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On the multipole expansion of the gravitational field

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Abstract. The multipole expansion (in general relativity) of the gravitational field generated by a slowly-moving isolated source is constructed. We introduce some definitions for the source multipole moments, valid to all orders in a post-Newtonian expansion, and depending in a well defined way on the total stress–energy pseudo-tensor of the material and gravitational fields. The source moments parametrize the linearized approximation of the gravitational field exterior to the source, as computed by means of a specific post-Minkowskian algorithm defined in a previous work. Since the radiative multipole moments parametrizing the radiation field far from the source can be obtained as nonlinear functionals of the source moments, the present paper allows one to relate the radiation field far from a slowly-moving source to the stress–energy pseudo-tensor of the source. This should be useful when comparing theory with the future observations of gravitational radiation by the LIGO and VIRGO experiments.

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1. Introduction

The multipole expansion is one of the most useful tools of theoretical physics. Extensively used in the past for dealing with wave emission and/or propagation problems in electromagnetism, the multipole expansion has also contributed much to gravitational radiation theory, starting with Einstein's pioneering work [1] (see also [2]) showing that gravitational radiation from a localized slowly moving source is dominantly quadrupolar, unlike electromagnetic radiation which is dipolar in general.

The multipole expansion of the field generated by an isolated gravitating source can be said to be fully understood at the *linearized* approximation of general relativity. Some early studies extended the Einstein quadrupole formula for the energy to include higher multipole contributions [3, 4], and obtained the corresponding formulae for linear momentum [3–6] and angular momentum [7, 8]. Then a number of authors [9–12] investigated the general structure of the *infinite* multipole expansion of the linearized field outside the source, and computed the associated fluxes of energy and momenta. It emerged from these works that the multipole expansion is entirely characterized by two and only two sets of multipole moments, which are the analogues of the electric and magnetic moments of electromagnetism and, in the gravitational case, are referred to as the mass and current moments. Notably, the full expression of the linearized vacuum metric as parametrized by symmetric and trace-free (STF) mass and current multipole moments was given by Thorne [12] (we shall refer below to this expression as 'canonical'). The use of STF harmonics is very convenient when performing computations of gravitational radiation [12].

Knowing the structure of the multipole expansion is not sufficient by itself, and one must relate the multipole moments to the material content of the source. In linearized gravity this relationship was achieved in a second series of papers [13–16], in which the multipole moments of the source were obtained as some explicit integrals extending over the stress–energy distribution of the matter fields, independently of any assumption concerning the source (such as being fast moving). The last of these papers, by Damour and Iyer [16], put the finishing touches to the theory by expressing the multipole expansion in STF form, and by correcting some errors in the previous paper [15].

The problem is more difficult, but more interesting, in the full nonlinear theory where the (source) multipole moments mix with each other causing nonlinear effects in the radiation field, for instance the so-called 'tail' effect—scattering of the quadrupolar waves off the spacetime generated by the mass monopole of the source. Several approaches have attempted to define a multipole expansion consistent with the nonlinear field equations. In particular, when the source is stationary (i.e. existence of a timelike Killing vector), the multipole moments in the far field are determined by the convergent expansion of the metric at spatial infinity ($r \to +\infty$, $t =$ constant). Several equivalent definitions of the far-field moments are named after Geroch and Hansen [17, 18], Thorne [12] and Simon and Beig [19]. In the non-stationary case, the expansion at future null infinity ($r \to +\infty$, $u =$ constant where u is a null coordinate) was constructed by Bondi *et al* [20] and Sachs [21]. By decomposing the so-called Bondi news function [20] in spherical harmonics (or in STF harmonics), one obtains the multipole moments which are actually 'measured' at infinity (radiative multipole moments). Another attempt at defining the multipole expansion for non-stationary sources is that of Bonnor and collaborators [22–25], who combined the multipole expansion with a weak-field or post-Minkowskian expansion (i.e. a power series in Newton's constant G). This type of approach, in which the multipole moments can be viewed as the source moments (distinct from the radiative moments), was later investigated by Thorne [12] and Blanchet and Damour [26] (we shall refer to [26] as paper I).

The main result of paper I was to show that one can define, starting from the Thorne 'canonical' linearized metric, an explicit *algorithm* for the computation of the exterior field up to any order in the post-Minkowskian expansion. The resulting metric represents the most general post-Minkowskian solution of the vacuum equations outside the source (up to a coordinate transformation). It is parametrized by two and only two sets of STF (source) multipole moments. Following paper I, we shall call this nonlinear metric MPM (multipolar post-Minkowskian). It was also proved [27] that the re-expansion of the MPM metric at future null infinity is consistent with the corresponding expansion constructed by Bondi *et al* [20]. This permits defining the radiative multipole moments within the framework of MPM metrics.

After dealing with the structure of the multipole expansion [12, 22, 26, 27], the next step is, like in the linearized theory, to express the multipole moments as explicit integrals extending over the source. At the leading order in a slow-motion expansion (speed of light $c \to \infty$), the multipole moments are given by the standard moments of the Newtonian mass and current densities in the source (see, e.g., [12]). At the first post-Newtonian (1PN) order, the source moments depend on the total stress–energy distribution of the matter fields *and* the gravitational field. Because of the contribution of the gravitational field, the support of the total stress–energy distribution is not spatially compact, and therefore the standard formulae used in the linearized theory [13–16] do not apply. If, nevertheless, one tries to define the source multipole moments using these formulae, one obtains some expressions which are (typically) divergent due to the behaviour of the integrals at spatial infinity. This was the approach followed initially by Epstein and Wagoner [28] (see also

[29]) and generalized formally by Thorne [12] to all post-Newtonian orders. The problem of divergences becomes very important starting at the 1.5PN order because of the appearance of nonlinear tails in the radiation field. Indeed one can show [30] that ignoring the divergent terms in the multipole moments as defined previously is equivalent to neglecting the latter tail effects. Recently this problem has been solved by Will and Wiseman [31] who showed how to compute the tails within the Epstein–Wagoner–Thorne approach by means of finite expressions for the multipole moments.

A satisfying derivation of the mass-type multipole moments of the source at 1PN order was done in [32] using a method of asymptotic matching to the interior field of the source. The moments were shown to be actually compact-supported, and to agree with the moments of Epstein and Wagoner modulo the formal discarding of infinite surface terms. The current-type multipole moments at 1PN order were obtained in [33] using a similar method of matching. At the 1PN order there is agreement between the radiative moments (at infinity) and the source moments. Only at the 1.5PN order do they start differing because of the contribution of tails in the wave zone [34].

To 2PN order the multipole moments were obtained using a matching by Blanchet [30] (we shall refer to [30] as paper II), with a rather transparent result which seemed to be amenable to generalization. Namely, it was found that the expressions of the multipole moments depend on the stress–energy (pseudo-)tensor of the matter and gravitational fields in the same way as would be obtained by using incorrectly the formulae valid for compact-supported sources (i.e. *à la* Epstein–Wagoner–Thorne), *but* that the multipole moments are endowed with a finite part operation based on analytic continuation which makes them perfectly well defined mathematically. The latter finite part operation was found to be the same as used in the construction of the exterior field in paper I; it was carried in paper II all the way from the definition of the exterior field to the final expression of the moments.

In this paper we shall show basically that the expressions of the multipole moments at 2PN order given in paper II, i.e. in terms of the total stress–energy pseudo-tensor, are actually valid up to *any* post-Newtonian order. We shall not perform a matching 'order by order' as was done in paper II, but rather construct directly the multipole expansion generated by the total stress–energy pseudo-tensor of the source. This entails finding a formula for the multipole expansion generated by a non-compact-supported source. The multipole moments we shall obtain are valid for slowly-moving sources, to all orders in the slow-motion (post-Newtonian) parameter. Of course, the lowest-order post-Newtonian terms in these general expressions agree with the previous results obtained in [32, 33, 30]. On the other hand, we have agreement in the limit of linearized gravity with the result of [16].

The multipole moments we obtain are fully consistent with the construction of MPM metrics in paper I as they parametrize the so-called 'particular' linearized metric of paper I (which differs from the 'canonical' metric by a gauge transformation). In a sense, the present paper realizes a complete matching of the external field of paper I to a slowly-moving source. Note that the multipole moments, though allowing correctly for all the nonlinearities in the near-zone field, parametrize the *linearized* (exterior) metric which needs to be iterated in order to obtain the radiative moments at infinity (see paper I and [27]). Thus the present results have to be combined with what we know about the relation between the source multipole moments and the radiative ones (see, e.g., [34, 35] and section 6).

Computing the moments to very high post-Newtonian order is part of a research program accompanying the development of the gravitational-wave detectors LIGO and VIRGO. Its motivation comes from the fact that in the case of the radiation emitted by compact binary systems an extremely precise prediction from general relativity will be necessary in order to extract the full potential information contained in the signal. By application of paper II, the theoretical prediction for binary systems was obtained at the 2PN order [36] (and subsequently at 2.5PN order [37]). Will and Wiseman [31] obtained independently the result at 2PN order by application of their improved Epstein–Wagoner formalism. The work at the 3PN order [38] will rely to a large part on the general expressions of the multipole moments as derived in the present paper.

The plan of this paper (besides this introduction and two appendices) is as follows. In section 2 we recall some material from paper I and state our basic assumptions. In section 3 we derive the expression of the multipole expansion of the gravitational field valid outside a general (slowly-moving) source. Rewriting the multipole expansion in a different form, we obtain in section 4 the 'linearized' metric (i.e. a solution of the linearized field equations) which is at the basis of the post-Minkowskian iteration of paper I. Decomposing the linearized metric into irreducible STF tensors yields in section 5 the general expressions for the moments. The paper ends with a discussion on the link between the source moments and the radiative moments (section 6).

2. Review and basic assumptions

The assumptions on which the present investigation is based are twofold. First we have the assumptions concerning the construction of *vacuum* metrics by means of the multipolar post-Minkowskian (MPM) method of paper I [26]. The so-called MPM metrics aim at describing the gravitational field in the region *exterior* to a general isolated system. Second we supplement the MPM method by other assumptions concerning the metric inside the isolated system. Essentially we assume that the metric is everywhere regular (smooth), and admits inside the system a post-Newtonian expansion, which matches in the exterior to the vacuum MPM metric. The matching is understood in the usual sense of matching of asymptotic expansions. Physically the formalism is restricted to slowly-moving sources, whose typical internal velocities define a small post-Newtonian parameter $\sim v/c$.

2.1. Review concerning the exterior field

The multipolar post-Minkowskian 'exterior' metric of paper I is defined in the open domain $\mathbb{R}^3_* \times \mathbb{R}$ (where $\mathbb{R}^3_* = \mathbb{R}^3 - \{\mathbf{0}\}$), i.e. in $\mathbb{R}^4$ deprived of the spatial origin $r \equiv |\boldsymbol{x}| = 0$. We denote it by $h^{\mu\nu}_{\rm ext} \equiv \sqrt{-g_{\rm ext}}\, g^{\mu\nu}_{\rm ext} - \eta^{\mu\nu}$, where $g^{\mu\nu}_{\rm ext}$ is the inverse and $g_{\rm ext}$ the determinant of the usual covariant metric, and where $\eta^{\mu\nu} = \eta_{\mu\nu} = \mathrm{diag}(-1, 1, 1, 1)$ is the Minkowski metric. The MPM metric $h^{\mu\nu}_{\rm ext}$ is in the form of a formal (infinite) post-Minkowskian expansion,

$$h^{\mu\nu}_{\rm ext} = G h^{\mu\nu}_1 + G^2 h^{\mu\nu}_2 + \cdots + G^n h^{\mu\nu}_n + \cdots \tag{2.1}$$

(G is Newton's constant), which is such that all the coefficients of the G^n's admit a *finite* multipolar expansion in symmetric and trace-free (STF) products of unit vectors $\boldsymbol{n} = \boldsymbol{x}/r$, i.e. $\hat{n}_L \equiv \mathrm{STF}\{n_L\}$ where $n_L = n_{i_1} n_{i_2} \cdots n_{i_l}$ (with $L = i_1 i_2 \cdots i_l$ a multi-index with l indices†). The decomposition on the tensors $\hat{n}_L$ is equivalent to the usual decomposition

† Our conventions and notation are the following: signature $-{+}{+}{+}$; Greek indices 0,1,2,3; Latin indices 1,2,3; $r = |\boldsymbol{x}| = (x_1^2 + x_2^2 + x_3^2)^{1/2}$; $n^i = n_i = x^i/r$; $\partial_i = \partial/\partial x^i$; $x^L = x_L = x_{i_1} x_{i_2} \cdots x_{i_l}$ and $\partial_L = \partial_{i_1} \partial_{i_2} \cdots \partial_{i_l}$, where $L = i_1 i_2 \cdots i_l$ is a multi-index with l indices; $x_{L-1} = x_{i_1} \cdots x_{i_{l-1}}$, $x_{aL-1} = x_a x_{i_1} \cdots x_{i_{l-1}}$, etc; $\hat{x}_L$ and $\hat{\partial}_L$ are the (symmetric) and trace-free parts of x_L and ∂_L, for instance $\hat{x}_{ij} = x_i x_j - \frac{1}{3}\delta_{ij} r^2$; more generally the STF part of a tensor T_L is denoted indifferently $\hat{T}_L = STF_L\, T_L = T_{<L>}$; $T_{(ij)} = \frac{1}{2}(T_{ij} + T_{ji})$; $\dot{A}$ is the time derivative and $\int A(u) = \int_{-\infty}^{u} \mathrm{d}u'\, A(u')$ the time antiderivative; $\mathbb{N}$, $\mathbb{Z}$, $\mathbb{R}$ and $\mathbb{C}$ are the usual sets of integers; $\mathbb{R}^3_* = \mathbb{R}^3 - \{\mathbf{0}\}$; $C^p(\Omega)$ is the set of p-times continuously differentiable functions on the open set Ω.

on the basis of spherical harmonics. Thus, $\forall n \in \mathbb{N}$,

$$h_n^{\mu\nu}(\boldsymbol{x}, t) = \sum_{l=0}^{l_{\max}} \hat{n}_L(\theta, \phi)\ {}_L h_n^{\mu\nu}(r, t)\,. \tag{2.2}$$

The expansion is considered to be finite at any given order n because in practical computations we construct the metric for each multipolar piece separately. Note, though, that $l_{\max}$ tends to infinity as $n \to \infty$. It is hoped that at the end of the construction it is possible to take the limit of an infinite number of multipole contributions.

The MPM metric satisfies the Einstein vacuum equations everywhere except at the origin $r = 0$ (i.e. $\forall(\boldsymbol{x}, t) \in \mathbb{R}^3_* \times \mathbb{R}$). Using harmonic coordinates, this means

$$\partial_\nu h^{\mu\nu}_{\rm ext} = 0\,, \tag{2.3}$$

$$\Box h^{\mu\nu}_{\rm ext} = \Lambda^{\mu\nu}_{\rm ext}\,, \tag{2.4}$$

where the box symbol denotes the flat d'Alembertian operator $\Box = \eta^{\rho\sigma}\partial_\rho\partial_\sigma$ (with $\partial_\rho = \partial/\partial x^\rho$), and where $\Lambda^{\mu\nu}_{\rm ext} \equiv \Lambda^{\mu\nu}(h_{\rm ext})$ is a gravitational source term, whose support extends out to infinity, and which encompasses all the nonlinearities, quadratic at least, of the field equations ($\Lambda^{\mu\nu}(h)$ depends on h and its spacetime derivatives ∂h and $\partial^2 h$). The relation between $\Lambda^{\mu\nu}(h)$ and the Landau–Lifshitz [2] pseudo-tensor is $\Lambda^{\mu\nu} = 16\pi G|g|t^{\mu\nu}_{\rm LL}/c^4 + \partial_\rho h^{\mu\sigma}\partial_\sigma h^{\nu\rho} - h^{\rho\sigma}\partial^2_{\rho\sigma}h^{\mu\nu}$. By the harmonic-coordinate condition (2.3) we have, $\forall(\boldsymbol{x}, t) \in \mathbb{R}^3_* \times \mathbb{R}$,

$$\partial_\nu \Lambda^{\mu\nu}_{\rm ext} = 0\,. \tag{2.5}$$

Following paper I, we further assume that the metric is stationary in a neighbourhood of past timelike infinity, i.e. that there exists a finite instant $-\mathcal{T}$ in the past such that

$$t \leqslant -\mathcal{T} \quad \Rightarrow \quad \partial/\partial t\ h^{\mu\nu}_{\rm ext}(\boldsymbol{x}, t) = 0\,. \tag{2.6}$$

This assumption permits one to implement in a simple way the condition of no incoming radiation, and, more technically, to avoid any problem of divergence at infinity of retarded integrals of nonlinear source terms (with non-compact support). The assumption (2.6) could presumably be weakened in order to allow for some always radiating matter systems. We assume also that the metric is asymptotically Minkowskian in the past, in the sense that

$$t \leqslant -\mathcal{T} \quad \Rightarrow \quad \lim_{|\boldsymbol{x}|\to\infty}\ h^{\mu\nu}_{\rm ext}(\boldsymbol{x}) = 0\,. \tag{2.7}$$

By substituting the MPM metric (2.1), (2.2) into the field equations (2.3), (2.4) and equating the coefficients of the G^n's on both sides, we obtain an infinite set of perturbation equations to be satisfied by all the $h_n^{\mu\nu}$, $\forall n \geqslant 1$,

$$\partial_\nu h_n^{\mu\nu} = 0\,, \tag{2.8}$$

$$\Box h_n^{\mu\nu} = N_n^{\mu\nu}\,. \tag{2.9}$$

For $n = 1$ we have $N_1 \equiv 0$. To any order $n \geqslant 2$ the source N_n is known from the previous iterations. For instance, if $\Lambda(h) = N_2(h) + \mathrm{O}(h^3)$ then $N_2 \equiv N_2(h_1)$. The solution $h^{\mu\nu}_{\rm ext}$ represents the infinite sum of the solutions of the perturbation equations (2.8), (2.9) to every post-Minkowskian order. Damour and Schmidt [39] have proved that the MPM expansion is 'reliable', in the sense that it is possible to construct smooth one-parameter families of solutions of the vacuum equations whose Taylor expansion when $G \to 0$ belongs to the class of MPM metrics.

The construction of the MPM metric proceeds iteratively starting from any linearized metric h_1, solution in $\mathbb{R}^3_* \times \mathbb{R}$ of the 'linearized' vacuum equations (i.e. (2.8), (2.9) where

$n = 1$ and $N_1 \equiv 0$) and the condition of retarded potentials. The most general linearized metric can be written as an explicit multipolar expansion parametrized by a set of STF time-varying multipole moments. This is the so-called 'particular' metric of paper I we present in (4.9)–(4.13) below. The 'particular' metric differs from the 'canonical' metric of Thorne [12] by a gauge transformation.

As h_1 is in the form of a multipole expansion, it is singular at $r = 0$, and so is N_2, and then successively $h_2, \ldots, N_n$. To deal with this problem, we apply when solving (2.9) the standard retarded integral $\Box_R^{-1}$ on the product of N_n and a factor $(r/r_0)^B$ where B is a complex number and r_0 a constant with the dimension of a length. The B-dependent retarded integral $\Box_R^{-1}[(r/r_0)^B N_n]$ then defines a function of B which is valid (by analytic continuation) for all values of B in $\mathbb{C} - \mathbb{Z}$. Near the value $B = 0$, this function admits a Laurent expansion, and it was shown in paper I that the *finite part* at $B = 0$ (denoted by $\mathrm{FP}_{B=0}$) of the Laurent expansion when $B \to 0$, i.e. the coefficient of the zeroth power of B, is a solution in $\mathbb{R}^3_* \times \mathbb{R}$ of the d'Alembertian equation with source N_n. That is, we set

$$u_n^{\mu\nu} = \mathrm{FP}_{B=0}\, \Box_R^{-1}\, [(r/r_0)^B N_n^{\mu\nu}]\,, \tag{2.10}$$

where $\Box_R^{-1}$ denotes the retarded integral

$$(\Box_R^{-1} N)(\boldsymbol{x}, t) \equiv -\frac{1}{4\pi} \int \frac{\mathrm{d}^3\boldsymbol{y}}{|\boldsymbol{x}-\boldsymbol{y}|} N(\boldsymbol{y}, t - |\boldsymbol{x}-\boldsymbol{y}|/c)\,. \tag{2.11}$$

Then we have, $\forall(\boldsymbol{x}, t) \in \mathbb{R}^3_* \times \mathbb{R}$, $\Box u_n^{\mu\nu} = N_n^{\mu\nu}$.

In order to satisfy the gauge condition (2.8), it is necessary to add to the particular solution u_n a certain homogeneous solution v_n whose divergence cancels out the divergence of u_n: $\partial_\nu v_n^{\mu\nu} = -\partial_\nu u_n^{\mu\nu}$. Following the algorithm proposed in paper I, we must decompose $\partial_\nu u_n^{\mu\nu}$ into four STF tensors A_L, B_L, C_L and D_L and apply the equations (4.13) in paper I. A slightly modified algorithm, which is more convenient for our purpose, has been defined in equation (2.12) of [35]. In appendix B we recall the definition of the modified algorithm for v_n in terms of the tensors A_L, B_L, C_L and D_L. Thus the solution of (2.8), (2.9) reads ($\forall n \geqslant 2$)

$$h_n^{\mu\nu} = u_n^{\mu\nu} + v_n^{\mu\nu}\,. \tag{2.12}$$

When starting from the 'particular' linearized metric $h_{\mathrm{part.1}}$ (see (4.11)–(4.13)), the previous algorithm for the construction of the MPM metric generates in fact the *most general* solution of the vacuum equations under the MPM assumptions. When starting from the 'canonical' metric $h_{\mathrm{can.1}}$, the MPM algorithm still generates the most general solution but modulo a coordinate transformation. See theorems 4.2 and 4.5 in paper I. Crucial to the construction of the MPM metric is the knowledge of the general structure of the singularity when $r \to 0$ of each of the post-Minkowskian coefficients h_n, which legitimizes the application of the operator $\mathrm{FP}\Box_R^{-1}$ at each post-Minkowskian order. The following result was proved in paper I.

Theorem. The expansion when $r \to 0$ of any post-Minkowskian coefficient h_n reads, $\forall(\boldsymbol{x}, t) \in \mathbb{R}^3_* \times \mathbb{R}$ and $\forall N \in \mathbb{N}$,

$$h_n(\boldsymbol{x}, t) = \sum_{a \leqslant N} \hat{n}_L r^a (\ln r)^p {}_L F_{n,a,p}(t) + R_{n,N}(\boldsymbol{x}, t)\,, \tag{2.13}$$

where $a \in \mathbb{Z}$ with $-a_0 \leqslant a \leqslant N$ (and $a_0 \in \mathbb{N}$), $p \in \mathbb{N}$ with $p \leqslant n - 1$, and still $l \leqslant l_{\max}$ (see (5.4) in paper I). Both a_0 and $l_{\max}$ depend on n, and tend to infinity when $n \to \infty$, as does the maximal power of the logarithms, $p_{\max} = n - 1$. (The logarithms in (2.13) should really be $\ln(r/r_0)$ but we include the constants $\ln r_0$ into the definition of ${}_L F_{n,a,p}$.)

The functions ${}_LF_{n,a,p}(t)$ are smooth (C^∞) functions of time (starting with C^∞ multipole moments at the linearized level), and constant in the remote past: ${}_LF_{n,a,p}(t) = \text{constant}$ when $t \leqslant -\mathcal{T}$. They are given by some complicated nonlinear functionals of the (source) multipole moments parametrizing the linear metric. The remainder $R_{n,N}(\boldsymbol{x}, t)$ belongs to the so-called class of functions $\mathrm{O}^N(r^N)$ introduced in paper I, i.e. basically all its time derivatives are zero in the past ($t \leqslant -\mathcal{T}$), are of class $C^N(\mathbb{R}^4)$, and are of order $\mathrm{O}(r^N)$ when $r \to 0$ (refer to paper I for full mathematical details).

To prove this theorem, one assumes as an induction hypothesis that N_n admits the same type of structure as (2.13), with the only exception that $p_{\max} = n - 2$. Then one shows that applying $\mathrm{FP}\Box_R^{-1}$ on each of the terms composing (2.13) makes sense (by analytic continuation), and that this is a stable operation in the sense that the 'solution' u_n, and then $h_n = u_n + v_n$, admits the same structure as the 'source' N_n, with merely an increase by one unit of the maximal power of the logarithms. Since (2.13) is manifestly correct for h_1 if $p_{\max} = 0$, one concludes that (2.13) is correct ($\forall n$) with the result that $p_{\max} = n - 2$ for N_n and $p_{\max} = n - 1$ for h_n.

2.2. Assumptions concerning the inner field

We now address the problem of the gravitational field inside an actual isolated system, described by a stress–energy tensor $T^{\mu\nu}(\boldsymbol{x}, t)$ in some coordinate system $(\boldsymbol{x}, t)$, with spatial origin $r \equiv |\boldsymbol{x}| = 0$ located within the system. We suppose that $T^{\mu\nu}$ is stationary in the past, and that its support is spatially compact, $T^{\mu\nu}(\boldsymbol{x}, t) = 0$ when $r > d$. The metric $h^{\mu\nu} \equiv \sqrt{-g}\, g^{\mu\nu} - \eta^{\mu\nu}$ satisfies the Einstein field equations throughout $\mathbb{R}^4$, thus (using harmonic coordinates like in (2.3))

$$\partial_\nu h^{\mu\nu} = 0\,, \tag{2.14}$$

$$\Box h^{\mu\nu} = \frac{16\pi G}{c^4} \tau^{\mu\nu}\,, \tag{2.15}$$

where we have defined the effective stress–energy (pseudo-)tensor

$$\tau^{\mu\nu} \equiv |g| T^{\mu\nu} + \frac{c^4}{16\pi G} \Lambda^{\mu\nu}\,. \tag{2.16}$$

The first term in (2.16) is the matter source term, which is of compact support. The second term is the non-compact-supported gravitational source term, which is the same as in (2.4) but is computed with h instead of h_{ext}, i.e. $\Lambda^{\mu\nu} = \Lambda^{\mu\nu}(h)$. From (2.14) the divergence of the (relaxed) Einstein equation (2.15) yields the equation of motion of the matter fields,

$$\partial_\nu \tau^{\mu\nu} = 0\,, \tag{2.17}$$

which is equivalent to the equation of conservation of $T^{\mu\nu}$ in the covariant sense ($\nabla_\nu T^{\mu\nu} = 0$). The retarded solution of the system of equations (2.14)–(2.16) is obtained by simple inversion of the flat d'Alembertian operator,

$$h^{\mu\nu} = \frac{16\pi G}{c^4} \Box_R^{-1} \tau^{\mu\nu}\,, \tag{2.18}$$

where $\Box_R^{-1}$ is given by (2.11). Since the pseudo-tensor $\tau^{\mu\nu}$ depends on h itself and its derivatives, the equation (2.18) (with the constraint equation (2.17)) should be viewed in fact as an integro-differential equation equivalent to (2.14)–(2.16) and the condition of retarded potentials.

We now assume that $h^{\mu\nu}$, as given by (2.18) with (2.17), satisfies certain mathematical properties complementing the properties of the exterior MPM metrics. Here we state our assumptions and comment on their adequacy in the case of the gravitational field. Based on these assumptions, we shall construct in section 3 the multipole expansion of $h^{\mu\nu}$.

Assumption 1 (Smoothness). The field h (dropping the spacetime indices) is a smooth function of the harmonic coordinates $(\boldsymbol{x}, t)$, namely

$$h(\boldsymbol{x}, t) \in C^{\infty}(\mathbb{R}^4)\,. \tag{2.19}$$

In particular, we assume no singularities in the distribution of matter fields (no point particles or black holes). The formalism will be *a priori* valid only for continuous matter distributions. However, there are indications that the formalism applies also to point particles modelling compact objects like black holes (see [34, 36, 38]). Simply, one needs to represent the matter fields by delta-functions and to use a regularization in order to deal with the field near the point particles.

Assumption 2 (Multipole expansion). h admits a multipolar expansion in the open domain exterior to the compact support of the matter source $T^{\mu\nu}$, in the sense that it agrees there with the MPM metric $h_{\rm ext}$ constructed in paper I. Throughout this paper, we denote the multipole expansion using the script letter $\mathcal{M}$. Thus, by definition,

$$\mathcal{M}(h) \equiv h_{\rm ext}\,. \tag{2.20}$$

Let the exterior domain be $r > \mathcal{R}$, where the constant $\mathcal{R}$ is such that $\mathcal{R} > d$ (with d the maximal radius of the compact support of $T^{\mu\nu}$). Our assumption reads

$$r > \mathcal{R} \quad \Rightarrow \quad \mathcal{M}(h) = h\,. \tag{2.21}$$

Note that (2.20) simply means that we have given to $h_{\rm ext}$ the *name* $\mathcal{M}(h)$ (multipole expansion of h); thus $\mathcal{M}(h)$ is a solution of the vacuum equations valid everywhere except at the spatial origin, i.e. $\forall(\boldsymbol{x}, t) \in \mathbb{R}^3_* \times \mathbb{R}$. In contrast, (2.21) states that at any field point located in the exterior of our physical system ($r > \mathcal{R} > d$), $\mathcal{M}(h)$ agrees *numerically* with h. Of course, inside the system h satisfies the non-vacuum field equations and therefore differs from the vacuum solution $\mathcal{M}(h)$. For instance, h is smooth throughout the system (assumption 1) while $\mathcal{M}(h)$ is singular at $r = 0$.

Summing all the post-Minkowskian contributions given by (2.13), let us write the theorem giving the structure of the expansion when $r \to 0$ of $\mathcal{M}(h)$: $\forall(\boldsymbol{x}, t) \in \mathbb{R}^3_* \times \mathbb{R}$, $\forall N \in \mathbb{N}$,

$$\mathcal{M}(h)(\boldsymbol{x}, t) = \sum_{a \leqslant N} \hat{n}_L r^a (\ln r)^p {}_L F_{a,p}(t) + R_N(\boldsymbol{x}, t)\,, \tag{2.22}$$

where the functions ${}_L F_{a,p}$ and R_N are of the type $\sum_n G^n {}_L F_{n,a,p}$ and $\sum_n G^n R_{n,N}$. Note that, in contrast to (2.13), the summation integers a, p and l in (2.22) take an infinite number of values: $a \in \mathbb{Z}$ is not bounded below (i.e. $-\infty \leqslant a \leqslant N$), and $p \in \mathbb{N}$, $l \in \mathbb{N}$ are not bounded above ($0 \leqslant p, l \leqslant +\infty$). We refer to the 'complete' expansion including all values of $a \in \mathbb{Z}$ in (2.22) as the 'near-zone' expansion ($r \to 0$) of the multipole expansion, and denote it by an overbar,

$$\overline{\mathcal{M}(h)}(\boldsymbol{x}, t) = \sum \hat{n}_L r^a (\ln r)^p {}_L F_{a,p}(t)\,. \tag{2.23}$$

This expansion should be considered in the sense of a formal series, i.e. as an infinite set of coefficients of $r^a (\ln r)^p$.

Assumption 3 (Post-Newtonian expansion). At any fixed spacetime position $(\boldsymbol{x}, t)$, h admits an asymptotic expansion when the speed of light $c \to +\infty$ along the basis of functions $c^{-i}(\ln c)^q$ where $i, q \in \mathbb{N}$ (h depends on c as a solution of the field equations). Thus, $\forall N \in \mathbb{N}$,

$$h(\boldsymbol{x}, t, c) = \sum_{i \leqslant N} c^{-i}(\ln c)^q \sigma_{i,q}(\boldsymbol{x}, t) + T_N(\boldsymbol{x}, t, c)\,, \tag{2.24}$$

where the coefficients $\sigma_{i,q}$ are smooth functions of $(\boldsymbol{x}, t)$, and where the remainder T_N is of order $\mathrm{O}(1/c^N)$ when $c \to \infty$ (non-uniformly in $|\boldsymbol{x}|$). The corresponding expansion up to any order $i \in \mathbb{N}$ is referred to as the post-Newtonian expansion of h and is also denoted by an overbar (this abuse of notation being justified by the matching below),

$$\overline{h}(\boldsymbol{x}, t, c) = \sum c^{-i}(\ln c)^q \sigma_{i,q}(\boldsymbol{x}, t)\,. \tag{2.25}$$

Like (2.23), this expansion is to be viewed in the sense of formal series. Clearly the assumed structure (2.24), (2.25) of the post-Newtonian expansion is consistent with (2.22), (2.23). Indeed it is known (paper I) that the MPM exterior field, namely the multipole expansion $\mathcal{M}(h)$, depends on the radial coordinate r only through the ratio r/c (when the multipole moments are considered to be independent functions of time). Thus the near-zone re-expansion of the multipole expansion, $\overline{\mathcal{M}(h)}$, can be equivalently viewed as an expansion when $c \to \infty$, which should be necessarily of the type $\sum c^{-i}(\ln c)^q$ (replacing r by r/c in (2.23)). Our assumption (2.25) means in fact that the multipole moments parametrizing the linearized metric admit themselves, when expressed in terms of the source parameters, the same type of expansion $\sum c^{-j}(\ln c)^s$. There are many indications that the post-Newtonian expansion involves, besides the usual powers of $1/c$, some (powers of) logarithms of c (see paper I and references therein). As usual (because all retardations r/c associated with the propagation of gravity at the speed of light tend to zero), the post-Newtonian expansion is valid only in a region surrounding the source which is of small extent as compared to one characteristic wavelength of the emitted radiation, i.e. $\overline{h}$ constitutes a good approximation to h only in the region $r \ll \lambda$.

Assumption 4 (Matching). A 'matching' region around the source exists, where the multipole expansion $\mathcal{M}(h)$ and the post-Newtonian expansion $\overline{h}$ are simultaneously valid. In this region one expects $r > \mathcal{R}$ *and* $r \ll \lambda$ so that $\mathcal{R} \ll \lambda$, which implies, since $\mathcal{R} > d$, that the size of the source is $d \ll \lambda$ or equivalently that the typical internal velocities within the source are $v \approx dc/\lambda \ll c$. Existence of the matching region implies therefore a *slowly-moving* source. In this region we have the numerical equality (from (2.21))

$$\mathcal{R} < r \ll \lambda \quad \Rightarrow \quad \mathcal{M}(h) = \overline{h}\,. \tag{2.26}$$

We now transform (2.26) into a matching equation, i.e. an equation relating two mathematical expressions of the same nature, by replacing in the left-hand side $\mathcal{M}(h)$ by its near-zone expansion $\overline{\mathcal{M}(h)}$ as given by (2.23), and in the right-hand side $\overline{h}$ by its multipole expansion $\mathcal{M}(\overline{h})$ obtained from (2.25) by substituting each of the coefficients $\sigma_{i,q}$ by their multipole expansion $\mathcal{M}(\sigma_{i,q})$. Actually we have not defined what we mean by $\mathcal{M}(\sigma_{i,q})$, as this would necessitate performing a 'multipolar post-Newtonian' iteration of the vacuum equations, analogous to the MPM iteration of paper I. We simply assume the existence of each $\mathcal{M}(\sigma_{i,q})$, and we shall determine its structure (2.30) as a consequence of the matching. Similarly to (2.21), the multipole expansion of $\overline{h}$ satisfies (*term by term* in the post-Newtonian expansion)

$$r > \mathcal{R} \quad \Rightarrow \quad \overline{h} = \mathcal{M}(\overline{h})\,. \tag{2.27}$$

Now the matching equation associated with (2.26) reads

$$\overline{\mathcal{M}(h)} = \mathcal{M}(\overline{h}) \,. \tag{2.28}$$

Formally this equation is true 'everywhere', in the sense that it represents an infinite set of *functional* equalities (valid $\forall(\boldsymbol{x},t) \in \mathbb{R}^3_* \times \mathbb{R}$) between each of the coefficients of the series on both sides. Of course, the two series should be rearranged as expansions along the same basis of functions, i.e. either $r^a(\ln r)^p$ or $c^{-i}(\ln c)^q$. Note that the matching equation shows that the right-hand side of (2.23) represents, equivalently, the 'near-zone' expansion ($r \to 0$) of $\mathcal{M}(h)$ and the 'far-zone' expansion ($r \to +\infty$) of $\overline{h}$. From (2.28) one deduces that the structure of the functions ${}_LF_{a,p}$ in (2.23) in terms of the independent variable c is

$${}_LF_{a,p}(t,c) = \sum c^{-i}(\ln c)^q \, {}_LG_{a,p,i,q}(t) \,, \tag{2.29}$$

and that the structure of each multipole expansion $\mathcal{M}(\sigma_{i,q})$ reads

$$\mathcal{M}(\sigma_{i,q})(\boldsymbol{x},t) = \sum \hat{n}_L r^a (\ln r)^p \, {}_LG_{a,p,i,q}(t) \,. \tag{2.30}$$

It is important to realize that although the matching is performed in the (exterior) near-zone where the post-Newtonian expansion is valid (because of the small retardation r/c), the multipole expansion (2.30) of the post-Newtonian coefficients is valid wherever $r > \mathcal{R}$, and *in particular* when $r \to \infty$. Actually (2.30) represents the far-zone expansion of $\sigma_{i,q}$ (at spatial infinity). Typically $\mathcal{M}(\sigma_{i,q})$ blows up in the far zone, as is clear from the positive powers of r in (2.30). But this is not a problem, as this simply reflects the fact that the post-Newtonian expansion is not valid in a neighbourhood of infinity (where it would constitute a very poor approximation of h).

Assumption 4 is an application of the method of matched asymptotic expansions. In the present context it permits one to 'anchor' the multipole expansion to the field inside the actual source. Matched asymptotic expansions were used in general relativity originally for dealing with radiation reaction problems [40, 41], and, within MPM expansions, in order to find the expression of the multipole moments with increasing post-Newtonian precision [32, 33, 30, 37]. Each time, the matching was found consistent in the sense that satisfying (2.28) determined all the desired information at the required order.

The above assumptions 1–4 are natural complements of the MPM framework; they should apply generically to the field generated by an isolated system. However, let us stress that we have made three physical restrictions on the system: that it be stationary in the remote past (before the instant $-\mathcal{T}$), slowly moving (existence of a small post-Newtonian parameter $\sim v/c$), and without singularities (no point particles or black holes).

3. Multipole expansion of the gravitational field

Having stated our assumptions, we now start our investigation. Notice that the assumptions 1–4 have been written for the field h, but we can readily prove, with the help of the field equation (2.15), that they also apply to the stress–energy pseudo-tensor τ. In particular, since $\mathcal{M}(\Lambda) = 16\pi G/c^4 \mathcal{M}(\tau)$ (because the matter stress–energy tensor T has compact support) we see that $\mathcal{M}(\Lambda)$, which is nothing other than the MPM source Λ_{ext} in (2.4), admits exactly the same structure (2.22) as $\mathcal{M}(h)$. Therefore we can apply on $\mathcal{M}(\Lambda)$ the finite part at $B = 0$ of the operator $\Box^{-1}_R (r/r_0)^B$ whose definition was given in (2.10), (2.11). We have seen that since $\mathcal{M}(\Lambda)$ involves an infinite sum of post-Minkowskian contributions, the summation in (2.22) is infinite (notably $-\infty \leqslant a \leqslant N$). Thus we mean by $\mathrm{FP}\,\Box^{-1}_R \,\mathcal{M}(\Lambda)$ the infinite sum of the $\mathrm{FP}\Box^{-1}_R$'s acting on each separate term composing $\mathcal{M}(\Lambda)$. Recall that we are working in the context of approximate solutions, which are constructed only up to a given (though arbitrary) post-Minkowskian order, and thus involve only a finite number of separate contributions. Here we assume that we can consider the *formal* series of approximations, but we do not control the precise mathematical nature of this series (see, however, [39]).

We can perform our derivation restricting our attention to the case where $\mathcal{M}(\Lambda)$ is not only constant in the past but *zero* in the past ($t \leqslant -\mathcal{T}$). Indeed we can check *a posteriori* that the derivation can be redone straightforwardly in the case of constant terms in $\mathcal{M}(\Lambda)$ by simply using the Poisson operator Δ^{-1} instead of the retarded integral $\Box_R^{-1}$. Thus the constant terms can be added to the result at the end with no modification except that $\Box_R^{-1}$ becomes Δ^{-1} when acting on such terms.

From paper I, the B-dependent retarded integral $\Box_R^{-1}[(r/r_0)^B\mathcal{M}(\Lambda)]$ admits a Laurent expansion when $B \to 0$ whose finite part solves the d'Alembertian equation with source $\mathcal{M}(\Lambda)$: $\forall(\boldsymbol{x}, t) \in \mathbb{R}^3_* \times \mathbb{R}$,

$$\Box \left\{\mathrm{FP}_{B=0}\ \Box_R^{-1}\left[\tilde{r}^B\mathcal{M}(\Lambda)\right]\right\} = \mathcal{M}(\Lambda)\,. \tag{3.1}$$

From now on we set $\tilde{r} = r/r_0$. Beyond the finite part, all possible multiple poles $\sim B^{-i}$ exist *a priori* ($\forall i \in \mathbb{N}$; indeed $i \leqslant i_{\max}$ to any post-Minkowskian order n but $i_{\max} \to \infty$ when $n \to \infty$).

We want to compute the multipole expansion of the field, namely $\mathcal{M}(h) \equiv 16\pi G/c^4\mathcal{M}(\Box_R^{-1}\tau)$. To do this we notice that because the multipole expansion satisfies $\Box\mathcal{M}(h) = \mathcal{M}(\Lambda)$ (throughout $\mathbb{R}^3_* \times \mathbb{R}$), we have for any B the relation $\Box[\tilde{r}^B\mathcal{M}(h)] = \tilde{r}^B\{\mathcal{M}(\Lambda) + 2Br^{-1}\partial_r\mathcal{M}(h) + B(B+1)r^{-2}\mathcal{M}(h)\}$, where $\partial_r \equiv n_i\partial_i$. Applying on both sides of this relation the retarded integral $\Box_R^{-1}$ and considering the finite part at $B = 0$, yields

$$\begin{aligned}\mathcal{M}(h) = {} & \mathrm{FP}_{B=0}\ \Box_R^{-1}\left[\tilde{r}^B\mathcal{M}(\Lambda)\right] \\ & + \mathrm{FP}_{B=0}\ \Box_R^{-1}\left[\tilde{r}^B\{2Br^{-1}\partial_r\mathcal{M}(h) + B(B+1)r^{-2}\mathcal{M}(h)\}\right].\end{aligned} \tag{3.2}$$

The first term will ensure that $\mathcal{M}(h)$ is a particular solution of the correct equation $\Box\mathcal{M}(h) = \mathcal{M}(\Lambda)$ (see (3.1)). This particular solution is already in the appropriate form, thus we concentrate our attention on the second term, which constitutes a homogeneous solution of the wave equation, as will follow from the fact that this second term in (3.2) involves an explicit factor B (so it appears like a residue rather than a finite part). Because of this factor B we see that the contribution of any term in (2.22) which is regular at $r = 0$ will be exactly zero. This is, in particular, the case for the remainder R_N in (2.22) which will give zero, and since this is true $\forall N \in \mathbb{N}$ we conclude that $\mathcal{M}(\Lambda)$ can be replaced by the infinite expansion $\overline{\mathcal{M}(\Lambda)}$ given by (2.23). Thus, writing the retarded integral in the form (2.11),

$$\begin{aligned}\mathcal{M}(h) = {} & \mathrm{FP}_{B=0}\ \Box_R^{-1}\left[\tilde{r}^B\mathcal{M}(\Lambda)\right] - \frac{1}{4\pi}\mathrm{FP}_{B=0}\int\frac{\mathrm{d}^3\boldsymbol{y}\,|\tilde{\boldsymbol{y}}|^B}{|\boldsymbol{x}-\boldsymbol{y}|} \\ & \times\left[2Br^{-1}\partial_r\overline{\mathcal{M}(h)} + B(B+1)r^{-2}\overline{\mathcal{M}(h)}\right](\boldsymbol{y}, t - |\boldsymbol{x}-\boldsymbol{y}|/c)\,,\end{aligned} \tag{3.3}$$

where $\tilde{\boldsymbol{y}} \equiv \boldsymbol{y}/r_0$. The factor B shows that the only contribution to the triple integral is due to a (simple) pole at $B = 0$, which in turn comes only from the integration on an infinitesimal neighbourhood of the spatial origin, $|\boldsymbol{y}| < \varepsilon$ where ε is an arbitrary small number ($0 < \varepsilon \ll |\boldsymbol{x}|$ say). (As we have assumed that the functions ${}_LF_{a,p}$ are zero in the past (we add the constant parts at the end) there is no problem at the upper bound $|\boldsymbol{y}| \to \infty$.) It is then legitimate to replace the integrand in (3.3) by its Taylor expansion when $|\boldsymbol{y}| \to 0$. The Taylor expansion of any $K(t - |\boldsymbol{x}-\boldsymbol{y}|/c)/|\boldsymbol{x}-\boldsymbol{y}|$ is given by

$\sum(-1)^l y_L \partial_L\{K(t-r/c)/r\}/l!$ (see the footnote in section 2 for our notation). In this way we obtain

$$\mathcal{M}(h) = \mathrm{FP}_{B=0}\, \Box_R^{-1}\, [\tilde{r}^B \mathcal{M}(\Lambda)] - \frac{4G}{c^4} \sum_{l=0}^{+\infty} \frac{(-1)^l}{l!} \partial_L \left\{ \frac{1}{r} \mathcal{K}_L(t-r/c) \right\} , \tag{3.4}$$

where the 'multipole moments' $\mathcal{K}_L(u)$ are given explicitly by ($u \equiv t - r/c$)

$$\mathcal{K}_L(u) = \frac{c^4}{16\pi G} \mathrm{FP}_{B=0} \int_{|\boldsymbol{y}|<\varepsilon} \mathrm{d}^3\boldsymbol{y}\, |\tilde{\boldsymbol{y}}|^B y_L \big[2Br^{-1}\partial_r \overline{\mathcal{M}(h)} + B(B+1)r^{-2}\overline{\mathcal{M}(h)} \big](\boldsymbol{y}, u) . \tag{3.5}$$

Next we perform a sequence of transformations of $\mathcal{K}_L(u)$. By (2.23), the structure of $\overline{\mathcal{M}(h)}(\boldsymbol{y}, u)$ is that of a series of terms of the type $\sum \hat{n}_L |\boldsymbol{y}|^a (\ln|\tilde{\boldsymbol{y}}|)^p{}_L F_{a,p}(u)$. This shows that after integrating over the angles, $\mathcal{K}_L(u)$ consists of a series of terms of the type $\mathrm{FP}_{B=0} B \int_0^\varepsilon \mathrm{d}|\boldsymbol{y}|\, |\tilde{\boldsymbol{y}}|^B |\boldsymbol{y}|^{2+l+a} (\ln|\tilde{\boldsymbol{y}}|)^p$ times a function of u. The latter radial integrals are defined by analytic continuation in B. Now the point is that the *complete* radial integral extending from 0 to $+\infty$, i.e. $\int_0^{+\infty} \mathrm{d}|\boldsymbol{y}|\, |\tilde{\boldsymbol{y}}|^B |\boldsymbol{y}|^{2+l+a} (\ln|\tilde{\boldsymbol{y}}|)^p$, is rigorously *zero* by analytic continuation, for all values of $B \in \mathbb{C}$. (We repeat the reasoning already made in paper II: the integral can be split into the sum of two integrals, namely $(\mathrm{d}/\mathrm{d}B)^p \int_0^{\mathcal{A}} \mathrm{d}|\boldsymbol{y}|\, |\tilde{\boldsymbol{y}}|^B |\boldsymbol{y}|^{2+l+a}$ and $(\mathrm{d}/\mathrm{d}B)^p \int_{\mathcal{A}}^{+\infty} \mathrm{d}|\boldsymbol{y}|\, |\tilde{\boldsymbol{y}}|^B |\boldsymbol{y}|^{2+l+a}$, where $\mathcal{A}$ is some positive constant. When the real part of B is a large positive number, the first integral reads $(\mathrm{d}/\mathrm{d}B)^p \{\tilde{\mathcal{A}}^B \mathcal{A}^{3+l+a}/(B+3+l+a)\}$; and when the real part of B is a large negative number, the second integral is equal to the opposite $-(\mathrm{d}/\mathrm{d}B)^p \{\tilde{\mathcal{A}}^B \mathcal{A}^{3+l+a}/(B+3+l+a)\}$. These expressions represent the unique analytic continuations of the two integrals for all values of B except $-(3+l+a)$. Now $\int_0^{+\infty} \mathrm{d}|\boldsymbol{y}|\, |\tilde{\boldsymbol{y}}|^B |\boldsymbol{y}|^{2+l+a} (\ln|\tilde{\boldsymbol{y}}|)^p$ is defined by analytic continuation to be the sum of the analytic continuations of the two separate integrals, and is therefore identically zero ($\forall B \in \mathbb{C}$).) Thus all the previous terms can be equivalently written as $-\mathrm{FP}_{B=0} B \int_\varepsilon^{+\infty} \mathrm{d}|\boldsymbol{y}|\, |\tilde{\boldsymbol{y}}|^B |\boldsymbol{y}|^{2+l+a} (\ln|\tilde{\boldsymbol{y}}|)^p$. Furthermore, using now the presence of the explicit factor B, we see that the only contribution to the finite part at $B = 0$ comes from an arbitrary neighbourhood of the upper bound $|\boldsymbol{y}| = +\infty$. In this paper it is sufficient to consider as a neighbourhood of infinity the domain $|\boldsymbol{y}| > \mathcal{R}$, where $\mathcal{R}$ denotes the radius giving the limit of validity of multipole expansions. Therefore $\mathcal{K}_L(u)$ is also given by a series of terms of the type $-\mathrm{FP}_{B=0} B \int_{\mathcal{R}}^{+\infty} \mathrm{d}|\boldsymbol{y}|\, |\tilde{\boldsymbol{y}}|^B |\boldsymbol{y}|^{2+l+a} (\ln|\tilde{\boldsymbol{y}}|)^p$, which shows that (3.5) is in fact equivalent to

$$\mathcal{K}_L(u) = -\frac{c^4}{16\pi G} \mathrm{FP}_{B=0} \int_{|\boldsymbol{y}|>\mathcal{R}} \mathrm{d}^3\boldsymbol{y}\, |\tilde{\boldsymbol{y}}|^B y_L \big[2Br^{-1}\partial_r \overline{\mathcal{M}(h)} + B(B+1)r^{-2}\overline{\mathcal{M}(h)} \big](\boldsymbol{y}, u) . \tag{3.6}$$

Note the minus sign with respect to the previous expression (3.5). The next step is to employ our assumption 4 of consistent matching between the post-Newtonian and multipole expansions, according to which one can commute the order of taking the post-Newtonian and multipole expansions: by (2.28), $\overline{\mathcal{M}(h)}$ and $\mathcal{M}(\overline{h})$ are functionally equal (term by term), so (3.6) can be rewritten as

$$\mathcal{K}_L(u) = -\frac{c^4}{16\pi G} \mathrm{FP}_{B=0} \int_{|\boldsymbol{y}|>\mathcal{R}} \mathrm{d}^3\boldsymbol{y}\, |\tilde{\boldsymbol{y}}|^B y_L \big[2Br^{-1}\partial_r \mathcal{M}(\overline{h}) + B(B+1)r^{-2}\mathcal{M}(\overline{h}) \big](\boldsymbol{y}, u) . \tag{3.7}$$

But now in the region $|\boldsymbol{y}| > \mathcal{R}$ one can replace the multipole expansion by the function itself (see (2.27)), thus

$$\mathcal{K}_L(u) = -\frac{c^4}{16\pi G}\mathrm{FP}_{B=0}\int_{|\boldsymbol{y}|>\mathcal{R}} \mathrm{d}^3\boldsymbol{y}\,|\tilde{\boldsymbol{y}}|^B y_L\big[2Br^{-1}\partial_r\overline{h} + B(B+1)r^{-2}\overline{h}\,\big](\boldsymbol{y},u)\,. \qquad (3.8)$$

Finally, we recall that by assumption 1 the field h and all the $\sigma_{i,q}$'s in the post-Newtonian expansion $\overline{h}$ are regular (C^∞) functions on $\mathbb{R}^4$, and in particular near the origin $r \to 0$. Therefore we see that the range of integration in (3.8) can be harmlessly extended to the whole three-dimensional space. Indeed, because of the factor B, the contribution to the integral due to any ball of finite radius, for instance the ball $|\boldsymbol{y}| \leqslant \mathcal{R}$, will be zero after taking the finite part at $B = 0$. Thus we have

$$\mathcal{K}_L(u) = -\frac{c^4}{16\pi G}\mathrm{FP}_{B=0}\int \mathrm{d}^3\boldsymbol{y}\,|\tilde{\boldsymbol{y}}|^B y_L\big[2Br^{-1}\partial_r\overline{h} + B(B+1)r^{-2}\overline{h}\,\big](\boldsymbol{y},u)\,. \qquad (3.9)$$

Since $\Box\overline{h} = 16\pi G\,\overline{\tau}/c^4$ we can re-combine the terms in the integrand and find the equivalent expression

$$\mathcal{K}_L(u) = \mathrm{FP}_{B=0}\int \mathrm{d}^3\boldsymbol{y}\,y_L\left[\tilde{r}^B\overline{\tau} - \frac{c^4}{16\pi G}\,\Box\,(\tilde{r}^B\overline{h})\right](\boldsymbol{y},u)\,. \qquad (3.10)$$

The point about (3.9) or (3.10) is that because of the factor B the numerical values of the multipole moments depend on the post-Newtonian expansion of the source as integrated formally in a neighbourhood of (spatial) infinity. This is despite the fact that the post-Newtonian expansion is expected *a priori* to be valid only near the origin. As we have shown here, this seemingly contradictory result is possible thanks to the properties of analytic continuation, which permit one to jump from a 'near-zone' integration range in (3.5) to the 'far-zone' in (3.6). However, we now remark that the contributions in the complete multipole expansion (involving all the $\mathcal{K}_L$'s) which are due to the second term in (3.10) actually sum up to give zero. Indeed we separate the d'Alembertian into a Laplacian and a second time derivative, $\Box(\tilde{r}^B\overline{h}) = \Delta(\tilde{r}^B\overline{h}) - \tilde{r}^B\partial_u^2\overline{h}/c^2$, and we integrate by parts the Laplacian using $\Delta y_L = l(l-1)\delta_{(i_l i_{l-1}}y_{L-2)}$, generating in this way a Kronecker symbol (the surface term during the integration by parts is zero by analytic continuation). Since the l indices $i_1\cdots i_l$ are contracted with the l indices carried by the spatial gradients $\partial_L \equiv \partial_{i_1}\cdots\partial_{i_l}$ present in the multipole expansion (see (3.4)), the latter Kronecker symbol $\delta_{i_l i_{l-1}}$ generates a Laplacian which is then equivalent (because it acts on a spherical retarded wave) to a second time derivative. It is not difficult to check that the sum of all these terms with second time derivatives cancels exactly the other second time derivatives issuing from the separation made above of the d'Alembertian into a Laplacian. Therefore we can rewrite the multipole decomposition by ignoring the second term in (3.10) and taking into account only the first term which really represents the multipole moment as generated by the (post-Newtonian expansion of the) source. Denoting by $\mathcal{H}_L$ the first term in (3.10), we obtain the main result of this paper (restoring the spacetime indices $\mu\nu$):

$$\mathcal{M}(h^{\mu\nu}) = \mathrm{FP}_{B=0}\,\Box_R^{-1}\,[\tilde{r}^B\mathcal{M}(\Lambda^{\mu\nu})] - \frac{4G}{c^4}\sum_{l=0}^{+\infty}\frac{(-1)^l}{l!}\partial_L\left\{\frac{1}{r}\mathcal{H}_L^{\mu\nu}(t-r/c)\right\}\,, \qquad (3.11)$$

where the (source) multipole moments are given by the simple expression

$$\mathcal{H}_L^{\mu\nu}(u) = \mathrm{FP}_{B=0}\int \mathrm{d}^3\boldsymbol{y}\,|\tilde{\boldsymbol{y}}|^B y_L\,\overline{\tau}^{\mu\nu}(\boldsymbol{y},u)\,, \qquad (3.12)$$

with $\overline{\tau}^{\mu\nu}$ the post-Newtonian expansion of $\tau^{\mu\nu}$. In this expression there is no explicit B-factor left out, so in contrast to (3.10) the integration over the whole three-dimensional space contributes to the multipole moment, including the regions at infinity, as well as the intermediate regions and most importantly the near zone. As we shall see in section 4, the first term in (3.11) represents the nonlinear corrections to be added in a post-Minkowskian expansion to the (linear-looking) multipole expansion as given by the second term. Due to the presence of this first term, the radiative multipole moments defined in the wave zone will differ from the source moments (see section 6). It can be shown that the multipole expansion $\mathcal{M}(h^{\mu\nu})$ is actually independent of the length scale r_0 entering in $\tilde{r} = r/r_0$ and $\tilde{\boldsymbol{y}} = \boldsymbol{y}/r_0$ (namely, the r_0's cancel out between the two terms of (3.11)).

Let us emphasize the interesting role played by the analytic continuation throughout the proof of (3.12). Witness in particular the crucial passage from (3.5) to (3.6), which permits *in fine* to get rid of the reference to the multipole expansion in the integrand of the multipole moments themselves. See also the last step from (3.10) to (3.12) where we discard some surface terms which are zero by analytic continuation, and which permit one to express the result in terms of an integral over the sole $\overline{\tau}$ (without explicit reference to $\overline{h}$). The result (3.11), (3.12) was already obtained (in STF form) in paper II at the 2PN order. What we have proved here is that the formulae for the multipole moments given in paper II admit a generalization to any post-Newtonian order. This is important for theoretical and also practical reasons, because the application to binary systems is currently done to very high order (3PN order in [38]). Furthermore, the method that we have employed—direct construction of the multipole expansion—is rather different from paper II, which obtained the result by means of a matching 'order by order' (looking for a coordinate transformation between the inner and outer fields). In appendix A we present an alternative proof of the result (3.11), (3.12), which is based (similarly to paper II) on the direct comparison between the field h and the multipole expansion $\mathrm{FP}_{B=0}\,\Box_R^{-1}\,[\tilde{r}^B\mathcal{M}(\Lambda)]$.

The multipole expansion (3.11), (3.12) has been written using non-trace-free multipole moments. Actually it is better to rewrite it using symmetric and trace-free (STF) moments, because the non-trace-free moments are not uniquely defined (for instance $\mathcal{K}_L$ and $\mathcal{H}_L$ yield the same multipole expansion). Here we simply report the result of the STF multipole expansion (which readily follows from equation (B.14*a*) in [32]):

$$\mathcal{M}(h^{\mu\nu}) = \mathrm{FP}_{B=0}\,\Box_R^{-1}\,[\tilde{r}^B\mathcal{M}(\Lambda^{\mu\nu})] - \frac{4G}{c^4}\sum_{l=0}^{+\infty}\frac{(-1)^l}{l!}\partial_L\left\{\frac{1}{r}\mathcal{F}_L^{\mu\nu}(t-r/c)\right\}, \tag{3.13}$$

where the STF multipole moments are given by

$$\mathcal{F}_L^{\mu\nu}(u) = \mathrm{FP}_{B=0}\int \mathrm{d}^3\boldsymbol{y}\,|\tilde{\boldsymbol{y}}|^B\,\hat{y}_L\int_{-1}^{1}\mathrm{d}z\,\delta_l(z)\overline{\tau}^{\mu\nu}(\boldsymbol{y}, u+z|\boldsymbol{y}|/c) \tag{3.14}$$

($\hat{y}_L$ denotes the trace-free part of y_L, see the footnote in section 2 for our notation). The STF moments (3.14) involve an integration over the z-dependent cone $t = u + z|\boldsymbol{y}|/c$ with weighting function

$$\delta_l(z) = \frac{(2l+1)!!}{2^{l+1}l!}(1-z^2)^l; \qquad \int_{-1}^{1}\mathrm{d}z\,\delta_l(z) = 1; \qquad \lim_{l\to\infty}\delta_l(z) = \delta(z)\,. \tag{3.15}$$

As a check of the result, let us consider the limit of linearized gravity where we can neglect $\Lambda^{\mu\nu}$ and $\mathcal{M}(\Lambda^{\mu\nu})$, and where $\tau^{\mu\nu}$ reduces to the matter stress–energy tensor $T^{\mu\nu}$ with compact support, i.e. $T^{\mu\nu}(\boldsymbol{x}, t) = 0$ when $|\boldsymbol{x}| > d$. In this limiting case the first term in (3.13) vanishes. Furthermore, replacing $\overline{\tau}^{\mu\nu}$ by $\overline{T}^{\mu\nu}$ in (3.14) we can remove the factor $|\tilde{\boldsymbol{y}}|^B$ and the finite part at $B = 0$ for compact-supported integrals. Finally, we can replace $\overline{T}^{\mu\nu}$ by $T^{\mu\nu}$ within the compact support of a slowly-moving source (since $d \ll \lambda$). Thus we recover exactly the expression of the multipole moments derived for compact-support sources in appendix B of [32] and used for studying the linearized gravity in [16] (in this case the result is valid also for fast-moving sources).

The multipole decomposition (3.11)–(3.15) appears to be quite general. Physically it should apply to any isolated slowly-moving source without singularities. Technically it does not make any reference, for instance, to the post-Minkowskian (or rather MPM) expansion which has been invoked in order to derive it. In a sense (3.11)–(3.15) represents a 'complete' matching equation (valid to any post-Newtonian and/or post-Minkowskian order), which is the general consequence of our matching assumption 4. We shall leave open the possibility that the multipole expansion (3.11)–(3.15), because of its generality, may have in fact a domain of validity larger than the one of MPM approximate solutions. For instance, it is plausible that (3.11)–(3.15) could be proved in a more general context of exact solutions (using perhaps an analysis similar to that of [39]).

4. The linearized multipolar metric

The most general solution of the vacuum field equations (off the time axis) was constructed within the MPM framework (see section 2). Therefore, if the present analysis makes sense, it should be possible to recast the general multipole expansion $\mathcal{M}(h^{\mu\nu})$ as given by (3.13), (3.14) into a form which shows clearly that it belongs to the class of MPM metrics. Namely, we would like to find a certain '*linearized*' multipolar metric such that $\mathcal{M}(h^{\mu\nu})$ appears to be the post-Minkowskian iteration of that metric (in the MPM sense). The advantage is that the multipole moments parametrizing this linearized metric, which constitute efficient tools in practical computations of gravitational radiation (see, e.g., [36–38]), will then be obtained with full generality as computable functionals of the matter fields in the source. This is what we shall do in this section and the following one, where the linearized multipolar metric associated with $\mathcal{M}(h^{\mu\nu})$ will be denoted by $h^{\mu\nu}_{\text{part.1}}$ following the notation of paper I.

Let us denote the first term in (3.13) by

$$u^{\mu\nu} \equiv \text{FP}_{B=0}\ \Box^{-1}_R\,[\tilde{r}^B \mathcal{M}(\Lambda^{\mu\nu})]\,. \tag{4.1}$$

The first step is to compute the divergence $\partial_\nu u^{\mu\nu}$ of this term. To do this, one notices first that $\mathcal{M}(h^{\mu\nu})$ is divergenceless by (2.14), and therefore that, by (3.13),

$$\partial_\nu u^{\mu\nu} = \frac{4G}{c^4}\partial_\nu\left(\sum_{l=0}^{+\infty}\frac{(-1)^l}{l!}\partial_L\left\{\frac{1}{r}\mathcal{F}^{\mu\nu}_L(t-r/c)\right\}\right). \tag{4.2}$$

If analytic continuation factors $|\tilde{\boldsymbol{y}}|^B$ were absent in the expression of $\mathcal{F}^{\mu\nu}_L$ given by (3.14), the right-hand side of (4.2) would be (formally) zero by virtue of $\partial_\nu\overline{\tau}^{\mu\nu} = 0$. With factors $|\tilde{\boldsymbol{y}}|^B$ included, it is straightforward to find (4.2). In fact, the computation is the same as the one yielding (4.1) in paper II. One must evaluate the time derivative of $\mathcal{F}^{0\mu}_L$ using $\partial_\nu\overline{\tau}^{\mu\nu} = 0$, and perform some integrations by parts both with respect to $\boldsymbol{y}$ and z. We differentiate where appropriate the factor $|\tilde{\boldsymbol{y}}|^B$. During the integrations by parts all the surface terms are zero by analytic continuation. Derivatives of the function $\delta_l(z)$ are computed using

$$\frac{\mathrm{d}}{\mathrm{d}z}[\delta_{l+1}(z)] = -(2l+3)z\,\delta_l(z) \tag{4.3a}$$

$$\frac{\mathrm{d}^2}{\mathrm{d}z^2}[\delta_{l+1}(z)] = -(2l+3)(2l+1)[\delta_l(z)-\delta_{l-1}(z)]\,. \tag{4.3b}$$

As a result, we find

$$\left(\frac{\mathrm{d}}{c\,\mathrm{d}u}\right)\mathcal{F}_L^{0\mu} = l\,\mathcal{F}_{L-1>}^{\mu<i_l} + \frac{1}{2l+3}\left(\frac{\mathrm{d}}{c\,\mathrm{d}u}\right)^2\mathcal{F}_{jL}^{j\mu} + \mathcal{G}_L^{\mu}\,, \tag{4.4}$$

where $\mathcal{F}_{L-1>}^{\mu<i_l}$ denotes the STF part of $\mathcal{F}_{L-1}^{\mu i_l}$, and where the function $\mathcal{G}_L^{\mu}$ is given by

$$\mathcal{G}_L^{\mu}(u) = \mathrm{FP}_{B=0}\int \mathrm{d}^3\boldsymbol{y}\,B|\tilde{\boldsymbol{y}}|^B|\boldsymbol{y}|^{-2}y_i\,\hat{y}_L\int_{-1}^{1}\mathrm{d}z\,\delta_l(z)\overline{\tau}^{\mu i}(\boldsymbol{y}, u+z|\boldsymbol{y}|/c)\,. \tag{4.5}$$

With (4.4) in hand, it is straightforward to transform (4.2) into

$$\partial_\nu u^{\mu\nu} = \frac{4G}{c^4}\sum_{l=0}^{+\infty}\frac{(-1)^l}{l!}\partial_L\left\{\frac{1}{r}\,\mathcal{G}_L^{\mu}(t-r/c)\right\}\,. \tag{4.6}$$

Another way of proving (4.5), (4.6) notices that the multipole expansion $\mathcal{M}(\Lambda^{\mu\nu}) \equiv \Lambda^{\mu\nu}_{\mathrm{ext}}$ is divergenceless by (2.5), and so from (4.1) we have $\partial_\nu u^{\mu\nu} = \mathrm{FP}_{B=0}\,\Box_R^{-1}[B\tilde{r}^B r^{-1}n_i\mathcal{M}(\Lambda^{\mu i})]$, where the factor B comes from the derivation of $\tilde{r}^B$. Thanks to this factor the finite part is actually a residue, and we can perform an analysis analogous to the one performed in going from (3.2) to (3.9) in the previous section. In this way we recover exactly the result (4.5), (4.6).

Having obtained the divergence of $u^{\mu\nu}$ in the form (4.6), we proceed similarly to paper I and construct from (4.6) a different object $v^{\mu\nu}$ which is, like (4.6), a retarded solution of the wave equation, and furthermore which is such that its divergence cancels out the divergence of $u^{\mu\nu}$: $\partial_\nu v^{\mu\nu} = -\partial_\nu u^{\mu\nu}$. In appendix B we recall from paper I (and [35]) the expression of $v^{\mu\nu}$ in terms of STF tensors A_L, B_L, C_L and D_L, and we show the equivalence to the different expression:

$$v^{00} = \frac{4G}{c^4}\left\{-\frac{c}{r}\int\mathcal{G}^0 + \partial_a\left(\frac{1}{r}\left[c\int\mathcal{G}_a^0 + c^2\int\!\!\int\mathcal{G}^a - \mathcal{G}_{ab}^b\right]\right)\right\}\,, \tag{4.7a}$$

$$v^{0i} = \frac{4G}{c^4}\left\{-\frac{1}{r}\left[c\int\mathcal{G}^i - \frac{1}{c}\dot{\mathcal{G}}_{ai}^a\right] + \frac{c}{2}\partial_a\left(\frac{1}{r}\left[\int\mathcal{G}_a^i - \int\mathcal{G}_i^a\right]\right)\right.$$
$$\left. - \sum_{l\geqslant 2}\frac{(-1)^l}{l!}\partial_{L-1}\left(\frac{1}{r}\mathcal{G}_{iL-1}^0\right)\right\}, \tag{4.7b}$$

$$v^{ij} = \frac{4G}{c^4}\left\{\frac{1}{r}\mathcal{G}_{j)}^{(i} + 2\sum_{l\geqslant 3}\frac{(-1)^l}{l!}\partial_{L-3}\left(\frac{1}{rc^2}\ddot{\mathcal{G}}_{ijaL-3}^a\right)\right.$$
$$+ \sum_{l\geqslant 2}\frac{(-1)^l}{l!}\left[\partial_{L-2}\left(\frac{1}{rc}\dot{\mathcal{G}}_{ijL-2}^0\right) + \partial_{aL-2}\left(\frac{1}{r}\mathcal{G}_{ijL-2}^a\right) + 2\delta_{ij}\partial_{L-1}\left(\frac{1}{r}\mathcal{G}_{aL-1}^a\right)\right.$$
$$\left.\left. - 4\partial_{L-2(i}\left(\frac{1}{r}\mathcal{G}_{j)aL-2}^a\right) - 2\partial_{L-1}\left(\frac{1}{r}\mathcal{G}_{j)L-1}^{(i}\right)\right]\right\}\,, \tag{4.7c}$$

where the $\mathcal{G}_L^{\mu}$'s are given by (4.5) (they are all evaluated at the retarded time $u = t - r/c$). The notation is $T_{(ij)} = \frac{1}{2}\left(T_{ij} + T_{ji}\right)$ and $T_{j)}^{(i} = \frac{1}{2}(T_j^i + T_i^j)$; $\mathcal{G}^{\mu} = (\mathcal{G}^0, \mathcal{G}^i)$ means $\mathcal{G}_L^{\mu}$ with $l = 0$; time derivatives are denoted by $\dot{\mathcal{G}}(u) = \mathrm{d}\mathcal{G}(u)/\mathrm{d}u$, and time antiderivatives by $\int\mathcal{G}(u) = \int_{-\infty}^{u}\mathrm{d}v\,\mathcal{G}(v)$, $\int\!\!\int\mathcal{G}(u) = \int_{-\infty}^{u}\mathrm{d}v\int\mathcal{G}(v)$. The main property of $v^{\mu\nu}$, i.e. $\partial_\nu v^{\mu\nu} = -\partial_\nu u^{\mu\nu}$, is checked directly on (4.7).

With the above construction of $v^{\mu\nu}$, we are able to define what will constitute a 'linearized' approximation to the multipolar expansion $\mathcal{M}(h^{\mu\nu})$ given by (3.13). Let us set

$$G h^{\mu\nu}_{\text{part.1}} = -\frac{4G}{c^4} \sum_{l=0}^{+\infty} \frac{(-1)^l}{l!} \partial_L \left\{ \frac{1}{r} \mathcal{F}^{\mu\nu}_L (t - r/c) \right\} - v^{\mu\nu} \,. \tag{4.8}$$

By the structure of $h^{\mu\nu}_{\text{part.1}}$ made out of retarded solutions of the source-free wave equation, and by the construction of $v^{\mu\nu}$ ensuring its zero divergence (see (4.2)), we have (in $\mathbb{R}^3_* \times \mathbb{R}$)

$$\Box h^{\mu\nu}_{\text{part.1}} = 0 \,, \tag{4.9}$$

$$\partial_\nu h^{\mu\nu}_{\text{part.1}} = 0 \,. \tag{4.10}$$

This means that (4.8) satisfies the linearized field equations in the exterior region, i.e. (2.8), (2.9) with $n = 1$. By theorem 2.1 of paper I (see also (4.7) in paper I), we know that the most general solution of the system of equations (4.9), (4.10) can always be written as the sum of a 'canonical' linearized metric $h^{\mu\nu}_{\text{can.1}}$ (introduced by Thorne [12]) and a linear *gauge* transformation. Therefore $h^{\mu\nu}_{\text{part.1}}$ (which is referred to in paper I as a 'particular' metric but which is in fact quite general) reads

$$h^{\mu\nu}_{\text{part.1}} = h^{\mu\nu}_{\text{can.1}} + \partial^\mu \varphi^\nu_1 + \partial^\nu \varphi^\mu_1 - \eta^{\mu\nu} \partial_\lambda \varphi^\lambda_1 \,. \tag{4.11}$$

The canonical metric is parametrized by *two* types of (STF) multipole moments I_L, J_L,

$$h^{00}_{\text{can.1}} = -\frac{4}{c^2} \sum_{l\geqslant 0} \frac{(-1)^l}{l!} \partial_L \left(\frac{1}{r} I_L(u) \right) , \tag{4.12a}$$

$$h^{0i}_{\text{can.1}} = \frac{4}{c^3} \sum_{l\geqslant 1} \frac{(-1)^l}{l!} \left\{ \partial_{L-1} \left(\frac{1}{r} \dot{I}_{iL-1}(u) \right) + \frac{l}{l+1} \varepsilon_{iab} \partial_{aL-1} \left(\frac{1}{r} J_{bL-1}(u) \right) \right\} , \tag{4.12b}$$

$$h^{ij}_{\text{can.1}} = -\frac{4}{c^4} \sum_{l\geqslant 2} \frac{(-1)^l}{l!} \left\{ \partial_{L-2} \left(\frac{1}{r} \ddot{I}_{ijL-2}(u) \right) + \frac{2l}{l+1} \partial_{aL-2} \left(\frac{1}{r} \varepsilon_{ab(i} \dot{J}_{j)bL-2}(u) \right) \right\} . \tag{4.12c}$$

On the other hand, the vector associated with the gauge transformation depends on *four* STF moments W_L, X_L, Y_L, Z_L:

$$\varphi^0_1 = \frac{4}{c^3} \sum_{l\geqslant 0} \frac{(-1)^l}{l!} \partial_L \left(\frac{1}{r} W_L(u) \right) , \tag{4.13a}$$

$$\varphi^i_1 = -\frac{4}{c^4} \sum_{l\geqslant 0} \frac{(-1)^l}{l!} \partial_{iL} \left(\frac{1}{r} X_L(u) \right) - \frac{4}{c^4} \sum_{l\geqslant 1} \frac{(-1)^l}{l!} \left\{ \partial_{L-1} \left(\frac{1}{r} Y_{iL-1}(u) \right) + \frac{l}{l+1} \varepsilon_{iab} \partial_{aL-1} \left(\frac{1}{r} Z_{bL-1}(u) \right) \right\} . \tag{4.13b}$$

All these multipole moments I_L, J_L, W_L, X_L, Y_L, Z_L will be computed in section 5. Using the previous definition of a linearized metric (4.8)–(4.13), we can thus rewrite our general multipole expansion (3.13) as

$$\mathcal{M}(h^{\mu\nu}) = G h^{\mu\nu}_{\text{part.1}} + u^{\mu\nu} + v^{\mu\nu} \tag{4.14}$$

(where we recall that $u^{\mu\nu}$ and $v^{\mu\nu}$ are functionals of $\mathcal{M}(\Lambda^{\mu\nu})$ given previously). Intuitively from this equation, the terms $u^{\mu\nu}$ and $v^{\mu\nu}$ should represent the nonlinear contributions (of order G^2 at least) to be added to the 'linearized' metric $h^{\mu\nu}_{\text{part.1}}$ in order to obtain the 'complete' (including all powers of G) multipole expansion $\mathcal{M}(h^{\mu\nu})$. We shall prove that this is true, i.e. that by performing the nonlinear iteration of $h^{\mu\nu}_{\text{part.1}}$ following exactly the MPM algorithm of paper I (the only difference being that we have slightly modified the algorithm for the computation of $v^{\mu\nu}$, see appendix B), we get an infinite power series in G which agrees with $\mathcal{M}(h^{\mu\nu})$ term by term in the post-Minkowskian expansion.

Consistently with the algorithm of paper I, we must consider that the first term in the post-Minkowskian expansion, i.e. $Gh_{\text{part.1}}$, is purely of order G, and thus that $h_{\text{part.1}}$ itself is of zeroth order in G. With this convention let us show that the terms u and v in (4.14) are of order G^2. Suppose they are not, so that $u = Gu_{\text{part.1}} + \mathrm{O}(G^2)$ and $v = Gv_{\text{part.1}} + \mathrm{O}(G^2)$ for some 'linearized' coefficients $u_{\text{part.1}}$ and $v_{\text{part.1}}$. From (4.14) the multipole expansion of the source, namely $\mathcal{M}(\Lambda) \equiv \Lambda(\mathcal{M}(h))$, is given by $\Lambda(Gh_{\text{part.1}} + u + v)$, and therefore, inserting the previous assumptions for u and v, by $\Lambda(G[h_{\text{part.1}} + u_{\text{part.1}} + v_{\text{part.1}}] + \mathrm{O}(G^2))$. Remember that Λ is quadratic in h, and set $\Lambda(h) = N_2(h) + \mathrm{O}(h^3)$ (see (2.8), (2.9)). Thus, obviously, $\Lambda(\mathcal{M}(h)) = G^2 N_2(h_{\text{part.1}} + u_{\text{part.1}} + v_{\text{part.1}}) + \mathrm{O}(G^3)$. Now, according to (4.1), the finite part of the retarded integral of the source $\Lambda(\mathcal{M}(h))$ is u itself. Using this fact and the fact that the operator $\mathrm{FP}\Box^{-1}_R$ does not depend on G (thus it does not mix up the powers of G), we obtain the equation $u = G^2 \mathrm{FP}\,\Box^{-1}_R\,[N_2(h_{\text{part.1}} + u_{\text{part.1}} + v_{\text{part.1}})] + \mathrm{O}(G^3)$. The right-hand side of the latter equation is of order G^2 so we deduce $u_{\text{part.1}} = 0$. Then from the 'linearization' of the formulae (4.7) we further deduce $v_{\text{part.1}} = 0$. Thus u and v are indeed of order $\mathrm{O}(G^2)$, and thus (4.14) shows that $\mathcal{M}(h^{\mu\nu})$ agrees with the 'particular' metric at linearized order:

$$\mathcal{M}(h^{\mu\nu}) = Gh^{\mu\nu}_{\text{part.1}} + \mathrm{O}(G^2)\,. \tag{4.15}$$

Now let us denote $u = G^2 u_{\text{part.2}} + \mathrm{O}(G^3)$ and $v = G^2 v_{\text{part.2}} + \mathrm{O}(G^3)$. By the equation used just before (4.15), in which we can now insert $u_{\text{part.1}} = v_{\text{part.1}} = 0$, we have $u = G^2 \mathrm{FP}\,\Box^{-1}_R\,[N_2(h_{\text{part.1}})] + \mathrm{O}(G^3)$, thus

$$u^{\mu\nu}_{\text{part.2}} = \mathrm{FP}_{B=0}\,\Box^{-1}_R\,[\tilde{r}^B N^{\mu\nu}_2(h_{\text{part.1}})]\,. \tag{4.16}$$

Next $v_{\text{part.2}}$ is obtained from $u_{\text{part.2}}$ by the formulae (4.7) (see also appendix B), which are precisely those used in the algorithm of paper I (as redefined in [35]). The definition of the quadratic part of the particular metric in paper I was $h_{\text{part.2}} = u_{\text{part.2}} + v_{\text{part.2}}$ (indeed see (2.10)–(2.12)), so we find from (4.14) that $\mathcal{M}(h^{\mu\nu})$ agrees with the particular metric to quadratic order,

$$\mathcal{M}(h^{\mu\nu}) = Gh^{\mu\nu}_{\text{part.1}} + G^2 h^{\mu\nu}_{\text{part.2}} + \mathrm{O}(G^3)\,. \tag{4.17}$$

The same reasoning is easily extended to all orders in G.

In conclusion, the 'particular' metric $h^{\mu\nu}_{\text{part}} = Gh^{\mu\nu}_{\text{part.1}} + G^2 h^{\mu\nu}_{\text{part.2}} + \mathrm{O}(G^3)$ which was defined in paper I agrees, in the sense of power series in G, with the general multipole expansion $\mathcal{M}(h^{\mu\nu})$. This result is mandatory because it was shown in paper I (theorem 4.2) that the general solution of the vacuum Einstein equations in $\mathbb{R}^3_* \times \mathbb{R}$ can be written as $h^{\mu\nu}_{\text{part}}$ for *some* set of moments I_L, J_L, W_L, X_L, Y_L, and Z_L. This is of course consistent with the fact that we have made no restriction when deriving the multipole expansion $\mathcal{M}(h^{\mu\nu})$ except that it should correspond to a slowly-moving isolated system (without singularities). What we have gained with respect to paper I is that we understand from section 3 the relation between $\mathcal{M}(h^{\mu\nu}) \equiv h^{\mu\nu}_{\text{part}}[I, J, \ldots, Z]$ and the matter distribution in the source, and therefore that we are able to *compute* the multipole moments $I_L, J_L, \ldots, Z_L$ (given a post-Newtonian algorithm for the computation of $\overline{\tau}^{\mu\nu}$).

5. The irreducible multipole moments

By (4.11)–(4.13), the linearized metric $h^{\mu\nu}_{\text{part.1}}$ is parametrized by six sets of irreducible (STF) multipole moments, with two sets I_L, J_L parametrizing the 'canonical' linear metric $h^{\mu\nu}_{\text{can.1}}$, and four sets W_L, X_L, Y_L, Z_L parametrizing a gauge transformation. We can refer to the moments I_L, J_L, W_L, X_L, Y_L, Z_L as the source multipole moments. Of course, since the moments W_L, X_L, Y_L, Z_L parametrize a gauge transformation, they do not play a physical role at the level of the *linearized* approximation. In this sense, these four moments are 'less important' than the moments I_L and J_L, which constitute the 'main' multipole moments of the source, respectively of mass type and current type. However, it is important to keep the moments $W_L, \ldots, Z_L$ as they start playing a role at the nonlinear level (at a high post-Newtonian approximation [37]). See the discussion in section 6 where we recall also that the six sets of moments I_L, J_L, $W_L, \ldots, Z_L$ are in fact equivalent physically to only two sets of other moments M_L and S_L.

In the present section we use the results of section 3 to compute explicitly the moments $I_L, J_L, \ldots, Z_L$. For this purpose it suffices to decompose into irreducible pieces the functions $\mathcal{F}^{\mu\nu}_L$ and $\mathcal{G}^{\mu}_L$ which parametrize the multipole expansions (3.13) and (4.2). We first decompose the components of $\mathcal{F}^{\mu\nu}_L$ according to

$$\mathcal{F}^{00}_L = R_L\,, \tag{5.1a}$$

$$\mathcal{F}^{0i}_L = {}^{(+)}T_{iL} + \varepsilon_{ai<i_l}\,{}^{(0)}T_{L-1>a} + \delta_{i<i_l}\,{}^{(-)}T_{L-1>}\,, \tag{5.1b}$$

$$\begin{aligned}\mathcal{F}^{ij}_L = {}^{(+2)}U_{ijL} + \underset{L}{\text{STF}}\,\underset{ij}{\text{STF}}\,[\varepsilon_{aii_l}\,{}^{(+1)}U_{ajL-1} + \delta_{ii_l}\,{}^{(0)}U_{jL-1}\\ + \delta_{ii_l}\varepsilon_{aji_{l-1}}\,{}^{(-1)}U_{aL-2} + \delta_{ii_l}\delta_{ji_{l-1}}\,{}^{(-2)}U_{L-2}] + \delta_{ij}V_L\,,\end{aligned} \tag{5.1c}$$

where the ten tensors R_L, ${}^{(+)}T_{L+1}, \ldots, {}^{(-2)}U_{L-2}$, V_L are STF in all their indices. We use the standard notation of [16, 26] (notably $<>$ denotes the STF projection, see the footnote in section 2 for our notation). These ten tensors are uniquely given in terms of the $\mathcal{F}^{\mu\nu}_L$'s by the inverse formulae

$$R_L = \mathcal{F}^{00}_L\,, \tag{5.2a}$$

$${}^{(+)}T_{L+1} = \mathcal{F}^{0<i_{l+1}}_{L>}\,, \tag{5.2b}$$

$${}^{(0)}T_L = \frac{l}{l+1}\mathcal{F}^{0a}_{b<L-1}\,\varepsilon_{i_l>ab}\,, \tag{5.2c}$$

$${}^{(-)}T_{L-1} = \frac{2l-1}{2l+1}\mathcal{F}^{0a}_{aL-1}\,, \tag{5.2d}$$

$${}^{(+2)}U_{L+2} = \mathcal{F}^{<i_{l+2}i_{l+1}}_{L>}\,, \tag{5.2e}$$

$${}^{(+1)}U_{L+1} = \frac{2l}{l+2}\,\underset{L+1}{\text{STF}}\,\mathcal{F}^{<ci_l>}_{dL-1}\,\varepsilon_{i_{l+1}cd}\,, \tag{5.2f}$$

$${}^{(0)}U_L = \frac{6l(2l-1)}{(l+1)(2l+3)}\,\underset{L}{\text{STF}}\,\mathcal{F}^{<ai_l>}_{aL-1}\,, \tag{5.2g}$$

$${}^{(-1)}U_{L-1} = \frac{2(l-1)(2l-1)}{(l+1)(2l+1)}\,\underset{L-1}{\text{STF}}\,\mathcal{F}^{<ac>}_{bcL-2}\varepsilon_{i_{l-1}ab}\,, \tag{5.2h}$$

$$ {}^{(-2)}U_{L-2} = \frac{2l-3}{2l+1}\mathcal{F}^{<ab>}_{abL-2}\,, \tag{5.2i} $$

$$ V_L = \tfrac{1}{3}\,\mathcal{F}^{aa}_L\,. \tag{5.2j} $$

See, for instance, (5.5)–(5.8) in [16]. Next we decompose the tensors $\mathcal{G}^{\mu}_L$ according to

$$ \mathcal{G}^0_L = P_L\,, \tag{5.3a} $$

$$ \mathcal{G}^i_L = {}^{(+)}Q_{iL} + \varepsilon_{ai<i_l}\,{}^{(0)}Q_{L-1>a} + \delta_{i<i_l}\,{}^{(-)}Q_{L-1>}\,, \tag{5.3b} $$

with inverse formulae

$$ P_L = \mathcal{G}^0_L\,, \tag{5.4a} $$

$$ {}^{(+)}Q_{L+1} = \mathcal{G}^{<i_{l+1}}_{L>}\,, \tag{5.4b} $$

$$ {}^{(0)}Q_L = \frac{l}{l+1}\mathcal{G}^a_{b<L-1}\,\varepsilon_{i_l>ab}\,, \tag{5.4c} $$

$$ {}^{(-)}Q_{L-1} = \frac{2l-1}{2l+1}\mathcal{G}^a_{aL-1}\,. \tag{5.4d} $$

The tensors parametrizing $\mathcal{G}^{\mu}_L$ are not independent of the tensors parametrizing $\mathcal{F}^{\mu\nu}_L$. This is because the metric $h^{\mu\nu}_{\rm part.1}$ is divergenceless by (4.10). The (four) relations linking these tensors are readily obtained from (4.4); we have

$$ P_L = \frac{1}{c}\,\dot{R}_L - l\,{}^{(+)}T_L - \frac{1}{c^2(2l+1)}\,{}^{(-)}\ddot{T}_L\,, \tag{5.5a} $$

$$ {}^{(+)}Q_L = \frac{1}{c}\,{}^{(+)}\dot{T}_L - (l-1)^{(+2)}U_L - \frac{(l+1)(2l+3)}{6c^2l(2l-1)(2l+1)}\,{}^{(0)}\ddot{U}_L - \frac{1}{c^2(2l+1)}\,\ddot{V}_L\,, \tag{5.5b} $$

$$ {}^{(0)}Q_L = \frac{1}{c}\,{}^{(0)}\dot{T}_L - \frac{l}{2}{}^{(+1)}U_L - \frac{l+2}{2c^2(l+1)(2l+1)}\,{}^{(-1)}\ddot{U}_L\,, \tag{5.5c} $$

$$ {}^{(-)}Q_L = \frac{1}{c}\,{}^{(-)}\dot{T}_L - \frac{l+1}{6}{}^{(0)}U_L - \frac{1}{c^2(2l+3)}\,{}^{(-2)}\ddot{U}_L - (l+1)V_L\,. \tag{5.5d} $$

These relations permit one to express the ten independent components of $h^{\mu\nu}_{\rm part.1}$ in terms of only six independent combinations of STF tensors. We substitute the decompositions (5.1) and (5.3) into the definition of the linearized metric (4.7), (4.8). After some manipulation of STF tensors, and use of the previous relations (5.5), we arrive at an expression that can be compared directly with the general decomposition of a linearized metric as given by equation (2.25) in paper I. Then the six sets of multipole moments I_L, J_L, W_L, X_L, Y_L, Z_L entering $h^{\mu\nu}_{\rm part.1}$ are obtained by applying the definitions (2.26) in paper I (actually our definitions differ from paper I by some constant factors). First the moments I_L and J_L are obtained as follows. In the particular cases where I_L has zero or one index ($l = 0, 1$) and where J_L has one index ($l = 1$), we have

$$ I = \frac{1}{c^2}(R+3V) - \frac{4}{c^3}{}^{(-)}\dot{T} + \frac{1}{c^4}{}^{(-2)}\ddot{U} - \frac{1}{c}\int P + \frac{3}{c^2}{}^{(-)}Q\,, \tag{5.6a} $$

$$ I_i = \frac{1}{c^2}(R_i+3V_i) - \frac{2}{c^3}{}^{(-)}\dot{T}_i + \frac{1}{3c^4}{}^{(-2)}\ddot{U}_i - \frac{1}{c}\int P_i - \iint{}^{(+)}Q_i + \frac{5}{3c^2}{}^{(-)}Q_i\,, \tag{5.6b} $$

$$ J_i = -\frac{2}{c}{}^{(0)}T_i + \frac{1}{2c^2}{}^{(-1)}\dot{U}_i + 2\int{}^{(0)}Q_i\,. \tag{5.6c} $$

Then, in the generic case where I_L and J_L have at least two indices ($l \geqslant 2$), we have

$$I_L = \frac{1}{c^2}(R_L + 3V_L) - \frac{4}{c^3(l+1)}{}^{(-)}\dot{T}_L + \frac{2}{c^4(l+1)(l+2)}{}^{(-2)}\ddot{U}_L\,, \qquad (5.7a)$$

$$J_L = -\frac{l+1}{lc}{}^{(0)}T_L + \frac{1}{2lc^2}{}^{(-1)}\dot{U}_L\,. \qquad (5.7b)$$

Secondly, the four tensors W_L, X_L, Y_L, Z_L are generically obtained as

$$W_L = \frac{1}{c(l+1)}{}^{(-)}T_L - \frac{1}{2c^2(l+1)(l+2)}{}^{(-2)}\dot{U}_L\,, \qquad (5.8a)$$

$$X_L = \frac{1}{2(l+1)(l+2)}{}^{(-2)}U_L\,, \qquad (5.8b)$$

$$Y_L = \frac{3}{c(l+1)}{}^{(-)}\dot{T}_L - \frac{2}{c^2(l+1)(l+2)}{}^{(-2)}\ddot{U}_L - 3\,V_L\,, \qquad (5.8c)$$

$$Z_L = -\frac{1}{2l}{}^{(-1)}U_L\,. \qquad (5.8d)$$

The expressions of all these moments are obtained first by substituting into their definitions (5.6)–(5.8) the inverse relations (5.2) and (5.4), and second by using the expressions (3.14) and (4.5) of the functions $\mathcal{F}_L^{\mu\nu}$ and $\mathcal{G}_L^{\mu}$.

We first investigate the lowest-order moments I, I_i and J_i defined by (5.6), which can, respectively, be called the mass monopole (or total ADM mass), mass dipole and current dipole of the source. As is readily checked using (5.5) (and the fact that ${}^{(0)}U = {}^{(+1)}U_i = 0$), we have the conservation laws appropriate for gravitational monopoles and dipoles, i.e.

$$\dot{I} = 0\,, \qquad \ddot{I}_i = 0\,, \qquad \dot{J}_i = 0\,. \qquad (5.9)$$

For simplicity we analyse only the case of the mass monopole I; the analysis of the dipoles I_i and J_i is similar. The expression for I deduced from (5.6*a*) is

$$Ic^2 = \mathrm{FP}_{B=0}\int \mathrm{d}^3\boldsymbol{y}\,|\tilde{\boldsymbol{y}}|^B \int_{-1}^{1}\mathrm{d}z\,\biggl\{\delta_0(\overline{\tau}^{00} + \overline{\tau}^{ii}) - \frac{4}{3c}\delta_1 y_i \dot{\overline{\tau}}^{0i} + \frac{1}{5c^2}\delta_2 \hat{y}_{ij}\ddot{\overline{\tau}}^{ij}$$
$$+ B|\boldsymbol{y}|^{-2}(\delta_1 y_{ij}\overline{\tau}^{ij} - c\delta_0 y_i {\textstyle\int}\overline{\tau}^{0i})\biggr\}(\boldsymbol{y}, u + z|\boldsymbol{y}|/c)\,, \qquad (5.10)$$

which can be transformed using (4.3) into the simpler form

$$Ic^2 = \mathrm{FP}_{B=0}\int \mathrm{d}^3\boldsymbol{y}\,|\tilde{\boldsymbol{y}}|^B \int_{-1}^{1}\mathrm{d}z\,\delta_0\left\{\overline{\tau}^{00} - z|\boldsymbol{y}|^{-1}y_i\overline{\tau}^{0i} - Bc|\boldsymbol{y}|^{-2}y_i{\textstyle\int}\overline{\tau}^{0i}\right\}(\boldsymbol{y}, u + z|\boldsymbol{y}|/c)\,. \qquad (5.11)$$

The latter expression looks unfamiliar for a conserved mass, but this is simply due to the unusual spacelike hypersurface $t - z|\boldsymbol{y}|/c = u =$ constant on which one integrates (see the discussion in [16]). The following technical identity is useful to transform (5.11):

$$\frac{\mathrm{d}}{\mathrm{d}z'}\left\{(\overline{\tau}^{00} - z'|\boldsymbol{y}|^{-1}y_i\overline{\tau}^{0i})(\boldsymbol{y}, u + z'|\boldsymbol{y}|/c)\right\} = -\partial_i\{|\boldsymbol{y}|\overline{\tau}^{0i}(\boldsymbol{y}, u + z'|\boldsymbol{y}|/c)\}\,. \qquad (5.12)$$

Multiplying this identity by $|\tilde{\boldsymbol{y}}|^B$, integrating over $\boldsymbol{y}$, and over z' from 0 to z, and then multiplying by δ_0 and integrating over z from -1 to 1, permits one to find the equivalent of (5.11) when one uses the usual spacelike hypersurface $t = u =$ constant,

$$Ic^2 = \mathrm{FP}_{B=0}\int \mathrm{d}^3\boldsymbol{y}\,|\tilde{\boldsymbol{y}}|^B\left[\overline{\tau}^{00} - Bc|\boldsymbol{y}|^{-2}y_i{\textstyle\int}\overline{\tau}^{0i}\right](\boldsymbol{y}, u)\,. \qquad (5.13)$$

The fact that I is constant is easily checked for this expression. Note that the second term in (5.13) (whose time derivative is associated with the flux of radiation at infinity) involves a factor B and therefore its value comes only from the poles of the integral at the upper bound $|\boldsymbol{y}| \to +\infty$.

We now deal with the 'dynamic' moments I_L and J_L having $l \geqslant 2$. In order to express them, it is convenient to use the following notation for combinations of components of the pseudo-tensor $\overline{\tau}^{\mu\nu}$:

$$\overline{\Sigma} = \frac{\overline{\tau}^{00} + \overline{\tau}^{ii}}{c^2}\,, \tag{5.14a}$$

$$\overline{\Sigma}_i = \frac{\overline{\tau}^{0i}}{c}\,, \tag{5.14b}$$

$$\overline{\Sigma}_{ij} = \overline{\tau}^{ij}\,, \tag{5.14c}$$

where $\overline{\tau}^{ii} \equiv \delta_{ij}\overline{\tau}^{ij}$. ($\overline{\Sigma}$, $\overline{\Sigma}_i$ and $\overline{\Sigma}_{ij}$ are of zeroth order in the post-Newtonian expansion.) Then, by (5.7), (5.2) and (3.14), we find

$$\begin{aligned} I_L(u) = \mathrm{FP}_{B=0} \int \mathrm{d}^3\boldsymbol{y}\, |\tilde{\boldsymbol{y}}|^B \int_{-1}^{1} \mathrm{d}z \biggl\{ &\delta_l \hat{y}_L \overline{\Sigma} - \frac{4(2l+1)}{c^2(l+1)(2l+3)} \delta_{l+1} \hat{y}_{iL} \dot{\overline{\Sigma}}_i \\ &+ \frac{2(2l+1)}{c^4(l+1)(l+2)(2l+5)} \delta_{l+2} \hat{y}_{ijL} \ddot{\overline{\Sigma}}_{ij} \biggr\}(\boldsymbol{y}, u + z|\boldsymbol{y}|/c)\,, \end{aligned} \tag{5.15}$$

$$\begin{aligned} J_L(u) = \varepsilon_{ab<i_l} \mathrm{FP}_{B=0} \int \mathrm{d}^3\boldsymbol{y}\, |\tilde{\boldsymbol{y}}|^B \int_{-1}^{1} \mathrm{d}z \biggl\{ &\delta_l \hat{y}_{L-1>a} \overline{\Sigma}_b \\ &- \frac{2l+1}{c^2(l+2)(2l+3)} \delta_{l+1} \hat{y}_{L-1>ac} \dot{\overline{\Sigma}}_{bc} \biggr\}(\boldsymbol{y}, u + z|\boldsymbol{y}|/c)\,. \end{aligned} \tag{5.16}$$

These expressions have been derived in the nonlinear theory (to all orders in the post-Newtonian expansion). As a check of I_L and J_L, we can compare their expressions to the corresponding expressions derived by Damour and Iyer [16] in the case of the linearized theory, where we can replace the pseudo-tensor $\overline{\tau}^{\mu\nu}$ by the matter stress–energy tensor $T^{\mu\nu}$ in flat spacetime (we have $\overline{T}^{\mu\nu} = T^{\mu\nu}$ inside the slowly-moving source), and then remove the analytic continuation factors since $T^{\mu\nu}$ is compact-supported. We find perfect agreement with equations (5.33) and (5.35) in [16]. In the nonlinear theory but at the 2PN order, expressions (5.15), (5.16) were already derived in paper II. To 1PN order these expressions are equivalent to some different expressions derived earlier in [32, 33] (see paper II for the proof).

Finally we write down the other four moments $W_L, \ldots, Z_L$. They are easily obtained as

$$\begin{aligned} W_L(u) = \mathrm{FP}_{B=0} \int \mathrm{d}^3\boldsymbol{y}\, |\tilde{\boldsymbol{y}}|^B \int_{-1}^{1} \mathrm{d}z \biggl\{ &\frac{2l+1}{(l+1)(2l+3)} \delta_{l+1} \hat{y}_{iL} \overline{\Sigma}_i \\ &- \frac{2l+1}{2c^2(l+1)(l+2)(2l+5)} \delta_{l+2} \hat{y}_{ijL} \dot{\overline{\Sigma}}_{ij} \biggr\}(\boldsymbol{y}, u + z|\boldsymbol{y}|/c)\,, \end{aligned} \tag{5.17}$$

$$X_L(u) = \mathrm{FP}_{B=0} \int \mathrm{d}^3\boldsymbol{y}\, |\tilde{\boldsymbol{y}}|^B \int_{-1}^{1} \mathrm{d}z \left\{ \frac{2l+1}{2(l+1)(l+2)(2l+5)} \delta_{l+2} \hat{y}_{ijL} \overline{\Sigma}_{ij} \right\} (\boldsymbol{y}, u + z|\boldsymbol{y}|/c)\,, \tag{5.18}$$

$$Y_L(u) = \mathrm{FP}_{B=0} \int \mathrm{d}^3\boldsymbol{y}\, |\tilde{\boldsymbol{y}}|^B \int_{-1}^{1} \mathrm{d}z \left\{ -\delta_l \hat{y}_L \overline{\Sigma}_{ii} + \frac{3(2l+1)}{(l+1)(2l+3)} \delta_{l+1} \hat{y}_{iL} \dot{\overline{\Sigma}}_i \right. \\ \left. - \frac{2(2l+1)}{c^2(l+1)(l+2)(2l+5)} \delta_{l+2} \hat{y}_{ijL} \ddot{\overline{\Sigma}}_{ij} \right\} (\boldsymbol{y}, u + z|\boldsymbol{y}|/c)\,, \tag{5.19}$$

$$Z_L(u) = \mathrm{FP}_{B=0} \int \mathrm{d}^3\boldsymbol{y}\, |\tilde{\boldsymbol{y}}|^B \int_{-1}^{1} \mathrm{d}z \left\{ -\frac{2l+1}{(l+2)(2l+3)} \varepsilon_{ab<i_l} \delta_{l+1} \hat{y}_{L-1>bc} \overline{\Sigma}_{ac} \right\} \\ \times (\boldsymbol{y}, u + z|\boldsymbol{y}|/c)\,. \tag{5.20}$$

To Newtonian order these expressions are in agreement with equations (4.17) in [37].

6. Discussion

Of what use are the expressions of the STF multipole moments I_L, J_L and W_L, X_L, Y_L, Z_L obtained in the previous equations (5.15)–(5.20)? From (4.11)–(4.13) these moments parametrize the linearized metric $h^{\mu\nu}_{\mathrm{part.1}}$ which is the 'seed' of the infinite nonlinear algorithm of paper I. Thus for a specific application the expressions (5.15)–(5.20) have to be computed in a post-Newtonian expansion up to a given order and for a specific matter model (i.e. a specific choice of $T^{\mu\nu}$), and then inserted into the so-called 'particular' algorithm of paper I for the computation of the field nonlinearities (essentially this is what has been done for compact binary systems in [36–38]). Actually the main moments to be computed are I_L and J_L because the other moments $W_L, \ldots, Z_L$ parametrize a gauge transformation and thus have no physical implications at the linearized order. In terms of a post-Newtonian expansion it was shown in [30, 37] that up to the 2PN order it is sufficient to compute I_L and J_L, and that the other moments $W_L, \ldots, Z_L$ start contributing at the 2.5PN order.

Now it was proved in paper I that the full nonlinear metric outside an isolated system can always be parametrized (modulo a coordinate transformation) by only *two* sets of STF multipole moments, say M_L and S_L (different from I_L and J_L). These moments parametrize the so-called 'canonical' algorithm of paper I, defined by the same formulae as for the 'particular' algorithm but starting with the canonical linearized metric $h^{\mu\nu}_{\mathrm{can.1}}$ given by (4.12) (but where I_L, J_L are replaced by M_L, S_L). By theorems 4.2 and 4.5 in paper I, the canonical and particular algorithms differ from each other by a coordinate transformation. Therefore the multipole moments M_L and S_L are necessarily given as some functionals of the other moments I_L, J_L and $W_L, \ldots, Z_L$, i.e.

$$M_L = M_L[I, J, W, X, Y, Z]\,, \tag{6.1a}$$

$$S_L = S_L[I, J, W, X, Y, Z]\,. \tag{6.1b}$$

These functionals are quite complicated in general but can be explicitly constructed up to any post-Minkowskian order by implementing the coordinate transformation between the two harmonic coordinate systems in which the 'particular' and 'canonical' metrics are defined (see section 4.3 in paper I). Since at the linearized level the canonical and particular metrics differ by a mere gauge transformation (see (4.11)), we have agreement at this level between M_L, S_L and I_L, J_L:

$$M_L = I_L + \mathrm{O}(G)\,, \tag{6.2a}$$

$$S_L = J_L + \mathrm{O}(G)\,, \tag{6.2b}$$

where $\mathrm{O}(G)$ symbolizes some nonlinear (quadratic at least) products of the source moments. Furthermore, the result of [37] is that in a post-Newtonian re-expansion $c \to +\infty$ we have

$$M_L = I_L + \frac{1}{c^5}\delta I_L + \mathrm{O}\left(\frac{1}{c^6}\right), \tag{6.3}$$

where δI_L is given in detail by equation (4.24) of [37].

On the other hand, it is known (see, e.g., [12]) that the transverse and tracefree (TT) part of the spatial metric at leading order $1/R$ in the distance can be parametrized by yet another double set of STF multipole moments, say U_L and V_L. These moments are called the radiative moments of the source, as they are the moments which would be measured at infinity. The radiative moments differ from the source moments because of the nonlinear terms $u^{\mu\nu}$ and $v^{\mu\nu}$ in (4.14). Since the exterior field is entirely determined by the moments M_L, S_L, the radiative moments U_L, V_L are necessarily given as some (fully nonlinear) functionals of them:

$$U_L = U_L[M, S], \tag{6.4a}$$

$$V_L = V_L[M, S]. \tag{6.4b}$$

U_L and V_L are conveniently chosen in such a way that at the linearized order they reduce to the lth time derivatives of the moments M_L, S_L:

$$U_L = \frac{\mathrm{d}^l M_L}{\mathrm{d}u^l} + \mathrm{O}(G), \tag{6.5a}$$

$$V_L = \frac{\mathrm{d}^l S_L}{\mathrm{d}u^l} + \mathrm{O}(G), \tag{6.5b}$$

where $\mathrm{O}(G)$ denotes the nonlinear terms. It was shown in [27] that the functionals (6.4) can be constructed to all orders in the post-Minkowskian expansion by implementing the coordinate transformation between the harmonic coordinates and some suitable 'radiative' coordinates in which the metric admits an expansion in powers of $1/R$ (without logarithms of R). Furthermore, once obtained in a post-Minkowskian expansion, the functionals (6.4) can be re-expanded when $c \to +\infty$. In this limit the dominant correction is of order $1/c^3$ (or, rather, G/c^3) and due to the so-called tails of waves [34]. For instance, we have, in the quadrupole case,

$$U_{ij}(u) = \frac{\mathrm{d}^2 M_{ij}}{\mathrm{d}u^2}(u) + \frac{2GM}{c^3}\int_{-\infty}^{u} \mathrm{d}v \left[\ln\left(\frac{u-v}{2b}\right) + \frac{11}{12}\right] \frac{\mathrm{d}^4 M_{ij}}{\mathrm{d}u^4}(v) + \mathrm{O}\left(\frac{1}{c^5}\right), \tag{6.6}$$

where M is the ADM mass of the source ($M \equiv I$), and where b is a constant time scale entering the relation between harmonic and radiative coordinates. The complete correction of order G involves other terms and can be found in [34, 35].

We now understand that the explicit expressions (5.15)–(5.20) of the 'source' multipole moments I_L, J_L and $W_L, \ldots, Z_L$ are to be inserted into the chain of functionals (6.1) and (6.4) giving the radiative moments U_L, V_L detected far away from the source. Since, in the present investigation, the source moments have been related to the stress–energy tensor of the source up to any (post-Newtonian) order, and since the functionals (6.1) and (6.4) can be 'algorithmically' computed up to any nonlinear order [26, 27] (and then be re-expanded when $c \to \infty$), we conclude that the radiative moments U_L, V_L can be computed up to any post-Newtonian order (in principle), in terms of the source parameters. Of course the resulting formalism becomes extremely complicated when going to high post-Newtonian orders, especially when computing the source multipole moments (5.15), (5.16). For the moment it has been investigated in the case of compact binary systems up to the 3PN level only [36–38].

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Appendix A. Alternative proof of the result

In this appendix we present an alternative derivation of the main result of the paper, which is the multipole decomposition (3.11), (3.12) or equivalently (3.13), (3.14). Though less elaborate than the proof presented in section 3, this alternative derivation (which generalizes the approach followed in paper II) permits a better understanding of why the multipole moments are given by (3.12).

We denote by Δ the *difference* between the field h, solution of the field equations (2.14)–(2.16), and the finite part of the retarded integral of $\mathcal{M}(\Lambda)$ as given by the first term in (3.11):

$$\Delta \equiv h - \mathrm{FP}_{B=0}\ \Box_R^{-1}\,[\tilde{r}^B \mathcal{M}(\Lambda)]\,. \tag{A.1}$$

Since h is given by (2.18), this difference reads

$$\Delta = \frac{16\pi G}{c^4}\,\Box_R^{-1}\,\tau - \mathrm{FP}_{B=0}\ \Box_R^{-1}\,[\tilde{r}^B \mathcal{M}(\Lambda)]\,. \tag{A.2}$$

In the second term the finite part at $B = 0$ is necessary because the multipole expansion $\mathcal{M}(\Lambda)$ is singular at the origin $r = 0$. On the other hand, τ in the first term of (A.2) is regular all over $\mathbb{R}^4$, and therefore one can conveniently add into this term the finite part at $B = 0$ without changing its numerical value (for convergent integrals the finite part simply gives back the value of the integral). Thus Δ can be equivalently rewritten in the more useful form

$$\Delta = \mathrm{FP}_{B=0}\ \Box_R^{-1}\left[\tilde{r}^B\left(\frac{16\pi G}{c^4}\tau - \mathcal{M}(\Lambda)\right)\right]. \tag{A.3}$$

In this form Δ appears to be the (finite part of a) retarded integral of a source with spatially *compact* support. This readily follows from our assumption that $\tau = \mathcal{M}(\tau)$ when $r > \mathcal{R}$ (indeed, apply $\Box$ on (2.21)), and the fact that $\mathcal{M}(\Lambda) = 16\pi G/c^4 \mathcal{M}(\tau)$ because T has compact support. Therefore the multipole expansion of Δ in the region $r > \mathcal{R}$ can be obtained directly from the standard formula valid for sources with compact support. This yields immediately

$$\mathcal{M}(\Delta) = -\frac{4G}{c^4}\sum_{l=0}^{+\infty}\frac{(-1)^l}{l!}\partial_L\left\{\frac{1}{r}\mathcal{H}_L(t - r/c)\right\}, \tag{A.4}$$

where the multipole moments are given by

$$\mathcal{H}_L = \mathrm{FP}_{B=0}\int \mathrm{d}^3\boldsymbol{y}\,|\tilde{\boldsymbol{y}}|^B y_L\left(\tau - \frac{c^4}{16\pi G}\mathcal{M}(\Lambda)\right). \tag{A.5}$$

Now in the case of a slowly-moving source, the zone of validity of the post-Newtonian expansion (or near-zone) covers the compact support of the source: $d < \mathcal{R} \ll \lambda$ in the notation of section 2. Therefore both τ and $\mathcal{M}(\Lambda)$ in (A.5) can be replaced by their post-Newtonian expansions, i.e.

$$\mathcal{H}_L = \mathrm{FP}_{B=0}\int \mathrm{d}^3\boldsymbol{y}\,|\tilde{\boldsymbol{y}}|^B y_L\left(\overline{\tau} - \frac{c^4}{16\pi G}\overline{\mathcal{M}(\Lambda)}\right). \tag{A.6}$$

Finally, thanks to the structure of the post-Newtonian (or near-zone) expansion $\overline{\mathcal{M}(\Lambda)}$ as given by the right-hand side of (2.23), we see that after integration over the angles the second term in (A.6) is a sum of terms of the type $\mathrm{FP}_{B=0} B \int_0^{+\infty} \mathrm{d}|\boldsymbol{y}|\, |\tilde{\boldsymbol{y}}|^B |\boldsymbol{y}|^{2+l+a} (\ln|\tilde{\boldsymbol{y}}|)^p$ multiplied by a function of time. All the latter radial integrals are zero by analytic continuation in B (see the discussion after (3.5)), thus we conclude that the second term in (A.6) vanishes identically, and we recover the same result as obtained in (3.12):

$$\mathcal{H}_L = \mathrm{FP}_{B=0} \int \mathrm{d}^3\boldsymbol{y}\, |\tilde{\boldsymbol{y}}|^B y_L \overline{\tau}\,. \tag{A.7}$$

Appendix B. The harmonicity algorithm

The role of the tensor $v^{\mu\nu}$ defined by (4.7) is to cancel out the divergence of the tensor $u^{\mu\nu}$ given by (4.1), i.e. $\partial_\nu(u^{\mu\nu} + v^{\mu\nu}) = 0$, while at the same time being a solution of the source-free wave equation, i.e. $\Box v^{\mu\nu} = 0$. We first show the agreement between (4.7) and the expression (2.12) in [35] which is itself a slight modification of the earlier definition (4.13) in paper I. The divergence $\partial_\nu u^{\mu\nu}$ is given by (4.6) in terms of the moments $\mathcal{G}_L^\mu$ which are themselves decomposed in (5.3) into STF tensors P_L, ${}^{(+)}Q_{L+1}$, ${}^{(0)}Q_L$ and ${}^{(-)}Q_{L-1}$. Setting

$$A_L = \frac{4G}{c^4}\frac{(-1)^l}{l!} P_L\,, \tag{B.1a}$$

$$B_L = \frac{4G}{c^4}\frac{(-1)^l}{l!}\frac{-1}{l+1}{}^{(-)}Q_L\,, \tag{B.1b}$$

$$C_L = \frac{4G}{c^4}\frac{(-1)^l}{l!}\left[-l^{(+)}Q_L + \frac{l}{c^2(l+1)(2l+1)}{}^{(-)}\ddot{Q}_L\right], \tag{B.1c}$$

$$D_L = \frac{4G}{c^4}\frac{(-1)^l}{l!}{}^{(0)}Q_L\,, \tag{B.1d}$$

we can rewrite (4.6) as

$$\partial_\nu u^{0\nu} = \sum_{l\geqslant 0} \partial_L\left(\frac{1}{r}A_L\right), \tag{B.2a}$$

$$\partial_\nu u^{i\nu} = \sum_{l\geqslant 0} \partial_{iL}\left(\frac{1}{r}B_L\right) + \sum_{l\geqslant 1}\left\{\partial_{L-1}\left(\frac{1}{r}C_{iL-1}\right) + \varepsilon_{iab}\partial_{aL-1}\left(\frac{1}{r}D_{bL-1}\right)\right\}. \tag{B.2b}$$

On the other hand, the tensor $v^{\mu\nu}$ given by (4.7) is easily transformed into

$$v^{00} = -\frac{c}{r}\int A + \partial_a\left(\frac{1}{r}\left[-c\int A_a + c^2\int\!\!\int C_a - 3B_a\right]\right), \tag{B.3a}$$

$$v^{0i} = \frac{1}{r}\left[-c\int C_i + \frac{3}{c}\dot{B}_i\right] - c\varepsilon_{iab}\partial_a\left(\frac{1}{r}\int D_b\right) - \sum_{l\geqslant 2}\partial_{L-1}\left(\frac{1}{r}A_{iL-1}\right), \tag{B.3b}$$

$$v^{ij} = -\delta_{ij}\frac{1}{r}B + \sum_{l\geqslant 2}\left\{2\delta_{ij}\partial_{L-1}\left(\frac{1}{r}B_{L-1}\right) - 6\partial_{L-2(i}\left(\frac{1}{r}B_{j)L-2}\right)\right.$$
$$+ \partial_{L-2}\left(\frac{1}{r}\left[\frac{1}{c}\dot{A}_{ijL-2} + \frac{3}{c^2}\ddot{B}_{ijL-2} - C_{ijL-2}\right]\right)$$
$$\left. - 2\partial_{aL-2}\left(\frac{1}{r}\varepsilon_{ab(i}D_{j)bL-2}\right)\right\}, \tag{B.3c}$$

whose spatial trace is simply monopolar:

$$v^{ii} = -\frac{3}{r}B\,. \tag{B.3d}$$

We thus have agreement with the definition proposed in equations (2.11), (2.12) in [35].

Next we compare $v^{\mu\nu}$ with the earlier definition proposed in paper I, that we denote here by $q^{\mu\nu}$. Using the same notation as for (B.3), we have

$$q^{00} = -\frac{c}{r}\int A + \partial_a\left(\frac{1}{r}\left[-c\int A_a + c^2\iint C_a\right]\right), \tag{B.4a}$$

$$q^{0i} = -\frac{c}{r}\int C_i - c\varepsilon_{iab}\partial_a\left(\frac{1}{r}\int D_b\right) - \sum_{l\geqslant 2}\partial_{L-1}\left(\frac{1}{r}A_{iL-1}\right), \tag{B.4b}$$

$$q^{ij} = -\delta_{ij}\left[\frac{1}{r}B + \partial_a\left(\frac{1}{r}B_a\right)\right] + \sum_{l\geqslant 2}\left\{2\delta_{ij}\partial_L\left(\frac{1}{r}B_L\right) - 6\partial_{L-1(i}\left(\frac{1}{r}B_{j)L-1}\right)\right.$$
$$+ \partial_{L-2}\left(\frac{1}{r}\left[\frac{1}{c}\dot{A}_{ijL-2} + \frac{3}{c^2}\ddot{B}_{ijL-2} - C_{ijL-2}\right]\right)$$
$$\left. - 2\partial_{aL-2}\left(\frac{1}{r}\varepsilon_{ab(i}D_{j)bL-2}\right)\right\}, \tag{B.4c}$$

with spatial trace

$$q^{ii} = -3\left[\frac{1}{r}B + \partial_a\left(\frac{1}{r}B_a\right)\right]. \tag{B.4d}$$

Subtracting (B.4) from (B.3), we obtain

$$v^{00} - q^{00} = -3\partial_a\left(\frac{1}{r}B_a\right), \tag{B.5a}$$

$$v^{0i} - q^{0i} = \frac{3}{rc}\dot{B}_i\,, \tag{B.5b}$$

$$v^{ij} - q^{ij} = 3\delta_{ij}\partial_a\left(\frac{1}{r}B_a\right) - 6\partial_{(i}\left(\frac{1}{r}B_{j)}\right), \tag{B.5c}$$

which can be re-expressed in the form of the gauge transformation

$$v^{\mu\nu} - q^{\mu\nu} = \partial^\mu\varepsilon^\nu + \partial^\nu\varepsilon^\mu - \eta^{\mu\nu}\partial_\lambda\varepsilon^\lambda\,, \tag{B.6}$$

associated with the vector

$$\varepsilon^0 = 0\,, \tag{B.7a}$$

$$\varepsilon^i = -\frac{3}{r} B_i\,. \tag{B.7b}$$

Thus, had we used in (4.8) the tensor $q^{\mu\nu}$ instead of the tensor $v^{\mu\nu}$, i.e. had we considered instead of $Gh^{\mu\nu}_{\mathrm{part.1}}$ the different linearized metric

$$Gh'^{\mu\nu}_{\mathrm{part.1}} = Gh^{\mu\nu}_{\mathrm{part.1}} + \partial^\mu \varepsilon^\nu + \partial^\nu \varepsilon^\mu - \eta^{\mu\nu}\partial_\lambda \varepsilon^\lambda\,, \tag{B.8}$$

we would have obtained a dipole moment Y'_i differing from the dipole Y_i as given by (5.19) (or (5.8*c*)) with $l = 1$ by the formula

$$Y'_i = Y_i - \frac{3c^4}{4G} B_i = Y_i - \tfrac{3}{2}\,{}^{(-)}Q_i\,, \tag{B.9}$$

with no other modifications whatsoever. Thus the expression of the dipole moment Y'_i would not be 'uniform' with the expressions of the other moments Y_L for arbitrary $l \geqslant 2$. In this paper we have opted for the definition $v^{\mu\nu}$ instead of $q^{\mu\nu}$ for this reason, and also because the spatial trace v^{ii} is simpler than the corresponding q^{ii} (compare (B.3*d*) and (B.4*d*)).

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