in the appendix an explicit proof that the integration formulae of the MPM formalism admit a well-defined limit when the multipole moments are continuously deformed into those of the BSS.

We are interested in the quadrupolar contribution in the full nonlinearity expansion (3.15), as seen from retarded infinity, $r \to +\infty$ with $u = \text{const}$ (this limit will be referred below to as $\mathcal{I}^+$). We base our investigation on the time–time component (00) of the metric, because this is that component which is necessary and sufficient in order to obtain the multipole moment itself, as opposed to some time derivative of it as would be deduced from the $0i$ and ij components (this is very important because we are looking for a term in M_{ij} which is a constant). At the linearized approximation, the 'far-field' quadrupole moment as seen from $\mathcal{I}^+$ simply reduces to the canonical moment M_{ij}, and from equation (3.16*a*) we get (with $\hat{n}^{ij} \equiv n^i n^j - \frac{1}{3}\delta^{ij}$)

$$G h^{00}_{\text{can}\,1} = \cdots - \frac{6G}{c^2 r^3} \hat{n}^{ij} M_{ij}(u) + \cdots. \tag{3.20}$$

Here we focus on the term of the form $\hat{n}^{ij} r^{-3} f(u)$, and the ellipsis refer to all the other terms, either involving some other multipolarities $\hat{n}_L$ with $\ell \neq 2$, or a power of $1/r$ different from 3. Consider now the corrections brought about by the nonlinear terms to be added to the linearized expression (3.20), and write

$$\begin{aligned} h^{00}_{\text{can}} &\equiv \sum_{n=1}^{+\infty} G^n h^{00}_{\text{can}\,n} \\ &= \cdots - \frac{6G}{c^2 r^3} \hat{n}^{ij} M^{\text{rad}}_{ij}(u) + \cdots, \end{aligned} \tag{3.21}$$

where the far-field or 'radiative' quadrupole moment (i.e., as seen from $\mathcal{I}^+$) is denoted by M^{rad}_{ij}. The nonlinear terms in (3.21) introduce many couplings between the different multipole moments, and there are a lot of possibilities in the general case. However, things are much simpler in the case of the BSS owing to the particular structure of the moments as determined in equation (3.19). Notably, the ℓ-th time derivative of the BSS moment M_L is always a constant. Then we find that at 3PN order the *most general* form of the allowed nonlinear terms in $M^{\rm rad}_{ij}$ reads as

$$M^{\rm rad}_{ij} = M_{ij} + \gamma \frac{GM}{c^3} \dot{M}_{ij} + \epsilon \frac{G}{c^3} M_{\langle i} \dot{M}_{j\rangle} + \sigma \frac{G}{c^5} \dot{M}_{k\langle i} \ddot{M}_{j\rangle k} + \phi \frac{G}{c^5} M_k \dddot{M}_{ijk} + \theta \frac{G}{c^5} \dot{M}_k \ddot{M}_{ijk}$$
$$+ \rho \frac{G^2 M^2}{c^6} \ddot{M}_{ij} + \eta \frac{G^2 M}{c^6} \dot{M}_{\langle i} \dot{M}_{j\rangle} + \mathcal{O}\left(\frac{1}{c^7}\right), \tag{3.22}$$

where $\gamma, \epsilon, \sigma, \phi, \theta, \rho, \eta$ represent some unknown (for the moment at least) coefficients, which are in general constant but as we shall see which can also depend on the logarithm of the distance r. Note that equation (3.22) might *a priori* contain also some non-local contributions, say $\int_{-\infty}^{u} \mathrm{d}v\, M \dddot{M}_{ij}(v)$, or (worse) $\int_{-\infty}^{u} \mathrm{d}v\, M^2 \ddot{M}_{ij}(v)$, but we shall discuss such terms in the appendix and show that they do not appear. The most important terms for the present purpose are the two last ones, with coefficients ρ and η, which involve the *cubic-order* multipole interactions $M^2 \times M_{ij}$ and $M \times M_i \times M_j$ (where M is the mass and M_i the mass dipole of the BSS).

The coefficient we are looking for in the 3PN BSS source-type quadrupole moment I_{ij} was denoted by $\mathcal{C}$, and we now write the corresponding term as

$$\delta_{\mathcal{C}} I_{ij} = \mathcal{C} \frac{G^2 M^3}{c^6} V^{\langle i} V^{j\rangle}. \tag{3.23}$$

The other terms in the BSS quadrupole have a different structure which has already been displayed in equation (3.12). The result $\mathcal{C} = 4/7$ of section 3.1 will be recovered by the following method. As we noted after equation (3.18), the coefficient $\mathcal{C}$ is necessarily the same for the source-type and canonical-type moments, hence

$$\delta_{\mathcal{C}} M_{ij} = \mathcal{C} \frac{G^2 M^3}{c^6} V^{\langle i} V^{j\rangle}. \tag{3.24}$$

Consider next the radiative-type moment (3.22), and look for the modification of $\mathcal{C}$ induced by nonlinearities. In order to do this we have to remember the fact (see [38]) that both the canonical and source moments M_L and I_L for general matter systems admit a PN expansion which is 'even' up to 2PN order, with the first 'odd' correction being at the 2.5PN level. This simple fact immediately shows that it is impossible that the 'odd' terms shown in equation (3.22), which carry explicitly in front the odd powers $1/c^3$ and $1/c^5$, contribute to a term at the 3PN order. So we conclude that the sought-for modification of $\mathcal{C}$ can come only from the two last terms, with coefficients ρ and η. Taking into account the Newtonian results $\ddot{M}_{ij} = 2MV^{\langle i} V^{j\rangle} + \mathcal{O}(c^{-2})$ and $\dot{M}_i = MV^i + \mathcal{O}(c^{-2})$ we find

$$\delta_{\mathcal{C}} M^{\rm rad}_{ij} = (\mathcal{C} + 2\rho + \eta) \frac{G^2 M^3}{c^6} V^{\langle i} V^{j\rangle}. \tag{3.25}$$

(Here our notation $\delta_{\mathcal{C}} M^{\rm rad}_{ij}$ means that we are considering the complete term having the above indicated structure.)

Next we come to the central part of this investigation, namely the computation of the two coefficients ρ and η. This task is not so easy because ρ and η are in factor of some cubically nonlinear terms. We have obtained them by straightforward application of the MPM algorithm of [6, 33]. Actually the value of ρ, corresponding to the interaction $M^2 \times M_{ij}$, is already contained in the result of the calculation of the gravitational wave 'tails-of-tails' in [7]. For convenience we relegate the details of this nonlinear iteration to the appendix, and simply quote here our end results:

$$\rho = \frac{1271}{735} - \frac{58}{21} \ln\left(\frac{r}{r_0}\right), \tag{3.26a}$$

$$\eta = \frac{2918}{735} + \frac{116}{21} \ln\left(\frac{r}{r_0}\right). \tag{3.26b}$$

For completeness we give also the known values of three other constants in (3.22):

$$\gamma = \tfrac{7}{2}, \qquad \epsilon = \tfrac{7}{3}, \qquad \sigma = \tfrac{20}{21}. \tag{3.27}$$

These values come from equation (2.8*a*) in [7] for γ, equation (A.9*a*) in the appendix below for ϵ, and table 2 in [33] for σ. (We have not computed the coefficients ϕ and θ.)

As we see from (3.26), both ρ and η depend on the logarithm of r/r_0, where r_0 is the same constant as in the source multipole moments (2.14), but we nicely find that the logarithms cancel out in the relevant combination of these coefficients which enters equation (3.25). In fact, we can argue that the cancellation of the logarithms must necessarily occur because nowhere in the far-zone expansion of the BSS metric (3.8) and (3.9) can such logarithms of r/r_0 be generated. Hence we get

$$\delta_{\mathcal{C}} M_{ij}^{\rm rad} = \left(\mathcal{C} + \frac{52}{7}\right) \frac{G^2 M^3}{c^6} V^{\langle i} V^{j\rangle}. \tag{3.28}$$

Let us now check that there are no gauge effects, linked to the non-geometrical nature of our definitions for the multipole moments, concerning the particular term we consider in (3.28), in the sense that the coordinate transformation between the canonical metric coefficient $h^{00}_{\rm can}$ and the corresponding BSS one, computed from the expansion of equations (3.8)–(3.9) and given by $\mathcal{M}(h^{00})$ in the MPM formalism, has no effect on this particular term. The proof goes by noting first that the coordinate transformation at the linearized level is given by the gauge vector (2.9) parametrized by the four source-type moments W_L, X_L, Y_L and Z_L. The latter moments were given in equations (5.17)–(5.20) of [2], and we have provided in equations (2.30) above their new forms in terms of surface integrals. An important point is that the moments W_L, X_L, Y_L, Z_L have been defined in such a way that they admit some nonzero *finite* limits when $c \to +\infty$; in other words, they 'start at Newtonian order' and their Newtonian limit is nonzero. By using this fact together with dimensional analysis, it is a simple matter to write down their structures in the case of the BSS, in a manner similar to what we did for the moment M_L in equation (3.19). We find

$$W_L = V^2 u \,\{\text{same structure as the one of } M_L \text{ given by (3.19)}\}, \tag{3.29a}$$

$$X_L = V^4 u^2 \,\{\text{same structure}\}, \tag{3.29b}$$

$$Y_L = V^2 \,\{\text{same structure}\}, \tag{3.29c}$$

while Z_L is a current-type moment so $Z_L = 0$ with our choice of origin for the BSS. The structures (3.29) imply that the potentially dangerous term, which is proportional to M^3, must necessarily appear in these moments at order $1/c^6$ relatively to the Newtonian order. Next one readily shows, again on dimensional grounds, that the only possible modifications of the quadrupole moment M_{ij} in (3.20) which are due to the gauge transformation, take the forms $\dot{W}_{ij}/c^2$, $\ddot{X}_{ij}/c^4$ or Y_{ij}/c^2. It is therefore impossible, because of the latter extra factors $1/c^2$ or $1/c^4$, that a dangerous term be generated in this way at the 3PN order. Similar arguments are even more easily applied at nonlinear order in the coordinate transformation: for instance we find that it is impossible that a nonlinear coupling of the type $M_L \times W_P$, or $W_L \times X_P$, has the correct structure at 3PN order in the quadrupole moment.

This check being done, we conclude that the coefficient we predicted for the relevant term in equation (3.28) represents exactly the quadrupolar contribution in the retarded far-zone expansion (at $\mathcal{I}^+$) of the BSS metric (and by the reasoning of section 3.1 we know that we can use the BSS metric in standard harmonic coordinates). Now the point is that we can also compute *directly* this contribution in the far-zone expansion of the BSS metric by using the formulae (3.8) and (3.9). Since the far-zone expansion is to be done at retarded time (and not at time $t = \text{const}$ as we did in section 3.1), we must for this calculation substitute t by $u + r/c$ in equations (3.10) and only afterwards take the limit $r \to +\infty$ (holding $u = \text{const}$). In this way we obtain the quadrupolar piece $\propto \hat{n}^{ij}/r^3$ in the 00 component of the metric, and as we have proved before we are allowed to *identify* the term therein having the correct structure with the one computed in (3.28). A simple Mathematica calculation reveals that the term in question has the coefficient: 8. Therefore $\mathcal{C} + \frac{52}{7} = 8$ and we obtain

$$\mathcal{C} = \frac{4}{7}, \tag{3.30}$$

in complete agreement with our previous finding (3.12), and in support of the value for the kinetic ambiguity parameter: $\zeta = -7/33$.

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Appendix. Nonlinear multipole interactions

This appendix is devoted to the computation of the cubically nonlinear coefficients ρ and η entering equation (3.22). Following the MPM algorithm of [6] we must first compute the nonlinear cubic source term, say $\Lambda^{\mu\nu}_{\text{can}\,3}$, which is composed of the sum of a quadratic-order piece made out of products between $h_{\text{can}\,1}$ and $h_{\text{can}\,2}$ (and of course their gradients), and of a purely cubic-order piece, involving three factors $h_{\text{can}\,1}$. The linearized metric has been given in (3.16), and from it all subsequent iterations are generated by the MPM algorithm, which is the same as in equations (2.11) and (2.12) except that, for the 'canonical' construction, the gauge vector is $\varphi^{\mu}_1 = 0$ and we use M_L and S_L as moments instead of I_L, J_L. The cubic source term $\Lambda^{\mu\nu}_{\text{can}\,3}[h_{\text{can}\,1}, h_{\text{can}\,2}]$ is inverted by means of the retarded d'Alembertian operator, regularized by the specific finite part $\text{FP}_{B=0}$ of equation (2.12),

$$u^{\mu\nu}_{\text{can}\,3} = \underset{B=0}{\text{FP}}\, \Box^{-1}_{\text{R}} \left[\left(\frac{r}{r_0} \right)^B \Lambda^{\mu\nu}_{\text{can}\,3} \right]. \tag{A.1}$$

The metric at cubic order reads then

$$h^{\mu\nu}_{\text{can}\,3} = u^{\mu\nu}_{\text{can}\,3} + v^{\mu\nu}_{\text{can}\,3}, \tag{A.2}$$

where $v^{\mu\nu}_{\text{can}\,3}$ represents a particular homogeneous solution of the wave equation, such that the harmonicity condition $\partial_\nu h^{\mu\nu}_{\text{can}\,3} = 0$ is satisfied. Below we shall not need to compute $v^{\mu\nu}_{\text{can}\,3}$, since we shall simply have to invoke the fact that its 00 component, $v^{00}_{\text{can}\,3}$, is made of multipolarities $\ell = 0$ or 1 only (see e.g. equation (2.12*a*) in [33]), and will thus always be zero for the quadrupole case $\ell = 2$ of concern to us here.

Consider first the cubic interaction $M \times M \times M_{ij}$. In this case $\Lambda^{\mu\nu}_{\text{can}\,3}$ is already known from previous work, equation (4.16*a*) of [7], which gives, for the needed 00 component[11],

$$\Lambda^{00}_{\text{can}\,3} = \hat{n}_{ab} M^2 [-516 r^{-7} M_{ab}(u) - 516 r^{-6} \dot{M}_{ab}(u) - 304 r^{-5} \ddot{M}_{ab}(u)]. \tag{A.3}$$

[11] In this appendix we pose $G = c = 1$.

To integrate we use the formulae given in appendix A of [33]. We limit ourselves to the computation of the part $u^{\mu\nu}_{\mathrm{can}\,3}$ of the algorithm since $v^{00}_{\mathrm{can}\,3} = 0$ for the particular multipole interaction we consider. Throughout this calculation we use the fact that the second time derivative of the BSS quadrupole moment is constant, hence $\dddot{M}_{ij} = 0$. First of all, by straightforward use of the integration formula (A.16*a*) of [33] we obtain

$$h^{00}_{\mathrm{can}\,3} = \hat{n}_{ab} M^2 \left[-\frac{516}{14} r^{-5} M_{ab}(u) - \frac{516}{14} r^{-4} \dot{M}_{ab}(u) - \frac{3354}{245} r^{-3} \ddot{M}_{ab}(u) \right] - \frac{580}{7} \mathop{\mathrm{FP}}_{B=0} \Box^{-1}_{\mathrm{R}} \left[\left(\frac{r}{r_0} \right)^B r^{-5} \hat{n}_{ab} M^2 \ddot{M}_{ab}(u) \right]. \tag{A.4}$$

The last term is *a priori* more delicate in the case of the BSS because we know from equation (A.13) of [33] that it could generate an integral depending on the whole time evolution of the system, such as a 'tail' integral of the type $J(u) \equiv \int_{-\infty}^{u} \mathrm{d}v\, M^2 \dddot{M}_{ij}(v)$. The value of an integral like J would be quite ambiguous within our MPM-based approach. Indeed, on the one hand, the boosted system that we finally consider has $M_{ij}(u)$ proportional to u^2, so that $\dddot{M}_{ij}(u)$ vanishes identically. We would therefore expect, from this point of view, that J, being the integral of a vanishing integrand, is zero: $J = 0$. On the other hand, we can perform the v-integral in J to get $J = M^2[\ddot{M}_{ij}(u) - \ddot{M}_{ij}(-\infty)]$. Now, as we recalled above, the MPM framework on which we base our discussion assumes that we start initially by considering systems such that the multipole moments become time independent in the remote past (before some finite instant $-\mathcal{T}$). For such systems, the second contribution in the latter expression for J vanishes, and we would get $J = M^2 \ddot{M}_{ij}(u)$, which does not vanish in the case of a boosted source.

However, this ambiguous situation does not appear in the present calculation. Indeed, the logic of our calculation is the following. To derive the MPM result (A.4) we had to initially assume that $M_{ij}(u)$ tends fast enough towards a constant in the infinite past. Then, after having done the MPM iteration we get the form (A.4). Now, starting from the explicit expression (A.4) we want to relax the original assumption about $M_{ij}(u)$, and consider a *deformation process* in which $M_{ij}(u)$ interpolates between an initial MPM-like $M_{ij}(u)$ (tending to a constant in the past) and a final $M^{\mathrm{BSS}}_{ij}(u)$ of the form of u^2. The question is then to know (i) whether the RHS of equation (A.4) admits any limit after this continuous deformation process, and (ii) what is the value of this limit. In mathematical terms the question is essentially a question of interchange of a limit with an integral operation, i.e. whether $\lim_n \int f_n(u)\,\mathrm{d}u = \int \lim_n f_n(u)\,\mathrm{d}u$ holds, for a sequence of functions $f_n(u)$. As we know the answer is positive under 'good' conditions, for instance of uniform convergence, or more generally (Lebesgue's theorem) of dominated convergence, which says essentially that if $|f_n(u)| < g(u)$ and if $\int g(u)\,\mathrm{d}u$ is finite, then we can interchange the limits. In our case, we can use Lebesgue's theorem of dominated convergence (see, e.g., the book [39]), which is both simple and powerful.

The only delicate term in the RHS of (A.4) is the last, integral term. We must interpolate between some initial $\ddot{M}_{ij}(u)$ which vanishes in the past to become a constant in the future, and a final $\ddot{M}^{\mathrm{BSS}}_{ij}(u)$ which is always constant. It is clear that we can do this interpolation in a way that $|\ddot{M}_{ij}(u)|$ remains *always bounded*. Using such a bound in the last term of (A.4), which is a three-dimensional (retarded) integral, we easily see that, under the assumption of a bounded $|\ddot{M}_{ij}(u)|$, the integrand is bounded by a (positive) function which is *integrable* (because of the fast convergence brought by the r^{-5} factor, together with the $1/r$ factor contained in the propagator). Therefore, we can indeed interchange the limiting process and the integration one, and conclude that the limit of the LHS of equation (A.4) exists, and is simply given by replacing $M_{ij}(u)$ in the RHS by its limiting expression for a BSS which is $M^{\mathrm{BSS}}_{ij}(u)$ proportional to u^2. As, under this limit, the integrand of the last term in (A.4) becomes time independent, we can explicitly compute the limit by replacing the retarded propagator by a Poisson integral. Hence, we get

$$\underset{B=0}{\mathrm{FP}}\,\Box_{\mathrm{R}}^{-1}\left[\left(\frac{r}{r_0}\right)^B r^{-5}\hat{n}_{ab}M^2\ddot{M}_{ab}\right] = \underset{B=0}{\mathrm{FP}}\,\Delta^{-1}\left[\left(\frac{r}{r_0}\right)^B r^{-5}\hat{n}_{ab}\right]M^2\ddot{M}_{ab}$$
$$= -\frac{1}{5}r^{-5}\left[\frac{1}{5}+\ln\left(\frac{r}{r_0}\right)\right]\hat{n}_{ab}M^2\ddot{M}_{ab}. \tag{A.5}$$

One easily checks that the result (A.5) agrees with the more formal way of doing the calculation which consists of applying the formula (A.13) of [33] to the BSS case. As we can see, the last term in the RHS of the latter formula, which is given by a 'tail' integral, vanishes when we insert the BSS quadrupole into the source term of the retarded integral, since this source term involves in fact the fifth time derivative of the BSS quadrupole which is zero. On the other hand, the logarithmic term in formula (A.13) of [33] does remain, and then we recover exactly equation (A.5)—with the same constant r_0 on both sides of the equation. The proof that we have detailed above, based on Lebesgue's theorem of dominated convergence, rigorously justifies (for the case at hand) that one can 'blindly' use the formulae in the appendix of [33] even for the case of the BSS.

Finally, gathering equations (A.4) and (A.5) we obtain

$$h^{00}_{\mathrm{can}\,3} = \hat{n}_{ab}M^2\left(-\frac{516}{14}r^{-5}M_{ab}(u) - \frac{516}{14}r^{-4}\dot{M}_{ab}(u) + \left[-\frac{2542}{245}+\frac{116}{7}\ln\left(\frac{r}{r_0}\right)\right]r^{-3}\ddot{M}_{ab}\right), \tag{A.6}$$

which shows by comparing to (3.21) and (3.22) that the coefficient we are seeking is

$$\rho = \frac{1271}{735} - \frac{58}{21}\ln\left(\frac{r}{r_0}\right). \tag{A.7}$$

The cubic multipole interaction $M \times M_i \times M_j$ takes much longer to obtain because we are obliged to compute it from scratch (no earlier results in the literature are available). At linearized order the metric reads

$$h^{00}_{\mathrm{can}\,1} = -4r^{-1}M + 4\partial_a[r^{-1}M_a(u)], \tag{A.8a}$$
$$h^{i0}_{\mathrm{can}\,1} = -4r^{-1}\dot{M}_i(u), \tag{A.8b}$$
$$h^{ij}_{\mathrm{can}\,1} = 0. \tag{A.8c}$$

Straightforward calculations following the MPM algorithm then yield the part of the metric at quadratic order corresponding to the multipole couplings $M \times M_i$ and $M_i \times M_j$,

$$h^{00}_{\mathrm{can}\,2} = n_aM(-14r^{-3}M_a - 14r^{-2}\dot{M}_a) + \hat{n}_{ab}(-7r^{-4}M_aM_b - 14r^{-3}M_a\dot{M}_b - 36r^{-2}\dot{M}_a\dot{M}_b)$$
$$+\left(-\tfrac{7}{3}r^{-4}M_aM_a - \tfrac{14}{3}r^{-3}M_a\dot{M}_a + \tfrac{23}{9}r^{-2}\dot{M}_a\dot{M}_a\right), \tag{A.9a}$$

$$h^{i0}_{\mathrm{can}\,2} = -\hat{n}_{ia}r^{-2}M\dot{M}_a - \tfrac{22}{3}r^{-2}M\dot{M}_i$$
$$+\hat{n}_{iab}(-2r^{-3}M_a\dot{M}_b - 2r^{-2}\dot{M}_a\dot{M}_b) + n_i\left(-\tfrac{17}{5}r^{-3}M_a\dot{M}_a - \tfrac{17}{5}r^{-2}\dot{M}_a\dot{M}_a\right)$$
$$+n_a\left(r^{-3}\left[-\tfrac{37}{5}\dot{M}_iM_a + \tfrac{18}{5}M_i\dot{M}_a\right] - \tfrac{19}{5}r^{-2}\dot{M}_i\dot{M}_a\right), \tag{A.9b}$$

$$h^{ij}_{\mathrm{can}\,2} = \hat{n}_{ija}M(-4r^{-3}M_a - 4r^{-2}\dot{M}_a) + \delta_{ij}n_aM\left(-\tfrac{4}{5}r^{-3}M_a - \tfrac{4}{5}r^{-2}\dot{M}_a\right)$$
$$+n_{(i}M\left(\tfrac{2}{5}r^{-3}M_{j)} + \tfrac{2}{5}r^{-2}\dot{M}_{j)}\right)$$
$$+\hat{n}_{ijab}\left(-\tfrac{9}{2}r^{-4}M_aM_b - 9r^{-3}M_a\dot{M}_b - 4r^{-2}\dot{M}_a\dot{M}_b\right)$$

$$+\delta_{ij}\hat{n}_{ab}\left(-\tfrac{1}{7}r^{-4}M_aM_b-\tfrac{2}{7}r^{-3}M_a\dot{M}_b+\tfrac{24}{7}r^{-2}\dot{M}_a\dot{M}_b\right)$$
$$+\hat{n}_{a(i}\left(-\tfrac{4}{7}r^{-4}M_aM_{j)}-\tfrac{4}{7}r^{-3}[\dot{M}_aM_{j)}+M_a\dot{M}_{j)}]-\tfrac{58}{7}r^{-2}\dot{M}_a\dot{M}_{j)}\right)$$
$$+\hat{n}_{ij}\left(\tfrac{6}{7}r^{-4}M_aM_a+\tfrac{12}{7}r^{-3}M_a\dot{M}_a+\tfrac{31}{7}r^{-2}\dot{M}_a\dot{M}_a\right)$$
$$+\delta_{ij}\left(-\tfrac{2}{15}r^{-4}M_aM_a-\tfrac{4}{15}r^{-3}M_a\dot{M}_a-\tfrac{19}{15}r^{-2}\dot{M}_a\dot{M}_a\right)$$
$$+\tfrac{1}{15}r^{-4}M_iM_j+\tfrac{2}{15}r^{-3}M_{(i}\dot{M}_{j)}-\tfrac{38}{15}r^{-2}\dot{M}_i\dot{M}_j. \qquad (A.9c)$$

Using such expressions (A.8) and (A.9) we next obtain the source term at the cubic-order approximation $M\times M_i\times M_j$. We are interested only in its 00 component which is then found to be

$$\Lambda^{00}_{\mathrm{can}\,3}=\hat{n}_{ab}(-324r^{-7}MM_aM_b-648r^{-6}MM_a\dot{M}_b-112r^{-5}M\dot{M}_a\dot{M}_b)$$
$$-160r^{-7}MM_aM_a-320r^{-6}MM_a\dot{M}_a-\frac{644}{9}r^{-5}M\dot{M}_a\dot{M}_a. \qquad (A.10)$$

The integration proceeds exactly in the same way as in equations (A.4) and (A.5). Again, the problem is that of interchanging a limiting process $\dot{M}_i(u)\longrightarrow\dot{M}_i^{\mathrm{BSS}}(u)$ with the retarded integration, and this can be proved by using the Lebesgue theorem, because the various powers of $1/r$ in the RHS of (A.10) ensure that, in a process where $|\dot{M}_i(u)|$ remains bounded, the integral is bounded by a positive convergent integral. And again the result agrees with the one we would formally obtain by using the formulae in the appendix of [33]. As before, we have $v^{00}_{\mathrm{can}\,3}=0$ from the argument concerning the multipolarity $\ell=0,1$ of this special piece. Our result is then

$$h^{00}_{\mathrm{can}\,3}=\hat{n}_{ab}\left(-\frac{162}{7}r^{-5}MM_aM_b-\frac{324}{7}r^{-4}MM_a\dot{M}_b\right.$$
$$\left.+\left[-\frac{5836}{245}-\frac{232}{7}\ln\left(\frac{r}{r_0}\right)\right]r^{-3}M\dot{M}_a\dot{M}_b\right)$$
$$-8r^{-5}MM_aM_a-16r^{-4}MM_a\dot{M}_a+\frac{110}{27}r^{-3}M\dot{M}_a\dot{M}_a, \qquad (A.11)$$

from which one recognizes, on comparison with (3.21) and (3.22), that

$$\eta=\frac{2918}{735}+\frac{116}{21}\ln\left(\frac{r}{r_0}\right). \qquad (A.12)$$

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