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Third post-Newtonian dynamics of compact binaries: Noetherian conserved quantities and equivalence between the harmonic-coordinate and ADM-Hamiltonian formalisms

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Abstract

A Lagrangian from which one can derive the third post-Newtonian (3PN) equations of motion of compact binaries (neglecting the radiation reaction damping) is obtained. The 3PN equations of motion were computed previously by Blanchet and Faye in harmonic coordinates. The Lagrangian depends on the harmonic-coordinate positions, velocities and accelerations of the two bodies. At the 3PN order, the appearance of one undetermined physical parameter λ reflects the incompleteness of the point-mass regularization used when deriving the equations of motion. In addition the Lagrangian involves two unphysical (gauge-dependent) constants r_1' and r_2' parametrizing some logarithmic terms. The expressions of the ten Noetherian conserved quantities, associated with the invariance of the Lagrangian under the Poincaré group, are computed. By performing an infinitesimal 'contact' transformation of the motion, we prove that the 3PN harmonic-coordinate Lagrangian is physically equivalent to the 3PN Arnowitt–Deser–Misner Hamiltonian obtained recently by Damour, Jaranowski and Schäfer.

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1. Motivation and relation to other works

The long-standing problem of the gravitational dynamics of compact bodies has become very important in recent years because of the need to construct accurate templates for detecting the gravitational waves from inspiralling compact binaries in future experiments such as LIGO and VIRGO [1–3]. Concerning the two-body problem, the current state of the art is the third post-Newtonian (3PN) approximation, corresponding to the inclusion of all the relativistic corrections up to the order $1/c^6$ (where c is the velocity of light) with respect to the Newtonian acceleration. Up to the 2.5PN or $1/c^5$ approximation the equations of motion are well known,

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as they have been derived by many different methods with complete agreement on the result [4–17]. They have already been used for constructing the 2.5PN-accurate templates of inspiralling compact binaries [18–20].

To the 3PN order, the problem of equations of motion has been pursued by two groups working independently with different methods: on one hand, Jaranowski and Schäfer [21, 22] and Damour, Jaranowski and Schäfer [23–25] employ the Arnowitt–Deser–Misner (ADM) Hamiltonian formulation of general relativity; on the other hand, Blanchet and Faye [26–29] work iteratively with the Einstein field equations in harmonic coordinates. Both groups use a regularization based on Hadamard's concept of 'partie finie' to overcome the problem of the infinite self-field of point-like particles. However, the details are actually different; notably the second group developed for this problem an extended version of the Hadamard regularization and a theory of generalized functions [27, 28]. Both groups found that there remains one and only one physical constant, ω_{static} in the ADM-Hamiltonian formalism [21–25] and λ in the harmonic-coordinate approach [26–29], that is left undetermined by the point-mass regularization. Furthermore, in the harmonic-coordinate approach, the equations of motion (obtained in [29]) depend on two additional constants r'_1 and r'_2 parametrizing some logarithmic terms, but these constants are not physical in the sense that they can be removed by a coordinate transformation. The aim of the present paper is threefold:

(a) to present the Lagrangian of the 3PN dynamics of the compact binary in harmonic coordinates;
(b) to obtain explicitly from it the ten Noetherian conserved integrals of the motion in harmonic coordinates;
(c) to exhibit a contact transformation of the harmonic-coordinate motion to some pseudo-ADM coordinates in order to compare our results [26–29] with those obtained by the other group [21–25].

Concerning (a), we find a generalized Lagrangian (i.e. depending on the positions, velocities and accelerations of the bodies) whose variation yields the conservative part of the 3PN equations of motion in harmonic coordinates as found in [29]. Our second point (b) is to use the fact that the Lagrangian incorporates the ten symmetries of the Poincaré group (notably the boost symmetry) to compute the ten integrals corresponding to the energy, the linear and angular momenta, and the centre-of-mass position. In particular, we find that the energy agrees with the previous result of [29]. As all of these integrals will probably be needed in future work we choose to display them explicitly, despite the length of the expressions. We also give the balance equations they satisfy when the radiation reaction effect is turned on. Finally, the result of point (c) is that there exists a unique contact transformation of the harmonic-coordinate dynamical variables that changes the generalized Lagrangian into an ordinary Lagrangian (depending on positions and velocities) whose associated 3PN Hamiltonian matches exactly that given by Damour *et al* [24]. This proves the complete equivalence of the results obtained from the two (rather different) methods followed by the two groups, and constitutes a strong support of the validity of both methods. This equivalence has also been shown independently by the other group [25] (who also presents the formulae needed for computing the conserved quantities). Note that it holds if and only if the undetermined constant λ in the harmonic-coordinate formalism and the ambiguity constant ω_{static} in the ADM Hamiltonian are related to each other by

$$\omega_{\text{static}} = -\tfrac{11}{3}\lambda - \tfrac{1987}{840}, \tag{1.1}$$

a result already obtained in [26] on the basis of the comparison of the invariant energy of binaries moving on circular orbits. The appearance of the unknown constant λ is probably not due to a real physical ambiguity, but is associated with an incompleteness of the point-mass regularization. It is probably related to the fact that, starting from the 3PN order, many separate integrals constituting the equations of motion of extended bodies would depend on the internal structure of the objects (e.g. their density profile), even in the limiting case where the radius of the objects tends to zero. Further work is needed to compute the precise value of λ. On the other hand, the constants r_1' and r_2' occurring in the harmonic-coordinate Lagrangian disappear from the ADM-Hamiltonian (where there are no logarithms), in accordance with the fact that they are pure gauge.

The plan of this paper is as follows. In section 2, motivated by the striking equivalence between the (regularization-related) unknown constants λ and $\omega_{\rm static}$, we discuss our method of point-mass regularization and contrast it with the method advocated in [21–25]. Section 3 is devoted to the theoretical investigations. First, we recall the theory of Noetherian conserved quantities in the case of a generalized Lagrangian, and next we show how to eliminate the accelerations in the harmonic-coordinate Lagrangian by a contact transformation at the 3PN order. The reader interested only in the results at the 3PN order can go directly to section 4, where we present the closed-form expressions of the Lagrangian and the conserved energy, momenta and centre of mass in harmonic coordinates, and give the result for the contact transformation as well as the final expressions for the Lagrangian and Hamiltonian in pseudo-ADM coordinates.

2. Discussion on the point-mass regularization

The equivalence between the respective formalisms of [21–25] and [26–29] is interesting because the two groups have adopted some different approaches regarding the point-mass regularization (chosen in both cases to be based on the Hadamard concept of 'partie finie' of a singular function or a divergent integral [30, 31]). Essentially, the group [21–25] introduced systematically some 'ambiguity' parameters in the ADM Hamiltonian whenever the standard Hadamard regularization yielded inconsistent results, while the group [26–29] looked for the most general solution allowed by some basic physical requirements and following from a new, mathematically consistent, Hadamard-type regularization.

More precisely, in our approach [26–29], we adopted some variants of the Hadamard regularization which were devised specifically for this problem [27, 28]. Let F be a function which is singular at two isolated points $\boldsymbol{y}_1$ and $\boldsymbol{y}_2$, and is smooth everywhere else; $\boldsymbol{y}_1$ and $\boldsymbol{y}_2$ are the positions of the particles in harmonic coordinates at some given instant t. The Hadamard partie finie of F at the point $\boldsymbol{y}_1$, denoted by $(F)_1$, is defined as the angular average over all directions of approach to $\boldsymbol{y}_1$ of the finite term (zeroth order) in the singular expansion of the function around this point. We found that this definition yields a natural extension of the notion of a Dirac distribution at the location of a singular point, that we constructed by means of the Riesz delta-function [32]. As a result, the 'partie finie delta-function' at point 1, denoted Pf δ_1 where $\delta_1 \equiv \delta(\boldsymbol{x}-\boldsymbol{y}_1)$, is the linear form defined on the set of singular functions of type F, that associates to any F the real number $(F)_1$ (see equation (6.9) in [27]). Using an integral notation this means that $\int \mathrm{d}^3\boldsymbol{x}\, F \cdot \mathrm{Pf}\,\delta_1 = (F)_1$. (The partie finie delta-function Pf δ_1 constitutes a mathematically well defined version of the so-called 'good delta function' of Infeld [33].) In our derivation of the equations of motion at 3PN order, this prescription is employed systematically to compute all the 'compact-support' integrals, whose integrand is made of the product of a singular potential with some mass density localized on the two particle worldlines.

By applying the latter definition to the product FG we obtain $\int \mathrm{d}^3\boldsymbol{x}\, FG \cdot \mathrm{Pf}\,\delta_1 = (FG)_1$, which permits us to give a sense to the more complicated object $F \cdot \mathrm{Pf}\,\delta_1 \equiv \mathrm{Pf}(F\delta_1)$, composed of the product of a delta-pseudo-function with a function which is singular on its support (such a product being ill-defined in the standard distribution theory). Namely, $\mathrm{Pf}(F\delta_1)$ is the linear form which associates to any function G the real number $(FG)_1$. It is important to realize that $\mathrm{Pf}(F\delta_1) \neq (F)_1\,\mathrm{Pf}\,\delta_1$ in general. This is an immediate consequence of the so-called 'non-distributivity' of the Hadamard partie finie, namely the fact that $(FG)_1 \neq (F)_1(G)_1$ for two singular functions F and G in general. As an example taken from [17], we have $(U^4)_1 = [(U)_1]^4 + 2[(U)_1]^2[(U)_2]^2$, where $U = Gm_1/r_1 + Gm_2/r_2$ denotes the Newtonian potential of two particles (with $r_1 = |\boldsymbol{x} - \boldsymbol{y}_1|$ and $r_2 = |\boldsymbol{x} - \boldsymbol{y}_2|$). In the post-Newtonian iteration one can check that the functions involved become singular enough so that the non-distributivity plays an actual role at the 3PN order: for instance, in the example above, U^4 will appear in the metric coefficient g_{00} with a factor $1/c^8$ in front, which indeed corresponds to the 3PN order. However, there is no problem linked with the non-distributivity in the equations of motion up to the 2.5PN approximation [17]. Therefore, from the 3PN order (but only from that order), it is a mathematically inconsistent regularization prescription to assume at once that $\int \mathrm{d}^3\boldsymbol{x}\, F \cdot \mathrm{Pf}\,\delta_1 = (F)_1$ and $\mathrm{Pf}(F\delta_1) = (F)_1\,\mathrm{Pf}\,\delta_1$. Faced with this problem, the authors [21–25] have advocated that the breakdown of the distributivity of the Hadamard regularization at the 3PN order is a source of ambiguities. (Actually, in their first paper, see appendix A in [21], these authors performed their basic computation using the inconsistent rule $\mathrm{Pf}(F\delta_1) = (F)_1\,\mathrm{Pf}\,\delta_1$. Later in [23] (see appendix A therein), they argued that their result was 'stable' against a possible violation of the latter rule). In contrast, the authors [26–29] have accepted the special features of the partie finie, such as its non-distributivity, and constructed by its mean a mathematically consistent regularization, which is able to give a precise sense to all computations at the 3PN order.

The Hadamard partie finie $(F)_1$ of a singular function involves a spherical average that is defined within the spatial hypersurface $t =$ constant of a global coordinate system such as the harmonic coordinates. Clearly, this definition is incompatible with the framework of a relativistic field theory, and we expect at some level a violation of the Lorentz invariance of the equations of motion due to this regularization. Remarkably, such a violation occurs only at the 3PN order; up to the 2.5PN order the equations of motion in harmonic coordinates, as computed using the regularization $(F)_1$, are Lorentz-invariant [17]. To overcome this problem at the 3PN order, it has been necessary to define a 'Lorentzian' regularization [28], which consists merely of applying the Hadamard partie finie within the spatial hypersurface orthogonal to the (Minkowskian) 4-velocity of a particle. It was shown in [29] that the Lorentzian regularization adds some new terms to the 3PN equations of motion (computed with the standard regularization $(F)_1$) which are mandatory in order to maintain their Lorentz invariance (see, for instance, equation (5.35) in [29]). The Lorentzian partie finie of a singular function F, denoted by $[F]_1$, enables one to define a 'Lorentzian' partie finie delta-function $\mathrm{Pf}\,\Delta_1$, namely a linear form whose action on any F gives the real number $[F]_1$. It also permits the precise definition, given by equation (5.11) in [28], of a model for the stress–energy tensor of point-particles in (post-Newtonian expansions of) general relativity.

Besides the compact-support integrals computed before, the equations of motion contain many 'non-compact' integrals, whose support extends up to infinity and which are divergent at the location of the particles. To them we assign systematically the value given by the Hadamard partie-finie of a divergent integral: $\mathrm{Pf} \int \mathrm{d}^3\boldsymbol{x}\, F$, see equation (3.1) in [27]. Furthermore, to any F in this class, we associate the pseudo-function $\mathrm{Pf}\,F$ which by definition is the linear form whose action on any other G gives the real number $\mathrm{Pf} \int \mathrm{d}^3\boldsymbol{x}\, FG$. Given then two pseudo-functions, their product is chosen to be the 'ordinary' one $\mathrm{Pf}\,F \cdot \mathrm{Pf}\,G = \mathrm{Pf}(FG)$.

An important feature of the Hadamard partie-finie integral is that the integral of a gradient is not zero in general, $\mathrm{Pf} \int \mathrm{d}^3\boldsymbol{x}\, \partial_i F \neq 0$, since it is equal to the sums of the parties finies of the surface integrals surrounding the singularities when the surface areas tend to zero (see equation (3.4) in [27]). This means that the ordinary derivative of singular functions shows a fundamental difference with the case of regular sources, since in this case the integral of a gradient is always zero (provided that the integrand decreases sufficiently fast at infinity). One can check that some non-vanishing integrals of a gradient start to appear precisely at the 3PN order. Confronted with this problem, the authors [21–25] have considered that this signals the presence of ambiguities at the 3PN order, notably because their ADM-Hamiltonian density is defined only modulo a total divergence, that one certainly does not want to contribute even in the case of singular sources. On the other hand, the authors [26–29] have accepted this feature and introduced a new kind of (spatial or temporal) distributional derivative acting on the pseudo-functions of type $\mathrm{Pf}\,F$ (for instance, $\partial_i\,\mathrm{Pf}\,F$) in order to ensure that the integral of a gradient is always zero. It was found [27] that it is impossible to define a derivative which satisfies the Leibniz rule for the derivation of a product, i.e. $\partial_i(\mathrm{Pf}\,FG) \neq F\partial_i\,\mathrm{Pf}\,G + G\partial_i\,\mathrm{Pf}\,F$ in general, but that when one replaces the Leibniz rule by the weaker rule of 'integration by parts', an interesting mathematical structure exists. By the rule of integration by parts, we refer to the relation $\int \mathrm{d}^3\boldsymbol{x}\,[F\partial_i\,\mathrm{Pf}\,G + G\partial_i\,\mathrm{Pf}\,F] = 0$, for F and G arbitrary functions (see equation (7.2) in [27] where we use a more appropriate bracket notation for the spatial integral). While the rule of integration by parts is nothing but an integrated version of the 'pointwise' Leibniz rule, the Leibniz rule itself is a stronger requirement, which is not satisfied in general as there are triplets of singular functions F, G, H for which $\int \mathrm{d}^3\boldsymbol{x}\,H[F\partial_i\,\mathrm{Pf}\,G + G\partial_i\,\mathrm{Pf}\,F] \neq \int \mathrm{d}^3\boldsymbol{x}\,H\partial_i(\mathrm{Pf}\,FG)$. The motivation for requiring the rule of integration by parts is that it is clearly valid in the case of regular fluid systems. Notably it implies that the integral of a gradient of any singular function of type F is zero. However, because it violates the Leibniz rule, the distributional derivative cannot be completely satisfying from the physical point of view.

Actually, two different distributional derivatives, and therefore two different regularizations, were introduced in [27]. A 'particular' derivative, defined by equation (7.7) in [27], was first chosen for its simplicity. The two main properties of this derivative are: that (a) it reduces to the ordinary derivative, i.e. $\partial_i\,\mathrm{Pf}\,F = \mathrm{Pf}(\partial_i F)$, whenever F is bounded near the singularities (in addition to being smooth everywhere else) and (b) it obeys the rule of integration by parts. Though the particular derivative is especially convenient to use in practical computations, it does not follow from some 'unicity' theorem. A more interesting derivative, from the mathematical point of view, is the so-called 'correct' derivative (we follow the terminology of [29]) which *does* satisfy a unicity theorem. Namely, this derivative is obtained in theorem 4 of [27] as the *unique* derivative satisfying properties (a) and (b) above, and, in addition, (c) the rule of commutation of successive derivatives (Schwarz lemma). As it turned out, the 'correct' derivative, given by equation (8.12) of [27], depends on one arbitrary numerical constant K. (Note that both the particular and correct derivatives reduce to the derivative of the standard distribution theory [31] when applied to smooth test functions with compact support.)

In summary, it is possible to construct a consistent regularization based on the Hadamard partie finie, thus one can give a precise meaning to any integral encountered in the computation, but there are several possible prescriptions associated with different distributional derivatives (and the Leibniz rule is not satisfied). Our strategy has been to perform two computations of the equations of motion, associated, respectively, with the 'particular' and 'correct' derivatives. Then the following were shown [29].

(I) The 3PN equations of motion, when computed by means of the Lorentzian regularization and the particular derivative, are in agreement with the known equations of motion up to the 2.5PN order, have the correct test-mass limit and most importantly are Lorentz invariant (in a perturbative post-Newtonian sense).

(II) Looking for the most general solution, allowed by the regularization, for the 3PN equations of motion to admit a conserved energy and a Lagrangian description, we find that they depend on two unphysical gauge-constants r_1' and r_2' (associated with the appearance of logarithms), and on one and only one physical constant λ which cannot be determined within the method. The equations of motion possess all the physical properties that we expect, but the presence of the unknown constant λ is somewhat baffling, as it probably reflects a physical incompleteness of the regularization.

(III) When the correct distributional derivative is used instead of the particular one, the equations of motion depend on K in addition to r_1', r_2' and λ. In this case we find that they are no longer Lorentz invariant in general, but that there is a unique value of K for which the Lorentz invariance is recovered: $K = \frac{41}{160}$. For this value the equations of motion have also all the physical properties we expect.

(IV) The different equations of motion as obtained by means of the 'particular' and 'correct' prescriptions (with $K = \frac{41}{160}$ in the second case) are *physically equivalent* in the sense that they differ from each other by an infinitesimal change of coordinates. This satisfying result indicates that the distributional derivatives introduced in [27] constitute merely some technical tools which are devoid of physical meaning.

In scenario (III) one may wonder why after having used the Lorentzian regularization defined in [28] one still has to adjust the constant K to a certain value in order to finally obtain the Lorentz invariance. The likely reason is that the distributional derivatives we use (the particular and correct ones) have not been defined in a Lorentz-invariant way, as their distributional terms are made of the delta-pseudo-function Pf δ_1 instead of the 'Lorentzian' delta-pseudo-function Pf Δ_1 (see equation (3.36) in [28]). As a result, we find in scenario (III) that although most of the terms satisfy the requirement of Lorentz invariance, notably the terms proportional to the combination of masses $m_1^2 m_2$ in the acceleration of particle one (these terms are shown to behave correctly thanks to the Lorentzian regularization), there still exists a limited class of terms, proportional to m_2^3, that do not obey the Lorentz invariance unless K is adjusted to the value $\frac{41}{160}$. (In scenario (I) where there is no constant to adjust the latter terms behave correctly.)

The problem of the Lorentz invariance of the equations of motion was solved in a quite different way by the other group [21–25]. We recall that the harmonic-coordinate equations of motion are manifestly Lorentz-invariant because the harmonic gauge condition preserves the Poincaré symmetry. In contrast, the coordinate conditions associated with the ADM Hamiltonian formalism do not respect the Poincaré group, and therefore the authors [21–25] had to prove that their Hamiltonian is compatible with the existence of generators in phase space such that the usual Poincaré algebra is satisfied. More precisely, they constructed a generic 'ambiguous' dynamics at the 3PN order, parametrized by some unknown ambiguity parameters associated notably with the non-distributivity of the Hadamard partie finie and to the fact that the integral of a gradient, in an ordinary sense, is not zero. They showed that there were only two ambiguity parameters which they denoted by ω_{kinetic} and ω_{static}. (Actually, in the first paper [21] they considered only the ambiguity constant ω_{kinetic} and obtained the value $\omega_{\text{static}} = \frac{1}{8}$. The static ambiguity was introduced in the second paper [22].) By imposing in an *ad hoc* manner the existence of the Poincaré generators for their ambiguous Hamiltonian, they showed [24] that the parameter ω_{kinetic} is fixed uniquely to the value $\frac{41}{24}$. This result was, in fact, obtained earlier [26] by comparing their expression of the energy of circular orbits [23] to the expression we obtained by means of the explicitly Lorentz-invariant formalism described in scenario (I) above. Finally, having fixed ω_{kinetic}, there still remained in the ADM-Hamiltonian formalism one and only one undetermined constant ω_{static}, that we shall find to be equivalent, in the sense of equation (1.1), to the constant λ appearing in harmonic coordinates. (Note that, despite the resemblance between the value $K = \frac{41}{160}$ in scenario (III) and the result $\omega_{\text{kinetic}} = \frac{41}{24}$, the constant K can be fixed to this unique value only if the sophisticated Lorentzian regularization is used before. Without such a regularization, several other terms not parametrized by K would not behave correctly under Lorentz transformations, and therefore no value of K could be chosen in order to restore the Lorentz invariance. In this sense the constant K is more 'specialized' than the constant ω_{kinetic}.)

Finally, choosing one or other of the two approaches advocated in [21–29] for the regularization is a matter of taste. In view of the equivalence of the final results, it is a good state of affairs that the two approaches are different conceptually and technically.

3. Theory

3.1. Noetherian conserved quantities for a generalized Lagrangian

At the 1PN order, the equations of motion of two compact objects in general relativity, as derived in [4, 5], can be deduced from an ordinary Lagrangian, depending on the positions and velocities of the bodies, which were obtained by Fichtenholz [6]. At the next 2PN order, the equations of motion in harmonic coordinates, as obtained in [8, 10, 11], can only be deduced from a 'generalized' Lagrangian, depending not only on the positions and velocities but also on the accelerations of the particles [8]. In particular, this confirmed a result of Martin and Sanz [34] that N-body systems cannot admit an ordinary Lagrangian description beyond the 1PN order, provided that the gauge conditions preserve the Lorentz invariance (as is the case for the harmonic gauge). However, it has been shown by Damour and Schäfer [12] that there exists a special class of coordinates, which includes those associated with the ADM formalism, such that the Lagrangian at the 2PN order expressed by means of such coordinates becomes ordinary, i.e. no longer depends on accelerations. This means that we can eliminate the accelerations in the harmonic-coordinate Lagrangian at the 2PN order by going to the ADM coordinates [12]. In this paper, we shall find that the 3PN terms in the Lagrangian in harmonic coordinates also depend on accelerations, and that, like at the 2PN order, these accelerations can be eliminated by a suitable coordinate transformation to some 'pseudo-ADM' coordinates, following the general method of redefinition of position variables [12, 35–37].

Strictly speaking, the dynamics of two compact bodies does not derive from a Lagrangian at the 3PN approximation because of the radiation reaction damping effect at the previous 2.5PN order. When speaking of a 3PN Lagrangian or Hamiltonian, we always refer to the conservative part of the dynamics, which corresponds to the 'even' post-Newtonian orders 1PN, 2PN and 3PN. As we shall see, the radiation reaction effect manifests itself in the non-conservation at the 2.5PN approximation of the conserved quantities associated with the conservative 3PN dynamics (see equations (4.7)).

Let us consider a harmonic-coordinate generalized 3PN Lagrangian

$$L^{\text{harmonic}} \equiv L[\boldsymbol{y}_A(t), \boldsymbol{v}_A(t), \boldsymbol{a}_A(t)], \tag{3.1}$$

depending on the instantaneous positions $y^i_A(t) \equiv \boldsymbol{y}_A(t)$ (with $A = 1, 2$ and $i = 1, 2, 3$), coordinate velocities $v^i_A(t) \equiv \boldsymbol{v}_A(t) = \mathrm{d}\boldsymbol{y}_A/\mathrm{d}t$, as well as coordinate accelerations $a^i_A(t) \equiv \boldsymbol{a}_A(t) = \mathrm{d}\boldsymbol{v}_A/\mathrm{d}t$. Our harmonic-coordinate 3PN Lagrangian is given by (4.1) below, but we do not need to be so specific in the present section, where most of the results hold, in fact, for N-body systems ($A = 1, \ldots, N$). We assume that the dependence of the Lagrangian

(3.1) upon the accelerations is linear. As a matter of fact, it is always possible to eliminate from a generalized post-Newtonian Lagrangian a contribution quadratic in the accelerations by rewriting it in the form of a so-called 'double-zero' term, which does not contribute to the equations of motion, plus a term linear in the acceleration [12] (this argument can be extended to any term polynomial in the accelerations).

The equations of motion of the Ath body are deduced from the Lagrangian by taking the functional derivative defined as

$$\frac{\delta L}{\delta y_A^i} \equiv \frac{\partial L}{\partial y_A^i} - \frac{\mathrm{d}}{\mathrm{d}t}\left(\frac{\partial L}{\partial v_A^i}\right) + \frac{\mathrm{d}^2}{\mathrm{d}t^2}\left(\frac{\partial L}{\partial a_A^i}\right) = 0. \tag{3.2}$$

We consider first, very generally, an infinitesimal transformation of the path of the particle A at some instant t, i.e. $\delta \boldsymbol{y}_A(t) = \boldsymbol{y}'_A(t) - \boldsymbol{y}_A(t)$. The corresponding variations of its velocity and acceleration are $\delta \boldsymbol{v}_A(t) = \mathrm{d}\delta \boldsymbol{y}_A/\mathrm{d}t$ and $\delta \boldsymbol{a}_A(t) = \mathrm{d}\delta \boldsymbol{v}_A/\mathrm{d}t$. Such a transformation of the motion induces a variation of the Lagrangian, namely $\delta L = L[\boldsymbol{y}'_A, \boldsymbol{v}'_A, \boldsymbol{a}'_A] - L[\boldsymbol{y}_A, \boldsymbol{v}_A, \boldsymbol{a}_A]$ which is readily found to be expressible, at the linearized order in $\delta \boldsymbol{y}_A$, in the form

$$\delta L = \frac{\mathrm{d}Q}{\mathrm{d}t} + \sum_A \frac{\delta L}{\delta y_A^i}\delta y_A^i + \mathcal{O}\left(\delta \boldsymbol{y}_A^2\right), \tag{3.3}$$

where the functional derivative $\delta L/\delta y_A^i$ is given by (3.2) (it is zero 'on-shell', i.e. when the equations of motion are satisfied), and where we have introduced the total time derivative of a function $Q \equiv Q[\delta \boldsymbol{y}_A, \delta \boldsymbol{v}_A]$ defined by

$$Q = \sum_A \left(p_A^i \delta y_A^i + q_A^i \delta v_A^i\right). \tag{3.4}$$

Here, p_A^i and q_A^i denote the momenta that are conjugate to the positions y_A^i and velocities v_A^i of the particle A, respectively, that is

$$p_A^i = \frac{\delta L}{\delta v_A^i} \equiv \frac{\partial L}{\partial v_A^i} - \frac{\mathrm{d}}{\mathrm{d}t}\left(\frac{\partial L}{\partial a_A^i}\right), \tag{3.5a}$$

$$q_A^i = \frac{\delta L}{\delta a_A^i} \equiv \frac{\partial L}{\partial a_A^i}. \tag{3.5b}$$

We now discuss the Noetherian conservation laws for generalized Lagrangians following [9, 11]. We know from [29] that the 3PN equations of motion in harmonic coordinates are manifestly invariant (in a perturbative post-Newtonian sense) under the Lorentz and more generally the Poincaré group. Thus the dynamics associated with our 3PN generalized Lagrangian (4.1) should stay the same after an infinitesimal Poincaré transformation of the dynamical variables $y_A^\mu = (ct, \boldsymbol{y}_A)$. In particular, this means that $\delta L = 0$ in the case of arbitrary infinitesimal constant spatial translations and rotations, $\delta y_A^i = \epsilon^i$ and $\delta y_A^i = \omega^i{}_j y_A^j$ with $\omega_{ij} = -\omega_{ji}$. In this case equation (3.3) implies the conservation on-shell (all the $\delta L/\delta y_A^i$s are zero) of the Noetherian linear and angular momenta given by

$$P^i = \sum_A p_A^i, \tag{3.6a}$$

$$J^i = \varepsilon_{ijk} \sum_A \left(y_A^j p_A^k + v_A^j q_A^k\right). \tag{3.6b}$$

Thus, $\mathrm{d}P^i/\mathrm{d}t = 0$ and $\mathrm{d}J^i/\mathrm{d}t = 0$ on-shell. On the other hand, we have $\delta L = \tau \mathrm{d}L/\mathrm{d}t$ in the case of an infinitesimal constant time-translation $\delta t = \tau$, hence the conservation on-shell of the Noetherian energy from equation (3.3),

$$E = \sum_A \left(v_A^i p_A^i + a_A^i q_A^i\right) - L. \tag{3.7}$$

Thus, $\mathrm{d}E/\mathrm{d}t = 0$. We shall give the explicit expressions of these Noetherian energy and momenta at the 3PN order in harmonic coordinates in the next section which is devoted to the results (see equations (4.2)–(4.4)).

Finally, let us consider the symmetry of the Lagrangian that is associated with the invariance under Lorentz special transformations or boosts. Clearly, since the dynamics must stay the same after an infinitesimal constant Lorentz boost, the corresponding variation of the Lagrangian has to take essentially the form of a total time derivative. At the linearized order in the boost velocity W^i, the transformation of the particle trajectories is given by $\delta y_A^i = -W^i t + \frac{1}{c^2} W^j y_A^j v_A^i + \mathcal{O}(W^i W^i)$. There should exist a certain functional Z^i of the positions, velocities and accelerations such that the 3PN Lagrangian variation reads $\delta L = W^i \,\mathrm{d}Z^i/\mathrm{d}t + \mathcal{O}(W^i W^i)$, plus some 'double-zero' terms at the 3PN order (which are zero on-shell when applying the Noether theorem). By applying equation (3.3), we readily find the conservation on-shell of the Noetherian integral $K^i = G^i - P^i t$, where P^i is the linear momentum (3.6*a*), and where G^i represents the centre-of-mass position:

$$G^i = -Z^i + \sum_A \left(-q_A^i + \frac{1}{c^2}\left[y_A^i p_A^j v_A^j + y_A^i q_A^j a_A^j + v_A^i q_A^j v_A^j \right] \right). \tag{3.8}$$

Thus, $\mathrm{d}K^i/\mathrm{d}t = 0$, or equivalently $\mathrm{d}^2 G^i/\mathrm{d}t^2 = 0$ (the centre-of-mass vector G^i is conserved in a frame where $P^i = 0$). The existence of the latter boost-symmetry of the Lagrangian is a confirmation of the Lorentz invariance of the 3PN equations of motion obtained in [29]. The Noetherian centre-of-mass G^i in harmonic coordinates at the 3PN order is given explicitly by equation (4.5) below.

The ten Noetherian quantities (3.6)–(3.8) have been found from our generalized Lagrangian as some functionals of the positions, velocities and accelerations of the particles. However, once they have been constructed, all the accelerations they involve can be order-reduced by using the fact that they take on-shell some definite expressions depending on the positions and velocities as given by the equations of motion. Our final results presented in section 4.1 have all been order-reduced consistently with the 3PN approximation.

3.2. Elimination of acceleration-dependent terms in a Lagrangian

We start from the harmonic coordinate system $x^\mu = (ct, \boldsymbol{x})$ and perform an infinitesimal coordinate transformation to a new coordinate system x'^μ, generally not obeying the harmonic gauge condition, of type

$$x'^\mu = x^\mu + \varepsilon^\mu(x), \tag{3.9}$$

where $\varepsilon^\mu(x)$ is a function of the spatial coordinates $\boldsymbol{x}$ as well as a (local-in-time) functional of the trajectories $\boldsymbol{y}_A(t)$ and velocities $\boldsymbol{v}_A(t)$ parametrized by the coordinate time $t = x^0/c$. Namely,

$$\varepsilon^\mu(\boldsymbol{x}, t) = \varepsilon^\mu[\boldsymbol{x}; \boldsymbol{y}_A(t), \boldsymbol{v}_A(t)]. \tag{3.10}$$

Since the accelerations in the harmonic-coordinate Lagrangian appear only at the 2PN order, we suppose that the coordinate transformation starts at the same level. This means that $\varepsilon^i = \mathcal{O}\left(\frac{1}{c^4}\right)$ and $\varepsilon^0 = \mathcal{O}\left(\frac{1}{c^3}\right)$. In particular, we can check that any term in the following that is at least quadratic in ε^μ is, in fact, of the order of $\mathcal{O}\left(\frac{1}{c^8}\right)$ and thus can be neglected in our study limited to the 3PN approximation. The trajectories and velocities in the new coordinates $x'^\mu = (ct', \boldsymbol{x}')$ are some functions $\boldsymbol{y}'_A(t')$ and $\boldsymbol{v}'_A(t')$ of the new coordinate time $t' = x'^0/c$. The 'contact' transformation of the particle variables induced by the coordinate transformation (3.9) and

(3.10) is defined by $\delta y^i_A(t) = y'^i_A(t) - y^i_A(t)$ (we use the same terminology as in [12]). Neglecting all the terms of the order of the square of ε^μ we obtain

$$\delta y^i_A(t) = \varepsilon^i(\boldsymbol{y}_A, t) - \frac{v^i_A}{c}\varepsilon^0(\boldsymbol{y}_A, t) + \mathcal{O}\left(\frac{1}{c^8}\right). \tag{3.11}$$

In this paper we shall construct a contact transformation δy^i_A, composed of 2PN and 3PN terms and neglecting $\mathcal{O}\left(\frac{1}{c^8}\right)$, which is issued from some infinitesimal coordinate transformation (3.9) and (3.10); however, we shall not be so much interested in the coordinate transformation itself, in particular this means that we shall not investigate to which coordinate conditions it corresponds to (non-harmonic and/or ADM type).

If the equations satisfied by the worldlines $\boldsymbol{y}_A(t)$ in some initial coordinate system derive from the Lagrangian L, then the equations satisfied by the new worldlines $\boldsymbol{y}'_A(t')$ in a new coordinate system will derive from the new Lagrangian L' that is such that

$$L'[\boldsymbol{y}'_A(t), \boldsymbol{v}'_A(t), \boldsymbol{a}'_A(t), \boldsymbol{b}'_A(t)] = L[\boldsymbol{y}_A(t), \boldsymbol{v}_A(t), \boldsymbol{a}_A(t)] \tag{3.12}$$

(see, e.g., equation (5) of Damour and Schäfer [12]). Since we assumed that the contact transformation $\delta\boldsymbol{y}_A$ depends on the velocities, the new Lagrangian necessarily depends on positions, velocities, accelerations and also derivatives of accelerations: $\boldsymbol{b}_A(t) = \mathrm{d}\boldsymbol{a}_A/\mathrm{d}t$. Now the same computation as the one leading to equation (3.3) shows that, at the linearized order in $\delta\boldsymbol{y}_A$,

$$L'[\boldsymbol{y}_A, \boldsymbol{v}_A, \boldsymbol{a}_A, \boldsymbol{b}_A] = L[\boldsymbol{y}_A, \boldsymbol{v}_A, \boldsymbol{a}_A] + \frac{\mathrm{d}Q}{\mathrm{d}t} + \sum_A \frac{\delta L}{\delta y^i_A}\delta y^i_A + \mathcal{O}\left(\frac{1}{c^8}\right). \tag{3.13}$$

Note that both sides of this relation are expressed in terms of the same 'dummy' variables, chosen to be the harmonic-coordinate ones, e.g. $\boldsymbol{y}_A$. At the end, when we obtain the new Lagrangian, we shall have to replace this dummy variable by the one corresponding to the new coordinate system, $\boldsymbol{y}'_A = \boldsymbol{y}_A + \delta\boldsymbol{y}_A$. The term with a total time derivative is the same as that found in equation (3.3), with Q given by (3.4). As one can see, the dependence of the Lagrangian L' upon derivatives of accelerations $\boldsymbol{b}_A$ comes only from this total time derivative. Therefore, by posing $L'' = L' - \frac{\mathrm{d}Q}{\mathrm{d}t}$ we obtain a Lagrangian which is dynamically equivalent to the Lagrangian L' and depends like L on positions, velocities and accelerations only,

$$L''[\boldsymbol{y}_A, \boldsymbol{v}_A, \boldsymbol{a}_A] = L[\boldsymbol{y}_A, \boldsymbol{v}_A, \boldsymbol{a}_A] + \sum_A \frac{\delta L}{\delta y^i_A}\delta y^i_A + \mathcal{O}\left(\frac{1}{c^8}\right). \tag{3.14}$$

We now show that there exists a contact transformation δy^i_A (actually, there exist infinitely many of them), together with a redefinition of the Lagrangian by the addition of a total time derivative, which eliminates all the accelerations in the Lagrangian up to the 3PN order. In other words, the 3PN Lagrangian that will follow is ordinary, i.e. depends on positions and velocities only. Damour and Schäfer [12] have already shown how to eliminate the accelerations at the 2PN level. We shall see how to do this at the next 3PN order, but, in fact, the method is a particular application of a general algorithm to eliminate higher-derivative terms in a Lagrangian [37]. Since the contact transformation (3.11) is assumed to start at 2PN order, i.e. $\delta y^i_A = \mathcal{O}\left(\frac{1}{c^4}\right)$, we must control the functional derivative $\frac{\delta L}{\delta y^i_A}$ appearing on the right-hand side of equation (3.14) at the relative 1PN order. The standard Newtonian contribution is then followed by a certain 1PN correction, denoted by $m_A C^i_A$, hence

$$\frac{\delta L}{\delta y^i_A} = m_A\left[-a^i_A - \sum_{B\neq A}\frac{Gm_B}{r^2_{AB}}n^i_{AB} + \frac{1}{c^2}C^i_A\right] + \mathcal{O}\left(\frac{1}{c^4}\right). \tag{3.15}$$

The 1PN term C_A^i can be computed straightforwardly from the Lagrangian (4.1). The point is that it does depend on accelerations, $C_A^i \equiv C_A^i[\boldsymbol{y}_B, \boldsymbol{v}_B, \boldsymbol{a}_B]$, with this dependence being *linear*. The presence of accelerations in C_A^i is the reason why the method used in [12] to deal with the problem at the 2PN order cannot be extended immediately at the 3PN approximation. We shall see that the method necessitates the introduction in the contact transformation at the 3PN order of some 'counter-term' X_A^i described below. Now, in view of the term $-m_A a_A^i$ present in equation (3.15), it is clear that we will be able to remove all the accelerations at the 2PN order if we choose for the contact transformation the term $\frac{1}{m_A} q_A^i$ (we recall that q_A^i is the conjugate momentum of the acceleration, $q_A^i = \frac{\partial L}{\partial a_A^i}$). Indeed, the only possible accelerations at the 2PN order in the Lagrangian L'' would be contained in the combination $L - \sum_A a_A^i q_A^i$, which clearly does *not* depend on accelerations because of the linearity of the original Lagrangian L upon a_A^i. Furthermore, as discussed in [12], once we have eliminated the accelerations at the 2PN order, we are free to add to the contact transformation any term of type

$$\frac{1}{m_A} \frac{\partial F}{\partial v_A^i},$$

where F is an arbitrary functional of the positions and velocities only, starting at the 2PN order. This follows immediately from the identity

$$\frac{\mathrm{d}F}{\mathrm{d}t} = \sum_A \left(v_A^i \frac{\partial F}{\partial y_A^i} + a_A^i \frac{\partial F}{\partial v_A^i} \right),$$

which shows that the further accelerations produced by this term are contained into the total time derivative of F, and so can be removed from the original Lagrangian without changing the dynamics. However, these procedures are no longer valid at the 3PN order because of the accelerations in the 1PN term C_A^i of (3.15), which will couple to the terms

$$\frac{1}{m_A} \left[q_A^i + \frac{\partial F}{\partial v_A^i} \right]$$

as suggested before and produce some new accelerations. The solution of the problem is to add to the contact transformation some correction term that we shall find to be adjustable in a unique way so that it works.

As a result, we look for a contact transformation of type

$$\delta y_A^i = \frac{1}{m_A} \left[q_A^i + \frac{\partial F}{\partial v_A^i} + \frac{1}{c^6} X_A^i \right] + \mathcal{O}\left(\frac{1}{c^8} \right), \tag{3.16}$$

where q_A^i is defined by equation (3.5*b*); F is a general functional of the positions and velocities, $F \equiv F[\boldsymbol{y}_A, \boldsymbol{v}_A]$, and X_A^i denotes some 'counter' term depending on positions and velocities only, $X_A^i \equiv X_A^i[\boldsymbol{y}_B, \boldsymbol{v}_B]$. We recall that q_A^i is composed of 2PN and 3PN terms, which are easily computed from the Lagrangian (4.1). The function F must start at the 2PN order; in addition, we assume that it contains all possible generic terms at 3PN. Finally, as explained above the counter term X_A^i is purely of the order of 3PN. We now replace both equations (3.15) and (3.16) into L'' given by (3.14) and investigate the occurrence of accelerations. Among the terms we recognize the combination $L - \sum_A a_A^i q_A^i$, which is free of any accelerations at the 3PN order. We also transfer several acceleration terms into the total time derivative of F as before. Finally, we find that the only remaining accelerations in L'' are contained in the particular combination of terms

$$\sum_A \left(\frac{1}{c^2} \left[q_A^i + \frac{\partial F}{\partial v_A^i} \right] C_A^i - \frac{1}{c^6} a_A^i X_A^i \right) + \mathcal{O}\left(\frac{1}{c^8} \right).$$

As all the terms in that combination are linear in the accelerations, we see that for *any* given function F there is a *unique* choice of the term X^i_A (for each particle) such that all the remaining accelerations are cancelled out, namely

$$\frac{1}{c^6}X^i_A = \sum_B \frac{1}{c^2}\left[q^j_B + \frac{\partial F}{\partial v^j_B}\right]\frac{\partial C^j_B}{\partial a^i_A} + \mathcal{O}\left(\frac{1}{c^8}\right). \tag{3.17}$$

With the latter choice, the contact transformation (3.16), defined for any F, yields a Lagrangian L'' whose only accelerations come from (minus) the total time derivative of F. Therefore, the 3PN Lagrangian $L''' = L'' + \frac{\mathrm{d}F}{\mathrm{d}t}$ is at once physically equivalent to L'', L' and L, and free of accelerations. Our result then reads

$$L'''[\boldsymbol{y}_A, \boldsymbol{v}_A] = L + \sum_A \frac{\delta L}{\delta y^i_A}\delta y^i_A + \frac{\mathrm{d}F}{\mathrm{d}t} + \mathcal{O}\left(\frac{1}{c^8}\right). \tag{3.18}$$

Recall the large freedom we still have on the definition of L''', since we constructed it for any functional F of the positions and velocities at the 2PN and 3PN orders.

In this paper we shall be able to determine uniquely the function F by the requirement that the Lagrangian L''' be exactly the ADM Lagrangian associated with the ADM (or ADM-type) Hamiltonian published by Damour *et al* [24]. We shall not give the details of the computation since it consists merely of parametrizing the most general function F, constructed with the dynamical variables of the problem and having a compatible dimension, by means of some arbitrary constant parameters, and showing that all these constants are uniquely fixed by the condition of matching to the ADM Hamiltonian. We find indeed, in complete agreement with [25], that there is a unique set of constants for which this works. In particular, the equivalence is possible if and only if the undetermined constant λ appearing in the harmonic-coordinate formalism [29] is related to the constant ω_{static} of Jaranowski and Schäfer [22] by equation (1.1). Note that the latter matching also shows that the logarithms $\ln\left(\frac{r_{12}}{r'_1}\right)$ and $\ln\left(\frac{r_{12}}{r'_2}\right)$ present in the harmonic-coordinate Lagrangian (4.1), where r'_1 and r'_2 denote some regularization constants, are eliminated by this contact transformation, in agreement with the fact proved in [29] that the logarithms, and the constants r'_1 and r'_2 therein, can be gauged away. See equation (4.9) below for the complete expression of the function F.

Finally, with F now fully specified by the equivalence with [24], we obtain the ordinary ADM-type Lagrangian

$$L^{\mathrm{ADM}} = L + \sum_A \frac{\delta L}{\delta y^i_A}\delta y^i_A + \frac{\mathrm{d}F}{\mathrm{d}t}, \tag{3.19}$$

given explicitly at the 3PN order by equation (4.11) below, in which, as mentioned above, we shall replace the 'dummy' variables used in the computation, y^i_A and v^i_A, by the real dynamical variables in pseudo-ADM coordinates, Y^i_A and V^i_A. The ADM momentum conjugate to the velocity is

$$P^i_A = \frac{\partial L^{\mathrm{ADM}}}{\partial v^i_A} = p^i_A + \frac{\delta}{\delta v^i_A}\left(\sum_B \delta y^j_B \frac{\delta L}{\delta y^j_B}\right) + \frac{\partial F}{\partial y^i_A}, \tag{3.20}$$

and the corresponding Hamiltonian follows from the ordinary Legendre transformation

$$H^{\mathrm{ADM}} = \sum_A P^i_A v^i_A - L^{\mathrm{ADM}}. \tag{3.21}$$

See equation (4.12) for the complete 3PN expression of this Hamiltonian (as a function of Y^i_A and P^i_A). (We have checked that the second equality in (3.20) is true at 3PN order.) Note that, strictly speaking, H^{ADM} is not the ADM one, as it differs from it by a shift in phase-space coordinates at the 3PN order which is given in [24]. Indeed, the ADM Hamiltonian at the 3PN order is not ordinary, as it depends on the positions and momenta as well as on their derivatives [21]. However, this is not a concern for our purpose, since we are interested in proving the equivalence between our approach [26–29] and that of [21–25], that is in finding the existence of a unique transformation connecting both works, in whatever coordinate systems the two approaches found it convenient to use. We think that the equivalence found in this paper and in [25] convincingly confirms the correctness of the result. This equivalence is especially important in view of the different procedures adopted by the two groups to treat the point-mass divergences (see section 2 for a discussion).

4. Results

4.1. Conserved quantities in harmonic coordinates at the 3PN order

We first exhibit a generalized Lagrangian from which we derive the 3PN equations of motion of two compact objects as they were obtained in harmonic coordinates; see equations (7.16) in [29]. The Lagrangian corresponds only to the conservative part of the equations, which excludes the radiation reaction term at the 2.5PN order. To compute it we proceed by guesswork, and find the occurrence of terms depending on accelerations at the 2PN and 3PN orders. The Lagrangian is chosen to be linear in the accelerations, and to agree at the 2PN approximation with the Lagrangian obtained in [11]. The result is

$$
\begin{aligned}
L = {} & \frac{Gm_1m_2}{2r_{12}} + \frac{m_1v_1^2}{2} \\
& + \frac{1}{c^2}\left\{ -\frac{G^2m_1^2m_2}{2r_{12}^2} + \frac{m_1v_1^4}{8} + \frac{Gm_1m_2}{r_{12}}\left(-\tfrac{1}{4}(n_{12}v_1)(n_{12}v_2) + \tfrac{3}{2}v_1^2 - \tfrac{7}{4}(v_1v_2)\right)\right\} \\
& + \frac{1}{c^4}\left\{ \frac{G^3m_1^3m_2}{2r_{12}^3} + \frac{19G^3m_1^2m_2^2}{8r_{12}^3} + \frac{G^2m_1^2m_2}{r_{12}^2}\left(\tfrac{7}{2}(n_{12}v_1)^2 - \tfrac{7}{2}(n_{12}v_1)(n_{12}v_2) \right.\right. \\
& \left. + \tfrac{1}{2}(n_{12}v_2)^2 + \tfrac{1}{4}v_1^2 - \tfrac{7}{4}(v_1v_2) + \tfrac{7}{4}v_2^2 \right) \\
& + \frac{Gm_1m_2}{r_{12}}\left(\tfrac{3}{16}(n_{12}v_1)^2(n_{12}v_2)^2 - \tfrac{7}{8}(n_{12}v_2)^2v_1^2 + \tfrac{7}{8}v_1^4 + \tfrac{3}{4}(n_{12}v_1)(n_{12}v_2)(v_1v_2) \right. \\
& \left. - 2v_1^2(v_1v_2) + \tfrac{1}{8}(v_1v_2)^2 + \tfrac{15}{16}v_1^2v_2^2 \right) + \frac{m_1v_1^6}{16} \\
& \left. + Gm_1m_2\left(-\tfrac{7}{4}(a_1v_2)(n_{12}v_2) - \tfrac{1}{8}(n_{12}a_1)(n_{12}v_2)^2 + \tfrac{7}{8}(n_{12}a_1)v_2^2 \right) \right\} \\
& + \frac{1}{c^6}\left\{ \frac{G^2m_1^2m_2}{r_{12}^2}\left(\tfrac{13}{18}(n_{12}v_1)^4 + \tfrac{83}{18}(n_{12}v_1)^3(n_{12}v_2) - \tfrac{35}{6}(n_{12}v_1)^2(n_{12}v_2)^2 \right.\right. \\
& - \tfrac{245}{24}(n_{12}v_1)^2v_1^2 + \tfrac{179}{12}(n_{12}v_1)(n_{12}v_2)v_1^2 - \tfrac{235}{24}(n_{12}v_2)^2v_1^2 + \tfrac{373}{48}v_1^4 \\
& + \tfrac{529}{24}(n_{12}v_1)^2(v_1v_2) - \tfrac{97}{6}(n_{12}v_1)(n_{12}v_2)(v_1v_2) - \tfrac{719}{24}v_1^2(v_1v_2) + \tfrac{463}{24}(v_1v_2)^2 \\
& - \tfrac{7}{24}(n_{12}v_1)^2v_2^2 - \tfrac{1}{2}(n_{12}v_1)(n_{12}v_2)v_2^2 + \tfrac{1}{4}(n_{12}v_2)^2v_2^2 + \tfrac{463}{48}v_1^2v_2^2 \\
& \left. - \tfrac{19}{2}(v_1v_2)v_2^2 + \tfrac{45}{16}v_2^4 \right) + \frac{5m_1v_1^8}{128}
\end{aligned}
$$

$$+Gm_1m_2\big(\tfrac{3}{8}(a_1v_2)(n_{12}v_1)(n_{12}v_2)^2+\tfrac{5}{12}(a_1v_2)(n_{12}v_2)^3$$
$$+\tfrac{1}{8}(n_{12}a_1)(n_{12}v_1)(n_{12}v_2)^3+\tfrac{1}{16}(n_{12}a_1)(n_{12}v_2)^4+\tfrac{11}{4}(a_1v_1)(n_{12}v_2)v_1^2$$
$$-(a_1v_2)(n_{12}v_2)v_1^2-2(a_1v_1)(n_{12}v_2)(v_1v_2)+\tfrac{1}{4}(a_1v_2)(n_{12}v_2)(v_1v_2)$$
$$+\tfrac{3}{8}(n_{12}a_1)(n_{12}v_2)^2(v_1v_2)-\tfrac{5}{8}(n_{12}a_1)(n_{12}v_1)^2v_2^2+\tfrac{15}{8}(a_1v_1)(n_{12}v_2)v_2^2$$
$$-\tfrac{15}{8}(a_1v_2)(n_{12}v_2)v_2^2-\tfrac{1}{2}(n_{12}a_1)(n_{12}v_1)(n_{12}v_2)v_2^2-\tfrac{5}{16}(n_{12}a_1)(n_{12}v_2)^2v_2^2\big)$$
$$+\frac{G^2m_1^2m_2}{r_{12}}\big(-\tfrac{235}{24}(a_2v_1)(n_{12}v_1)-\tfrac{29}{24}(n_{12}a_2)(n_{12}v_1)^2-\tfrac{235}{24}(a_1v_2)(n_{12}v_2)$$
$$-\tfrac{17}{6}(n_{12}a_1)(n_{12}v_2)^2+\tfrac{185}{16}(n_{12}a_1)v_1^2-\tfrac{235}{48}(n_{12}a_2)v_1^2-\tfrac{185}{8}(n_{12}a_1)(v_1v_2)$$
$$+\tfrac{20}{3}(n_{12}a_1)v_2^2\big)$$
$$+\frac{Gm_1m_2}{r_{12}}\big(-\tfrac{5}{32}(n_{12}v_1)^3(n_{12}v_2)^3+\tfrac{1}{8}(n_{12}v_1)(n_{12}v_2)^3v_1^2+\tfrac{5}{8}(n_{12}v_2)^4v_1^2$$
$$-\tfrac{11}{16}(n_{12}v_1)(n_{12}v_2)v_1^4+\tfrac{1}{4}(n_{12}v_2)^2v_1^4+\tfrac{11}{16}v_1^6-\tfrac{15}{32}(n_{12}v_1)^2(n_{12}v_2)^2(v_1v_2)$$
$$+(n_{12}v_1)(n_{12}v_2)v_1^2(v_1v_2)+\tfrac{3}{8}(n_{12}v_2)^2v_1^2(v_1v_2)-\tfrac{13}{16}v_1^4(v_1v_2)$$
$$+\tfrac{5}{16}(n_{12}v_1)(n_{12}v_2)(v_1v_2)^2+\tfrac{1}{16}(v_1v_2)^3-\tfrac{5}{8}(n_{12}v_1)^2v_1^2v_2^2$$
$$-\tfrac{23}{32}(n_{12}v_1)(n_{12}v_2)v_1^2v_2^2+\tfrac{1}{16}v_1^4v_2^2-\tfrac{1}{32}v_1^2(v_1v_2)v_2^2\big)$$
$$-\frac{3G^4m_1^4m_2}{8r_{12}^4}+\frac{G^4m_1^3m_2^2}{r_{12}^4}\left(-\frac{5809}{280}+\frac{11}{3}\lambda+\frac{22}{3}\ln\left(\frac{r_{12}}{r_1'}\right)\right)$$
$$+\frac{G^3m_1^2m_2^2}{r_{12}^3}\big(\tfrac{383}{24}(n_{12}v_1)^2-\tfrac{889}{48}(n_{12}v_1)(n_{12}v_2)-\tfrac{123}{64}(n_{12}v_1)^2\pi^2$$
$$+\tfrac{123}{64}(n_{12}v_1)(n_{12}v_2)\pi^2-\tfrac{305}{72}v_1^2+\tfrac{41}{64}\pi^2v_1^2+\tfrac{439}{144}(v_1v_2)-\tfrac{41}{64}\pi^2(v_1v_2)\big)$$
$$+\frac{G^3m_1^3m_2}{r_{12}^3}\left(-\frac{8243}{210}(n_{12}v_1)^2+\frac{15\,541}{420}(n_{12}v_1)(n_{12}v_2)+\frac{3}{2}(n_{12}v_2)^2\right.$$
$$+\frac{15\,611}{1260}v_1^2-\frac{17\,501}{1260}(v_1v_2)+\frac{5}{4}v_2^2+22(n_{12}v_1)^2\ln\left(\frac{r_{12}}{r_1'}\right)$$
$$\left.-22(n_{12}v_1)(n_{12}v_2)\ln\left(\frac{r_{12}}{r_1'}\right)-\frac{22}{3}v_1^2\ln\left(\frac{r_{12}}{r_1'}\right)+\frac{22}{3}(v_1v_2)\ln\left(\frac{r_{12}}{r_1'}\right)\right)\Bigg\}$$
$$+1\leftrightarrow 2+\mathcal{O}\left(\frac{1}{c^7}\right). \tag{4.1}$$

In our notation, $r_{12}=|\boldsymbol{y}_1-\boldsymbol{y}_2|$, $\boldsymbol{n}_{12}=(\boldsymbol{y}_1-\boldsymbol{y}_2)/r_{12}$, and the scalar products are written, for example, as $(n_{12}v_2)=\boldsymbol{n}_{12}\cdot\boldsymbol{v}_2$. To the terms given explicitly above, we have to add the terms corresponding to the relabelling $1\leftrightarrow 2$, including those that are symmetric under the label exchange. Note the presence of the constant λ which is the only unknown physical parameter in this Lagrangian, and of the two unknown gauge constants r_1' and r_2' (we follow exactly the notation of [29]). The Lagrangian presented here is not the only admissible one, as we can always add to it an arbitrary total time derivative (double-zero terms would make the Lagrangian nonlinear in the accelerations). We have checked that our Lagrangian (4.1) does indeed differ from the one given by equations (5.4)–(5.10) in [25] by a total time derivative.

Next we present the expressions of the conserved integrals of the 3PN harmonic-coordinate motion as constructed in section 3.1. These expressions involve only the relativistic 1PN, 2PN and 3PN terms corresponding to the conservative part of the dynamics at the 3PN order. The radiation reaction damping effect is added afterwards. All the quantities we present depend only on the positions and velocities, because all accelerations therein have been systematically order-reduced by means of the equations of motion. The energy E reads

$$
\begin{aligned}
E = {} & \frac{m_1 v_1^2}{2} - \frac{G m_1 m_2}{2 r_{12}} \\
& + \frac{1}{c^2}\left\{ \frac{G^2 m_1^2 m_2}{2 r_{12}^2} + \frac{3 m_1 v_1^4}{8} + \frac{G m_1 m_2}{r_{12}}\left(-\tfrac{1}{4}(n_{12}v_1)(n_{12}v_2) + \tfrac{3}{2}v_1^2 - \tfrac{7}{4}(v_1v_2)\right)\right\} \\
& + \frac{1}{c^4}\left\{ -\frac{G^3 m_1^3 m_2}{2 r_{12}^3} - \frac{19 G^3 m_1^2 m_2^2}{8 r_{12}^3} + \frac{5 m_1 v_1^6}{16} \right. \\
& + \frac{G m_1 m_2}{r_{12}}\left(\tfrac{3}{8}(n_{12}v_1)^3(n_{12}v_2) + \tfrac{3}{16}(n_{12}v_1)^2(n_{12}v_2)^2 - \tfrac{9}{8}(n_{12}v_1)(n_{12}v_2)v_1^2\right. \\
& - \tfrac{13}{8}(n_{12}v_2)^2 v_1^2 + \tfrac{21}{8}v_1^4 + \tfrac{13}{8}(n_{12}v_1)^2(v_1v_2) + \tfrac{3}{4}(n_{12}v_1)(n_{12}v_2)(v_1v_2) \\
& \left. - \tfrac{55}{8}v_1^2(v_1v_2) + \tfrac{17}{8}(v_1v_2)^2 + \tfrac{31}{16}v_1^2v_2^2\right) \\
& \left. + \frac{G^2 m_1^2 m_2}{r_{12}^2}\left(\tfrac{29}{4}(n_{12}v_1)^2 - \tfrac{13}{4}(n_{12}v_1)(n_{12}v_2) + \tfrac{1}{2}(n_{12}v_2)^2 - \tfrac{3}{2}v_1^2 + \tfrac{7}{4}v_2^2\right)\right\} \\
& + \frac{1}{c^6}\left\{ \frac{35 m_1 v_1^8}{128} + \frac{G m_1 m_2}{r_{12}}\left(-\tfrac{5}{16}(n_{12}v_1)^5(n_{12}v_2) - \tfrac{5}{16}(n_{12}v_1)^4(n_{12}v_2)^2\right.\right. \\
& - \tfrac{5}{32}(n_{12}v_1)^3(n_{12}v_2)^3 + \tfrac{19}{16}(n_{12}v_1)^3(n_{12}v_2)v_1^2 + \tfrac{15}{16}(n_{12}v_1)^2(n_{12}v_2)^2v_1^2 \\
& + \tfrac{3}{4}(n_{12}v_1)(n_{12}v_2)^3v_1^2 + \tfrac{19}{16}(n_{12}v_2)^4v_1^2 - \tfrac{21}{16}(n_{12}v_1)(n_{12}v_2)v_1^4 - 2(n_{12}v_2)^2v_1^4 \\
& + \tfrac{55}{16}v_1^6 - \tfrac{19}{16}(n_{12}v_1)^4(v_1v_2) - (n_{12}v_1)^3(n_{12}v_2)(v_1v_2) \\
& - \tfrac{15}{32}(n_{12}v_1)^2(n_{12}v_2)^2(v_1v_2) + \tfrac{45}{16}(n_{12}v_1)^2v_1^2(v_1v_2) + \tfrac{5}{4}(n_{12}v_1)(n_{12}v_2)v_1^2(v_1v_2) \\
& + \tfrac{11}{4}(n_{12}v_2)^2v_1^2(v_1v_2) - \tfrac{139}{16}v_1^4(v_1v_2) - \tfrac{3}{4}(n_{12}v_1)^2(v_1v_2)^2 \\
& + \tfrac{5}{16}(n_{12}v_1)(n_{12}v_2)(v_1v_2)^2 + \tfrac{41}{8}v_1^2(v_1v_2)^2 + \tfrac{1}{16}(v_1v_2)^3 - \tfrac{45}{16}(n_{12}v_1)^2v_1^2v_2^2 \\
& \left. - \tfrac{23}{32}(n_{12}v_1)(n_{12}v_2)v_1^2v_2^2 + \tfrac{79}{16}v_1^4v_2^2 - \tfrac{161}{32}v_1^2(v_1v_2)v_2^2\right) \\
& + \frac{G^2 m_1^2 m_2}{r_{12}^2}\left(-\tfrac{49}{8}(n_{12}v_1)^4 + \tfrac{75}{8}(n_{12}v_1)^3(n_{12}v_2) - \tfrac{187}{8}(n_{12}v_1)^2(n_{12}v_2)^2\right. \\
& + \tfrac{247}{24}(n_{12}v_1)(n_{12}v_2)^3 + \tfrac{49}{8}(n_{12}v_1)^2v_1^2 + \tfrac{81}{8}(n_{12}v_1)(n_{12}v_2)v_1^2 - \tfrac{21}{4}(n_{12}v_2)^2v_1^2 \\
& + \tfrac{11}{2}v_1^4 - \tfrac{15}{2}(n_{12}v_1)^2(v_1v_2) - \tfrac{3}{2}(n_{12}v_1)(n_{12}v_2)(v_1v_2) + \tfrac{21}{4}(n_{12}v_2)^2(v_1v_2) \\
& - 27v_1^2(v_1v_2) + \tfrac{55}{2}(v_1v_2)^2 + \tfrac{49}{4}(n_{12}v_1)^2v_2^2 - \tfrac{27}{2}(n_{12}v_1)(n_{12}v_2)v_2^2 \\
& \left. + \tfrac{3}{4}(n_{12}v_2)^2v_2^2 + \tfrac{55}{4}v_1^2v_2^2 - 28(v_1v_2)v_2^2 + \tfrac{135}{16}v_2^4\right) \\
& + \frac{3G^4 m_1^4 m_2}{8 r_{12}^4} + \frac{G^4 m_1^3 m_2^2}{r_{12}^4}\left(\frac{5809}{280} - \frac{11}{3}\lambda - \frac{22}{3}\ln\left(\frac{r_{12}}{r'_1}\right)\right)
\end{aligned}
$$

$$+\frac{G^3m_1^2m_2^2}{r_{12}^3}\left(\tfrac{547}{12}(n_{12}v_1)^2-\tfrac{3115}{48}(n_{12}v_1)(n_{12}v_2)-\tfrac{123}{64}(n_{12}v_1)^2\pi^2\right.$$
$$\left.+\tfrac{123}{64}(n_{12}v_1)(n_{12}v_2)\pi^2-\tfrac{575}{18}v_1^2+\tfrac{41}{64}\pi^2v_1^2+\tfrac{4429}{144}(v_1v_2)-\tfrac{41}{64}\pi^2(v_1v_2)\right)$$
$$+\frac{G^3m_1^3m_2}{r_{12}^3}\left(-\frac{44\,627}{840}(n_{12}v_1)^2+\frac{32\,027}{840}(n_{12}v_1)(n_{12}v_2)+\frac{3}{2}(n_{12}v_2)^2\right.$$
$$+\frac{24\,187}{2520}v_1^2-\frac{27\,967}{2520}(v_1v_2)+\frac{5}{4}v_2^2+22(n_{12}v_1)^2\ln\left(\frac{r_{12}}{r'_1}\right)$$
$$\left.\left.-22(n_{12}v_1)(n_{12}v_2)\ln\left(\frac{r_{12}}{r'_1}\right)-\frac{22}{3}v_1^2\ln\left(\frac{r_{12}}{r'_1}\right)+\frac{22}{3}(v_1v_2)\ln\left(\frac{r_{12}}{r'_1}\right)\right)\right\}$$
$$+1\leftrightarrow 2+\mathcal{O}\left(\frac{1}{c^7}\right). \tag{4.2}$$

We find that this energy is in agreement with the expression obtained in [29] by guesswork, starting directly from the equations of motion. The logarithms $\ln\left(\frac{r_{12}}{r'_1}\right)$ and $\ln\left(\frac{r_{12}}{r'_2}\right)$ take the form of a gauge transformation of the energy (see equation (6.16) in [29]). Accordingly, they will never enter a physical result such as the circular-orbit energy when expressed in terms of the orbital frequency of the circular motion (see [26]). Such is not the case for the constant λ which does enter the invariant energy. The total linear momentum P^i at the 3PN order is given by

$$P^i=m_1v_1^i+\frac{1}{c^2}\left\{-n_{12}^i\frac{Gm_1m_2}{2r_{12}}(n_{12}v_1)+v_1^i\left(-\frac{Gm_1m_2}{2r_{12}}+\frac{m_1v_1^2}{2}\right)\right\}$$
$$+\frac{1}{c^4}\left\{n_{12}^i\left(\frac{G^2m_1^2m_2}{r_{12}^2}\left(\tfrac{29}{4}(n_{12}v_1)-\tfrac{9}{4}(n_{12}v_2)\right)+\frac{Gm_1m_2}{r_{12}}\left(\tfrac{3}{8}(n_{12}v_1)^3\right.\right.\right.$$
$$\left.\left.+\tfrac{3}{8}(n_{12}v_1)^2(n_{12}v_2)-\tfrac{9}{8}(n_{12}v_1)v_1^2-\tfrac{7}{8}(n_{12}v_2)v_1^2+\tfrac{7}{4}(n_{12}v_1)(v_1v_2)\right)\right)$$
$$+v_1^i\left(-\frac{3G^2m_1^2m_2}{r_{12}^2}+\frac{7G^2m_1m_2^2}{2r_{12}^2}+\frac{3m_1v_1^4}{8}\right.$$
$$+\frac{Gm_1m_2}{r_{12}}\left(\tfrac{13}{8}(n_{12}v_1)^2-\tfrac{1}{4}(n_{12}v_1)(n_{12}v_2)\right.$$
$$\left.\left.\left.-\tfrac{13}{8}(n_{12}v_2)^2+\tfrac{5}{8}v_1^2-\tfrac{7}{4}(v_1v_2)+\tfrac{7}{8}v_2^2\right)\right)\right\}$$
$$+\frac{1}{c^6}\left\{n_{12}^i\left(\frac{G^2m_1^2m_2}{r_{12}^2}\left(-\tfrac{45}{8}(n_{12}v_1)^3+\tfrac{59}{8}(n_{12}v_1)^2(n_{12}v_2)-\tfrac{179}{8}(n_{12}v_1)(n_{12}v_2)^2\right.\right.\right.$$
$$+\tfrac{247}{24}(n_{12}v_2)^3+\tfrac{135}{8}(n_{12}v_1)v_1^2-\tfrac{87}{8}(n_{12}v_2)v_1^2-\tfrac{53}{2}(n_{12}v_1)(v_1v_2)$$
$$\left.+\tfrac{87}{4}(n_{12}v_2)(v_1v_2)+\tfrac{53}{4}(n_{12}v_1)v_2^2-12(n_{12}v_2)v_2^2\right)$$
$$+\frac{Gm_1m_2}{r_{12}}\left(-\tfrac{5}{16}(n_{12}v_1)^5-\tfrac{5}{16}(n_{12}v_1)^4(n_{12}v_2)-\tfrac{5}{16}(n_{12}v_1)^3(n_{12}v_2)^2\right.$$
$$+\tfrac{19}{16}(n_{12}v_1)^3v_1^2+\tfrac{15}{16}(n_{12}v_1)^2(n_{12}v_2)v_1^2+\tfrac{3}{4}(n_{12}v_1)(n_{12}v_2)^2v_1^2+\tfrac{5}{8}(n_{12}v_2)^3v_1^2$$

$$
\begin{aligned}
&-\tfrac{21}{16}(n_{12}v_1)v_1^4-\tfrac{11}{16}(n_{12}v_2)v_1^4-\tfrac{5}{4}(n_{12}v_1)^3(v_1v_2)-\tfrac{9}{8}(n_{12}v_1)^2(n_{12}v_2)(v_1v_2)\\
&+\tfrac{15}{8}(n_{12}v_1)v_1^2(v_1v_2)+(n_{12}v_2)v_1^2(v_1v_2)-\tfrac{1}{8}(n_{12}v_1)(v_1v_2)^2-\tfrac{15}{16}(n_{12}v_1)v_1^2v_2^2\big)\\
&-\frac{175G^3m_1^2m_2^2}{8r_{12}^3}(n_{12}v_1)+\frac{G^3m_1^3m_2}{r_{12}^3}\left(-\frac{46\,517}{840}(n_{12}v_1)+\frac{34\,547}{840}(n_{12}v_2)\right.\\
&\left.\left.+22(n_{12}v_1)\ln\left(\frac{r_{12}}{r_1'}\right)-22(n_{12}v_2)\ln\left(\frac{r_{12}}{r_1'}\right)\right)\right)\\
&+v_1^i\left(\frac{5m_1v_1^6}{16}+\frac{G^2m_1^2m_2}{r_{12}^2}\big(-\tfrac{39}{4}(n_{12}v_1)^2+22(n_{12}v_1)(n_{12}v_2)-\tfrac{21}{4}(n_{12}v_2)^2\right.\\
&-2v_1^2+(v_1v_2)-\tfrac{1}{2}v_2^2\big)\\
&+\frac{G^2m_1m_2^2}{r_{12}^2}\big(\tfrac{23}{4}(n_{12}v_1)^2-\tfrac{101}{4}(n_{12}v_1)(n_{12}v_2)+17(n_{12}v_2)^2\\
&+\tfrac{9}{4}v_1^2-(v_1v_2)+\tfrac{1}{2}v_2^2\big)\\
&+\frac{Gm_1m_2}{r_{12}}\big(-\tfrac{19}{16}(n_{12}v_1)^4+\tfrac{1}{4}(n_{12}v_1)^3(n_{12}v_2)+\tfrac{3}{16}(n_{12}v_1)^2(n_{12}v_2)^2\\
&+\tfrac{1}{8}(n_{12}v_1)(n_{12}v_2)^3+\tfrac{19}{16}(n_{12}v_2)^4+\tfrac{45}{16}(n_{12}v_1)^2v_1^2-\tfrac{5}{8}(n_{12}v_1)(n_{12}v_2)v_1^2\\
&-2(n_{12}v_2)^2v_1^2+\tfrac{23}{16}v_1^4-\tfrac{3}{4}(n_{12}v_1)^2(v_1v_2)+\tfrac{3}{4}(n_{12}v_1)(n_{12}v_2)(v_1v_2)\\
&+\tfrac{19}{8}(n_{12}v_2)^2(v_1v_2)-\tfrac{31}{8}v_1^2(v_1v_2)+\tfrac{17}{8}(v_1v_2)^2+\tfrac{3}{8}(n_{12}v_1)^2v_2^2\\
&-\tfrac{1}{2}(n_{12}v_1)(n_{12}v_2)v_2^2-\tfrac{45}{16}(n_{12}v_2)^2v_2^2+\tfrac{31}{16}v_1^2v_2^2-3(v_1v_2)v_2^2+\tfrac{19}{16}v_2^4\big)\\
&-\frac{19G^3m_1^2m_2^2}{8r_{12}^3}+\frac{G^3m_1^3m_2}{r_{12}^3}\left(\frac{20\,407}{2520}-\frac{22}{3}\ln\left(\frac{r_{12}}{r_1'}\right)\right)\\
&\left.\left.+\frac{G^3m_1m_2^3}{r_{12}^3}\left(-\frac{21\,667}{2520}+\frac{22}{3}\ln\left(\frac{r_{12}}{r_2'}\right)\right)\right)\right\}+1\leftrightarrow2+\mathcal{O}\left(\frac{1}{c^7}\right).
\end{aligned}
\tag{4.3}
$$

Next, the 3PN angular momentum J^i is

$$
\begin{aligned}
J^i=\varepsilon_{ijk}m_1y_1^jv_1^k&+\frac{1}{c^2}\varepsilon_{ijk}\left\{y_1^jv_1^k\left(\frac{3Gm_1m_2}{r_{12}}+\frac{m_1v_1^2}{2}\right)-y_1^jv_2^k\frac{7Gm_1m_2}{2r_{12}}\right.\\
&\left.+y_1^jy_2^k\frac{Gm_1m_2}{2r_{12}^2}(n_{12}v_1)\right\}\\
&+\frac{1}{c^4}\varepsilon_{ijk}\left\{-v_1^jv_2^k\frac{7Gm_1m_2}{4}(n_{12}v_1)+y_1^jv_1^k\left(-\frac{5G^2m_1^2m_2}{4r_{12}^2}+\frac{7G^2m_1m_2^2}{2r_{12}^2}+\frac{3m_1v_1^4}{8}\right.\right.\\
&\left.+\frac{Gm_1m_2}{r_{12}}\big(-\tfrac{3}{2}(n_{12}v_2)^2+\tfrac{7}{2}v_1^2-4(v_1v_2)+2v_2^2\big)\right)\\
&+y_1^jv_2^k\left(-\frac{7G^2m_1m_2^2}{4r_{12}^2}+\frac{Gm_1m_2}{r_{12}}\big(-\tfrac{1}{8}(n_{12}v_1)^2-\tfrac{1}{4}(n_{12}v_1)(n_{12}v_2)+\tfrac{13}{8}(n_{12}v_2)^2\right.\\
&\left.-\tfrac{9}{8}v_1^2+\tfrac{9}{4}(v_1v_2)-\tfrac{23}{8}v_2^2\big)\right)
\end{aligned}
$$

$$
\begin{aligned}
&+y_1^j y_2^k\left(\frac{G^2m_1^2m_2}{r_{12}^3}\left(-\tfrac{29}{4}(n_{12}v_1)+\tfrac{9}{4}(n_{12}v_2)\right)+\frac{Gm_1m_2}{r_{12}^2}\left(-\tfrac{3}{8}(n_{12}v_1)^3\right.\right.\\
&\left.\left.\left.-\tfrac{3}{8}(n_{12}v_1)^2(n_{12}v_2)+\tfrac{9}{8}(n_{12}v_1)v_1^2+\tfrac{7}{8}(n_{12}v_2)v_1^2-\tfrac{7}{4}(n_{12}v_1)(v_1v_2)\right)\right)\right\}\\
&+\frac{1}{c^6}\varepsilon_{ijk}\left\{v_1^jv_2^k\left(\frac{G^2m_1^2m_2}{r_{12}}\left(\tfrac{235}{24}(n_{12}v_1)-\tfrac{235}{24}(n_{12}v_2)\right)+Gm_1m_2\left(\tfrac{5}{12}(n_{12}v_1)^3\right.\right.\right.\\
&\left.\left.+\tfrac{3}{8}(n_{12}v_1)^2(n_{12}v_2)-\tfrac{15}{8}(n_{12}v_1)v_1^2-(n_{12}v_2)v_1^2+\tfrac{1}{4}(n_{12}v_1)(v_1v_2)\right)\right)\\
&+y_1^jv_1^k\left(\frac{5m_1v_1^6}{16}+\frac{Gm_1m_2}{r_{12}}\left(\tfrac{9}{8}(n_{12}v_2)^4-\tfrac{7}{4}(n_{12}v_2)^2v_1^2+\tfrac{33}{8}v_1^4+2(n_{12}v_2)^2(v_1v_2)\right.\right.\\
&\left.-6v_1^2(v_1v_2)+2(v_1v_2)^2-\tfrac{5}{2}(n_{12}v_2)^2v_2^2+3v_1^2v_2^2-4(v_1v_2)v_2^2+2v_2^4\right)\\
&+\frac{G^2m_1^2m_2}{r_{12}^2}\left(-\tfrac{161}{12}(n_{12}v_1)^2+\tfrac{223}{12}(n_{12}v_1)(n_{12}v_2)-\tfrac{29}{12}(n_{12}v_2)^2+\tfrac{41}{24}v_1^2\right.\\
&\left.-\tfrac{14}{3}(v_1v_2)+\tfrac{7}{3}v_2^2\right)\\
&+\frac{G^2m_1m_2^2}{r_{12}^2}\left(\tfrac{1}{2}(n_{12}v_1)^2-(n_{12}v_1)(n_{12}v_2)-3(n_{12}v_2)^2+\tfrac{45}{4}v_1^2\right.\\
&\left.-19(v_1v_2)+\tfrac{19}{2}v_2^2\right)\\
&+\frac{5G^3m_1m_2^3}{2r_{12}^3}+\frac{G^3m_1^2m_2^2}{r_{12}^3}\left(-\frac{2605}{72}+\frac{41}{32}\pi^2\right)\\
&\left.+\frac{G^3m_1^3m_2}{r_{12}^3}\left(\frac{55\,409}{2520}-\frac{44}{3}\ln\left(\frac{r_{12}}{r_1'}\right)\right)\right)\\
&+y_1^jy_2^k\left(\frac{G^2m_1^2m_2}{r_{12}^3}\left(\tfrac{45}{8}(n_{12}v_1)^3-\tfrac{59}{8}(n_{12}v_1)^2(n_{12}v_2)+\tfrac{179}{8}(n_{12}v_1)(n_{12}v_2)^2\right.\right.\\
&-\tfrac{247}{24}(n_{12}v_2)^3-\tfrac{135}{8}(n_{12}v_1)v_1^2+\tfrac{87}{8}(n_{12}v_2)v_1^2+\tfrac{53}{2}(n_{12}v_1)(v_1v_2)\\
&\left.-\tfrac{87}{4}(n_{12}v_2)(v_1v_2)-\tfrac{53}{4}(n_{12}v_1)v_2^2+12(n_{12}v_2)v_2^2\right)\\
&+\frac{Gm_1m_2}{r_{12}^2}\left(\tfrac{5}{16}(n_{12}v_1)^5+\tfrac{5}{16}(n_{12}v_1)^4(n_{12}v_2)+\tfrac{5}{16}(n_{12}v_1)^3(n_{12}v_2)^2\right.\\
&-\tfrac{19}{16}(n_{12}v_1)^3v_1^2-\tfrac{15}{16}(n_{12}v_1)^2(n_{12}v_2)v_1^2-\tfrac{3}{4}(n_{12}v_1)(n_{12}v_2)^2v_1^2-\tfrac{5}{8}(n_{12}v_2)^3v_1^2\\
&+\tfrac{21}{16}(n_{12}v_1)v_1^4+\tfrac{11}{16}(n_{12}v_2)v_1^4+\tfrac{5}{4}(n_{12}v_1)^3(v_1v_2)+\tfrac{9}{8}(n_{12}v_1)^2(n_{12}v_2)(v_1v_2)\\
&\left.-\tfrac{15}{8}(n_{12}v_1)v_1^2(v_1v_2)-(n_{12}v_2)v_1^2(v_1v_2)+\tfrac{1}{8}(n_{12}v_1)(v_1v_2)^2+\tfrac{15}{16}(n_{12}v_1)v_1^2v_2^2\right)\\
&+\frac{175G^3m_1^2m_2^2}{8r_{12}^4}(n_{12}v_1)+\frac{G^3m_1^3m_2}{r_{12}^4}\left(\frac{46\,517}{840}(n_{12}v_1)-\frac{34\,547}{840}(n_{12}v_2)\right.\\
&\left.\left.-22(n_{12}v_1)\ln\left(\frac{r_{12}}{r_1'}\right)+22(n_{12}v_2)\ln\left(\frac{r_{12}}{r_1'}\right)\right)\right)
\end{aligned}
$$

$$+y_1^j v_2^k\left(\frac{G^2m_1^2m_2}{r_{12}^2}\big(20(n_{12}v_1)^2-\tfrac{97}{4}(n_{12}v_1)(n_{12}v_2)+\tfrac{21}{4}(n_{12}v_2)^2-9v_1^2\right.$$
$$+18(v_1v_2)-9v_2^2\big)$$
$$+\frac{G^2m_1m_2^2}{r_{12}^2}\big(-\tfrac{17}{6}(n_{12}v_1)^2+\tfrac{41}{12}(n_{12}v_1)(n_{12}v_2)+\tfrac{11}{3}(n_{12}v_2)^2-\tfrac{17}{6}v_1^2$$
$$+\tfrac{17}{3}(v_1v_2)-\tfrac{89}{24}v_2^2\big)$$
$$+\frac{Gm_1m_2}{r_{12}}\big(\tfrac{1}{16}(n_{12}v_1)^4+\tfrac{1}{8}(n_{12}v_1)^3(n_{12}v_2)+\tfrac{3}{16}(n_{12}v_1)^2(n_{12}v_2)^2$$
$$+\tfrac{1}{4}(n_{12}v_1)(n_{12}v_2)^3-\tfrac{19}{16}(n_{12}v_2)^4-\tfrac{5}{16}(n_{12}v_1)^2v_1^2-\tfrac{1}{2}(n_{12}v_1)(n_{12}v_2)v_1^2$$
$$+\tfrac{3}{8}(n_{12}v_2)^2v_1^2-\tfrac{13}{16}v_1^4+\tfrac{3}{8}(n_{12}v_1)^2(v_1v_2)+\tfrac{3}{4}(n_{12}v_1)(n_{12}v_2)(v_1v_2)$$
$$-\tfrac{3}{4}(n_{12}v_2)^2(v_1v_2)+v_1^2(v_1v_2)+\tfrac{1}{8}(v_1v_2)^2-\tfrac{1}{4}(n_{12}v_1)^2v_2^2-\tfrac{5}{8}(n_{12}v_1)(n_{12}v_2)v_2^2$$
$$+\tfrac{45}{16}(n_{12}v_2)^2v_2^2-\tfrac{17}{16}v_1^2v_2^2+\tfrac{17}{8}(v_1v_2)v_2^2-\tfrac{43}{16}v_2^4\big)$$
$$+\frac{G^3m_1^2m_2^2}{r_{12}^3}\left(\frac{1217}{36}-\frac{41}{32}\pi^2\right)+\frac{G^3m_1^3m_2}{r_{12}^3}\left(-\frac{27\,967}{2520}+\frac{22}{3}\ln\left(\frac{r_{12}}{r_1'}\right)\right)$$
$$\left.\left.+\frac{G^3m_1m_2^3}{r_{12}^3}\left(-\frac{17\,501}{1260}+\frac{22}{3}\ln\left(\frac{r_{12}}{r_2'}\right)\right)\right)\right\}+1\leftrightarrow 2+\mathcal{O}\left(\frac{1}{c^7}\right). \tag{4.4}$$

The last constant of the motion is the vector K^i. We prefer to present the vector $G^i = P^i t + K^i$ which represents the centre-of-mass position and varies linearly with time,

$$G^i = m_1y_1^i+\frac{1}{c^2}\left\{y_1^i\left(-\frac{Gm_1m_2}{2r_{12}}+\frac{m_1v_1^2}{2}\right)\right\}+\frac{1}{c^4}\left\{v_1^iGm_1m_2\big(-\tfrac{7}{4}(n_{12}v_1)-\tfrac{7}{4}(n_{12}v_2)\big)\right.$$
$$+y_1^i\left(-\frac{5G^2m_1^2m_2}{4r_{12}^2}+\frac{7G^2m_1m_2^2}{4r_{12}^2}+\frac{3m_1v_1^4}{8}\right.$$
$$+\frac{Gm_1m_2}{r_{12}}\big(-\tfrac{1}{8}(n_{12}v_1)^2-\tfrac{1}{4}(n_{12}v_1)(n_{12}v_2)+\tfrac{1}{8}(n_{12}v_2)^2+\tfrac{19}{8}v_1^2$$
$$\left.\left.-\tfrac{7}{4}(v_1v_2)-\tfrac{7}{8}v_2^2\big)\right)\right\}$$
$$+\frac{1}{c^6}\left\{v_1^i\left(\frac{235G^2m_1^2m_2}{24r_{12}}((n_{12}v_1)-(n_{12}v_2))\right.\right.$$
$$-\frac{235G^2m_1m_2^2}{24r_{12}}((n_{12}v_1)-(n_{12}v_2))$$
$$+Gm_1m_2\big(\tfrac{5}{12}(n_{12}v_1)^3+\tfrac{3}{8}(n_{12}v_1)^2(n_{12}v_2)+\tfrac{3}{8}(n_{12}v_1)(n_{12}v_2)^2+\tfrac{5}{12}(n_{12}v_2)^3$$
$$-\tfrac{15}{8}(n_{12}v_1)v_1^2-(n_{12}v_2)v_1^2+\tfrac{1}{4}(n_{12}v_1)(v_1v_2)+\tfrac{1}{4}(n_{12}v_2)(v_1v_2)$$
$$\left.-(n_{12}v_1)v_2^2-\tfrac{15}{8}(n_{12}v_2)v_2^2\big)\right)$$

$$+y_1^i\bigg(\frac{5m_1v_1^6}{16}+\frac{Gm_1m_2}{r_{12}}\big(\tfrac{1}{16}(n_{12}v_1)^4+\tfrac{1}{8}(n_{12}v_1)^3(n_{12}v_2)+\tfrac{3}{16}(n_{12}v_1)^2(n_{12}v_2)^2$$
$$+\tfrac{1}{4}(n_{12}v_1)(n_{12}v_2)^3-\tfrac{1}{16}(n_{12}v_2)^4-\tfrac{5}{16}(n_{12}v_1)^2v_1^2-\tfrac{1}{2}(n_{12}v_1)(n_{12}v_2)v_1^2$$
$$-\tfrac{11}{8}(n_{12}v_2)^2v_1^2+\tfrac{53}{16}v_1^4+\tfrac{3}{8}(n_{12}v_1)^2(v_1v_2)+\tfrac{3}{4}(n_{12}v_1)(n_{12}v_2)(v_1v_2)$$
$$+\tfrac{5}{4}(n_{12}v_2)^2(v_1v_2)-5v_1^2(v_1v_2)+\tfrac{17}{8}(v_1v_2)^2-\tfrac{1}{4}(n_{12}v_1)^2v_2^2$$
$$-\tfrac{5}{8}(n_{12}v_1)(n_{12}v_2)v_2^2+\tfrac{5}{16}(n_{12}v_2)^2v_2^2+\tfrac{31}{16}v_1^2v_2^2-\tfrac{15}{8}(v_1v_2)v_2^2-\tfrac{11}{16}v_2^4\big)$$
$$+\frac{G^2m_1^2m_2}{r_{12}^2}\big(\tfrac{79}{12}(n_{12}v_1)^2-\tfrac{17}{3}(n_{12}v_1)(n_{12}v_2)+\tfrac{17}{6}(n_{12}v_2)^2-\tfrac{175}{24}v_1^2$$
$$+\tfrac{40}{3}(v_1v_2)-\tfrac{20}{3}v_2^2\big)$$
$$+\frac{G^2m_1m_2^2}{r_{12}^2}\big(-\tfrac{7}{3}(n_{12}v_1)^2+\tfrac{29}{12}(n_{12}v_1)(n_{12}v_2)+\tfrac{2}{3}(n_{12}v_2)^2+\tfrac{101}{12}v_1^2$$
$$-\tfrac{40}{3}(v_1v_2)+\tfrac{139}{24}v_2^2\big)$$
$$-\frac{19G^3m_1^2m_2^2}{8r_{12}^3}+\frac{G^3m_1^3m_2}{r_{12}^3}\left(\frac{13\,721}{1260}-\frac{22}{3}\ln\left(\frac{r_{12}}{r_1'}\right)\right)$$
$$+\frac{G^3m_1m_2^3}{r_{12}^3}\left(-\frac{14\,351}{1260}+\frac{22}{3}\ln\left(\frac{r_{12}}{r_2'}\right)\right)\bigg)\bigg\}+1\leftrightarrow 2+\mathcal{O}\left(\frac{1}{c^7}\right). \tag{4.5}$$

We checked that this expression of the harmonic-coordinate centre of mass is changed under the contact transformation into the ADM-coordinate expression which is given by equations (16)–(22) in [24]. Note that the energy E is the only one among these integrals of the 3PN motion that depends on the unknown constant λ. The other integrals P^i, J^i and G^i do not depend on λ and therefore are entirely determined.

The latter Noetherian quantities are no longer conserved when we take into account the radiation reaction effect at the 2.5PN order. In order to express the resulting balance equations in the best way, we modify of all these quantities by certain terms of the order of 2.5PN and find that the right-hand sides of the equations take the form appropriate for a radiative flux at infinity. We pose

$$\widetilde{E}=E+\frac{4G^2m_1^2m_2}{5c^5r_{12}^2}(n_{12}v_1)\left[(v_1-v_2)^2+\frac{2G(m_2-m_1)}{r_{12}}\right]+1\leftrightarrow 2, \tag{4.6a}$$

$$\widetilde{P}^i=P^i+\frac{4G^2m_1^2m_2}{5c^5r_{12}^2}n_{12}^i\left((v_1-v_2)^2-\frac{2Gm_1}{r_{12}}\right)+1\leftrightarrow 2, \tag{4.6b}$$

$$\widetilde{J}^i=J^i+\frac{4Gm_1m_2}{5c^5}\varepsilon_{ijk}\bigg[v_1^2v_1^jv_2^k+2\frac{Gm_1}{r_{12}}v_1^jv_2^k-\frac{2Gm_1}{r_{12}^2}(n_{12}v_1)\big(v_1^jy_2^k+y_1^jv_1^k\big)$$
$$-\frac{Gm_1}{r_{12}^3}(v_1-v_2)^2y_1^jy_2^k+\frac{2G^2m_1^2}{r_{12}^4}y_1^jy_2^k\bigg]+1\leftrightarrow 2, \tag{4.6c}$$

$$\widetilde{G}^i=G^i+\frac{4Gm_1m_2}{5c^5}v_1^i\left((v_1-v_2)^2-\frac{2G(m_1+m_2)}{r_{12}}\right)+1\leftrightarrow 2 \tag{4.6d}$$

as well as $\widetilde{K}^i = \widetilde{G}^i - t\widetilde{P}^i$. Then, the 3PN balance equations are given by

$$\frac{\mathrm{d}\widetilde{E}}{\mathrm{d}t} = -\frac{G}{5c^5}\frac{\mathrm{d}^3 Q_{ij}}{\mathrm{d}t^3}\frac{\mathrm{d}^3 Q_{ij}}{\mathrm{d}t^3} + \mathcal{O}\left(\frac{1}{c^7}\right), \tag{4.7a}$$

$$\frac{\mathrm{d}\widetilde{P}^i}{\mathrm{d}t} = \mathcal{O}\left(\frac{1}{c^7}\right), \tag{4.7b}$$

$$\frac{\mathrm{d}\widetilde{J}^i}{\mathrm{d}t} = -\frac{2G}{5c^5}\varepsilon_{ijk}\frac{\mathrm{d}^2 Q_{jl}}{\mathrm{d}t^2}\frac{\mathrm{d}^3 Q_{kl}}{\mathrm{d}t^3} + \mathcal{O}\left(\frac{1}{c^7}\right), \tag{4.7c}$$

$$\frac{\mathrm{d}\widetilde{K}^i}{\mathrm{d}t} = \mathcal{O}\left(\frac{1}{c^7}\right), \tag{4.7d}$$

where the Newtonian trace-free quadrupole moment is $Q_{ij} = m_1(y_1^i y_1^j - \frac{1}{3}\delta^{ij}y_1^2) + 1 \leftrightarrow 2$.

4.2. *Contact transformation and the ADM Hamiltonian at the 3PN order*

Our final result for the contact transformation (3.16) is as follows. The first term in (3.16) is composed of the conjugate momentum of the acceleration and is readily obtained by differentiating (4.1):

$$\begin{aligned}
q_1^i = {}& \frac{1}{c^4}\left\{n_{12}^i\left(-\tfrac{1}{8}Gm_1m_2(n_{12}v_2)^2 + \tfrac{7}{8}Gm_1m_2v_2^2\right) - \tfrac{7}{4}Gm_1m_2(n_{12}v_2)v_2^i\right\} \\
& + \frac{1}{c^6}\left\{n_{12}^i\left(\frac{G^2m_1^2m_2}{r_{12}}\left(-\tfrac{17}{6}(n_{12}v_2)^2 + \tfrac{185}{16}v_1^2 - \tfrac{185}{8}(v_1v_2) + \tfrac{20}{3}v_2^2\right)\right.\right. \\
& + \frac{G^2m_1m_2^2}{r_{12}}\left(\tfrac{29}{24}(n_{12}v_2)^2 + \tfrac{235}{48}v_2^2\right) + Gm_1m_2\left(\tfrac{1}{8}(n_{12}v_1)(n_{12}v_2)^3 + \tfrac{1}{16}(n_{12}v_2)^4\right. \\
& \left.\left. + \tfrac{3}{8}(n_{12}v_2)^2(v_1v_2) - \tfrac{5}{8}(n_{12}v_1)^2v_2^2 - \tfrac{1}{2}(n_{12}v_1)(n_{12}v_2)v_2^2 - \tfrac{5}{16}(n_{12}v_2)^2v_2^2\right)\right) \\
& + v_1^i\left(Gm_1m_2\left(\tfrac{11}{4}(n_{12}v_2)v_1^2 - 2(n_{12}v_2)(v_1v_2) + \tfrac{15}{8}(n_{12}v_2)v_2^2\right)\right) \\
& + v_2^i\left(-\frac{235G^2m_1^2m_2}{24r_{12}}(n_{12}v_2) + \frac{235G^2m_1m_2^2}{24r_{12}}(n_{12}v_2)\right. \\
& + Gm_1m_2\left(\tfrac{3}{8}(n_{12}v_1)(n_{12}v_2)^2 + \tfrac{5}{12}(n_{12}v_2)^3 - (n_{12}v_2)v_1^2\right. \\
& \left.\left.\left. + \tfrac{1}{4}(n_{12}v_2)(v_1v_2) - \tfrac{15}{8}(n_{12}v_2)v_2^2\right)\right)\right\} + \mathcal{O}\left(\frac{1}{c^7}\right).
\end{aligned} \tag{4.8}$$

The second term in (3.16) involves the function F that constitutes the only possible freedom to adjust in order to match the harmonic-coordinate and ADM Hamiltonian formalisms. This F was determined uniquely as

$$\begin{aligned}
F = {}& \frac{1}{c^4}\left\{\frac{G^2m_1^2m_2}{r_{12}}\left(\tfrac{7}{4}(n_{12}v_1) - \tfrac{1}{4}(n_{12}v_2)\right) + \frac{Gm_1m_2}{4}(n_{12}v_2)v_1^2\right\} \\
& + \frac{1}{c^6}\left\{\frac{G^2m_1^2m_2}{r_{12}}\left(-\tfrac{91}{144}(n_{12}v_1)^3 + \tfrac{21}{16}(n_{12}v_1)^2(n_{12}v_2) - \tfrac{113}{24}(n_{12}v_1)v_1^2\right.\right. \\
& \left. + \tfrac{35}{8}(n_{12}v_2)v_1^2 + \tfrac{195}{16}(n_{12}v_1)(v_1v_2) - \tfrac{3}{4}(n_{12}v_1)v_2^2 - \tfrac{1}{8}(n_{12}v_2)v_2^2\right)
\end{aligned}$$

$$
\begin{aligned}
&+Gm_1m_2\left(-\tfrac{1}{16}(n_{12}v_1)(n_{12}v_2)^2v_1^2-\tfrac{5}{24}(n_{12}v_2)^3v_1^2-\tfrac{1}{2}(n_{12}v_2)v_1^4\right.\\
&\left.+\tfrac{1}{8}(n_{12}v_2)v_1^2(v_1v_2)+\tfrac{5}{16}(n_{12}v_1)v_1^2v_2^2\right)\\
&+\frac{G^3m_1^2m_2^2}{r_{12}^2}\left(\tfrac{245}{18}(n_{12}v_1)-\tfrac{21}{32}(n_{12}v_1)\pi^2\right)\\
&\left.+\frac{G^3m_1^3m_2}{r_{12}^2}\left(-\frac{25\,867}{2520}(n_{12}v_1)-\frac{3}{4}(n_{12}v_2)+\frac{22}{3}(n_{12}v_1)\ln\left(\frac{r_{12}}{r'_1}\right)\right)\right\}\\
&+1\leftrightarrow 2+\mathcal{O}\left(\frac{1}{c^7}\right).
\end{aligned}
\tag{4.9}
$$

Note the dependence of F on the logarithms, namely

$$
\frac{22}{3}\frac{G^3m_1^3m_2}{c^6r_{12}^2}(n_{12}v_1)\ln\left(\frac{r_{12}}{r'_1}\right)-\frac{22}{3}\frac{G^3m_1m_2^3}{c^6r_{12}^2}(n_{12}v_2)\ln\left(\frac{r_{12}}{r'_2}\right),
$$

which is necessary in order for the contact transformation to remove the logarithms of the harmonic-coordinate Lagrangian (4.1). This result can be checked to be in agreement with the coordinate transformation given by equations (7.2) in [29]. The third term in (3.16) involves a correction term, purely of the order of 3PN, which is defined by (3.17). For this term we obtain

$$
\begin{aligned}
\frac{1}{c^6}X_1^i=\frac{1}{c^6}\Bigg\{&n_{12}^i\left(-\frac{G^3m_1^3m_2}{r_{12}^2}-\frac{49}{4}\frac{G^3m_1^2m_2^2}{r_{12}^2}-\frac{3}{4}\frac{G^3m_1m_2^3}{r_{12}^2}\right.\\
&+\frac{G^2m_1^2m_2}{r_{12}}\left(\tfrac{11}{8}(n_{12}v_1)^2-\tfrac{1}{4}(n_{12}v_1)(n_{12}v_2)-\tfrac{27}{8}v_1^2\right)\\
&\left.+\frac{G^2m_1m_2^2}{r_{12}}\left(\tfrac{3}{8}(n_{12}v_2)^2-\tfrac{1}{8}v_1^2-\tfrac{15}{8}v_2^2\right)+Gm_1m_2\left(\tfrac{1}{16}(n_{12}v_2)^2v_1^2-\tfrac{5}{16}v_1^2v_2^2\right)\right)\\
&+v_1^i\left(\frac{35G^2m_1^2m_2}{8r_{12}}(n_{12}v_1)+\frac{G^2m_1m_2^2}{r_{12}}\left(-\tfrac{1}{4}(n_{12}v_1)-\tfrac{3}{2}(n_{12}v_2)\right)\right.\\
&\left.+Gm_1m_2\left(\tfrac{1}{8}(n_{12}v_1)(n_{12}v_2)^2-\tfrac{3}{4}(n_{12}v_2)v_1^2+\tfrac{7}{4}(n_{12}v_2)(v_1v_2)-\tfrac{5}{8}(n_{12}v_1)v_2^2\right)\right)\\
&+v_2^i\left(-\frac{7G^2m_1^2m_2}{4r_{12}}(n_{12}v_1)+\frac{21G^2m_1m_2^2}{4r_{12}}(n_{12}v_2)+\frac{7Gm_1m_2}{8}(n_{12}v_2)v_1^2\right)\Bigg\}\\
&+\mathcal{O}\left(\frac{1}{c^7}\right).
\end{aligned}
\tag{4.10}
$$

The term X_2^i is obtained by relabelling $1 \leftrightarrow 2$. With those results we obtain the ADM Lagrangian (3.19) which is an ordinary Lagrangian, not containing any accelerations, and furthermore not containing any logarithms. Though the investigations in section 3.2 were done with the harmonic-coordinate quantities taken as 'dummy' variables, we must present here the ADM Lagrangian in terms of the variables corresponding to the motion in ADM coordinates. We denote them exactly like in harmonic coordinates but with uppercase letters, e.g. $R_{12} = |\boldsymbol{Y}_1 - \boldsymbol{Y}_2|$, $\boldsymbol{N}_{12} = (\boldsymbol{Y}_1 - \boldsymbol{Y}_2)/R_{12}$, $(N_{12}V_2) = \boldsymbol{N}_{12}\cdot\boldsymbol{V}_2$,

$$\begin{aligned}
L^{\mathrm{ADM}} = {}& \frac{Gm_1m_2}{2R_{12}} + \frac{1}{2}m_1V_1^2 + \frac{1}{c^2}\Bigg\{-\frac{G^2m_1^2m_2}{2R_{12}^2} + \frac{1}{8}m_1V_1^4 \\
&+\frac{Gm_1m_2}{R_{12}}\left(-\tfrac{1}{4}(N_{12}V_1)(N_{12}V_2) + \tfrac{3}{2}V_1^2 - \tfrac{7}{4}(V_1V_2)\right)\Bigg\} \\
&+\frac{1}{c^4}\Bigg\{\frac{G^3m_1^3m_2}{4R_{12}^3} + \frac{5G^3m_1^2m_2^2}{8R_{12}^3} + \frac{m_1V_1^6}{16} \\
&+\frac{G^2m_1^2m_2}{R_{12}^2}\left(\tfrac{15}{8}(N_{12}V_1)^2 + \tfrac{11}{8}V_1^2 - \tfrac{15}{4}(V_1V_2) + 2V_2^2\right) \\
&+\frac{Gm_1m_2}{R_{12}}\left(\tfrac{3}{16}(N_{12}V_1)^2(N_{12}V_2)^2 - \tfrac{1}{4}(N_{12}V_1)(N_{12}V_2)V_1^2 - \tfrac{5}{8}(N_{12}V_2)^2V_1^2\right. \\
&\left.+\tfrac{7}{8}V_1^4 + \tfrac{3}{4}(N_{12}V_1)(N_{12}V_2)(V_1V_2) - \tfrac{7}{4}V_1^2(V_1V_2) + \tfrac{1}{8}(V_1V_2)^2 + \tfrac{11}{16}V_1^2V_2^2\right)\Bigg\} \\
&+\frac{1}{c^6}\Bigg\{-\frac{G^4m_1^4m_2}{8R_{12}^4} + \frac{G^4m_1^3m_2^2}{R_{12}^4}\left(-\tfrac{993}{140} + \tfrac{11}{3}\lambda + \tfrac{21}{32}\pi^2\right) + \frac{5m_1V_1^8}{128} \\
&+\frac{Gm_1m_2}{R_{12}}\left(-\tfrac{5}{32}(N_{12}V_1)^3(N_{12}V_2)^3 + \tfrac{3}{16}(N_{12}V_1)^2(N_{12}V_2)^2V_1^2\right. \\
&+\tfrac{9}{16}(N_{12}V_1)(N_{12}V_2)^3V_1^2 - \tfrac{3}{16}(N_{12}V_1)(N_{12}V_2)V_1^4 - \tfrac{5}{16}(N_{12}V_2)^2V_1^4 \\
&+\tfrac{11}{16}V_1^6 - \tfrac{15}{32}(N_{12}V_1)^2(N_{12}V_2)^2(V_1V_2) + \tfrac{3}{4}(N_{12}V_1)(N_{12}V_2)V_1^2(V_1V_2) \\
&-\tfrac{1}{16}(N_{12}V_2)^2V_1^2(V_1V_2) - \tfrac{21}{16}V_1^4(V_1V_2) + \tfrac{5}{16}(N_{12}V_1)(N_{12}V_2)(V_1V_2)^2 \\
&+\tfrac{1}{8}V_1^2(V_1V_2)^2 + \tfrac{1}{16}(V_1V_2)^3 - \tfrac{5}{16}(N_{12}V_1)^2V_1^2V_2^2 - \tfrac{9}{32}(N_{12}V_1)(N_{12}V_2)V_1^2V_2^2 \\
&\left.+\tfrac{7}{8}V_1^4V_2^2 - \tfrac{15}{32}V_1^2(V_1V_2)V_2^2\right) \\
&+\frac{G^2m_1^2m_2}{R_{12}^2}\left(-\tfrac{5}{12}(N_{12}V_1)^4 - \tfrac{13}{8}(N_{12}V_1)^3(N_{12}V_2) - \tfrac{23}{24}(N_{12}V_1)^2(N_{12}V_2)^2\right. \\
&+\tfrac{13}{16}(N_{12}V_1)^2V_1^2 + \tfrac{1}{4}(N_{12}V_1)(N_{12}V_2)V_1^2 + \tfrac{5}{6}(N_{12}V_2)^2V_1^2 + \tfrac{21}{16}V_1^4 \\
&-\tfrac{1}{2}(N_{12}V_1)^2(V_1V_2) + \tfrac{1}{3}(N_{12}V_1)(N_{12}V_2)(V_1V_2) - \tfrac{97}{16}V_1^2(V_1V_2) + \tfrac{341}{48}(V_1V_2)^2 \\
&\left.+\tfrac{29}{24}(N_{12}V_1)^2V_2^2 - (N_{12}V_1)(N_{12}V_2)V_2^2 + \tfrac{43}{12}V_1^2V_2^2 - \tfrac{71}{8}(V_1V_2)V_2^2 + \tfrac{47}{16}V_2^4\right) \\
&+\frac{G^3m_1^2m_2^2}{R_{12}^3}\left(\tfrac{73}{16}(N_{12}V_1)^2 - 11(N_{12}V_1)(N_{12}V_2) + \tfrac{3}{64}\pi^2(N_{12}V_1)^2\right. \\
&\left.-\tfrac{3}{64}\pi^2(N_{12}V_1)(N_{12}V_2) - \tfrac{265}{48}V_1^2 - \tfrac{1}{64}\pi^2V_1^2 + \tfrac{59}{8}(V_1V_2) + \tfrac{1}{64}\pi^2(V_1V_2)\right) \\
&+\frac{G^3m_1^3m_2}{R_{12}^3}\left(-5(N_{12}V_1)^2 - \tfrac{1}{8}(N_{12}V_1)(N_{12}V_2) + \tfrac{173}{48}V_1^2 - \tfrac{27}{8}(V_1V_2) + \tfrac{13}{8}V_2^2\right)\Bigg\} \\
&+1\leftrightarrow 2 + \mathcal{O}\left(\frac{1}{c^7}\right). \qquad (4.11)
\end{aligned}$$

The corresponding ADM (or, rather, ADM-type [24]) Hamiltonian is given by the ordinary Legendre transformation (3.21) as

$$
\begin{aligned}
H^{\mathrm{ADM}} = &-\frac{Gm_1m_2}{2R_{12}} + \frac{P_1^2}{2m_1} + \frac{1}{c^2}\left\{-\frac{P_1^4}{8m_1^3} + \frac{G^2m_1^2m_2}{2R_{12}^2}\right.\\
&\left.+\frac{Gm_1m_2}{R_{12}}\left(\frac{1}{4}\frac{(N_{12}P_1)(N_{12}P_2)}{m_1m_2} - \frac{3}{2}\frac{P_1^2}{m_1^2} + \frac{7}{4}\frac{(P_1P_2)}{m_1m_2}\right)\right\}\\
&+\frac{1}{c^4}\left\{\frac{P_1^6}{16m_1^5} - \frac{G^3m_1^3m_2}{4R_{12}^3} - \frac{5G^3m_1^2m_2^2}{8R_{12}^3}\right.\\
&+\frac{G^2m_1^2m_2}{R_{12}^2}\left(-\frac{3}{2}\frac{(N_{12}P_1)(N_{12}P_2)}{m_1m_2} + \frac{19}{4}\frac{P_1^2}{m_1^2} - \frac{27}{4}\frac{(P_1P_2)}{m_1m_2} + \frac{5P_2^2}{2m_2^2}\right)\\
&+\frac{Gm_1m_2}{R_{12}}\left(-\frac{3}{16}\frac{(N_{12}P_1)^2(N_{12}P_2)^2}{m_1^2m_2^2} + \frac{5}{8}\frac{(N_{12}P_2)^2P_1^2}{m_1^2m_2^2} + \frac{5}{8}\frac{P_1^4}{m_1^4}\right.\\
&\left.\left.-\frac{3}{4}\frac{(N_{12}P_1)(N_{12}P_2)(P_1P_2)}{m_1^2m_2^2} - \frac{1}{8}\frac{(P_1P_2)^2}{m_1^2m_2^2} - \frac{11}{16}\frac{P_1^2P_2^2}{m_1^2m_2^2}\right)\right\}\\
&+\frac{1}{c^6}\left\{-\frac{5P_1^8}{128m_1^7} + \left(\frac{G^4m_1^4m_2}{8R_{12}^4} + \frac{G^4m_1^3m_2^2}{R_{12}^4}\left(\tfrac{993}{140} - \tfrac{11}{3}\lambda - \tfrac{21}{32}\pi^2\right)\right)\right.\\
&+\frac{G^3m_1^2m_2^2}{R_{12}^3}\left(-\frac{43}{16}\frac{(N_{12}P_1)^2}{m_1^2} + \frac{119}{16}\frac{(N_{12}P_1)(N_{12}P_2)}{m_1m_2} - \frac{3}{64}\pi^2\frac{(N_{12}P_1)^2}{m_1^2}\right.\\
&+\frac{3}{64}\pi^2\frac{(N_{12}P_1)(N_{12}P_2)}{m_1m_2} - \frac{473}{48}\frac{P_1^2}{m_1^2} + \frac{1}{64}\pi^2\frac{P_1^2}{m_1^2} + \frac{143}{16}\frac{(P_1P_2)}{m_1m_2}\\
&\left.-\frac{1}{64}\pi^2\frac{(P_1P_2)}{m_1m_2}\right)\\
&+\frac{G^3m_1^3m_2}{R_{12}^3}\left(\frac{5}{4}\frac{(N_{12}P_1)^2}{m_1^2} + \frac{21}{8}\frac{(N_{12}P_1)(N_{12}P_2)}{m_1m_2} - \frac{425}{48}\frac{P_1^2}{m_1^2}\right.\\
&\left.+\frac{77}{8}\frac{(P_1P_2)}{m_1m_2} - \frac{25P_2^2}{8m_2^2}\right)\\
&+\frac{G^2m_1^2m_2}{R_{12}^2}\left(\frac{5}{12}\frac{(N_{12}P_1)^4}{m_1^4} - \frac{3}{2}\frac{(N_{12}P_1)^3(N_{12}P_2)}{m_1^3m_2} + \frac{10}{3}\frac{(N_{12}P_1)^2(N_{12}P_2)^2}{m_1^2m_2^2}\right.\\
&+\frac{17}{16}\frac{(N_{12}P_1)^2P_1^2}{m_1^4} - \frac{15}{8}\frac{(N_{12}P_1)(N_{12}P_2)P_1^2}{m_1^3m_2} - \frac{55}{12}\frac{(N_{12}P_2)^2P_1^2}{m_1^2m_2^2} + \frac{P_1^4}{16m_1^4}\\
&-\frac{11}{8}\frac{(N_{12}P_1)^2(P_1P_2)}{m_1^3m_2} + \frac{125}{12}\frac{(N_{12}P_1)(N_{12}P_2)(P_1P_2)}{m_1^2m_2^2} - \frac{115}{16}\frac{P_1^2(P_1P_2)}{m_1^3m_2}\\
&\left.+\frac{25}{48}\frac{(P_1P_2)^2}{m_1^2m_2^2} - \frac{193}{48}\frac{(N_{12}P_1)^2P_2^2}{m_1^2m_2^2} + \frac{371}{48}\frac{P_1^2P_2^2}{m_1^2m_2^2} - \frac{27}{16}\frac{P_2^4}{m_2^4}\right)\\
&+\frac{Gm_1m_2}{R_{12}}\left(\frac{5}{32}\frac{(N_{12}P_1)^3(N_{12}P_2)^3}{m_1^3m_2^3} + \frac{3}{16}\frac{(N_{12}P_1)^2(N_{12}P_2)^2P_1^2}{m_1^4m_2^2}\right.
\end{aligned}
$$

$$-\frac{9}{16}\frac{(N_{12}P_1)(N_{12}P_2)^3P_1^2}{m_1^3m_2^3}-\frac{5}{16}\frac{(N_{12}P_2)^2P_1^4}{m_1^4m_2^2}-\frac{7}{16}\frac{P_1^6}{m_1^6}$$
$$+\frac{15}{32}\frac{(N_{12}P_1)^2(N_{12}P_2)^2(P_1P_2)}{m_1^3m_2^3}+\frac{3}{4}\frac{(N_{12}P_1)(N_{12}P_2)P_1^2(P_1P_2)}{m_1^4m_2^2}$$
$$+\frac{1}{16}\frac{(N_{12}P_2)^2P_1^2(P_1P_2)}{m_1^3m_2^3}-\frac{5}{16}\frac{(N_{12}P_1)(N_{12}P_2)(P_1P_2)^2}{m_1^3m_2^3}+\frac{1}{8}\frac{P_1^2(P_1P_2)^2}{m_1^4m_2^2}$$
$$-\frac{1}{16}\frac{(P_1P_2)^3}{m_1^3m_2^3}-\frac{5}{16}\frac{(N_{12}P_1)^2P_1^2P_2^2}{m_1^4m_2^2}+\frac{7}{32}\frac{(N_{12}P_1)(N_{12}P_2)P_1^2P_2^2}{m_1^3m_2^3}+\frac{1}{2}\frac{P_1^4P_2^2}{m_1^4m_2^2}$$
$$\left.+\frac{1}{32}\frac{P_1^2(P_1P_2)P_2^2}{m_1^3m_2^3}\right)\Bigg\}+1\leftrightarrow 2+\mathcal{O}\left(\frac{1}{c^7}\right). \tag{4.12}$$

This result is in perfect agreement with the expression obtained by Damour *et al* [24]. (Note that in their published result, equation (12) in [24], the following terms are missing:

$$\frac{G^2}{c^6r_{12}^2}\left(-\tfrac{55}{12}m_1-\tfrac{193}{48}m_2\right)\frac{(N_{12}P_2)^2P_1^2}{m_1m_2}+1\leftrightarrow 2.$$

This is a misprint which has been corrected in an erratum [24].) Finally, we recall that the agreement works if and only if our undetermined constant λ is related to their static-ambiguity constant ω_{static} by equation (1.1), and their kinetic-ambiguity constant takes the value $\omega_{\text{kinetic}}=\frac{41}{24}$. This completes the proof of the equivalence of the harmonic-coordinate and ADM-Hamiltonian approaches to the equations of motion of compact binaries at the 3PN order.

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