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The 2.5PN gravitational wave polarizations from inspiralling compact binaries in circular orbits

K G Arun[1], Luc Blanchet[2], Bala R Iyer[1] and Moh'd S S Qusailah[1,3]

[1] Raman Research Institute, Bangalore 560 080, India
[2] $\mathcal{G}\mathbb{R}\varepsilon\mathbb{C}\mathcal{O}$, Institut d'Astrophysique de Paris—CNRS, 98[bis] boulevard Arago, 75014 Paris, France

E-mail: arun@rri.res.in, blanchet@iap.fr, bri@rri.res.in and mssq@rri.res.in

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Abstract
Using the multipolar post-Minkowskian and matching formalism, we compute the gravitational wave form of inspiralling compact binaries moving in quasi-circular orbits at the second and a half post-Newtonian (2.5PN) approximation to general relativity. The inputs we use include notably the mass-type quadrupole at the 2.5PN order, the mass octupole and current quadrupole at the 2PN order, the mass 2^5-pole and current 2^4-pole at 1PN. The nonlinear hereditary terms come from the monopole–quadrupole multipole interactions or tails, present at the 1.5PN, 2PN and 2.5PN orders, and the quadrupole–quadrupole interaction arising at the 2.5PN level. In particular, the specific effect of nonlinear memory is computed using a simplified model of binary evolution in the past. The 'plus' and 'cross' wave polarizations at the 2.5PN order are obtained in ready-to-use form, extending the 2PN results calculated earlier by Blanchet, Iyer, Will and Wiseman.

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1. Introduction

1.1. On gravitational-wave astronomy

With the ongoing scientific runs of the Laser Interferometric Gravitational Wave Observatory (LIGO) [1] and the recent commissioning of the French–Italian VIRGO detector [2], there is tremendous excitement among physicists and astronomers about the possibility of the first ever direct detection of gravitational waves. The Japanese detector TAMA300 is already taking

[3] On leave from University of Sana, Yemen.

data [3] and the British–German joint venture GEO600 will be operational soon [4]. These detectors will form a worldwide network capable of coincident observations and joint data analysis. The design sensitivity of the three LIGO interferometers together with that of VIRGO and GEO600 is sufficient for the detection of GWs from powerful astrophysical sources such as binary black holes in the mass range of about $40M_{\odot}$ [5–7]. These detectors are ground based and will be sensitive to the high frequency regime of the spectrum of GW, i.e. between 10 Hz and 10 kHz. The proposed space-borne detector LISA (Laser Interferometric Space Antenna) will probe the universe at low frequencies (1 Hz–10^{-4} Hz) [8]. Even if LIGO-1 does not detect any gravitational waves, it will be able to put upper bounds on the fluxes and populations of different types of sources. The recent discovery of a binary pulsar PSR J0737-3039 [9] has brought double neutron star coalescence rate estimates within an astrophysically relevant regime for LIGO-1 and VIRGO. The estimated event rate for compact binary merger for LIGO-1 can now go as high as 1 event per 1.5 yr [10]. LIGO-2 with its ten-fold improved sensitivity should definitely be able to detect gravitational waves from different astrophysical sources.

GW signals can be broadly classified into four categories: inspiral, periodic, burst and stochastic. The strategies for data analysis vary for different types of signals. Deterministic signals from inspiral and periodic sources will be probed by matched filtering (to be discussed later), whereas statistical signals such as stochastic sources will be searched for by cross-correlating pairs of interferometers, seeking correlations and statistical variations. Unmodelled signals such as those from supernova explosion or gamma ray bursts (GRBs) cannot, in principle, be looked for by any parametric techniques. One needs to monitor power excesses in the frequency domain or have notable amplitude variations in time to detect them. In all these studies, coincidence between multiple detectors improves the efficiency of detection significantly.

The joint observation of astrophysical phenomena in electromagnetic as well as non-electromagnetic windows (such as GWs and neutrinos) will have a huge impact on observational astronomy in the future. The current theories of GRBs involve either the merger of a compact binary or a collapse. In either case, the GRBs are associated with the emission of gravitational radiation [11]. Detection of a merger GW signal would require a lower signal-to-noise ratio, if it is coincident with a GRB [12]. The data analysis techniques for such phenomena will include comparison of the correlated output of two gravitational wave detectors, one prior to the GRB and the other at times not associated with the GRB [13, 14]. The spectral flux distribution of the neutrinos from GRBs associated with supernovae can test for the possible time delays between the supernovae and the GRB down to much shorter time scales than can be resolved with only photons [15].

In the cosmological context, the primordial gravitational waves resulting from inflation contribute to the polarization of the cosmic microwave background radiation (CMBR). Their observations could provide us valuable information about the early universe as well as important checks on the inflationary cosmological model [16–18]. The proposed Decihertz Interferometer Gravitational Wave Observatory (DECIGO) [19] could even probe the acceleration of the universe, and hence provide an independent determination of the spatial curvature of the cosmological model. This is important as projects such as SuperNova/Acceleration Probe (SNAP) [20] would also probe the acceleration of the universe by electromagnetic means.

Pulsars can spin up by accreting matter from a neighbouring star. Recent observations point towards a possibility of GW emission balancing the accretion torque putting upper bounds on pulsar spin [21]. If these gravitational waves are due to r-mode or CFS instability in the pulsar, as it is presently understood, they can be detected by the advanced LIGO detector

[22]. The detection of GWs from instabilities in relativistic stars [23] by advanced LIGO would be able to test models of stellar interiors.

The 'chirp' from the inspiral of two compact objects is one of the most plausible GW signals the ground-based detectors would look for. The early inspiral will fall in the sensitivity band of the space-based detectors, whereas the late inspiral will be a good candidate source for the ground-based detectors. Though these GWs are extremely weak and buried deep in the detector noise, the large number of precisely predictable cycles in the detector bandwidth would push the signal up to the level of detection. One can then use the technique of matched filtering first for the detection of GW and later (i.e. off-line) for the estimation of the parameters of the binary. In order to have a good detection, it is extremely important to cross-correlate the detector output with a number of copies of the theoretically predicted signal (corresponding to different signal parameters) which is as precise as possible, and which remains in accurate phase with the signal. This has made general relativistic modelling of the inspiralling compact binary (ICB) one of the most demanding requirements for GW data analysis [6, 24–30].

1.2. An overview of current calculations of radiation from ICBs

Currently, post-Newtonian (PN) theory provides the most satisfactory description of the dynamics of ICBs and gravitational radiation emitted by them. Starting from the gravitational-wave-generation formalism based on multipolar post-Minkowskian expansions (see the following section), the gravitational wave form and energy flux at the 2PN order[4] were computed by Blanchet *et al* [31]. This incorporated the tail contribution at 1.5PN order both in the wave form and in the energy flux; the polarization states corresponding to the 1.5PN wave form were calculated in [32] (note that some algebraic errors in this reference are corrected in [35]). The 2PN results have been independently obtained using a direct integration of the relaxed Einstein field equation [33, 34]. The associated polarization states (i.e. the 'plus' and 'cross' polarization wave forms) were obtained in [35]. These works provided accurate theoretical templates which are currently used for data analysis in all the laser interferometric GW detectors such as LIGO/VIRGO. Extending the wave-generation formalism, the 2.5PN term in the energy flux, which arises from a subdominant tail effect, was added in [36]. In the case of binaries moving in quasi-elliptical orbits, the instantaneous parts of the wave form, energy flux and angular momentum flux at 2PN order were computed by Gopakumar and Iyer [37]. The polarizations of the wave form at this order (in the adiabatic approximation) have been obtained more recently [38], and the phasing of binaries in inspiralling eccentric orbits has also been discussed [39].

The extension of the gravitational wave-generation formalism to third post-Newtonian order, and computation of the energy flux up to 3.5PN accuracy was achieved in [40]. To this order, in addition to the 'instantaneous' contributions, coming from relativistic corrections in the multipole moments of the source, the results include several effects of tails, and tails generated by tails. But, unlike at the 2PN or 2.5PN order, where the calculation is free from ambiguities, at the 3PN order the incompleteness of the Hadamard self-field regularization leads to some undetermined constants in the mass quadrupole moment of point-particle binaries (we comment more on this below). On the other hand, the computation of the binary's flux crucially requires the 3PN equations of motion (EOM). These were obtained earlier by two independent calculations, one based on the ADM Hamiltonian formalism of general relativity [41–44], and the other on the direct 3PN iteration of the Einstein field equations in

[4] As usual the nPN order refers either to the terms $\sim 1/c^{2n}$ in the equations of motion, with respect to the usual Newtonian acceleration, or in the radiation field, relative to the standard quadrupolar wave form.

harmonic coordinates [45–48]. Both approaches lead to an undetermined constant parameter at 3PN when using a Hadamard regularization, but this constant has now been fixed using a dimensional regularization [49, 50]. An independent method [51–53], using surface integrals together with a strong field point-particle limit, has yielded results for the 3PN EOM in agreement with those of the first two methods. In particular, the EOM so obtained are independent of any ambiguity parameter, and consistent with the end result of dimensional regularization [49, 50]. The conserved 3PN energy is thus uniquely determined, and then the 3.5PN energy flux, together with the usual energy balance argument, leads to the expression for the evolution for orbital phase and frequency at 3.5PN order [54].

Once the gravitational wave polarizations are available, one proceeds to use them for constructing templates for gravitational-wave data analysis. Templates for detection generally use the so-called restricted wave form (RWF), where the phase is at the highest available PN order, but the amplitude is retained to be Newtonian, involving only the main signal harmonics at twice the orbital frequency. Though for detection (which will be done online) the RWF approximation may be enough [6, 7, 25, 55], the complete wave form will be useful for the parameter estimation (off-line). Recently, studies using the complete wave form, which includes the contribution from higher harmonics besides the dominant one, have shown that it may play a vital role in parameter estimation [56–58]. The complete wave form carries information which can break the degeneracy of the model, and allow one to estimate the otherwise badly correlated parameters. In the case of a chirping neutron star binary, the masses of the individual stars can be separated because of the mass dependence of the higher harmonics [56]. In the case of black-hole binaries, whose frequencies are too low to be seen in the detector sensitivity window for long, higher harmonics compensate for the information lost when the signal does not last long enough to be apparent in the data [56]. An independent study [57] about the angular resolution of the space-based LISA-type gravitational wave detectors with a time domain 2PN wave form showed the importance of including higher PN corrections to the wave amplitude in predicting the angular resolution of the elliptic-plane detector configuration.

In the present work, we provide the gravitational wave form from ICBs to still higher accuracy, namely 2.5PN, which should in consequence be useful for future improved studies in GW data analysis, for both LIGO-type and LISA-type detectors. We shall include in the 2.5PN wave form instantaneous as well as ‘hereditary’ terms, exactly as they are predicted by general relativity, completing therefore the 2.5PN generation problem for binaries moving in quasi-circular orbits initiated in [36]. Using the wave form, we next obtain the two ‘plus’ and ‘cross’ GW polarizations at 2.5PN extending the results of [35]. We shall verify that the 2.5 PN waveform is in perfect agreement, in the test-mass limit for one of the bodies, with the result of linear black hole perturbations [59].

The plan of this paper is as follows. In section 2, we outline the post-Newtonian generation formalism, based on multipolar post-Minkowskian (MPM) expansions and matching to a general post-Newtonian source. In section 3, we discuss the structure of the gravitational wave form up to the 2.5PN order for general sources. Then we present our list of results for the mass and current-type multipole moments required for the computation of the wave form in the case of point-particle binaries. In section 4, we compute all the hereditary contributions relevant to the 2.5PN wave form, which are made at this order of tails and the nonlinear memory integral. The results for the 2.5PN wave form are given in section 5, ready to be implemented as 2.5PN GW template in LIGO/VIRGO experiments. Section 5 also contains some short discussion, concerning the multipole moments and future issues related to the 3PN wave form.

2. The post-Newtonian wave-generation formalism

The wave-generation formalism relates the gravitational waves observed at a detector in the far-zone of the source to the stress–energy tensor of the source. Successful wave-generation formalisms mix and match approximation techniques from currently available collections. These include post-Minkowskian (PM) methods, post-Newtonian (PN) methods, multipole (M) expansions and perturbations around curved backgrounds. A recent review [60] discusses in detail the formalism we follow in the computation of the gravitational field; we summarize below the main features of this approach. This formalism has two independent aspects addressing two different problems. The first aspect is the general method applicable to extended or fluid sources with compact support, based on the mixed PM and multipole expansion (we call it a MPM expansion), and matching to some PN source. The second aspect is the application to point-particle binaries modelling ICBs.

2.1. The MPM expansion and matching to a post-Newtonian source

To define the solution in the exterior of the source within the complete nonlinear theory we follow [61–66], which are based on earlier seminal work of Bonnor [67] and Thorne [68], to set up the multipolar post-Minkowskian expansion. Starting from the general solution to the linearized Einstein's equations in the form of a multipolar expansion (valid in the external region), we perform a PM iteration and treat individually each multipolar piece at any PM order. In addition to terms evaluated at one retarded time, the expression for the gravitational field also contains terms integrated over the entire past 'history' of the source or hereditary terms. For the external field, the general method is not limited *a priori* to PN sources. However, closed form expressions for the multipole moments can presently only be obtained for PN sources, because the exterior field may be connected to the inner field only if there exists an 'overlapping' region where both the MPM and PN expansions are valid and can be matched together. For PN sources, this region always exists and is the exterior ($r > a$) near ($r \ll \lambda$) zone. After matching, it is found that the multipole moments have a non-compact support owing to the gravitational field stress–energy distributed everywhere up to spatial infinity. To correctly include these contributions coming from infinity, the definition of the multipole moments involves a finite part operation, based on analytic continuation. This process is also equivalent to a Hadamard 'partie finie' of the integrals at the bound at infinity.

The formalism, notably the asymptotic matching procedure therein, has been explored in detail and extended in a systematic way to higher PN orders [69–72]. The final result of this analysis is that the physical post-Newtonian (slowly moving) source is characterized by six symmetric and trace free (STF) time-varying moments, denoted by $\{I_L, J_L, W_L, X_L, Y_L, Z_L\}$,[5] which are specified for each source in the form of functionals of the formal PN expansion, up to any PN order, of the stress–energy pseudo-tensor $\tau^{\mu\nu}$ of the material and gravitational fields [72]. These moments parametrize the linear approximation to the vacuum metric outside the source, which is the first approximation in the MPM algorithm. In the linearized gravity case $\tau^{\mu\nu}$ reduces to the compact-support matter stress–energy tensor $T^{\mu\nu}$ and the expressions match perfectly with those derived in [73].

Starting from the complete set of six STF *source moments* $\{I_L, J_L, W_L, X_L, Y_L, Z_L\}$, for which general expressions can be given valid to any PN order, one can define a different set of only two '*canonical*' source moments, denoted by $\{M_L, S_L\}$, such that the two sets of moments $\{I_L, \ldots, Z_L\}$ and $\{M_L, S_L\}$ are physically equivalent. By this we mean that they describe the

[5] As usual $L = i_1 i_2, \ldots, i_l$ denotes a multi-index made of l spatial indices (ranging from 1 to 3). The integer l is referred to as the multipolar order.

same physical source, i.e. the two metrics, constructed respectively out of $\{I_L, \ldots, Z_L\}$ and $\{M_L, S_L\}$, differ by a mere coordinate transformation (are isometric). However, the six general source moments $\{I_L, \ldots, Z_L\}$ are closer rooted to the source because we know their expressions as integrals over $\tau^{\mu\nu}$. On the other hand, the canonical source moments $\{M_L, S_L\}$ are also necessary because their use simplifies the calculation of the external nonlinearities. In addition, their existence shows that any radiating isolated source is characterized by two and only two sets of time-varying multipole moments [68, 61].

The MPM formalism is valid all over the weak field region outside the source including the wave zone (up to future null infinity). It is defined in harmonic coordinates. The far zone expansion at Minkowskian future null infinity contains logarithms in the distance which are artefacts of the harmonic coordinates. One can define, step by step in the PM expansion, some *radiative* coordinates by a coordinate transformation so that the log-terms are eliminated [62] and one recovers the standard (Bondi-type) radiative form of the metric, from which the *radiative moments*, denoted by $\{U_L, V_L\}$, can be extracted in the usual way [68]. The wave-generation formalism resulting from the exterior MPM field and matching to the PN source is able to take into account, in principle, any PN correction in both the source and radiative multipole moments. Nonlinearities in the external field are computed by a post-Minkowskian algorithm. This allows one to obtain the radiative multipole moments $\{U_L, V_L\}$ as some nonlinear functional of the canonical moments $\{M_L, S_L\}$, and then of the actual source moments $\{I_L, \ldots, Z_L\}$. These relations between radiative and source moments include many nonlinear multipole interactions as the source moments mix with each other as the waves propagate from the source to the detector. The dominant nonlinear effect is due to the tails of waves, made of coupling between non-static moments and the total mass of the source, occurring at 1.5PN order ($\sim 1/c^3$) relative to the leading quadrupole radiation [63]. There is a corresponding tail effect in the equations of motion of the source, occurring at 1.5PN order relative to the leading 2.5PN radiation reaction, hence at 4PN order ($\sim 1/c^8$) beyond the Newtonian acceleration [64]. At higher PN orders, there are different types of nonlinear multipole interactions that are responsible for the presence of some important hereditary (i.e. past-history dependent) contributions to the wave form and energy flux.

A different wave-generation formalism from isolated sources, based on direct retarded integration of Einstein's equations in harmonic coordinates, is due to Will and Wiseman [33], and provided major improvement and elucidation of earlier investigations in the same line [68, 74]. This formalism is based on different source multipole moments (defined by integrals extending over the near zone only), together with a different scheme for computing the nonlinearities in the external field. It has currently been completed up to the 2PN order. At the most general level, i.e. for any PN extended source and in principle at any PN order, the Will–Wiseman formalism is completely equivalent to the present formalism based on MPM expansions with asymptotic matching (see section 5.3 in [60] for the proof).

2.2. Application to compact binary systems

This application represents the second aspect of our approach. To this end, in the first instance, the compact objects (neutron stars or black holes) are modelled as point particles represented by Dirac δ-functions. Indeed one can argue that for compact objects, the effects of finite size and quadrupole distortion induced by tidal interactions are higher order in the PN approximation. However, the general formalism outlined in section 2.1 is set up for a continuous (smooth) matter distribution, with continuous $T^{\mu\nu}$, and cannot be directly applied to point particles, since they lead to divergent integrals at the location of the particles, when $T^{\mu\nu}_{\text{point-particle}}$ is substituted into the source moments $\{I_L, \ldots, Z_L\}$. The calculation needs to be supplemented by a prescription for removing the infinite part of the integrals. Hadamard regularization, based on Hadamard's notion of partie finie, is what we employ. This is our ansatz for applying a well-defined general 'fluid' formalism to an initially ill-defined point-particle source.

To summarize, a systematic analytical approximation scheme has been set up for the calculation of wave forms and associated quantities from point particles to the PN order required (or permitted by given resources). A technical cost is the need to handle δ-functions in a nonlinear theory, which is dealt with by the Hadamard regularization scheme or a variant of it. However, we already mentioned that at the 3PN order, subtleties arise notably due to the so-called non-distributivity of the Hadamard partie finie, which resulted, as shown in [40], in some 'ambiguities' when computing the 3PN mass-type quadrupole moment, which could be entirely encoded into three undetermined numerical coefficients ξ, κ and ζ. These combined into the unique quantity $\theta = \xi + 2\kappa + \zeta$ in the 3PN energy flux for circular orbits.

The latter ambiguities are the analogues of the undetermined parameters found in the binary EOM at 3PN order, namely ω_k and ω_s in the canonical ADM approach [41, 42], and λ in the harmonic-coordinates formalism [45, 46]. The parameter λ is related to the 'static' ambiguity ω_s by $\lambda = -\frac{3}{11}\omega_s - \frac{1987}{3080}$, while the 'kinetic' ambiguity ω_k has been determined [45] to the value $\frac{41}{24}$ (see [44, 47] for details). The presence of the static ambiguity ω_s or, equivalently, λ, is a consequence of the Hadamard regularization scheme which happens to become physically incomplete at the 3PN order. Recently, Damour *et al* [49] proposed to use a better regularization: *dimensional* regularization. This led them to a unique determination of ω_s, namely $\omega_s = 0$. More recently [50], the application of dimensional regularization to the computation of the EOM in harmonic coordinates has led to the equivalent result for λ, which is $\lambda = -\frac{1987}{3080}$. The EOM are thus completely determined to the 3PN order within both the ADM and harmonic-coordinate approaches using Hadamard regularization supplemented by a crucial argument from dimensional regularization in order to fix the last parameter[6]. All the 3PN conserved quantities are determined in [44, 47]. A 3PN accurate centre of mass has been constructed and used to reduce the conserved energy and angular momentum [48].

The numerical values of the radiation-field-related ambiguity coefficients ξ, κ and ζ introduced in [40] have also been determined [75] using dimensional regularization, so that the PN corrections to phasing are completely determined to 3.5PN accuracy. However, as we shall work in the present paper on the 2.5PN wave form, i.e. one half order before 3PN, these ambiguities will not concern us here, and we shall find that no ambiguity shows up in any of our calculations based on Hadamard's regularization. In fact, it can be shown that up to the 2.5PN order Hadamard's regularization as we shall employ here gives the same result as would dimensional regularization. The reason is that, in the source multipole moments up to this order, there are no logarithmic divergences occurring at the particles' locations, which correspond in d dimensions to poles in $\varepsilon = d - 3$.

In the present work we will require, for the computation of the time-derivatives of multipole moments, the EOM for the case of circular orbits to 2.5PN accuracy. We denote the PN parameter in harmonic coordinates by

$$\gamma \equiv \frac{Gm}{rc^2}, \tag{2.1}$$

where $r = |\mathbf{x}|$ is the binary separation ($\mathbf{x} \equiv \mathbf{y}_1 - \mathbf{y}_2$ is the vectorial separation between the particles), and $m = m_1 + m_2$ is the total mass of the binary system. We pose $X_1 = \frac{m_1}{m}$, $X_2 = \frac{m_2}{m}$

[6] Note that both calculations [49, 50] are performed in the limit $\varepsilon \to 0$, where the dimension of space is $d = 3 + \varepsilon$. The principle is to add to the end results given by the Hadamard regularization [41, 42, 45, 46] the *difference* between the dimensional and Hadamard regularizations, which is specifically due to the poles $\sim 1/\varepsilon$ and their associated finite part.

and $\nu = X_1 X_2$. Occasionally we also employ $\delta m = m_1 - m_2$ so that $\frac{\delta m}{m} = X_1 - X_2$. Then the 2.5PN binary acceleration reads ($\mathbf{v} \equiv \mathbf{v}_1 - \mathbf{v}_2$) [7]

$$\frac{\mathrm{d}\mathbf{v}}{\mathrm{d}t} = -\omega^2 \mathbf{x} - \frac{32G^3}{5c^5} \frac{m^3 \nu}{r^4} \mathbf{v} + \mathcal{O}(6), \tag{2.2}$$

where the explicit 2.5PN term ($\sim 1/c^5$) is the radiation-reaction force in the harmonic coordinate system used here. The radiation-reaction force plays an important role in our calculation of the wave form—it must be consistently included in all replacements of accelerations at 2.5PN order (however, the reaction force yields no contribution to the energy flux at 2.5PN order for circular orbits [36]). In equation (2.2), the orbital frequency $\omega \equiv 2\pi/P$ (where P is the orbital period) is related to the binary separation r in harmonic coordinates with 2PN accuracy by [76]

$$\omega^2 = \frac{Gm}{r^3} \left\{ 1 + [-3 + \nu]\gamma + \left[6 + \frac{41}{4}\nu + \nu^2 \right] \gamma^2 + \mathcal{O}(6) \right\}. \tag{2.3}$$

In the following we shall also need the inverse of equation (2.3), i.e. γ in terms of ω, which can conveniently be written in the form

$$\gamma = x \left\{ 1 + \left[1 - \frac{\nu}{3} \right] x + \left[1 - \frac{65}{12}\nu \right] x^2 + \mathcal{O}(6) \right\}, \tag{2.4}$$

in which we have introduced the gauge invariant frequency-dependent parameter

$$x \equiv \left(\frac{Gm\,\omega}{c^3} \right)^{2/3}. \tag{2.5}$$

3. The 2.5PN gravitational wave form

3.1. Wave form as a functional of multipole moments

In an appropriate radiative coordinate system $X^\mu = (cT, X^i)$, the transverse-traceless (TT) projection of the deviation of the metric of an isolated body from a flat metric defines the asymptotic wave form h^{TT}_{km} (lower-case latin indices take the values 1, 2, 3). The leading-order $1/R$ part of h^{TT}_{km} (where $R = |\mathbf{X}|$ is the distance to the body) can be uniquely decomposed [68] into its *radiative* multipole contributions introduced in section 2. Furthermore, the PN order of the asymptotic wave form scales with the multipolar order l. Hence, at any PN order only a finite number of multipoles is required, and we have, with 2.5PN accuracy,

$$\begin{aligned} h^{\mathrm{TT}}_{km} = \frac{2G}{c^4 R} \mathcal{P}_{ijkm}(\mathbf{N}) \Bigg\{ & U_{ij} + \frac{1}{c} \left[\frac{1}{3} N_a U_{ija} + \frac{4}{3} \varepsilon_{ab(i} V_{j)a} N_b \right] \\ & + \frac{1}{c^2} \left[\frac{1}{12} N_{ab} U_{ijab} + \frac{1}{2} \varepsilon_{ab(i} V_{j)ac} N_{bc} \right] \\ & + \frac{1}{c^3} \left[\frac{1}{60} N_{abc} U_{ijabc} + \frac{2}{15} \varepsilon_{ab(i} V_{j)acd} N_{bcd} \right] \\ & + \frac{1}{c^4} \left[\frac{1}{360} N_{abcd} U_{ijabcd} + \frac{1}{36} \varepsilon_{ab(i} V_{j)acde} N_{bcde} \right] \\ & + \frac{1}{c^5} \left[\frac{1}{2520} N_{abcde} U_{ijabcde} + \frac{1}{210} \varepsilon_{ab(i} V_{j)acdef} N_{bcdef} \right] + \mathcal{O}(6) \Bigg\}. \end{aligned} \tag{3.1}$$

[7] We systematically use the shorthand $\mathcal{O}(n)$ to mean a small post-Newtonian remainder term of the order of $\mathcal{O}(c^{-n})$.

The U_L and V_L (with $L = ij\cdots$ a multi-index composed of l indices) appearing in the above wave form are respectively called the mass-type and the current-type radiative multipole moments (see discussion in section 2.1). They are functions of the retarded time $T_R \equiv T - R/c$ in radiative coordinates, $U_L(T_R)$ and $V_L(T_R)$. We denote by $\mathbf{N} \equiv \mathbf{X}/R$ the unit vector pointing along the direction of the source located at distance R from the detector. A product of components of $\mathbf{N} = (N_i)_{i=1,2,3}$ is generally denoted by $N_L \equiv N_i N_j \cdots$. The Levi-Civita antisymmetric symbol reads ε_{abi}, such that $\varepsilon_{123} = +1$. The operator $\mathcal{P}_{ijkm}$ represents the usual TT algebraic projector which is given by

$$\mathcal{P}_{ijkm} = \tfrac{1}{2}(\mathcal{P}_{ik}\mathcal{P}_{jm} + \mathcal{P}_{im}\mathcal{P}_{jk} - \mathcal{P}_{ij}\mathcal{P}_{km}), \tag{3.2a}$$

$$\mathcal{P}_{ij} \equiv \delta_{ij} - N_i N_j. \tag{3.2b}$$

Given an orthonormal triad ($\mathbf{N}$, $\mathbf{P}$, $\mathbf{Q}$), consisting of the radial direction $\mathbf{N}$ to the observer, and two unit polarization vectors $\mathbf{P}$ and $\mathbf{Q}$, transverse to the direction of propagation, we define the two 'plus' and 'cross' polarization wave forms by

$$h_+ = \tfrac{1}{2}(P_i P_j - Q_i Q_j) h_{ij}^{\mathrm{TT}}, \tag{3.3a}$$

$$h_\times = \tfrac{1}{2}(P_i Q_j + Q_i P_j) h_{ij}^{\mathrm{TT}}, \tag{3.3b}$$

in which the projector $\mathcal{P}_{ijkm}$ present in front of the TT wave form may be omitted.

In the case of circular binary systems we shall adopt for $\mathbf{P}$ the vector lying along the intersection of the orbital plane with the plane of the sky in the direction of the 'ascending node' $\mathcal{N}$, i.e. the point at which the bodies cross the plane of the sky moving towards the detector, and $\mathbf{Q} = \mathbf{N} \times \mathbf{P}$. Following the convention of [35], the unit vector joining the particle 2 to the particle 1, i.e. $\mathbf{n} = (\mathbf{y}_1 - \mathbf{y}_2)/r$ where $r = |\mathbf{y}_1 - \mathbf{y}_2|$, is given by $\mathbf{n} = \mathbf{P}\cos\phi + (\mathbf{Q}\cos i + \mathbf{N}\sin i)\sin\phi$, where i denotes the orbit's inclination angle and ϕ is the orbital phase, namely the angle between the ascending node and the direction of the body one[8]. The unit direction of the velocity, i.e. $\boldsymbol{\lambda}$ such that $\mathbf{v} = r\omega\boldsymbol{\lambda}$ (for circular orbits), is given by $\boldsymbol{\lambda} = -\mathbf{P}\sin\phi + (\mathbf{Q}\cos i + \mathbf{N}\sin i)\cos\phi$. (See figure 7 and equation (7.4) of [33].)

Using the MPM formalism, the radiative moments entering equation (3.1) can be expressed in terms of the source variables with sufficient accuracy, that is a fractional accuracy of $\mathcal{O}(6) \equiv \mathcal{O}(c^{-6})$ relative to the lowest-order quadrupolar wave form. For this approximation to be complete, one must compute: mass-type radiative quadrupole U_{ij} with 2.5PN accuracy; current-type radiative quadrupole V_{ij} and mass-type radiative octupole U_{ijk} with 2PN accuracy; mass-type hexadecapole U_{ijkl} and current-type octupole V_{ijk} with 1.5PN precision; U_{ijklm} and V_{ijkl} up to 1PN order; U_{ijklmn}, V_{ijklm} at 0.5PN and finally $U_{ijklmno}$, V_{ijklmn} to Newtonian precision. The relations connecting the radiative moments U_L and V_L to the corresponding 'canonical' moments M_L and S_L (see section 2.1 for a short recall of their meaning) are given as follows [63, 65, 66]. For the mass-type moments we have

$$\begin{aligned} U_{ij}(T_R) = M_{ij}^{(2)}(T_R) &+ \frac{2Gm}{c^3}\int_{-\infty}^{T_R} \mathrm{d}V \left[\ln\left(\frac{T_R - V}{2b}\right) + \frac{11}{12}\right] M_{ij}^{(4)}(V) \\ &+ \frac{G}{c^5}\left\{-\frac{2}{7}\int_{-\infty}^{T_R} \mathrm{d}V\, M_{a\langle i}^{(3)}(V) M_{j\rangle a}^{(3)}(V) + \frac{1}{7} M_{a\langle i}^{(5)} M_{j\rangle a}\right. \\ &\left. - \frac{5}{7} M_{a\langle i}^{(4)} M_{j\rangle a}^{(1)} - \frac{2}{7} M_{a\langle i}^{(3)} M_{j\rangle a}^{(2)} + \frac{1}{3}\varepsilon_{ab\langle i} M_{j\rangle a}^{(4)} S_b\right\} + \mathcal{O}(6), \end{aligned} \tag{3.4a}$$

[8] The angle ϕ in our convention differs by $\frac{\pi}{2}$ from the same in [31, 38]. We follow here the convention of BIWW [35] that is related to the BDI one [31, 38] by $\phi_{\mathrm{BDI}} = \phi_{\mathrm{BIWW}} - \frac{\pi}{2}$.

$$U_{ijk}(T_R) = M^{(3)}_{ijk}(T_R) + \frac{2Gm}{c^3}\int_{-\infty}^{T_R} \mathrm{d}V \left[\ln\left(\frac{T_R - V}{2b}\right) + \frac{97}{60}\right] M^{(5)}_{ijk}(V) + \mathcal{O}(5), \tag{3.4b}$$

$$\begin{aligned} U_{ijkm}(T_R) = M^{(4)}_{ijkm}(T_R) + \frac{G}{c^3}\Bigg\{2m\int_{-\infty}^{T_R} \mathrm{d}V \left[\ln\left(\frac{T_R - V}{2b}\right) + \frac{59}{30}\right] M^{(6)}_{ijkm}(V) \\ + \frac{2}{5}\int_{-\infty}^{T_R} \mathrm{d}V\, M^{(3)}_{\langle ij}(V) M^{(3)}_{km\rangle}(V) - \frac{21}{5} M^{(5)}_{\langle ij} M_{km\rangle} \\ - \frac{63}{5} M^{(4)}_{\langle ij} M^{(1)}_{km\rangle} - \frac{102}{5} M^{(3)}_{\langle ij} M^{(2)}_{km\rangle}\Bigg\} + \mathcal{O}(4), \end{aligned} \tag{3.4c}$$

where the brackets $\langle\,\rangle$ denote the STF projection, while, for the necessary current-type moments,

$$V_{ij}(T_R) = S^{(2)}_{ij}(T_R) + \frac{2Gm}{c^3}\int_{-\infty}^{T_R} \mathrm{d}V \left[\ln\left(\frac{T_R - V}{2b}\right) + \frac{7}{6}\right] S^{(4)}_{ij}(V) + \mathcal{O}(5), \tag{3.5a}$$

$$\begin{aligned} V_{ijk}(T_R) = S^{(3)}_{ijk}(T_R) + \frac{G}{c^3}\Bigg\{2m\int_{-\infty}^{T_R} \mathrm{d}V \left[\ln\left(\frac{T_R - V}{2b}\right) + \frac{5}{3}\right] S^{(5)}_{ijk}(V) \\ + \frac{1}{10}\varepsilon_{ab\langle i} M^{(5)}_{j\underline{a}} M_{k\rangle b} - \frac{1}{2}\varepsilon_{ab\langle i} M^{(4)}_{j\underline{a}} M^{(1)}_{k\rangle b} - 2 S_{\langle i} M^{(4)}_{jk\rangle}\Bigg\} + \mathcal{O}(4). \end{aligned} \tag{3.5b}$$

(The underlined index $\underline{a}$ means that it should be excluded from the STF projection.) For all the other needed moments we are allowed to simply write

$$U_L(T_R) = M^{(l)}_L(T_R) + \mathcal{O}(3), \tag{3.6a}$$

$$V_L(T_R) = S^{(l)}_L(T_R) + \mathcal{O}(3). \tag{3.6b}$$

In the above formulae, m is the total ADM mass of the binary system, which agrees with the mass monopole moment, $M = m$. The M_L and S_L are the mass and current-type canonical source moments, and $M^{(p)}_L$, $S^{(p)}_L$ denote their pth time derivatives.

The parameter b appearing in the logarithms of equations (3.4) and (3.5) is a freely specifiable constant, having the dimension of time, entering the relation between the retarded time $T_R = T - R/c$ in radiative coordinates and the corresponding one $t - \rho/c$ in harmonic coordinates (where ρ is the distance of the source in harmonic coordinates). More precisely we have

$$T_R = t - \frac{\rho}{c} - \frac{2Gm}{c^3}\ln\left(\frac{\rho}{cb}\right). \tag{3.7}$$

The constant b can be chosen at will because it simply corresponds to a choice of the origin of radiative time with respect to harmonic time.

As recalled in section 2.1, the 'canonical' moments M_L, S_L do not represent the best definition for what should be referred to as the 'source' moments. This is why we now replace the M_L and S_L by adequate source multipole moments I_L, J_L, $W_L, \ldots$, which admit closed-form expressions in terms of the source's stress–energy tensor. At the 2.5PN order it turns out that we need to only take into account a correction in the 2.5PN canonical mass-type quadrupole moment and given, in a centre of mass frame, by (see [36, 40])[9]

$$M_{ij} = I_{ij} + \frac{4G}{c^5}\left[W^{(2)} I_{ij} - W^{(1)} I^{(1)}_{ij}\right] + \mathcal{O}(7), \tag{3.8}$$

[9] Equation (11.7*a*) in [40] contains a sign error, but with no consequence for any of the results of that reference. The correct sign is reproduced here.

where I_{ij} is the source mass quadrupole, and W denotes the 'monopole' corresponding to the set of moments W_L. (We shall need W only at the Newtonian order where it will be given by (3.18*b*); see section 3.3 for the expressions of all the source moments in the case of circular binary systems.) Note that a formula generalizing equation (3.8) to all PN orders (and multipole interactions) is not possible at present and needs to be investigated anew for specific cases. This is why it is more convenient to define the source moments to be I_L and J_L (and also the other ones $W_L, \ldots, Z_L$, but in view of, e.g., equation (3.8) these appear to be much less important than I_L, J_L) rather than M_L and S_L. For all the other moments needed here, besides the mass quadrupole (3.8), we can write, with the required precision, that M_L agrees with the corresponding I_L and that similarly S_L agrees with J_L. Namely we always have

$$M_L = I_L + \mathcal{O}(5), \tag{3.9a}$$

$$S_L = J_L + \mathcal{O}(5), \tag{3.9b}$$

and we can neglect in the 2.5PN wave form all the remainders in (3.9) except for the case of M_{ij} where the required result is provided by equation (3.8). Thus, from now on, the wave form will be considered a function of the 'main' source moments I_L, J_L, and also, of the 'auxiliary' moment W appearing in equation (3.8).

3.2. Instantaneous versus hereditary contributions

From equations (3.4) and (3.5) it is clear that the radiative moments contain two types of terms, those which depend on the source moments at a single instant, namely the retarded time $T_R \equiv T - R/c$, referred to as *instantaneous* terms, and those which are sensitive to the entire 'past history' of the system, i.e. which depend on all previous times ($V < T_R$), and are referred to as the *hereditary* terms.

In this work, we find it convenient to further subclassify the instantaneous terms in the radiative moments into three types based on their structure. The leading instantaneous contribution to the radiative moment U_L (respectively V_L) from the source moment I_L (respectively J_L), is of the form $I_L^{(l)}$ or $J_L^{(l)}$. We refer to these as instantaneous contributions from the *source* moments and denote them by the subscript 'inst(s)'. Starting at 2.5PN order, additional instantaneous terms arise, of the form $I_{ij}^{(n)} I_{km}^{(p)}$ or $I_{ij}^{(n)} J_k$, in the expressions relating radiative moments to source moments (see equations (3.4) and (3.5)). We call such additional terms, the instantaneous terms in the *radiative* moment and denote them by the subscript 'inst(r)'. Thirdly, from equation (3.8) we see that the replacement of the canonical moments by the source moments induces also some new terms at 2.5PN level, of the form $I_{ij}^{(n)} W^{(p)}$, which we shall call the instantaneous terms in the *canonical* moment denoted by the subscript 'inst(c)'.

At the 1.5PN order, the hereditary terms are due to the interaction of the mass quadrupole moment with mass monopole (ADM mass $m = M$) and cause the effect of wave tails [63]. Physically, this effect can be visualized as the backscattering of the linear waves (described by I_{ij}) off the constant spacetime curvature generated by the mass energy m. This can be viewed as a part of the gravitational field propagating inside the light cone (e.g. [64]). At higher PN orders there are similar tails due to the interaction between m and higher moments $I_{ijk}, J_{ij}, \cdots$. In addition, at the 3PN order (however, negligible for the present study), there is an effect of tails generated by tails, because of the cubic interaction between the quadrupole moment and two mass monopoles, $m \times m \times I_{ij}$ [66]. The hereditary term arising at the 2.5PN order in the radiative quadrupole (3.4*a*) is different in nature. It is made of the quadrupole–quadrupole interaction, $I_{ij} \times I_{kl}$, and can be physically thought of as due to the re-radiation of the stress–energy tensor of the linearized quadrupolar gravitational waves. It is responsible for the so-called 'nonlinear memory' or Christodoulou effect [77–79] (investigated within the present approach in [63, 65]). So far, all these effects are taken into account in the calculation of the wave form up to 2PN order [31] and in the energy flux up to 3.5PN [36, 40, 54]. The two different types of hereditary terms will be denoted by subscripts 'tail' and 'memory'.

Summarizing, with the present notation, the total 2.5PN wave form may be written as

$$h_{km}^{\mathrm{TT}} = \left(h_{km}^{\mathrm{TT}}\right)_{\mathrm{inst(s)}} + \left(h_{km}^{\mathrm{TT}}\right)_{\mathrm{inst(r)}} + \left(h_{km}^{\mathrm{TT}}\right)_{\mathrm{inst(c)}} + \left(h_{km}^{\mathrm{TT}}\right)_{\mathrm{tail}} + \left(h_{km}^{\mathrm{TT}}\right)_{\mathrm{memory}} + \mathcal{O}(6). \tag{3.10}$$

We give each of the above contributions explicitly. The instantaneous part of type (s) reads

$$\begin{aligned}
\left(h_{km}^{\mathrm{TT}}\right)_{\mathrm{inst(s)}} = \frac{2G}{c^4 R}\mathcal{P}_{ijkm}\Bigg\{ & I_{ij}^{(2)} + \frac{1}{c}\left[\frac{1}{3}N_a I_{ija}^{(3)} + \frac{4}{3}\varepsilon_{ab(i}J_{j)a}^{(2)}N_b\right] \\
& + \frac{1}{c^2}\left[\frac{1}{12}N_{ab}I_{ijab}^{(4)} + \frac{1}{2}\varepsilon_{ab(i}J_{j)ac}^{(3)}N_{bc}\right] \\
& + \frac{1}{c^3}\left[\frac{1}{60}N_{abc}I_{ijabc}^{(5)} + \frac{2}{15}\varepsilon_{ab(i}J_{j)acd}^{(4)}N_{bcd}\right] \\
& + \frac{1}{c^4}\left[\frac{1}{360}N_{abcd}I_{ijabcd}^{(6)} + \frac{1}{36}\varepsilon_{ab(i}J_{j)acde}^{(5)}N_{bcde}\right] \\
& + \frac{1}{c^5}\left[\frac{1}{2520}N_{abcde}I_{ijabcde}^{(7)} + \frac{1}{210}\varepsilon_{ab(i}J_{j)acdef}^{(6)}N_{bcdef}\right]\Bigg\},
\end{aligned} \tag{3.11}$$

where all the source moments are evaluated at the current time T_R. The type (r) is

$$\begin{aligned}
\left(h_{km}^{\mathrm{TT}}\right)_{\mathrm{inst(r)}} = \frac{2G}{c^4 R}\mathcal{P}_{ijkm}\frac{G}{c^5}\Bigg\{ & \frac{1}{7}I_{a\langle i}^{(5)}I_{j\rangle a} - \frac{5}{7}I_{a\langle i}^{(4)}I_{j\rangle a}^{(1)} - \frac{2}{7}I_{a\langle i}^{(3)}I_{j\rangle a}^{(2)} + \frac{1}{3}\varepsilon_{ab\langle i}I_{j\rangle a}^{(4)}J_b \\
& + \frac{1}{12}N_{ab}\left[-\frac{21}{5}I_{\langle ij}^{(5)}I_{ab\rangle} - \frac{63}{5}I_{\langle ij}^{(4)}I_{ab\rangle}^{(1)} - \frac{102}{5}I_{\langle ij}^{(3)}I_{ab\rangle}^{(2)}\right] \\
& + \frac{1}{2}N_{bc}\varepsilon_{abi}\left[\frac{1}{10}\varepsilon_{pq\langle j}I_{a\underline{p}}^{(5)}I_{c\rangle q} - \frac{1}{2}\varepsilon_{pq\langle j}I_{a\underline{p}}^{(4)}I_{c\rangle q}^{(1)} - 2J_{\langle j}I_{ac\rangle}^{(3)}\right]\Bigg\}.
\end{aligned} \tag{3.12}$$

Apart from two terms involving the source dipole moment J_i or angular momentum, these terms are made of quadrupole–quadrupole couplings coming from U_{ij}, U_{ijk} and V_{ijk} in equations (3.4) and (3.5) and computed in [65] (though, using dimensional and parity arguments, their structure can easily be written, the computation of the numerical coefficients in front of each inst(r) term needs a detailed study). The inst(c) terms refer to the instantaneous terms in the 'canonical' moment and can be written as

$$\left(h_{km}^{\mathrm{TT}}\right)_{\mathrm{inst(c)}} = \frac{2G}{c^4 R}\mathcal{P}_{ijkm}\frac{G}{c^5}\left\{4\left[W^{(4)}I_{ij} + W^{(3)}I_{ij}^{(1)} - W^{(2)}I_{ij}^{(2)} - W^{(1)}I_{ij}^{(3)}\right]\right\}, \tag{3.13}$$

where W is the particular 'monopole' moment introduced in (3.8). Both the inst(r) and inst(c) terms represent new features of the 2.5PN wave form. Concerning hereditary parts, we have the tail integrals (dominantly 1.5PN) which read

$$\begin{aligned}
\left(h_{km}^{\mathrm{TT}}\right)_{\mathrm{tail}} = \frac{2G}{c^4 R}\mathcal{P}_{ijkm}\frac{2Gm}{c^3}\int_{-\infty}^{T_R}\mathrm{d}V\,\Bigg\{ & \left[\ln\left(\frac{T_R - V}{2b}\right) + \frac{11}{12}\right]I_{ij}^{(4)}(V) \\
& + \frac{N_a}{3c}\left[\ln\left(\frac{T_R - V}{2b}\right) + \frac{97}{60}\right]I_{ija}^{(5)}(V) \\
& + \frac{4N_b}{3c}\left[\ln\left(\frac{T_R - V}{2b}\right) + \frac{7}{6}\right]\varepsilon_{ab(i}J_{j)a}^{(4)}(V)
\end{aligned}$$

$$+\frac{N_{ab}}{12c^2}\left[\ln\left(\frac{T_R-V}{2b}\right)+\frac{59}{30}\right]I^{(6)}_{ijab}(V)$$
$$+\frac{N_{bc}}{2c^2}\left[\ln\left(\frac{T_R-V}{2b}\right)+\frac{5}{3}\right]\varepsilon_{ab(i}J^{(5)}_{j)ac}(V)\Bigg\}, \tag{3.14}$$

and the nonlinear memory integral (which is purely of the order of 2.5PN) given by

$$\left(h^{\rm TT}_{km}\right)_{\rm memory}=\frac{2G}{c^4R}\mathcal{P}_{ijkm}\frac{G}{c^5}\int_{-\infty}^{T_R}\mathrm{d}V\left\{-\frac{2}{7}I^{(3)}_{a\langle i}(V)I^{(3)}_{j\rangle a}(V)+\frac{N_{ab}}{30}I^{(3)}_{\langle ij}(V)I^{(3)}_{ab\rangle}(V)\right\}. \tag{3.15}$$

The latter expression is in complete agreement with the results of [63, 65, 78].

3.3. Source multipole moments required at the 2.5PN order

Evidently the above formulae remain empty unless we feed them with the explicit expressions of the source multipole moments, essentially the mass-type I_L and current-type J_L, appropriate for a specific choice of matter model. In the present section, we list the I_L and J_L needed for the 2.5PN accurate wave form in the case of point-particle binaries in circular orbits. This is the extension of the list of moments given in equations (4.4) of [31] for the computation of the 2PN accurate wave form. We do not give any details on this calculation because it follows exactly the same techniques as in [40].

Up to 2.5PN order in the wave form the mass moments are

$$I_{ij}=\nu m\,\mathrm{STF}_{ij}\left\{x^{ij}\left[1+\gamma\left(-\frac{1}{42}-\frac{13}{14}\nu\right)+\gamma^2\left(-\frac{461}{1512}-\frac{18\,395}{1512}\nu-\frac{241}{1512}\nu^2\right)\right]\right.$$
$$\left.+\frac{r^2}{c^2}v^{ij}\left[\frac{11}{21}-\frac{11}{7}\nu+\gamma\left(\frac{1607}{378}-\frac{1681}{378}\nu+\frac{229}{378}\nu^2\right)\right]+\frac{48}{7}\frac{r}{c}x^iv^j\nu\gamma^2\right\}$$
$$+\mathcal{O}(6), \tag{3.16a}$$

$$I_{ijk}=\nu m(X_2-X_1)\,\mathrm{STF}_{ijk}\left\{x^{ijk}\left[1-\gamma\nu-\gamma^2\left(\frac{139}{330}+\frac{11\,923}{660}\nu+\frac{29}{110}\nu^2\right)\right]\right.$$
$$\left.+\frac{r^2}{c^2}x^iv^{jk}\left[1-2\nu-\gamma\left(-\frac{1066}{165}+\frac{1433}{330}\nu-\frac{21}{55}\nu^2\right)\right]\right\}+\mathcal{O}(5), \tag{3.16b}$$

$$I_{ijkl}=\nu m\,\mathrm{STF}_{ijkl}\left\{x^{ijkl}\left[1-3\nu+\gamma\left(\frac{3}{110}-\frac{25}{22}\nu+\frac{69}{22}\nu^2\right)\right]\right.$$
$$\left.+\frac{78}{55}\frac{r^2}{c^2}v^{ij}x^{kl}(1-5\nu+5\nu^2)\right\}+\mathcal{O}(4), \tag{3.16c}$$

$$I_{ijklm}=\nu m(X_2-X_1)\,\mathrm{STF}_{ijklm}\left\{x^{ijklm}\left[1-2\nu+\gamma\left(\frac{2}{39}-\frac{47}{39}\nu+\frac{28}{13}\nu^2\right)\right]\right.$$
$$\left.+\frac{70}{39}\frac{r^2}{c^2}x^{ijk}v^{lm}\left(1-4\nu+3\nu^2\right)\right\}+\mathcal{O}(3), \tag{3.16d}$$

$$I_{ijklmn}=\nu m\,\mathrm{STF}_{ijklmn}\{x^{ijklmn}(1-5\nu+5\nu^2)\}+\mathcal{O}(2), \tag{3.16e}$$

$$I_{ijklmno}=\nu m(X_2-X_1)(1-4\nu+3\nu^2)\,\mathrm{STF}_{ijklmno}\{x^{ijklmno}\}+\mathcal{O}(1). \tag{3.16f}$$

Further, the current moments are given by

$$J_{ij} = \nu m(X_2 - X_1)\,\mathrm{STF}_{ij}\left\{\varepsilon_{abi}x^{ja}v^b\left[1+\gamma\left(\frac{67}{28}-\frac{2}{7}\nu\right)\right.\right.$$
$$\left.\left. +\gamma^2\left(\frac{13}{9}-\frac{4651}{252}\nu-\frac{1}{168}\nu^2\right)\right]\right\} + \mathcal{O}(5), \quad (3.17a)$$

$$J_{ijk} = \nu m\,\mathrm{STF}_{ijk}\left\{\varepsilon_{kab}x^{aij}v^b\left[1-3\nu+\gamma\left(\frac{181}{90}-\frac{109}{18}\nu+\frac{13}{18}\nu^2\right)\right]\right.$$
$$\left. +\frac{7}{45}\frac{r^2}{c^2}\varepsilon_{kab}x^a v^{bij}(1-5\nu+5\nu^2)\right\} + \mathcal{O}(4), \quad (3.17b)$$

$$J_{ijkl} = \nu m(X_2 - X_1)\,\mathrm{STF}_{ijkl}\left\{\varepsilon_{lab}x^{ijka}v^b\left[1-2\nu+\gamma\left(\frac{20}{11}-\frac{155}{44}\nu+\frac{5}{11}\nu^2\right)\right]\right.$$
$$\left. +\frac{4}{11}\frac{r^2}{c^2}\varepsilon_{lab}x^{ia}v^{jkb}\left(1-4\nu+3\nu^2\right)\right\} + \mathcal{O}(3), \quad (3.17c)$$

$$J_{ijklm} = \nu m\,\mathrm{STF}_{ijklm}\{\varepsilon_{mab}x^{aijkl}v^b(1-5\nu+5\nu^2)\} + \mathcal{O}(2), \quad (3.17d)$$

$$J_{ijklmn} = \nu m(X_2 - X_1)(1-4\nu+3\nu^2)\,\mathrm{STF}_{ijklmn}\{\varepsilon_{nab}x^{aijklm}v^b\} + \mathcal{O}(1). \quad (3.17e)$$

(We recall that $X_1 = \frac{m_1}{m}$, $X_2 = \frac{m_2}{m}$ and $\nu = X_1X_2$; the PN parameter γ is defined by (2.1); the STF projection is mentioned explicitly in front of each term.)

In addition, the current dipole J_i in (3.12) is the constant binary total angular momentum which needs to be given only at Newtonian order, and we also need to give the monopolar moment W which appears inside the inst(c) terms of (3.13) and comes from the relation (3.8) between canonical and source quadrupoles. We have

$$J_i = \nu m\varepsilon_{iab}x^a v^b + \mathcal{O}(2), \quad (3.18a)$$

$$W = \tfrac{1}{3}\nu m\mathbf{x}\cdot\mathbf{v} + \mathcal{O}(2). \quad (3.18b)$$

With all the latter source moments valid for a specific matter system (compact binary in circular orbit), the gravitational wave form is fully specified up to the 2.5PN order.

4. Hereditary terms at the 2.5PN order

We now come to the computation of the hereditary terms, i.e. made of integrals extending on all the past history of the non-stationary source, at any time V from $-\infty$ in the past up to $T_R = T - R/c$. In the following we shall refer to T_R as the *current* time—the one at which the observation of the radiation field occurs. As we have seen in section 3.2, at the 2.5PN order the wave form contains two types of hereditary terms: tail integrals, given by equation (3.14), and the nonlinear memory integral (3.15).[10]

Evidently, in order to evaluate the hereditary terms, one must take into account the fact that the binary's orbit will have evolved, by gravitational radiation reaction, from early time on. However we shall show, following [32, 80], that the tails can basically be computed using the binary *current* dynamics at time T_R, i.e. a circular orbit travelled at the current orbital frequency $\omega(T_R)$ (this will be true modulo negligible 4PN terms in the wave form). Concerning the memory integral, one definitely needs to take into account a model of binary evolution in the past.

[10] Note that in the gravitational-wave *flux* (in contrast to the wave form), the nonlinear memory integral is instantaneous—it is made of a simple time anti-derivative and the flux depends on the time-derivative of the wave form.

4.1. Model for adiabatic inspiral in the remote past

In this paper we adopt a simplified model of binary's past evolution in which the orbit decays adiabatically because of gravitational radiation damping according to the standard quadrupolar (i.e. Newtonian) approximation. We shall justify later that such a model is sufficient for our purpose—because we shall have to take into account the PN corrections in the tails only at the current epoch. The orbit will be assumed to remain circular, apart from the gradual inspiral, at any time $V < T_R$. We shall ignore any astrophysical (non-gravitational) processes such as the binary formation by a capture process in some dense stellar cluster, the successive supernova explosions and associated core collapses leading to the formation of the two compact objects, etc.

Let us recall the expressions of the binary's orbital parameters as explicit functions of time V in the quadrupolar circular-orbit approximation [81]. The orbital separation $r(V)$ evolves according to a power law, namely

$$r(V) = 4\left[\frac{G^3m^3\nu}{5c^5}(T_c - V)\right]^{1/4}, \tag{4.1}$$

where T_c denotes the coalescence instant, at which the two bodies merge together and the orbital frequency formally tends to infinity. The factor $1/c^5$ therein represents the 2.5PN order of radiation reaction. We assume that our current detection of the binary takes place before the coalescence instant, $T_R < T_c$, in a regime where the binary inspiral is *adiabatic* and the approximation valid[11].

The orbital frequency ω in this model ($\omega = 2\pi/P$, where P is the orbital period) is related at any time to the orbital separation (4.1) by Kepler's (Newtonian) law $Gm = r^3\omega^2$. (Again we shall justify later our use of a Newtonian model for the early-time inspiral.) Hence,

$$\omega(V) = \frac{1}{8}\left[\frac{G^{5/3}m^{5/3}\nu}{5c^5}(T_c - V)\right]^{-3/8}. \tag{4.2}$$

Instead of ω it is convenient to make use of the traditional frequency-related post-Newtonian parameter $x \equiv (Gm\omega/c^3)^{2/3}$ already considered in (2.5). It is given by

$$x(V) = \frac{1}{4}\left[\frac{\nu c^3}{5Gm}(T_c - V)\right]^{-1/4}. \tag{4.3}$$

The orbital phase $\phi = \int \omega\,\mathrm{d}t = \frac{c^3}{Gm}\int x^{3/2}\,\mathrm{d}t$, namely the angle between the binary's separation and the ascending node $\mathcal{N}$, reads

$$\phi(V) = \phi_c - \frac{1}{\nu}\left[\frac{\nu c^3}{5Gm}(T_c - V)\right]^{5/8}, \tag{4.4}$$

where ϕ_c is the value of the phase at the coalescence instant.

The latter expressions are inserted into the various hereditary terms, and integrated from $-\infty$ in the past up to now. In order to better understand the structure of the integrals, it is advisable to re-express the above quantities (4.1)–(4.4) in terms of their values at the *current* time T_R. A simple way to achieve this is to introduce, following [80], the new time-related variable

$$y \equiv \frac{T_R - V}{T_c - T_R}, \tag{4.5}$$

[11] The formal PN order of the time interval left until coalescence is the inverse of the order of radiation reaction, $T_c - T_R = \mathcal{O}(c^5) = \mathcal{O}(-5)$.

and to make use of the power-law dependence in time of equations (4.1)–(4.4). This leads immediately, for the orbital radius $r(V)$ and similarly for $\omega(V)$ and $x(V)$, to

$$r(V) = r(T_R)(1+y)^{1/4}, \tag{4.6}$$

where $r(T_R)$ refers to the current value of the radius. For the orbital phase we get

$$\phi(V) = \phi(T_R) - \frac{1}{\nu}\left[\frac{\nu c^3}{5Gm}(T_c - T_R)\right]^{5/8}[(1+y)^{5/8} - 1]. \tag{4.7}$$

The latter form is, however, not the one we are looking for. Instead we want to make explicit the fact that the phase difference between T_R and some early time V will become larger when the inspiral rate gets slower, i.e. when the *relative* change of the orbital frequency ω in one corresponding period P becomes smaller.

To this end we introduce a dimensionless 'adiabatic parameter' associated with the inspiral rate *at the current time* T_R. This is properly defined as the ratio between the current period and the time left until coalescence. We adopt the definition[12]

$$\xi(T_R) \equiv \frac{1}{(T_c - T_R)\omega(T_R)}. \tag{4.8}$$

The adiabatic parameter ξ is of the order of 2.5PN. Written in terms of the PN variable x defined by (4.3) it reads

$$\xi(T_R) = \frac{256\nu}{5}x^{5/2}(T_R). \tag{4.9}$$

Now equation (4.7) can be expressed with the help of $\xi(T_R)$ in the more interesting form

$$\phi(V) = \phi(T_R) - \frac{8}{5\xi(T_R)}[(1+y)^{5/8} - 1], \tag{4.10}$$

which makes it clear that the phase difference $\Delta\phi = \phi(T_R) - \phi(V)$, which is 2π times the number of orbital cycles between V and T_R, tends to infinity when $\xi(T_R) \to 0$ on any 'remote-past' time interval for which y is bounded from below, for instance $y > 1$. What is important is that (4.10) depends on the *current* value of the adiabatic parameter, so we shall be able to compute the hereditary integrals in the relevant limit where $\xi(T_R) \to 0$, appropriate to the current adiabatic regime. Note that, at recent time, when $V \to T_R$ or equivalently $y \to 0$, we have

$$\phi(V) = \phi(T_R) - \frac{y}{\xi(T_R)} + \mathcal{O}(y^2), \tag{4.11}$$

which is of course the same as the Taylor expansion

$$\phi(V) = \phi(T_R) - (T_R - V)\omega(T_R) + \mathcal{O}[(T_R - V)^2]. \tag{4.12}$$

4.2. The nonlinear memory integral

We tackle the computation of the novel hereditary term at the 2.5PN order, namely the nonlinear memory integral given by (3.15). As we shall see the computation boils down to the evaluation of only two 'elementary' hereditary integrals, below denoted by $I(T_R)$ and $J(T_R)$. A third type of elementary integral, $K(T_R)$, will be necessary to compute the tail integrals in section 4.3.

[12] We have $\xi = \frac{8}{3}\dot{\omega}/\omega^2$, so our definition agrees with the actual relative frequency change $\propto \dot{\omega}/\omega^2$ in one period. It is also equivalent to the one adopted in [80]: $\xi = \xi_{\rm BS}/\pi$.

The two wave polarizations corresponding to equation (3.15), calculated with our conventions and notation explained after (3.3), are readily obtained from the Newtonian approximation to the quadrupole moment I_{ij} (first term in (3.16a)), and cast in the form

$$(h_+)_{\rm memory}(T_R) = \frac{2G}{c^4 R}\frac{G^4 m^5 \nu^2 \sin^2 i}{c^5}\int_{-\infty}^{T_R}\frac{{\rm d}V}{r^5(V)}\left\{-\frac{12}{5}+\frac{2}{15}\sin^2 i + \left(\frac{4}{15}-\frac{2}{15}\sin^2 i\right)\cos[4\phi(V)]\right\}, \tag{4.13a}$$

$$(h_\times)_{\rm memory}(T_R) = \frac{2G}{c^4 R}\frac{G^4 m^5 \nu^2 \sin^2 i}{c^5}\int_{-\infty}^{T_R}\frac{{\rm d}V}{r^5(V)}\left\{\frac{4\cos i}{15}\sin[4\phi(V)]\right\}. \tag{4.13b}$$

In our model of binary evolution the radius of the orbit, $r(V)$, and orbital phase, $\phi(V)$, are given by equations (4.1) and (4.4), at any time such that $V < T_R < T_c$. Because $r(V) \propto (T_c - V)^{1/4}$ we see that the integrals in (4.13) are perfectly well defined, and in fact absolutely convergent at the bound $V \to -\infty$. There are two distinct types of terms in (4.13). A term, present only in the *plus* polarization (4.13a), is independent of the orbital phase ϕ, and given by a steadily varying function of time, having an amplitude increasing like some power law but without any oscillating behaviour. This 'steadily increasing' term is specifically responsible for the memory effect. The other terms, present in both polarizations, oscillate with time like some sine or cosine of the phase, in addition to having a steadily increasing maximal amplitude.

Consider first the steadily growing, non-oscillating term. Its computation simply relies, as is clear from (4.13a), on the single elementary integral

$$I(T_R) \equiv \frac{(Gm)^4}{c^7}\int_{-\infty}^{T_R}\frac{{\rm d}V}{r^5(V)}, \tag{4.14}$$

where we find convenient to factorize an appropriate coefficient in order to make it dimensionless. The calculation of (4.14) is easily done directly, but it is useful to perform our change of variable (4.5), as an exercise to prepare the treatment of the (somewhat less easy) oscillating terms. Thus, we write in a first stage

$$I(T_R) = \frac{(Gm)^4}{c^7}\frac{T_c - T_R}{r^5(T_R)}\int_0^{+\infty}\frac{{\rm d}y}{(1+y)^{5/4}}. \tag{4.15}$$

The factor in front is best expressed in terms of the dimensionless PN parameter $x(T_R)$, and of course the remaining integral is trivially integrated. We get

$$I(T_R) = \frac{5}{256\nu}x(T_R)\int_0^{+\infty}\frac{{\rm d}y}{(1+y)^{5/4}} = \frac{5}{64\nu}x(T_R). \tag{4.16}$$

With this result we obtain the steadily increasing or memory term in equation (4.13a). However, as its name indicates, this term keeps a 'memory' of the past activity of the system. As a test of the numerical influence of the binary's past history on equation (4.16), let us suppose that the binary was created *ex nihilo* at some finite initial time T_0 in a circular orbit state. This premise is not very realistic—we should more realistically assume, e.g., a binary capture in some stellar cluster and/or consider an initially eccentric orbit — but it should give an estimate of how sensitive is the memory term on initial conditions. In this crude model, we have to consider the integral $I_0(T_R)$ extending from T_0 up to T_R. We find that the ratio of $I_0(T_R)$ and our earlier model $I(T_R)$ is

$$\frac{I_0(T_R)}{I(T_R)} = 1 - \left(\frac{T_c - T_R}{T_c - T_0}\right)^{1/4} = 1 - \frac{r(T_R)}{r(T_0)}. \tag{4.17}$$

Let us choose the observation time T_R such that the binary is visible by the VIRGO/LIGO detectors. At the entry of the detectors' frequency bandwidth, say $\omega_{\rm seismic} \simeq 30$ Hz, we obtain $T_c - T_R \simeq 10^3$ s and $r(T_R) \simeq 700$ km in the case of two neutron stars ($m = 2.8 M_\odot$). For a binary system initially formed on an orbit of the size of the Sun, $r(T_0) \simeq 10^6$ km (corresponding to $T_c - T_0 \simeq 10^8$ yr), we find that the fractional difference between our two models I_0 and I amounts to about 10^{-3}. For an initial orbit of the size of a white dwarf, $r(T_0) \simeq 10^4$ km, the fractional difference is of the order of 10%—rather large indeed. So we conclude that indeed the memory term in (4.13*a*) depends rather severely on detailed assumptions concerning the past evolution of the binary system. We shall have to keep this feature in mind when we present our final results for this term.

We next turn our attention to the phase-dependent, oscillating terms in equations (4.13). Clearly these terms are obtained once we know the elementary integral

$$J(T_R) \equiv \frac{(Gm)^4}{c^7} \int_{-\infty}^{T_R} \mathrm{d}V \frac{\mathrm{e}^{4\mathrm{i}\phi(V)}}{r^5(V)}. \tag{4.18}$$

Inserting (4.6) and (4.10) into it we are led to the form (which exactly parallels (4.16))

$$J(T_R) = \frac{5}{256\nu} x(T_R)\, \mathrm{e}^{4\mathrm{i}\phi(T_R)} \int_0^{+\infty} \frac{\mathrm{d}y}{(1+y)^{5/4}}\, \mathrm{e}^{-\frac{32\mathrm{i}}{5\xi(T_R)}[(1+y)^{5/8}-1]}. \tag{4.19}$$

We shall compute this integral in the form of an approximation series, valid in the adiabatic limit $\xi(T_R) \to 0$. The easiest way to obtain successive approximations is to integrate by parts. We obtain

$$\int_0^{+\infty} \frac{\mathrm{d}y}{(1+y)^{5/4}}\, \mathrm{e}^{-\frac{32\mathrm{i}}{5\xi}[(1+y)^{5/8}-1]} = \frac{\xi}{4\mathrm{i}} \left\{ 1 - \frac{7}{8} \int_0^{+\infty} \frac{\mathrm{d}y}{(1+y)^{15/8}}\, \mathrm{e}^{-\frac{32\mathrm{i}}{5\xi}[(1+y)^{5/8}-1]} \right\}. \tag{4.20}$$

A further integration by parts shows that the integral in the curly brackets of (4.20) is itself of the order of ξ, so we have the result[13]

$$\int_0^{+\infty} \frac{\mathrm{d}y}{(1+y)^{5/4}}\, \mathrm{e}^{-\frac{32\mathrm{i}}{5\xi}[(1+y)^{5/8}-1]} = \frac{\xi}{4\mathrm{i}} \{1 + \mathcal{O}(\xi)\}. \tag{4.21}$$

A standard way of understanding it is to remark that, when y is different from zero, the phase of the integrand of (4.21) oscillates very rapidly when $\xi \to 0$, so the integral is made of a sum of alternatively positive and negative terms and is essentially zero. Consequently, the value of the integral is essentially given by the contribution due to the bound at $y = 0$, which can be approximated by

$$\int_0 \mathrm{d}y\, \mathrm{e}^{-\frac{4\mathrm{i}y}{\xi}} = \frac{\xi}{4\mathrm{i}}. \tag{4.22}$$

Because $\xi(T_R)$ is of the order of 2.5PN the result (4.21) is sufficient for the control of the 2.5PN wave form, thus our elementary integral reads, with the required precision,

$$J(T_R) = x^{7/2}(T_R) \frac{\mathrm{e}^{4\mathrm{i}\phi(T_R)}}{4\mathrm{i}} \{1 + \mathcal{O}(\xi)\}. \tag{4.23}$$

We find that the 'oscillating' integral $J(T_R)$ is an order 2.5PN smaller than the 'steadily growing' or 'memory' integral $I(T_R)$. This can be interpreted by saying that the cumulative (secular) effect of the integration over the whole binary inspiral in $I(T_R)$ is comparable to

[13] By successive integration by parts one generates the asymptotic series (divergent for any value of ξ)

$$\int_0^{+\infty} \frac{\mathrm{d}y}{(1+y)^{5/4}}\, \mathrm{e}^{-\frac{32\mathrm{i}}{5\xi}[(1+y)^{5/8}-1]} \sim -8 \sum_{n=1}^{+\infty} (5n-3)(5n-8)\cdots(12)(7) \left(\frac{\mathrm{i}\xi}{32}\right)^n,$$

where we have used the standard notation $\sim$ for equalities valid in the sense of asymptotic series.

the inverse of the order of radiation-reaction forces—a quite natural result. By contrast the oscillations in J, due to the sequence of orbital cycles in the entire life of the binary system, compensate each other more or less yielding a net result which is 2.5PN smaller than for I. Furthermore, the argument leading to the evaluation of (4.22) shows that J, in contrast to I, is quite insensitive to the details of the binary's past evolution.

Substituting equations (4.16) and (4.23) into (4.13) we finally obtain the hereditary memory-type contributions to the polarization wave forms as

$$(h_+)_{\text{memory}} = \frac{2Gm\nu x}{c^2 R}\sin^2 i\left\{-\frac{17+\cos^2 i}{96}+\frac{\nu}{30}x^{5/2}(1+\cos^2 i)\sin(4\phi)\right\}, \tag{4.24a}$$

$$(h_\times)_{\text{memory}} = \frac{2Gm\nu x}{c^2 R}\sin^2 i\left\{-\frac{\nu}{15}x^{5/2}\cos i\cos(4\phi)\right\}, \tag{4.24b}$$

where, of course, all quantities correspond to the current time T_R. The phase-independent term is the nonlinear memory or Christodoulou effect [63, 65, 77–79] in the case of inspiralling compact binaries. Our calculation, leading to a factor $\propto \sin^2 i\,(17+\cos^2 i)$, agrees with the result of Wiseman and Will [78][14]. The nonlinear memory term *stricto-sensu* affects the plus polarization but not the cross polarization, for which it is exactly *zero*. As we have said, it represents a part of the wave form whose amplitude grows with time, but which is nearly constant over one orbital period. It is therefore essentially a *zero-frequency* (dc) effect, which has rather poor observational consequences in the case of the LIGO–VIRGO detectors, whose frequency bandwidth is always limited from below by $\omega_{\text{seismic}} > 0$. In addition, we know that detecting and analysing the ICBs relies essentially on monitoring the phase evolution, which in turn is determined by the total gravitational-wave flux. But we have already noted that the nonlinear memory integral (3.15) is instantaneous in the flux (and in fact it does not contribute to the phase in the circular orbit case [36]). It seems thus that the net cumulative 'memory' change in the wave form of ICBs is hardly detectable. On the other hand, the frequency-dependent terms found in equation (4.24) form an integral part of the 2.5PN wave form.

An important comment is in order. As we have seen the memory effect, i.e. the dc term in equation (4.24*a*), is 'Newtonian' because the cumulative integration over the binary's past just compensates the formal 2.5PN order of the hereditary integral. Thus we expect that some formal PN terms strictly higher that 2.5PN will actually contribute to the 2.5PN wave form *via* a similar cumulative integration. For instance, at the 3.5PN level there will be a memory-type integral in the radiative quadrupole moment U_{ij}, which is quadratic in the mass octupole moment (of the symbolic form $I_{ab\langle i} \times I_{j\rangle ab}$). After integration over the past using our model of adiabatic inspiral, we expect that the 'steadily growing' part of the integral should yield a dc contribution to the wave form at the relative 1PN order. In the present paper we do not consider such higher-order post-Newtonian dc contributions, and leave their computation to future work.

4.3. Gravitational-wave tails

The tails up to 2.5PN order are given by equation (3.14). Because of the logarithmic kernel they involve, the tails are more complicated than simple time anti-derivatives, and they constitute a crucial part of both the wave form and the energy flux, and in particular of the orbital phase with important observational consequences (for a review see [60]).

[14] The difference of a factor -2 between their result and our result here is probably due to a different convention for the polarization wave forms.

The computation of tails reduces to the computation of a new type of 'elementary' integral, differing from $J(T_R)$ given in (4.18) by the presence of an extra logarithmic factor in the integrand, and given by[15]

$$K(T_R) \equiv \frac{(Gm)^4}{c^7} \int_{-\infty}^{T_R} \mathrm{d}V \frac{\mathrm{e}^{4\mathrm{i}\phi(V)}}{r^5(V)} \ln\left(\frac{T_R - V}{T_c - T_R}\right). \tag{4.25}$$

For convenience, we have inserted into the logarithm of (4.25) the constant time scale $T_c - T_R$ instead of the normalization $2b$ more appropriate for tails (or, for instance, $2b\,\mathrm{e}^{-11/12}$, see (3.14)), but we can do this at the price of adding another term which will be proportional to $J(T_R)$ already computed in section 4.2.

With the help of the y-variable (4.5) we transform the latter integral into

$$K(T_R) = \frac{5}{256\nu} x(T_R)\,\mathrm{e}^{4\mathrm{i}\phi(T_R)} \int_0^{+\infty} \frac{\mathrm{d}y \ln y}{(1+y)^{5/4}}\,\mathrm{e}^{-\frac{32\mathrm{i}}{5\xi(T_R)}[(1+y)^{5/8}-1]}. \tag{4.26}$$

Because of the factor $\ln y$ we are not able to directly integrate by parts as we did to compute $J(T_R)$. Instead we must split the integral into some 'recent-past' contribution, say $y \in \left]0, 1\right[$, and the 'remote-past' contribution, $y \in \left]1, +\infty\right[$. In the remote-past integral, whose lower bound at $y = 1$ allows for differentiating the factor $\ln y$, we perform the integrations by parts in a way similar to (4.20). This leads to

$$\int_1^{+\infty} \frac{\mathrm{d}y \ln y}{(1+y)^{5/4}}\,\mathrm{e}^{-\frac{32\mathrm{i}}{5\xi}[(1+y)^{5/8}-1]} = \mathcal{O}(\xi^2), \tag{4.27}$$

which is found to be of the order of $\mathcal{O}(\xi^2)$ instead of $\mathcal{O}(\xi)$ in equation (4.21). This is a consequence of the $\ln y$ which is zero at the bound $y = 1$, and thus kills the all-integrated term. Hence we deduce from (4.27) that the contribution from the remote past in the tail integrals is in fact quite small. The details concerning the remote-past activity of the source are negligible when computing the tails. More precisely, because $\xi = \mathcal{O}(5)$, we can check that terms such as (4.27) do not contribute to the wave form before the 4PN order, so the tail integrals can be legitimately approximated, with 2.5PN accuracy, by their 'recent-past' history (in agreement with the findings of [32, 80]).

Now, in the recent-past integral, i.e. $y \in \left]0, 1\right[$, we are allowed to replace the integrand by its equivalent when $y \to 0$, modulo terms of the same magnitude as (4.27). This fact has been proved rigorously in appendix B of [80]. Here we shall not reproduce the proof but simply state the end result, which reads

$$\int_0^1 \frac{\mathrm{d}y \ln y}{(1+y)^{5/4}}\,\mathrm{e}^{-\frac{32\mathrm{i}}{5\xi}[(1+y)^{5/8}-1]} = \int_0^1 \mathrm{d}y \ln y\,\mathrm{e}^{-\frac{4\mathrm{i}y}{\xi}} + \mathcal{O}(\xi^2 \ln \xi). \tag{4.28}$$

As we see, the remainder is $\mathcal{O}(\xi^2 \ln \xi)$, instead of being merely $\mathcal{O}(\xi^2)$ as in equation (4.27). The integral in the rhs of (4.28) gives the main contribution to the tail integral (the only one to

[15] Actually in order to describe the tails we should consider the more general integrals

$$K_{n,p}(T_R) = \frac{(Gm)^{p-1}}{c^{2p-3}} \int_{-\infty}^{T_R} \mathrm{d}V \frac{\mathrm{e}^{\mathrm{i}n\phi(V)}}{r^p(V)} \ln\left(\frac{T_R - V}{T_c - T_R}\right).$$

The dominant tails at the 1.5PN order correspond to $p = 4$ and $n = 2$, the tails at the 2PN order have $p = 9/2$ and $n = 1, 3$ and those at the 2.5PN order have $p = 5$ and $n = 2, 4$. Here we deal with the particular case $p = 5, n = 4$ because the calculation parallels the one of $I(T_R)$ and $J(T_R)$ in section 4.2. The other cases are treated in a similar way.

be taken into account up to very high 4PN order). It is easily computed by using a standard formula[16],

$$\int_0^1 \mathrm{d}y\,\ln y\,\mathrm{e}^{-\frac{4\mathrm{i}y}{\xi}} = \frac{\xi}{4\mathrm{i}}\left[\frac{\pi}{2\mathrm{i}} - \ln\left(\frac{4}{\xi}\right) - C\right] + \mathcal{O}(\xi^2), \tag{4.29}$$

where $C = 0.577\cdots$ is the Euler constant.

Finally our needed result is

$$K(T_R) = x^{7/2}(T_R)\frac{\mathrm{e}^{4\mathrm{i}\phi(T_R)}}{4\mathrm{i}}\left\{\frac{\pi}{2\mathrm{i}} - \ln\left(\frac{4}{\xi(T_R)}\right) - C + \mathcal{O}\left(\xi\ln\xi\right)\right\}, \tag{4.30}$$

where it is crucial that all the binary's parameters are evaluated at the current time T_R. It is interesting to compare (4.30) with our earlier result for $J(T_R)$ given by (4.23), in order to see the effect of adding a logarithmic-type kernel into an oscillating, phase-dependent integral. Equation (4.30), together with its trivial extension to $K_{n,p}$, is used for the computation of all the tails in the wave form at the 2.5PN order (and it could in fact be used up to the order 3.5PN included). Actually we still have to justify this because during the derivation of (4.30), we employed a *Newtonian* model for the binary inspiral in the past, e.g., we assumed the Kepler law $Gm = r^3(V)\omega^2(V)$ at any time $V < T_R$. In the case of the memory term (3.15) this is okay because it needs to be evaluated with Newtonian accuracy. But in the case of tails, equation (3.14), the dominant effect is at the 1.5PN order, so we have to take into account a 1PN relative correction in order to control the 2.5PN wave form. Nevertheless, our model of Newtonian inspiral in the past *is* compatible with taking into account 1PN effects, for the basic reason that for tails the past behaviour of the source is negligible, so the 1PN effects have only to be included in the *current* values of the parameters, i.e. $x(T_R)$, $\phi(T_R)$ and $\xi(T_R)$, in equation (4.30).

To see more precisely how this works, suppose that we want to replace the 'Newtonian' inspiral (4.1) by the more accurate 1PN law,

$$r(V) = 4\left[\frac{G^3m^3\nu}{5c^5}(T_c - V)\right]^{1/4} + \frac{Gm\eta}{c^2}, \tag{4.31}$$

where $\eta \equiv -\frac{1751}{1008} - \frac{7}{12}\nu$ (see, e.g., [60]). Substituting the variable y we obtain the 1PN equivalent of (4.6) as

$$r(V) = r(T_R)(1+y)^{1/4}\{1 + \eta[(1+y)^{-1/4} - 1]x(T_R)\}, \tag{4.32}$$

where we parametrized the 1PN correction term by means of the variable x evaluated at *current* time T_R (we consistently neglect higher PN terms). For the orbital phase we get

$$\phi(V) = \phi(T_R) - \frac{8}{5\xi(T_R)}\{[(1+y)^{5/8} - 1](1 + \zeta x(T_R)) + \tau[(1+y)^{3/8} - 1]x(T_R)\}, \tag{4.33}$$

where $\zeta = -\frac{743}{672} - \frac{11}{8}\nu$, $\tau = \frac{3715}{2016} + \frac{55}{24}\nu$. We insert those expressions into our basic integral (4.25), and split it into recent-past and remote-past contributions. Exactly as in (4.27) we can prove that the remote past of that integral, for which $y \in\,]1, +\infty[$, is negligible—of the order of $\mathcal{O}(\xi^2)$. Next, in the recent-past integral, $y \in\,]0, 1[$, we expand at the 1PN order

[16] For any real number ϵ we have

$$\epsilon\int_0^1 \mathrm{d}y\,\ln y\,\mathrm{e}^{\mathrm{i}\epsilon y} + \mathrm{i}\int_1^{+\infty}\frac{\mathrm{d}y}{y}\,\mathrm{e}^{\mathrm{i}\epsilon y} = -\frac{\pi}{2}\mathrm{sgn}(\epsilon) - \mathrm{i}(\ln|\epsilon| + C),$$

where $\mathrm{sgn}(\epsilon)$ and $|\epsilon|$ denote the sign and absolute value of ϵ.

(i.e. $x(T_R) \to 0$), to obtain some 1PN correction term, with respect to the previous calculation, of the form

$$\int_0^1 \frac{\mathrm{d}y \ln y}{(1+y)^{5/4}} [(1+y)^\alpha - 1]\, \mathrm{e}^{-\frac{32\mathrm{i}}{5\xi}[(1+y)^{5/8}-1]}, \tag{4.34}$$

where α can take the values $-\frac{1}{4}$, $\frac{3}{8}$ or $\frac{5}{8}$. Now the point is that this integral, like the remote-past one, is *also* of the order of $\mathcal{O}(\xi^2)$ or, rather, $\mathcal{O}(\xi^2 \ln \xi)$. Indeed, the new factor $(1+y)^\alpha - 1$ in the integrand of (4.34) is crucial in that it adds (after taking the equivalent when $y \to 0$) an extra factor y, and we have to thus treat the following equivalent:

$$\int_0^1 \mathrm{d}y\, y \ln y\, \mathrm{e}^{-\frac{4\mathrm{i}y}{\xi}} = \mathcal{O}(\xi^2 \ln \xi), \tag{4.35}$$

which is *smaller* by a factor $\xi = \mathcal{O}(5)$ than the integral (4.29), as easily seen by integrating by parts. This means that the order of magnitude of the correction induced by our more sophisticated 1PN model for inspiral in the past is negligible. In conclusion, even at the relative PN order, one can use equation (4.30) for computing the tails, but of course the current values of the binary's orbital parameters $x(T_R)$, $\phi(T_R)$ and $\xi(T_R)$ must consistently include their relevant PN corrections. This is what we do in the present paper, following the computation in [36, 66] of the higher-order tails up to relative 2PN order (i.e. 3.5PN beyond quadrupolar radiation).

Finally our results for the 2.5PN-accurate tail terms are as follows. It is convenient, following [35], to introduce, in place of the 'natural' constant time-scale b entering the tails and defined by (3.7), a constant *frequency*-scale ω_0 given by

$$\omega_0 \equiv \frac{\mathrm{e}^{\frac{11}{12}-C}}{4b}. \tag{4.36}$$

Like b, it can be chosen at will, for instance to be equal, as suggested in [35], to the seismic cut-off frequency of some interferometric detector, $\omega_0 = \omega_{\mathrm{seismic}}$. Then we find that all the dependence of the tails in the *logarithm* of the orbital frequency, i.e. the terms involving $\ln\omega$ and coming from the logarithm present in the rhs of (4.29), can be factorized out, up to the 2.5PN order, in the way

$$(h_{+,\times})_{\mathrm{tail}} = (k_{+,\times})_{\mathrm{tail}} - 2x^{3/2} \frac{\partial h_{+,\times}}{\partial\phi} \ln\left(\frac{\omega}{\omega_0}\right), \tag{4.37}$$

where all the dependence upon $\ln(\omega/\omega_0)$ is given as indicated (i.e. the $(k_{+,\times})_{\mathrm{tail}}$ are independent of $\ln(\omega/\omega_0)$). Because we are computing the tails with 1PN relative precision, this means that the factor of $\ln(\omega/\omega_0)$, namely $\partial h_{+,\times}/\partial\phi$ and therefore also $h_{+,\times}$ itself, is given at the relative 1PN order. The existence of this structure implies an elegant formulation of the 2.5PN wave form in terms of a new phase variable ψ given by equation (5.6). The phase ψ was already introduced in [35], and we have shown here that it is also valid, interestingly enough, for tails at the relative 1PN order. The 'main' tail contributions are then given, up to 2.5PN order, by

$$\begin{aligned}
(k_+)_{\mathrm{tail}} = \frac{2Gm\nu x}{c^2 R} \bigg\{ & -2\pi x^{3/2}\big(1+c_i^2\big)\cos 2\phi + \frac{s_i}{40}\frac{\delta m}{m} x^2 \big[\big(11 + 7c_i^2 + 10\big(5+c_i^2\big)\ln 2\big)\sin\phi \\
& - 5\pi\big(5+c_i^2\big)\cos\phi - 27[7 - 10\ln(3/2)]\big(1+c_i^2\big)\sin 3\phi + 135\pi\big(1+c_i^2\big)\cos 3\phi\big] \\
& + x^{5/2}\bigg[\frac{\pi}{3}\big(19 + 9c_i^2 - 2c_i^4 + \nu\big(-19 + 11c_i^2 + 6c_i^4\big)\big)\cos 2\phi \\
& + \frac{1}{5}\big(-9 + 14c_i^2 + 7c_i^4 + \nu\big(27 - 42c_i^2 - 21c_i^4\big)\big)\sin 2\phi
\end{aligned}$$

$$- \frac{16\pi}{3}(1 - c_i^4)(1 - 3\nu)\cos 4\phi$$
$$+ \frac{8}{15}\big(1 - c_i^4\big)(1 - 3\nu)(21 - 20\ln 2)\sin 4\phi\Big]\Big\}, \quad (4.38a)$$

$$(k_\times)_{\rm tail} = \frac{2Gm\nu x}{c^2 R}\Big\{-4\pi x^{3/2} c_i \sin 2\phi - \frac{3 s_i c_i}{20}\frac{\delta m}{m} x^2\,[(3 + 10\ln 2)\cos\phi + 5\pi\sin\phi$$
$$- 9[7 - 10\ln(3/2)]\cos 3\phi - 45\pi\sin 3\phi]$$
$$+ x^{5/2}\Big[\frac{2\pi}{3}c_i\big(13 + 4s_i^2 + \nu\big(-1 - 12 s_i^2\big)\big)\sin 2\phi$$
$$+ \frac{2}{5}c_i(1 - 3\nu)\left(-6 + 11 s_i^2\right)\cos 2\phi - \frac{32\pi}{3}c_i s_i^2(1 - 3\nu)\sin 4\phi$$
$$+ \frac{16}{15}c_i s_i^2(1 - 3\nu)\left(-21 + 20\ln 2\right)\cos 4\phi\Big]\Big\}. \quad (4.38b)$$

Here c_i and s_i denote the cosine and sine of the inclination angle i, and $\delta m = m_1 - m_2$ is the mass difference (so that $\delta m/m = X_1 - X_2$). Up to the 2PN order, we have agreement with the results of [35].

5. Results for the 2.5PN polarization wave forms

5.1. Instantaneous terms at the 2.5PN order

In section 3.3, we have given the list of source multipole moments needed for controlling the wave form at the 2.5PN order. In this section, we proceed to calculate the instantaneous terms (of types (s), (r) and (c)) in the 2.5PN wave form of circular compact binaries. The first step towards it is the computation of time derivatives of different moments I_L, J_L (and also W) using the binary's EOM up to 2.5PN order as given in equations (2.2)–(2.4). These time derivatives are then contracted with the unit direction **N**, for insertion into equations (3.11)–(3.13). We can write the inst(s) wave form schematically as,

$$\left(h^{\rm TT}_{km}\right)_{\rm inst(s)} = \frac{2G\nu m}{c^4 R}\mathcal{P}_{ijkm}\big\{\xi^{(0)}_{ij} + (X_2 - X_1)\xi^{(1/2)}_{ij} + \xi^{(1)}_{ij}$$
$$+ (X_2 - X_1)\xi^{(3/2)}_{ij} + \xi^{(2)}_{ij} + (X_2 - X_1)\xi^{(5/2)}_{ij} + \rho^{(5/2)}_{ij}\big\}. \quad (5.1)$$

The instantaneous terms up to 2PN order were already reported in equations (4.5) of [31]. We have reproduced them in our present computation. At the 2.5PN order, most of the terms vanish when the two masses are equal ($X_1 = X_2$) like at the previous 'odd approximations' 0.5PN and 1.5PN, but there is also an extra contribution, denoted by $\rho^{(5/2)}_{ij}$ in equation (5.1). This consists of two parts: $\big(\rho^{(5/2)}_{ij}\big)_{\rm reac} = -\frac{64}{5}\nu\frac{Gm}{r}\frac{\gamma^2}{c}n^{(i}v^{j)}$ which arises directly from the 2.5PN radiation-reaction term in the EOM given by equation (2.2), and $\big(\rho^{(5/2)}_{ij}\big)_{\rm quad} = -\frac{192}{7}\nu\frac{Gm}{r}\frac{\gamma^2}{c}n^{(i}v^{j)}$ which comes from the 2.5PN contribution in the mass quadrupole (3.16*a*). We find

$$\rho^{(5/2)}_{ij} = -\frac{1408}{35}\nu\frac{Gm}{r}\frac{\gamma^2}{c}n^{(i}v^{j)}. \quad (5.2)$$

The other contributions follow from a long but straightforward computation starting from the multipole moments listed earlier, and read

$$
\begin{aligned}
\xi_{ij}^{(5/2)} = &\Bigg\{ n^{ij}\left[\frac{\gamma}{c^3}\left\{\left(\frac{1}{3}-\frac{4}{3}\nu+\nu^2\right)(\mathbf{N}\cdot\mathbf{v})^5\right\}+\frac{\gamma^2}{c}\left\{\left(\frac{1199}{180}-\frac{539}{45}\nu-\frac{101}{60}\nu^2\right)(\mathbf{N}\cdot\mathbf{v})^3\right.\right.\\
&+\left.\left(\frac{263}{10}-\frac{526}{5}\nu+\frac{789}{10}\nu^2\right)(\mathbf{N}\cdot\mathbf{n})^2(\mathbf{N}\cdot\mathbf{v})^3\right\}\\
&+\frac{Gm}{r}\frac{\gamma^2}{c}\left\{\left(-\frac{263}{72}+\frac{553}{90}\nu+\frac{17}{120}\nu^2\right)(\mathbf{N}\cdot\mathbf{v})\right.\\
&+\left(-\frac{757}{12}+\frac{8237}{60}\nu-\frac{433}{20}\nu^2\right)(\mathbf{N}\cdot\mathbf{n})^2(\mathbf{N}\cdot\mathbf{v})\\
&+\left.\left.\left(-\frac{2341}{72}+\frac{2341}{18}\nu-\frac{2341}{24}\nu^2\right)(\mathbf{N}\cdot\mathbf{n})^4(\mathbf{N}\cdot\mathbf{v})\right\}\right]\\
&+n^{(i}v^{j)}\left[\frac{\gamma}{c^3}\left\{\left(-\frac{74}{3}+\frac{296}{3}\nu-74\nu^2\right)(\mathbf{N}\cdot\mathbf{n})(\mathbf{N}\cdot\mathbf{v})^4\right\}\right.\\
&+\frac{\gamma^2}{c}\left\{\left(\frac{5161}{90}-\frac{5612}{45}\nu+\frac{461}{30}\nu^2\right)(\mathbf{N}\cdot\mathbf{n})(\mathbf{N}\cdot\mathbf{v})^2\right.\\
&+\left.\left(\frac{1811}{15}-\frac{7244}{15}\nu+\frac{1811}{5}\nu^2\right)(\mathbf{N}\cdot\mathbf{n})^3(\mathbf{N}\cdot\mathbf{v})^2\right\}\\
&+\frac{Gm}{r}\frac{\gamma^2}{c}\left\{\left(-\frac{479}{60}+\frac{187}{6}\nu-\frac{9}{4}\nu^2\right)(\mathbf{N}\cdot\mathbf{n})\right.\\
&+\left(-\frac{5587}{90}+\frac{1282}{9}\nu-\frac{65}{2}\nu^2\right)(\mathbf{N}\cdot\mathbf{n})^3\\
&+\left.\left.\left(-\frac{3787}{180}+\frac{3787}{45}\nu-\frac{3787}{60}\nu^2\right)(\mathbf{N}\cdot\mathbf{n})^5\right\}\right]+v^{ij}\left[\frac{1}{c^5}(2-8\nu+6\nu^2)(\mathbf{N}\cdot\mathbf{v})^5\right.\\
&+\frac{\gamma}{c^3}\left\{\left(-\frac{4}{3}+\frac{10}{3}\nu\right)(\mathbf{N}\cdot\mathbf{v})^3+\left(-\frac{158}{3}+\frac{632}{3}\nu-158\nu^2\right)(\mathbf{N}\cdot\mathbf{n})^2(\mathbf{N}\cdot\mathbf{v})^3\right\}\\
&+\frac{\gamma^2}{c}\left\{\left(-\frac{536}{45}-\frac{1531}{45}\nu-\frac{1}{15}\nu^2\right)(\mathbf{N}\cdot\mathbf{v})\right.\\
&+\left(\frac{1345}{36}-\frac{799}{9}\nu+\frac{245}{12}\nu^2\right)(\mathbf{N}\cdot\mathbf{n})^2(\mathbf{N}\cdot\mathbf{v})\\
&+\left.\left.\left(\frac{2833}{60}-\frac{2833}{15}\nu+\frac{2833}{20}\nu^2\right)(\mathbf{N}\cdot\mathbf{n})^4(\mathbf{N}\cdot\mathbf{v})\right\}\right]\Bigg\},
\end{aligned}
\tag{5.3}
$$

where we recall that $\mathbf{n} = \mathbf{x}/r$ and the parameter γ is defined by (2.1).

In addition, we must compute the instantaneous (r) and (c) parts of the wave form for the compact binaries in circular orbits. These parts are purely of the order of 2.5PN. The inst(c) part is computed starting from the expression for W in equation (3.18*a*), but it turns out to be zero for circular orbits. We find

$$
\begin{aligned}
\left(h_{km}^{\mathrm{TT}}\right)_{\mathrm{inst(r)}} = &\frac{2G\nu m}{c^4 R}\mathcal{P}_{ijkm}\frac{\nu\gamma^2}{c}\left\{-\frac{84}{5}\frac{Gm}{r}(\mathbf{N}\cdot\mathbf{n})(\mathbf{N}\cdot\mathbf{v})n^{ij}+28(\mathbf{N}\cdot\mathbf{n})(\mathbf{N}\cdot\mathbf{v})v^{ij}\right.\\
&+\left.\frac{Gm}{r}\left(\frac{64}{35}-\frac{192}{5}(\mathbf{N}\cdot\mathbf{n})^2\right)n^{(i}v^{j)}+16(\mathbf{N}\cdot\mathbf{v})^2 n^{(i}v^{j)}\right\},
\end{aligned}
\tag{5.4a}
$$

$$
\left(h_{km}^{\mathrm{TT}}\right)_{\mathrm{inst(c)}} = 0. \tag{5.4b}
$$

5.2. The complete plus and cross polarizations

We have computed all the five different contributions to the wave form contained in equation (3.10). This together with the results of [31] provides a complete 2.5PN accurate wave form for the circular orbit case. In this section, from the 2.5PN wave form, we present the result for the two gravitational-wave polarizations, extending a similar analysis at the 2PN order in [35].

The polarizations corresponding to the instantaneous terms are computed using equations (5.3) and (5.4), while those corresponding to the hereditary terms were obtained in equations (4.24), (4.37) and (4.38). As in the earlier work, these polarizations are represented in terms of the gauge invariant parameter $x \equiv (Gm\omega/c^3)^{2/3}$, where ω represents the orbital frequency of the circular orbit, accurate up to 2.5PN order. This requires the relation between γ and x, which has already been given in equation (2.4). The final form of the 2.5PN polarizations may now be written as,

$$h_{+,\times} = \frac{2Gm\nu x}{c^2 R}\left\{H^{(0)}_{+,\times} + x^{1/2}H^{(1/2)}_{+,\times} + xH^{(1)}_{+,\times} + x^{3/2}H^{(3/2)}_{+,\times} + x^2 H^{(2)}_{+,\times} + x^{5/2}H^{(5/2)}_{+,\times}\right\}. \tag{5.5}$$

In particular, we shall recover the 2PN results of [35]. However, for the comparison we have to employ the same phase variable as in [35], which means introducing an auxiliary phase variable ψ, shifted away from the actual orbital phase ϕ we have used up to now, by equation (5) of [35]. Furthermore, the phase ψ given in [35] is *a priori* adequate up to only the 2PN order, but we have proved it to be also correct at the higher 2.5PN order. Indeed, the motivation for the shift $\phi \longrightarrow \psi$ is to 'remove' all the logarithms of the frequency (i.e. $\ln\omega$ or, rather, $\ln(\omega/\omega_0)$) in the two polarization wave forms and to absorb them into the definition of the new phase angle ψ. As a result, the two polarization wave forms, when expressed in terms of ψ instead of ϕ, become substantially simpler. From equation (4.37), we see that if we re-express the wave form by means of the phase [35]

$$\psi = \phi - 2x^{3/2}\ln\left(\frac{\omega}{\omega_0}\right), \tag{5.6}$$

we are able to move all the $\ln\omega$-terms into the phase angle. Note that the possibility of this move is interesting because it shows that the $\ln\omega$-terms, which were originally computed as some modification of the wave *amplitude* at orders 1.5PN, 2PN and 2.5PN, now appear as a modulation of the *phase* of the wave at the relative orders 4PN, 4.5PN and 5PN. The reason is that the lowest-order phase evolution is at the inverse of the order of radiation reaction, i.e. $c^5 = \mathcal{O}(-5)$, so as usual there is a difference of 2.5PN order between amplitude and phase. This shows therefore that the modification of the phase in (5.6) is presently negligible (it is of the same order of magnitude as unknown 4PN terms in the orbital phase evolution when it is given as a function of time). It could be ignored in practice, but it is probably better to keep it as it stands into the definition of templates of ICBs. The phase shift (5.6) corresponds to some spreading of the different frequency components of the wave, i.e. the 'wave packets' composing it, along the line of sight from the source to the detector (see [80] for a discussion).

With this above choice of phase variable, the same as in [35], all terms up to 2PN match with those listed in equations (3) and (4) of [35], though we recast them in a slightly different form for our convenience to present the 2.5PN terms. We find,

$$H^{(0)}_{+} = -\left(1+c_i^2\right)\cos 2\psi - \frac{1}{96}s_i^2\left(17+c_i^2\right), \tag{5.7a}$$

$$H^{(0.5)}_{+} = -s_i\frac{\delta m}{m}\left[\cos\psi\left(\frac{5}{8}+\frac{1}{8}c_i^2\right) - \cos 3\psi\left(\frac{9}{8}+\frac{9}{8}c_i^2\right)\right], \tag{5.7b}$$

$$H_+^{(1)} = \cos 2\psi \left[\frac{19}{6} + \frac{3}{2}c_i^2 - \frac{1}{3}c_i^4 + \nu\left(-\frac{19}{6} + \frac{11}{6}c_i^2 + c_i^4\right)\right] - \cos 4\psi \left[\frac{4}{3}s_i^2(1+c_i^2)(1-3\nu)\right], \tag{5.7c}$$

$$H_+^{(1.5)} = s_i\frac{\delta m}{m}\cos\psi\left[\frac{19}{64} + \frac{5}{16}c_i^2 - \frac{1}{192}c_i^4 + \nu\left(-\frac{49}{96} + \frac{1}{8}c_i^2 + \frac{1}{96}c_i^4\right)\right] + \cos 2\psi\left[-2\pi(1+c_i^2)\right] + s_i\frac{\delta m}{m}\cos 3\psi\left[-\frac{657}{128} - \frac{45}{16}c_i^2 + \frac{81}{128}c_i^4 + \nu\left(\frac{225}{64} - \frac{9}{8}c_i^2 - \frac{81}{64}c_i^4\right)\right] + s_i\frac{\delta m}{m}\cos 5\psi\left[\frac{625}{384}s_i^2(1+c_i^2)(1-2\nu)\right], \tag{5.7d}$$

$$H_+^{(2)} = \pi s_i\frac{\delta m}{m}\cos\psi\left[-\frac{5}{8} - \frac{1}{8}c_i^2\right] + \cos 2\psi\left[\frac{11}{60} + \frac{33}{10}c_i^2 + \frac{29}{24}c_i^4 - \frac{1}{24}c_i^6 + \nu\left(\frac{353}{36} - 3c_i^2 - \frac{251}{72}c_i^4 + \frac{5}{24}c_i^6\right) + \nu^2\left(-\frac{49}{12} + \frac{9}{2}c_i^2 - \frac{7}{24}c_i^4 - \frac{5}{24}c_i^6\right)\right] + \pi s_i\frac{\delta m}{m}\cos 3\psi\left[\frac{27}{8}(1+c_i^2)\right] + \cos 4\psi\left[\frac{118}{15} - \frac{16}{5}c_i^2 - \frac{86}{15}c_i^4 + \frac{16}{15}c_i^6 + \nu\left(-\frac{262}{9} + 16c_i^2 + \frac{166}{9}c_i^4 - \frac{16}{3}c_i^6\right) + \nu^2\left(14 - 16c_i^2 - \frac{10}{3}c_i^4 + \frac{16}{3}c_i^6\right)\right] + \cos 6\psi\left[-\frac{81}{40}s_i^4(1+c_i^2)(1-5\nu+5\nu^2)\right] + s_i\frac{\delta m}{m}\sin\psi\left[\frac{11}{40} + \frac{5\ln 2}{4} + c_i^2\left(\frac{7}{40} + \frac{\ln 2}{4}\right)\right] + s_i\frac{\delta m}{m}\sin 3\psi\left[\left(-\frac{189}{40} + \frac{27}{4}\ln(3/2)\right)(1+c_i^2)\right], \tag{5.7e}$$

$$H_\times^{(0)} = -2c_i\sin 2\psi, \tag{5.8a}$$

$$H_\times^{(0.5)} = s_i c_i\frac{\delta m}{m}\left[-\frac{3}{4}\sin\psi + \frac{9}{4}\sin 3\psi\right], \tag{5.8b}$$

$$H_\times^{(1)} = c_i\sin 2\psi\left[\frac{17}{3} - \frac{4}{3}c_i^2 + \nu\left(-\frac{13}{3} + 4c_i^2\right)\right] + c_i s_i^2\sin 4\psi\left[-\frac{8}{3}(1-3\nu)\right], \tag{5.8c}$$

$$H_\times^{(1.5)} = s_i c_i\frac{\delta m}{m}\sin\psi\left[\frac{21}{32} - \frac{5}{96}c_i^2 + \nu\left(-\frac{23}{48} + \frac{5}{48}c_i^2\right)\right] - 4\pi c_i\sin 2\psi + s_i c_i\frac{\delta m}{m}\sin 3\psi\left[-\frac{603}{64} + \frac{135}{64}c_i^2 + \nu\left(\frac{171}{32} - \frac{135}{32}c_i^2\right)\right] + s_i c_i\frac{\delta m}{m}\sin 5\psi\left[\frac{625}{192}(1-2\nu)s_i^2\right], \tag{5.8d}$$

$$H_\times^{(2)} = s_i c_i\frac{\delta m}{m}\cos\psi\left[-\frac{9}{20} - \frac{3}{2}\ln 2\right] + s_i c_i\frac{\delta m}{m}\cos 3\psi\left[\frac{189}{20} - \frac{27}{2}\ln(3/2)\right] - s_i c_i\frac{\delta m}{m}\left[\frac{3\pi}{4}\right]\sin\psi + c_i\sin 2\psi\left[\frac{17}{15} + \frac{113}{30}c_i^2 - \frac{1}{4}c_i^4\right.$$

$$+\nu\left(\frac{143}{9}-\frac{245}{18}c_i^2+\frac{5}{4}c_i^4\right)+\nu^2\left(-\frac{14}{3}+\frac{35}{6}c_i^2-\frac{5}{4}c_i^4\right)\Bigg]$$
$$+s_i c_i\frac{\delta m}{m}\sin 3\psi\left[\frac{27\pi}{4}\right]+c_i\sin 4\psi\left[\frac{44}{3}-\frac{268}{15}c_i^2+\frac{16}{5}c_i^4\right.$$
$$\left.+\nu\left(-\frac{476}{9}+\frac{620}{9}c_i^2-16c_i^4\right)+\nu^2\left(\frac{68}{3}-\frac{116}{3}c_i^2+16c_i^4\right)\right]$$
$$+c_i\sin 6\psi\left[-\frac{81}{20}s_i^4(1-5\nu+5\nu^2)\right]. \tag{5.8e}$$

Note a difference with the results of [35], in that we have included the specific effect of nonlinear memory into the polarization wave form at the *Newtonian* order, cf the term proportional to $s_i^2(17+c_i^2)$ in $H_+^{(0)}$ given by (5.7*a*). This is consistent with the order of magnitude of this effect, calculated in section 4.2. However, note that the memory effect is rather sensitive to the details of the whole time evolution of the binary prior to the current detection, so the zero-frequency (dc) term we have included in (5.7*a*) may change depending on the binary past history (see our discussion in section 4.2). Nevertheless, we feel that it is a good point to include the 'Newtonian' nonlinear memory effect, exactly as it is given in equation (5.7*a*), for the detection and analysis of ICBs[17].

The purely 2.5PN contributions, in the plus and cross polarizations, constitute, together with the memory term in (5.7*a*), the final result of this paper. They read,

$$H_+^{(2.5)}=s_i\frac{\delta m}{m}\cos\psi\left[\frac{1771}{5120}-\frac{1667}{5120}c_i^2+\frac{217}{9216}c_i^4-\frac{1}{9216}c_i^6\right.$$
$$+\nu\left(\frac{681}{256}+\frac{13}{768}c_i^2-\frac{35}{768}c_i^4+\frac{1}{2304}c_i^6\right)$$
$$\left.+\nu^2\left(-\frac{3451}{9216}+\frac{673}{3072}c_i^2-\frac{5}{9216}c_i^4-\frac{1}{3072}c_i^6\right)\right]$$
$$+\pi\cos 2\psi\left[\frac{19}{3}+3c_i^2-\frac{2}{3}c_i^4+\nu\left(-\frac{19}{3}+\frac{11}{3}c_i^2+2c_i^4\right)\right]$$
$$+s_i\frac{\delta m}{m}\cos 3\psi\left[\frac{3537}{1024}-\frac{22\,977}{5120}c_i^2-\frac{15\,309}{5120}c_i^4+\frac{729}{5120}c_i^6\right.$$
$$+\nu\left(-\frac{23\,829}{1280}+\frac{5529}{1280}c_i^2+\frac{7749}{1280}c_i^4-\frac{729}{1280}c_i^6\right)$$
$$\left.+\nu^2\left(\frac{29\,127}{5120}-\frac{27\,267}{5120}c_i^2-\frac{1647}{5120}c_i^4+\frac{2187}{5120}c_i^6\right)\right]$$
$$+\cos 4\psi\left[-\frac{16\pi}{3}\left(1+c_i^2\right)s_i^2(1-3\nu)\right]$$
$$+s_i\frac{\delta m}{m}\cos 5\psi\left[-\frac{108\,125}{9216}+\frac{40\,625}{9216}c_i^2+\frac{83\,125}{9216}c_i^4-\frac{15\,625}{9216}c_i^6\right.$$
$$+\nu\left(\frac{8125}{256}-\frac{40\,625}{2304}c_i^2-\frac{48\,125}{2304}c_i^4+\frac{15\,625}{2304}c_i^6\right)$$
$$\left.+\nu^2\left(-\frac{119\,375}{9216}+\frac{40\,625}{3072}c_i^2+\frac{44\,375}{9216}c_i^4-\frac{15\,625}{3072}c_i^6\right)\right]$$

[17] We already remarked that we have not computed the dc terms possibly present in the higher-order harmonics of the 2.5PN wave form.

$$
\begin{aligned}
&+\frac{\delta m}{m}\cos 7\psi\left[\frac{117\,649}{46\,080}s_i^5\left(1+c_i^2\right)(1-4\nu+3\nu^2)\right]\\
&+\sin 2\psi\left[-\frac{9}{5}+\frac{14}{5}c_i^2+\frac{7}{5}c_i^4+\nu\left(\frac{96}{5}-\frac{8}{5}c_i^2-\frac{28}{5}c_i^4\right)\right]\\
&+s_i^2\left(1+c_i^2\right)\sin 4\psi\left[\frac{56}{5}-\frac{32\ln 2}{3}-\nu\left(\frac{1193}{30}-32\ln 2\right)\right].
\end{aligned}
\tag{5.9}
$$

$$
\begin{aligned}
H_\times^{(2.5)} = {}&\frac{6}{5}s_i^2c_i\nu+c_i\cos 2\psi\left[2-\frac{22}{5}c_i^2+\nu\left(-\frac{154}{5}+\frac{94}{5}c_i^2\right)\right]\\
&+c_is_i^2\cos 4\psi\left[-\frac{112}{5}+\frac{64}{3}\ln 2+\nu\left(\frac{1193}{15}-64\ln 2\right)\right]\\
&+s_ic_i\frac{\delta m}{m}\sin\psi\left[-\frac{913}{7680}+\frac{1891}{11\,520}c_i^2-\frac{7}{4608}c_i^4\right.\\
&\left.+\nu\left(\frac{1165}{384}-\frac{235}{576}c_i^2+\frac{7}{1152}c_i^4\right)+\nu^2\left(-\frac{1301}{4608}+\frac{301}{2304}c_i^2-\frac{7}{1536}c_i^4\right)\right]\\
&+\pi c_i\sin 2\psi\left[\frac{34}{3}-\frac{8}{3}c_i^2-\nu\left(\frac{26}{3}-8c_i^2\right)\right]+s_ic_i\frac{\delta m}{m}\sin 3\psi\left[\frac{12\,501}{2560}\right.\\
&-\frac{12\,069}{1280}c_i^2+\frac{1701}{2560}c_i^4+\nu\left(-\frac{19\,581}{640}+\frac{7821}{320}c_i^2-\frac{1701}{640}c_i^4\right)\\
&\left.+\nu^2\left(\frac{18\,903}{2560}-\frac{11\,403}{1280}c_i^2+\frac{5103}{2560}c_i^4\right)\right]+s_i^2c_i\sin 4\psi\left[-\frac{32\pi}{3}(1-3\nu)\right]\\
&+\frac{\delta m}{m}s_ic_i\sin 5\psi\left[-\frac{101\,875}{4608}+\frac{6875}{256}c_i^2-\frac{21\,875}{4608}c_i^4\right.\\
&+\nu\left(\frac{66\,875}{1152}-\frac{44\,375}{576}c_i^2+\frac{21\,875}{1152}c_i^4\right)\\
&\left.+\nu^2\left(-\frac{100\,625}{4608}+\frac{83\,125}{2304}c_i^2-\frac{21\,875}{1536}c_i^4\right)\right]\\
&+\frac{\delta m}{m}s_i^5c_i\sin 7\psi\left[\frac{117\,649}{23\,040}(1-4\nu+3\nu^2)\right].
\end{aligned}
\tag{5.10}
$$

Note that the latter cross polarization contains a zero-frequency term (first term in equation (5.10)), which comes from the inst(r) contribution given by (5.4). We employ the same notation as in [35], except that c_i and s_i denote respectively the cosine and sine of the inclination angle i (which is defined as the angle between the vector $\mathbf{N}$, along the line of sight from the binary to the detector, and the normal to the orbital plane, chosen to be right handed with respect to the sense of motion, so that $0 \leqslant i \leqslant \pi$). In particular, the mass difference reads $\delta m = m_1 - m_2$. Like in [35], our results are in terms of the phase variable ψ defined by (5.6) in function of the actual orbital phase ϕ (namely the angle oriented in the sense of motion between the ascending node $\mathcal{N}$ and direction of body one—i.e. $\phi = 0$ mod 2π when the two bodies lie along $\mathbf{P}$). We have verified that the plus and cross polarizations (5.9)–(5.10) reduce in the limit $\nu \to 0$ to the result of black hole perturbation theory as given in appendix B of Tagoshi and Sasaki [59] (the phase variable used in [59] differs from ours by $\psi_{\rm TS} = \psi + \pi/2 + 2x^{3/2}[\ln 2 - 17/12]$ and we have $\theta_{\rm TS} = \pi - i$).[18]

[18] We spotted a misprint in appendix B of [59], namely the sign of the harmonic coefficient $\zeta^\times_{7,3}$ (i.e. having $l = 7$, $m = 3$, and corresponding to the cross polarization) should be changed, so that one should read $\zeta^\times_{7,3} = +\frac{729}{10\,250\,240}\cos\theta(167+\cdots)\sin\theta(v^5\cos 3\psi - \cdots)$.

Equations (5.9) and (5.10), together with (5.7) and (5.8), provide the 2.5PN accurate template for the ICBs moving on quasi-circular orbits, extending the results of [35] by half a PN order. They are complete except for the possible inclusion of memory-type (zero-frequency or dc) contributions in higher PN amplitudes (1PN and 2PN). These wave polarizations together with the phasing formula of [54], i.e. the crucial time variation of the phase: $\phi(t)$, constitute the currently best available templates for the data analysis of ICB for ground based as well as space-borne GW interferometers.

5.3. Comments on the 3PN wave form

In section 3.3, we have given the list of source multipole moments needed for controlling the wave form at the 2.5PN order. The computation of the 3PN wave form obviously requires more accurate versions of these moments as well as new moments, which all together would constitute the basis of the computation of the 3PN wave form. Thus, even though the 3PN mass quadrupole moment [40] and 3PN accurate EOM [41–50] are available, our present level of accuracy is *not* sufficient enough to compute the 3PN wave form[19].

The source multipole moments at 3PN order would yield the control of the inst(s) part of the wave form, as well as the tail terms, but we have also to consider other types of contributions, which are not all under control. The main reason for the present incompleteness at 3PN order is that the instantaneous terms of type (r) and (c), generalizing (3.12) and (3.13) to the 3PN order, are not computed.

Recall that the inst(r) terms are those whose sum constitutes the instantaneous part of the relationships between the radiative moments U_L, V_L and the 'canonical' ones M_L, S_L, see (3.4) and (3.5). Though one can guess the structure of these terms at the 3PN order, using dimensional and parity arguments, the numerical coefficients in front of each of them require detailed (and generally tedious) work. For instance, in the 3PN wave form we shall need the radiative mass-type octupole moment U_{ijk} at the 2.5PN order, and therefore we have to know what is the remainder term $\mathcal{O}(5)$ in equation (3.4*b*) which we do not (such a calculation would notably entail controlling the quadratic interactions between one mass and one current quadrupole, $M_{ij} \times S_{kl}$, and between one mass quadrupole and one octupole, $M_{ij} \times M_{klm}$). And similarly for the radiative current-type quadrupole V_{ij} given by (3.5*a*). We have to also compute the remainder terms $\mathcal{O}(3)$ in the corresponding expressions of U_{ijkl} and V_{ijk}.

Concerning the inst(c) terms, which are the instantaneous terms coming from the difference between the canonical moments M_L, S_L and the general source ones I_L, $J_L, \ldots$ (cf equations (3.8) and (3.9)), it does not seem to be obvious to guess even their structure. The crucial new input we would need at 3PN order concerns the relation between the canonical mass octupole M_{ijk} and current quadrupole S_{ij} to the corresponding source moments I_{ijk} and J_{ij} at 2.5PN order, using for instance an analysis similar to the one in [36].

Finally, at 3PN order we would have to extend the present computation of hereditary terms. In the case of quadratic tails, like in (3.14), the computation would probably be straightforward (indeed we have seen in section 4.3 that the complications due to the influence of the model of adiabatic inspiral in the past appear only at the 4PN order), but we have also to take into account the tail-of-tail cubic contribution in the mass-quadrupole moment at 3PN order, given in equation (4.13) of [66]. In addition, the analysis should be extended to the nonlinear memory terms. The complete 3PN wave form and polarizations can be computed only after all the points listed above are addressed.

[19] We are speaking here of the 3PN *wave form*. The computation of the 3PN *flux* is less demanding, because each multipolar order brings in a new factor $c^{-2} = \mathcal{O}(2)$ instead of $\mathcal{O}(1)$ in the case of the wave form, which explains why it is possible to control it up to the 3.5PN order in [40].

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