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Gravitational-wave tails of tails

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Abstract. The tails of gravitational waves are caused by the scattering of linear waves onto the spacetime curvature generated by the total mass–energy of the source. Quite naturally, the tails of tails are caused by curvature scattering of the tails of waves themselves. The tails of tails are associated with the cubic nonlinear interaction between two mass monopole moments and, dominantly, the mass quadrupole of the source. In this paper we determine the radiation field at large distances from the source for this particular monopole–monopole–quadrupole interaction. We find that the tails of tails appear at the third post-Newtonian (3PN) order beyond the usual quadrupole radiation. Motivated by the need for accurate templates to be used in the data analysis of future detectors of gravitational waves, we compute the contribution of tails, and of tails of tails, up to the 3.5PN order in the energy flux generated by in-spiralling compact binaries.

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1. Introduction

1.1. Motivation and overview

Gravitational waves propagating through vacuum from their source to infinity share all possible contributions associated with products of multipole moments—indeed, this is a consequence of the infinite nonlinearity of the field equations.

At the quadratic nonlinear order, two multipole interactions play a prominent role (and yield effects which are representative of that order). The first interaction is that of the (mass-type) quadrupole moment $M_{pq}(t)$, which dominates the radiation field for slowly-moving sources, with itself. See [1] and references therein for the computation and discussion of this interaction (henceforth we shall refer to [1] as paper I). The second interaction is between $M_{pq}(t)$ and the static mass monopole moment M of the source (the Schwarzschild mass, or, more precisely, the ADM mass of the source). Such an interaction $M \times M_{pq}(t)$ is physically due to the propagation of quadrupole waves on the Schwarzschild background associated with M . In particular, the scattering of waves onto the potential barrier of the Schwarzschild metric produces the so-called tails, which are pieces of the field depending on the parameters of the source at all instants from $-\infty$ in the past up to the retarded time $t - r/c$.

The numerous works related or devoted to tails include some mathematical investigations of curved spacetime wave equations [2–9], several investigations and constructions of post-Minkowskian expansions [10–17], the studies of the linear perturbations of the Schwarzschild metric by fields of various spins [18–24], some discussions of physical properties of tails [25–31, 51], and the development of accurate wave-generation formalisms

[52, 53, 32–35]. Not only do the tails exist as theoretical objects predicted by general relativity, but they should exist in the future as real observed phenomena. Indeed the presence of tails in the gravitational-wave signals generated by in-spiralling compact binaries should be deciphered by the planned experiments VIRGO and LIGO (see [30, 31, 36, 37]).

In the present paper we develop this subject further by computing the *cubic* interaction between the quadrupole $M_{pq}(t)$ and *two* monopoles M . Physically this ‘monopole–monopole–quadrupole’ interaction is responsible for the scattering of the linear quadrupole waves M_{pq} onto the second-order (M^2) potential barrier of the Schwarzschild metric, and for the scattering of the quadratic tails $M \times M_{pq}$ themselves onto the first-order (M) potential barrier. The latter effect produces the so-called ‘tails of tails’ of gravitational waves, as they can pictorially be referred to. In fact we shall employ this crude appellation for the whole $M^2 \times M_{pq}$ interaction.

Like the quadratic tails, the cubic tails of tails could be computed using black-hole perturbation techniques. In this paper, we instead employ the particular post-Minkowskian approximation method proposed in [16], and which is used to compute the quadratic metric $M_{pq} \times M_{rs}$ in paper I. In section 2 we recall from previous work some relevant results on linear and quadratic metrics, and we determine the cubic source term in the field equations (in vacuum) corresponding to the looked-for interaction $M^2 \times M_{pq}$. Then we compute in section 3 the (finite part of the) retarded integral of this cubic source term, restricting ourselves to the leading-order $1/r$ in the distance to the source. The complete radiation field for the $M^2 \times M_{pq}$ interaction is obtained, and discussed, in section 4.

We find that the tails of tails carry a supplementary factor $1/c^6$ with respect to the dominant quadrupole radiation, and therefore contribute to the radiation field at the so-called third post-Newtonian (3PN) order. Such a high post-Newtonian order is *a priori* quite small in absolute magnitude, but it is still relevant to the observations of in-spiralling compact binaries by VIRGO and LIGO [36–40, 56]. Essentially the post-Newtonian corrections in the field affect, through gravitational radiation reaction, the evolution of the binary’s orbital phase, and the latter observable will be monitored very accurately in future detectors thanks to the large number of observed periods of rotation. In the author’s opinion, it is remarkable that such a cubically nonlinear effect as a tail of tail should be known in advance for comparison with real observations—and, therefore, should in principle be *detected* at the same time. In section 5 we compute the tails of tails at 3PN order occurring in the total energy flux generated by in-spiralling compact binaries (the energy flux is the crucial quantity to predict because it yields, via an energy balance argument, the effects of radiation reaction). We also compute the contribution of quadratic tails at the 3.5PN order (extending previous results at the 1.5PN and 2.5PN orders). Besides the tails of tails at 3PN in the energy flux of binaries, there are also some ‘instantaneous’ contributions, the computation of which is in progress [54, 55].

In the particular case where the mass of one body is very small compared with the other mass, the radiation field of compact binaries has been computed analytically up to a very high post-Newtonian approximation [46–49]: notably up to the 4PN order [48] and, more recently, 5.5PN order [49]. Our results in section 5 are in perfect agreement, wherever the comparison can be made, with these latter works.

The notation and conventions are essentially the same as in paper I†. In order to reduce

† Our notation is the following: signature $-+++$; Greek indices = 0, 1, 2, 3; Latin indices = 1, 2, 3; $g = \det(g_{\mu\nu})$; $\eta_{\mu\nu} = \eta^{\mu\nu} = \text{flat metric} = \text{diag}(-1, 1, 1, 1)$; $r = |\mathbf{x}| = (x_1^2 + x_2^2 + x_3^2)^{1/2}$; $n^i = n_i = x^i/r$; $\partial_i = \partial/\partial x^i$; $n^L = n_L = n_{i_1} n_{i_2} \cdots n_{i_\ell}$ and $\partial_L = \partial_{i_1} \partial_{i_2} \cdots \partial_{i_\ell}$, where $L = i_1 i_2 \cdots i_\ell$ is a multi-index with ℓ indices; $n_{L-1} = n_{i_1} \cdots n_{i_{\ell-1}}$, $n_{\alpha L-1} = n_\alpha n_{L-1}$, etc; \hat{n}_L and $\hat{\partial}_L$ are the (symmetric) and trace-free (STF) parts of n_L and ∂_L , also denoted by $n_{(L)}$, $\partial_{(L)}$; the superscript (n) denotes n time derivatives; $T_{(\alpha\beta)} = \frac{1}{2}(T_{\alpha\beta} + T_{\beta\alpha})$ and $T_{(ij)} = \frac{1}{2}(T_{ij} + T_{ji})$.

clutter we set $G = c = 1$ whenever their presence is not essential. For a short review on the post-Minkowskian method we refer the reader to section 2 of paper I.

1.2. Notation for the field equations

Our basic field variable is $h^{\alpha\beta} = \sqrt{-g} g^{\alpha\beta} - \eta^{\alpha\beta}$, with $g^{\alpha\beta}$ the contravariant metric, g the determinant of the covariant metric, and $\eta^{\alpha\beta}$ the Minkowski metric in Minkowskian coordinates (signature $-+++$). Subject to the condition of harmonic coordinates,

$$\partial_\beta h^{\alpha\beta} = 0, \quad (1.1)$$

the vacuum field equations read

$$\square h^{\alpha\beta} = N^{\alpha\beta}(h, h) + M^{\alpha\beta}(h, h, h) + \mathcal{O}(h^4), \quad (1.2)$$

where \square denotes the flat d'Alembertian operator, and where the source term in the right-hand side is made of an infinite sum of quadratic, cubic, and so on, functionals of $h^{\alpha\beta}$ and its first and second derivatives. The quadratic and cubic terms are given by

$$\begin{aligned} N^{\alpha\beta}(h, h) = & -h^{\mu\nu} \partial_\mu \partial_\nu h^{\alpha\beta} + \frac{1}{2} \partial^\alpha h_{\mu\nu} \partial^\beta h^{\mu\nu} - \frac{1}{4} \partial^\alpha h \partial^\beta h \\ & - 2\partial^{(\alpha} h_{\mu\nu} \partial^{\mu} h^{\beta)v} + \partial_\nu h^{\alpha\mu} (\partial^\nu h_\mu^\beta + \partial_\mu h^{\beta\nu}) \\ & + \eta^{\alpha\beta} \left[-\frac{1}{4} \partial_\rho h_{\mu\nu} \partial^\rho h^{\mu\nu} + \frac{1}{8} \partial_\mu h \partial^\mu h + \frac{1}{2} \partial_\mu h_{\nu\rho} \partial^\nu h^{\mu\rho} \right], \end{aligned} \quad (1.3)$$

$$\begin{aligned} M^{\alpha\beta}(h, h, h) = & -h^{\mu\nu} (\partial^\alpha h_{\mu\rho} \partial^\beta h_\nu^\rho + \partial_\rho h_\mu^\alpha \partial^\rho h_\nu^\beta - \partial_\mu h_\rho^\alpha \partial_\nu h^{\beta\rho}) \\ & + h^{\alpha\beta} \left[-\frac{1}{4} \partial_\rho h_{\mu\nu} \partial^\rho h^{\mu\nu} + \frac{1}{8} \partial_\mu h \partial^\mu h + \frac{1}{2} \partial_\mu h_{\nu\rho} \partial^\nu h^{\mu\rho} \right] + \frac{1}{2} h^{\mu\nu} \partial^{(\alpha} h_{\mu\nu} \partial^{\beta)} h \\ & + 2h^{\mu\nu} \partial_\rho h_\mu^{(\alpha} \partial^{\beta)} h_\nu^\rho + h^{\mu(\alpha} (\partial^{\beta)} h_{\nu\rho} \partial_\mu h^{\nu\rho} - 2\partial_\nu h_\rho^{(\beta)} \partial_\mu h^{\nu\rho} - \frac{1}{2} \partial^{\beta)} h \partial_\mu h) \\ & + \eta^{\alpha\beta} \left[\frac{1}{8} h^{\mu\nu} \partial_\mu h \partial_\nu h - \frac{1}{4} h^{\mu\nu} \partial_\rho h_{\mu\nu} \partial^\rho h - \frac{1}{4} h^{\rho\sigma} \partial_\rho h_{\mu\nu} \partial_\sigma h^{\mu\nu} \right. \\ & \left. - \frac{1}{2} h^{\rho\sigma} \partial_\mu h_{\rho\nu} \partial^\nu h_\sigma^\mu + \frac{1}{2} h^{\rho\sigma} \partial_\mu h_\rho^\nu \partial^\mu h_{\sigma\nu} \right]. \end{aligned} \quad (1.4)$$

The higher-order terms are even more complicated but will not be needed in this paper.

2. The monopole–monopole–quadrupole source term

2.1. The linear and quadratic metrics

We look for a solution of the equations (1.1)–(1.4) in the form of a post-linear (or post-Minkowskian) expansion

$$h^{\alpha\beta} = G h_1^{\alpha\beta} + G^2 h_2^{\alpha\beta} + G^3 h_3^{\alpha\beta} + \dots, \quad (2.1)$$

where G denotes Newton's constant. The harmonic coordinate condition (1.1) implies that all the coefficients of the G^n are divergenceless. On the other hand, the field equations (1.2) imply that the coefficients of any G^n obey a d'Alembertian equation whose source is known from the previous coefficients, i.e. coefficients of the G^m 's where $1 \leq m \leq n-1$ (see, e.g., section 2 in paper I).

Our starting point is the linearized metric $h_1^{\alpha\beta}$ defined by equation (2.3) of paper I. This metric is in the form of a multipolar series parametrized by symmetric and trace-free (STF) mass-type multipole moments M_L ($\ell \geq 0$) and current-type multipole moments S_L ($\ell \geq 1$). These moments reduce, in the Newtonian limit $c \rightarrow \infty$, to the usual Newtonian multipole moments [14]. As we are ultimately interested only in the cubic interaction between two

monopoles M and the quadrupole M_{pq} , we retain in the linearized metric only the terms involving M and M_{pq} . Accordingly we denote

$$h_1^{\alpha\beta} = h_{(M)}^{\alpha\beta} + h_{(M_{pq})}^{\alpha\beta}. \quad (2.2)$$

The monopole term reads

$$h_{(M)}^{00} = -4r^{-1}M, \quad (2.3a)$$

$$h_{(M)}^{0i} = h_{(M)}^{ij} = 0. \quad (2.3b)$$

This is simply the linearized piece of the Schwarzschild metric in harmonic coordinates, for which only the 00 component of our field variable is non-zero. The quadrupole term in (2.2) is

$$h_{(M_{pq})}^{00} = -2\partial_{ab} [r^{-1}M_{ab}(t-r)], \quad (2.4a)$$

$$h_{(M_{pq})}^{0i} = 2\partial_a [r^{-1}M_{ai}^{(1)}(t-r)], \quad (2.4b)$$

$$h_{(M_{pq})}^{ij} = -2r^{-1}M_{ij}^{(2)}(t-r). \quad (2.4c)$$

We use the same notation as in (2.3) of paper I, which gives the complete linearized metric including all multipole terms. In the following we need the metric (2.4) in expanded form, where the spatial derivatives acting on both r^{-1} and $t-r$ are worked out. We have

$$h_{(M_{pq})}^{00} = -2n_{ab}r^{-3} \{3M_{ab} + 3rM_{ab}^{(1)} + r^2M_{ab}^{(2)}\}, \quad (2.5a)$$

$$h_{(M_{pq})}^{0i} = -2n_a r^{-2} \{M_{ai}^{(1)} + rM_{ai}^{(2)}\}, \quad (2.5b)$$

$$h_{(M_{pq})}^{ij} = -2r^{-1}M_{ij}^{(2)}. \quad (2.5c)$$

Henceforth, we generally do not indicate the dependence of the moments on $t-r$.

Consider the quadratic metric $h_2^{\alpha\beta}$ generated by the linear metric (2.2)–(2.5). It is clear that $h_2^{\alpha\beta}$ involves a term proportional to M^2 , the mixed term corresponding to the interaction $M \times M_{pq}$, and the term corresponding to the self-interaction of M_{pq} , thus

$$h_2^{\alpha\beta} = h_{(M^2)}^{\alpha\beta} + h_{(MM_{pq})}^{\alpha\beta} + h_{(M_{pq}M_{rs})}^{\alpha\beta}. \quad (2.6)$$

The first term is the quadratic piece of the Schwarzschild metric in harmonic coordinates,

$$h_{(M^2)}^{00} = -7r^{-2}M^2, \quad (2.7a)$$

$$h_{(M^2)}^{0i} = 0, \quad (2.7b)$$

$$h_{(M^2)}^{ij} = -n_{ij}r^{-2}M^2, \quad (2.7c)$$

with four-dimensional trace ($h \equiv \eta_{\alpha\beta}h^{\alpha\beta}$)

$$h_{(M^2)} = 6r^{-2}M^2. \quad (2.7d)$$

The term $h_{(MM_{pq})}^{\alpha\beta}$, which constitutes the dominant non-static multipole interaction at the quadratic order, obeys a d'Alembertian equation whose source is given by (1.3) where (2.3) and (2.5) are inserted, and which can be written, with obvious notation, as $N^{\alpha\beta}(h_{(M)}, h_{(M_{pq})}) + N^{\alpha\beta}(h_{(M_{pq})}, h_{(M)})$. The solution of this d'Alembertian equation, and, then, the complete metric $h_{(MM_{pq})}^{\alpha\beta}$ itself, is obtained using the present method. The monopole–quadrupole metric is the analogue of the 2–2 metric in Bonnor's double series

method [10]. We simply report the result of its computation, which can be found in appendix B of [32]:

$$M^{-1}h_{(MM_{pq})}^{00} = n_{ab}r^{-4} \left\{ -21M_{ab} - 21rM_{ab}^{(1)} + 7r^2M_{ab}^{(2)} + 10r^3M_{ab}^{(3)} \right\} \\ + 8n_{ab} \int_1^{+\infty} dx Q_2(x)M_{ab}^{(4)}(t-rx), \quad (2.8a)$$

$$M^{-1}h_{(MM_{pq})}^{0i} = n_{iab}r^{-3} \left\{ -M_{ab}^{(1)} - rM_{ab}^{(2)} - \frac{1}{3}r^2M_{ab}^{(3)} \right\} \\ + n_ar^{-3} \left\{ -5M_{ai}^{(1)} - 5rM_{ai}^{(2)} + \frac{19}{3}r^2M_{ai}^{(3)} \right\} \\ + 8n_a \int_1^{+\infty} dx Q_1(x)M_{ai}^{(4)}(t-rx), \quad (2.8b)$$

$$M^{-1}h_{(MM_{pq})}^{ij} = n_{ijab}r^{-4} \left\{ -\frac{15}{2}M_{ab} - \frac{15}{2}rM_{ab}^{(1)} - 3r^2M_{ab}^{(2)} - \frac{1}{2}r^3M_{ab}^{(3)} \right\} \\ + \delta_{ij}n_{ab}r^{-4} \left\{ -\frac{1}{2}M_{ab} - \frac{1}{2}rM_{ab}^{(1)} - 2r^2M_{ab}^{(2)} - \frac{11}{6}r^3M_{ab}^{(3)} \right\} \\ + n_{a(i}r^{-4} \left\{ 6M_{j)a} + 6rM_{j)a}^{(1)} + 6r^2M_{j)a}^{(2)} + 4r^3M_{j)a}^{(3)} \right\} \\ + r^{-4} \left\{ -M_{ij} - rM_{ij}^{(1)} - 4r^2M_{ij}^{(2)} - \frac{11}{3}r^3M_{ij}^{(3)} \right\} \\ + 8 \int_1^{+\infty} dx Q_0(x)M_{ij}^{(4)}(t-rx), \quad (2.8c)$$

with four-dimensional trace

$$M^{-1}h_{(MM_{pq})} = n_{ab}r^{-4} \left\{ 18M_{ab} + 18rM_{ab}^{(1)} - 10r^2M_{ab}^{(2)} - 12r^3M_{ab}^{(3)} \right\} \\ - 8n_{ab} \int_1^{+\infty} dx Q_2(x)M_{ab}^{(4)}(t-rx). \quad (2.8d)$$

The metric is composed of two types of terms, instantaneous terms depending on the quadrupole moment at time $t-r$ only, and non-local (or hereditary) integrals depending on all times from $-\infty$ in the past to $t-r$. The usual tail effects are contained in the non-local integrals of (2.8). Note that the non-local integrals come exclusively from the source terms whose radial dependence is r^{-2} (see section 3 of paper I and (3.1) below). The integrals are expressed in (2.8) by means of the Legendre function of the second kind Q_ℓ (with branch cut from $-\infty$ to 1), which is related to the Legendre polynomial P_ℓ by

$$Q_\ell(x) = \frac{1}{2} \int_{-1}^1 P_\ell(y) \frac{dy}{x-y} \quad (2.9a)$$

$$= \frac{1}{2} P_\ell(x) \ln \left(\frac{x+1}{x-1} \right) - \sum_{j=1}^{\ell} \frac{1}{j} P_{\ell-j}(x) P_{j-1}(x) \quad (2.9b)$$

(the first of these relations is known as Neumann's formula for the Legendre function, see, e.g., [50]); see also (A.15) in appendix A for still another expression of the Legendre function. For future reference we quote here the expansion of Q_ℓ when $x \rightarrow 1$ (with $x > 1$),

$$Q_\ell(x) = -\frac{1}{2} \ln \left(\frac{x-1}{2} \right) - \sum_{j=1}^{\ell} \frac{1}{j} + \mathcal{O}[(x-1) \ln(x-1)]. \quad (2.9c)$$

(On the other hand, recall that the Legendre function behaves like $1/x^{\ell+1}$ when $x \rightarrow +\infty$.) Finally, the term $h_{(M_{pq}M_{rs})}^{\alpha\beta}$ in (2.6) is the quadrupole–quadrupole metric whose computation was the subject of paper I (we do not need this term in the present paper).

Now, the cubic metric $h_3^{\alpha\beta}$ is made out of all possible interactions of three moments chosen (with repetition) among M and the quadrupole M_{pq} , and is therefore constituted of four terms,

$$h_3^{\alpha\beta} = h_{(M^3)}^{\alpha\beta} + h_{(M^2M_{pq})}^{\alpha\beta} + h_{(MM_{pq}M_{rs})}^{\alpha\beta} + h_{(M_{pq}M_{rs}M_{tu})}^{\alpha\beta}. \quad (2.10)$$

Of these terms only the first one is known within the present approach (before this paper): this is the cubic piece of the Schwarzschild metric that we give here for completeness,

$$h_{(M^3)}^{00} = -8r^{-3}M^3, \quad (2.11a)$$

$$h_{(M^3)}^{0i} = h_{(M^3)}^{ij} = 0. \quad (2.11b)$$

The term $h_{(M^2M_{pq})}^{\alpha\beta}$ is the monopole–monopole–quadrupole metric, which will be dealt with in the present paper. The two last terms, which involve at least the interaction of two quadrupole moments, will be left undetermined for the time being.

2.2. Expression of the cubic source

The metric $h_{(M^2M_{pq})}^{\alpha\beta}$ obeys the harmonic-coordinates condition $\partial_\beta h_{(M^2M_{pq})}^{\alpha\beta} = 0$ and a d'Alembertian equation whose source exhausts all possibilities of generating the multipole interaction $M^2 \times M_{pq}$ by means of linear and quadratic metrics. From (1.2), one has

$$\square h_{(M^2M_{pq})}^{\alpha\beta} = \Lambda_{(M^2M_{pq})}^{\alpha\beta}, \quad (2.12)$$

where the source term $\Lambda_{(M^2M_{pq})}^{\alpha\beta}$ is obtained from $N^{\alpha\beta}$ and $M^{\alpha\beta}$ defined in (1.3) and (1.4) as

$$\begin{aligned} \Lambda_{(M^2M_{pq})}^{\alpha\beta} &= N^{\alpha\beta} (h_{(M^2)}, h_{(M_{pq})}) + N^{\alpha\beta} (h_{(M_{pq})}, h_{(M^2)}) \\ &+ N^{\alpha\beta} (h_{(M)}, h_{(MM_{pq})}) + N^{\alpha\beta} (h_{(MM_{pq})}, h_{(M)}) \\ &+ M^{\alpha\beta} (h_{(M)}, h_{(M)}, h_{(M_{pq})}) + M^{\alpha\beta} (h_{(M)}, h_{(M_{pq})}, h_{(M)}) \\ &+ M^{\alpha\beta} (h_{(M_{pq})}, h_{(M)}, h_{(M)}). \end{aligned} \quad (2.13)$$

In the first and second lines a linear metric is coupled to a quadratic one, while in the third and fourth lines three linear metrics are coupled together. The metrics have been distributed with all possible combinations of the nonlinear sources (1.3) and (1.4).

Using the explicit formulae (2.3), (2.5), (2.7) and (2.8), one obtains the cubic source term $\Lambda_{(M^2M_{pq})}^{\alpha\beta}$ after a tedious but straightforward computation. As the monopole–quadrupole metric (2.8) involves some non-local integrals, so does $\Lambda_{(M^2M_{pq})}^{\alpha\beta}$, which can thus be split into a local (instantaneous) part, say $I_{(M^2M_{pq})}^{\alpha\beta}$, and a non-local (tail) part, $T_{(M^2M_{pq})}^{\alpha\beta}$:

$$\Lambda_{(M^2M_{pq})}^{\alpha\beta} = I_{(M^2M_{pq})}^{\alpha\beta} + T_{(M^2M_{pq})}^{\alpha\beta}. \quad (2.14)$$

The result for the instantaneous part is

$$\begin{aligned} M^{-2}I_{(M^2M_{pq})}^{00} &= n_{ab}r^{-7} \{ -516M_{ab} - 516rM_{ab}^{(1)} - 304r^2M_{ab}^{(2)} \\ &- 76r^3M_{ab}^{(3)} + 108r^4M_{ab}^{(4)} + 40r^5M_{ab}^{(5)} \}, \end{aligned} \quad (2.15a)$$

$$\begin{aligned}
M^{-2}I_{(M^2M_{pq})}^{0i} &= \hat{n}_{iab}r^{-6} \left\{ 4M_{ab}^{(1)} + 4rM_{ab}^{(2)} - 16r^2M_{ab}^{(3)} + \frac{4}{3}r^3M_{ab}^{(4)} - \frac{4}{3}r^4M_{ab}^{(5)} \right\} \\
&\quad + n_a r^{-6} \left\{ -\frac{372}{5}M_{ai}^{(1)} - \frac{372}{5}rM_{ai}^{(2)} - \frac{232}{5}r^2M_{ai}^{(3)} - \frac{84}{5}r^3M_{ai}^{(4)} + \frac{124}{5}r^4M_{ai}^{(5)} \right\},
\end{aligned} \tag{2.15b}$$

$$\begin{aligned}
M^{-2}I_{(M^2M_{pq})}^{ij} &= \hat{n}_{ijabr}^{-5} \left\{ -190M_{ab}^{(2)} - 118rM_{ab}^{(3)} - \frac{92}{3}r^2M_{ab}^{(4)} - 2r^3M_{ab}^{(5)} \right\} \\
&\quad + \delta_{ij}n_{abr}^{-5} \left\{ \frac{160}{7}M_{ab}^{(2)} + \frac{176}{7}rM_{ab}^{(3)} - \frac{596}{21}r^2M_{ab}^{(4)} - \frac{160}{21}r^3M_{ab}^{(5)} \right\} \\
&\quad + \hat{n}_{a(i}r^{-5} \left\{ -\frac{312}{7}M_{j)a}^{(2)} - \frac{248}{7}rM_{j)a}^{(3)} + \frac{400}{7}r^2M_{j)a}^{(4)} + \frac{104}{7}r^3M_{j)a}^{(5)} \right\} \\
&\quad + r^{-5} \left\{ -12M_{ij}^{(2)} - \frac{196}{15}rM_{ij}^{(3)} - \frac{56}{5}r^2M_{ij}^{(4)} - \frac{48}{5}r^3M_{ij}^{(5)} \right\}
\end{aligned} \tag{2.15c}$$

(we recall, e.g., that in our notation \hat{n}_{ijab} denotes the STF projection of $n_{ijab} \equiv n_i n_j n_a n_b$). The tail part is composed of sums of products of local terms with derivatives of tail terms. Using some elementary properties of the Legendre function (namely $xQ_1(x) = \frac{2}{3}Q_2(x) + \frac{1}{3}Q_0(x)$ and $xQ_2(x) = \frac{3}{5}Q_3(x) + \frac{2}{5}Q_1(x)$), we obtain

$$M^{-2}T_{(M^2M_{pq})}^{00} = n_{abr}^{-3} \int_1^{+\infty} dx \left\{ 96Q_0M_{ab}^{(4)} + \left[\frac{272}{5}Q_1 + \frac{168}{5}Q_3 \right] rM_{ab}^{(5)} + 32Q_2r^2M_{ab}^{(6)} \right\}, \tag{2.16a}$$

$$\begin{aligned}
M^{-2}T_{(M^2M_{pq})}^{0i} &= \hat{n}_{iab}r^{-3} \int_1^{+\infty} dx \left\{ -32Q_1M_{ab}^{(4)} + \left[-\frac{32}{3}Q_0 + \frac{8}{3}Q_2 \right] rM_{ab}^{(5)} \right\} \\
&\quad + n_a r^{-3} \int_1^{+\infty} dx \left\{ \frac{96}{5}Q_1M_{ai}^{(4)} + \left[\frac{192}{5}Q_0 + \frac{112}{5}Q_2 \right] rM_{ai}^{(5)} + 32Q_1r^2M_{ai}^{(6)} \right\},
\end{aligned} \tag{2.16b}$$

$$\begin{aligned}
M^{-2}T_{(M^2M_{pq})}^{ij} &= \hat{n}_{ijabr}^{-3} \int_1^{+\infty} dx \left\{ -32Q_2M_{ab}^{(4)} + \left[-\frac{32}{5}Q_1 - \frac{48}{5}Q_3 \right] rM_{ab}^{(5)} \right\} \\
&\quad + \delta_{ij}n_{abr}^{-3} \int_1^{+\infty} dx \left\{ -\frac{32}{7}Q_2M_{ab}^{(4)} + \left[-\frac{208}{7}Q_1 + \frac{24}{7}Q_3 \right] rM_{ab}^{(5)} \right\} \\
&\quad + \hat{n}_{a(i}r^{-3} \int_1^{+\infty} dx \left\{ \frac{96}{7}Q_2M_{j)a}^{(4)} + \left[\frac{2112}{35}Q_1 - \frac{192}{35}Q_3 \right] rM_{j)a}^{(5)} \right\} \\
&\quad + r^{-3} \int_1^{+\infty} dx \left\{ \frac{32}{5}Q_2M_{ij}^{(4)} + \left[\frac{1536}{75}Q_1 - \frac{96}{75}Q_3 \right] rM_{ij}^{(5)} + 32Q_0r^2M_{ij}^{(6)} \right\},
\end{aligned} \tag{2.16c}$$

(where the Legendre functions are computed at x and the moments at $t - rx$). At this stage we have a good check that the computation is going well. Indeed $\Lambda_{(M^2M_{pq})}^{\alpha\beta}$ is the source of the third-order field equations in harmonic coordinates and therefore should be divergenceless. The divergence of the tail piece (2.16) is computed using the Legendre equation $(1-x^2)Q''_\ell(x) - 2xQ'_\ell(x) + \ell(\ell+1)Q_\ell(x) = \delta_+(x-1)$, where the distribution δ_+ is defined by $\int_1^{+\infty} dx \delta_+(x-1)\phi(x) = \phi(1)$ for any test function ϕ . We find that all the tail integrals disappear, and get

$$M^{-2}\partial_\beta T_{(M^2M_{pq})}^{0\beta} = n_{abr}^{-4} \left\{ -48M_{ab}^{(4)} - 48rM_{ab}^{(5)} - 16r^2M_{ab}^{(6)} \right\}, \tag{2.17a}$$

$$M^{-2}\partial_\beta T_{(M^2M_{pq})}^{i\beta} = n_{iab}r^{-4} \left\{ -8M_{ab}^{(4)} - 16rM_{ab}^{(5)} \right\} - 32n_a r^{-2} M_{ai}^{(6)}. \tag{2.17b}$$

On the other hand, the divergence of the instantaneous piece is computed directly from (2.15). We readily obtain that it cancels exactly (2.17) so that the required condition

$$\partial_\beta \Lambda_{(M^2 M_{pq})}^{\alpha\beta} = 0 \quad (2.18)$$

is fulfilled.

3. The retarded integral of the source term

The previous check being done, we confidently tackle the difficult part of the analysis, namely to find the inversion of the d'Alembert equation with source term given by (2.14)–(2.16). We shall limit ourselves to the computation of the metric at large distances from the source ($r \rightarrow \infty$ with $t - r = \text{constant}$), keeping only the dominant $1/r$ term at infinity (possibly multiplied by some powers of $\ln r$). This is sufficient in view of applications to astrophysical sources of gravitational radiation.

3.1. Integrating the instantaneous terms

The general form of the terms composing (2.15) is $\hat{n}_L r^{-k} F(t - r)$, where \hat{n}_L is equivalent to a spherical harmonics of order ℓ , and where the radial dependence is such that $k \geq 2$. So let us review the formulae required to compute the $1/r$ (and $\ln r/r$) terms at infinity of the retarded integral of any source term $\hat{n}_L r^{-k} F(t - r)$. Recall that in general the retarded integral cannot be applied directly to $\hat{n}_L r^{-k} F(t - r)$ because of the singular behaviour when $r \rightarrow 0$. We follow the procedure proposed in [16] to obtain a particular retarded solution (also singular when $r \rightarrow 0$) of the wave equation. It consists of first multiplying the source term by a factor $(r/r_0)^B$, where B is a complex number and r_0 a constant length scale, thereby defining a B -dependent fictitious source which, for large values of the real part of B , is regular, and in fact tends to zero, when $r \rightarrow 0$ (i.e. the singularity at $r = 0$ is killed). For large values of $\text{Re}(B)$ one is thus allowed to apply the retarded integral to the fictitious source (there is no problem at the bound $r \rightarrow \infty$ of the integral because the moments are assumed to be constant in the remote past). In this way one defines a function of B , *a priori* only for $\text{Re}(B)$ large enough, but the point is that this function is extendible by analytic continuation to all complex values of B , except at some integer values (including, in general, the value of interest $B = 0$), where it admits a Laurent expansion with some poles. Now it has been shown [16] that near the value $B = 0$ the finite part of the Laurent expansion is a particular solution of the d'Alembertian equation we wanted to solve. We call the operator giving this solution the finite part of the retarded integral, and denote it by $\text{FP}_{B=0} \square_R^{-1} (r/r_0)^B$, or more simply by $\text{FP} \square_R^{-1}$. The finite part procedure is especially convenient when doing practical computations. See appendix A in paper I for a compendium of formulae, obtained with this procedure, enabling the computation of the quadratic nonlinearities. More complicated formulae for computing the cubic nonlinearities are reported in appendix A of the present paper.

For a source term of the type $\hat{n}_L r^{-k} F(t - r)$ three cases must be distinguished (as we have already $k \geq 2$): $k = 2$, $3 \leq k \leq \ell + 2$, and $\ell + 3 \leq k$ (see appendix A of paper I). The case $k = 2$ corresponds to a retarded integral, which is convergent (so the finite part at $B = 0$ is unnecessary), and given by a non-local integral admitting an expansion when $r \rightarrow \infty$, $t - r = \text{constant}$ in powers of $1/r$ with a logarithm of r . The formula, already used in (2.8) to express the tail integrals, reads

$$\square_R^{-1} [\hat{n}_L r^{-2} F(t - r)] = -\hat{n}_L \int_1^{+\infty} dx Q_\ell(x) F(t - rx), \quad (3.1)$$

where Q_ℓ is the Legendre function (2.9). The leading term at infinity is obtained by using the expansion of Q_ℓ as given by (2.9c). We obtain [32]

$$\square_R^{-1} [\hat{n}_L r^{-2} F(t-r)] = \frac{\hat{n}_L}{2r} \int_0^{+\infty} d\tau F(t-r-\tau) \left[\ln \left(\frac{\tau}{2r} \right) + \sum_{j=1}^{\ell} \frac{2}{j} \right] + \mathcal{O} \left(\frac{\ln r}{r^2} \right). \quad (3.2)$$

In the case $3 \leq k \leq \ell + 2$ (thus $\ell \geq 1$), we find that the B -dependent integral is finite (no pole at $B = 0$), and is given by a simple local expression, without logarithms. The $1/r$ term at infinity is

$$\begin{aligned} \square_R^{-1} [(r/r_0)^B \hat{n}_L r^{-k} F(t-r)] \Big|_{B=0} \\ = - \frac{2^{k-3} (k-3)! (\ell+2-k)! \hat{n}_L}{(\ell+k-2)! r} F^{(k-3)}(t-r) + \mathcal{O} \left(\frac{1}{r^2} \right). \end{aligned} \quad (3.3)$$

In the last case, $k \geq \ell + 3$, the B -dependent retarded integral admits a truly polar part, and its finite part is given, as in (3.2), by a non-local integral, but now the expansion at infinity involves no logarithms of r (instead it involves the logarithm of the constant r_0). We have, for the $1/r$ term,

$$\begin{aligned} \text{FP} \square_R^{-1} [\hat{n}_L r^{-k} F(t-r)] &= \frac{(-1)^{k+\ell} 2^{k-3} (k-3)! \hat{n}_L}{(k+\ell-2)! (k-\ell-3)! r} \\ &\times \int_0^{+\infty} d\tau F^{(k-2)}(t-r-\tau) \left[\ln \left(\frac{\tau}{2r_0} \right) + \sum_{j=1}^{k-\ell-3} \frac{1}{j} + \sum_{j=k-2}^{k+\ell-2} \frac{1}{j} \right] + \mathcal{O} \left(\frac{1}{r^2} \right). \end{aligned} \quad (3.4)$$

With the formulae (3.2)–(3.4) it is straightforward to obtain the $1/r$ and $\ln r/r$ terms in the finite part of the retarded integral (in short $\text{FP} \square_R^{-1}$) of (2.15):

$$\begin{aligned} \text{FP} \square_R^{-1} [M^{-2} I_{(M^2 M_{pq})}^{00}] &= \frac{n_{ab}}{r} \int_0^{+\infty} d\tau M_{ab}^{(5)} \left\{ 20 \ln \left(\frac{\tau}{2r} \right) + \frac{116}{21} \ln \left(\frac{\tau}{2r_0} \right) + \frac{106.054}{2205} \right\} \\ &+ \mathcal{O} \left(\frac{\ln r}{r^2} \right), \end{aligned} \quad (3.5a)$$

$$\begin{aligned} \text{FP} \square_R^{-1} [M^{-2} I_{(M^2 M_{pq})}^{0i}] &= \frac{\hat{n}_{iab}}{r} \int_0^{+\infty} d\tau M_{ab}^{(5)} \left\{ -\frac{2}{3} \ln \left(\frac{\tau}{2r} \right) - \frac{4}{105} \ln \left(\frac{\tau}{2r_0} \right) - \frac{26.044}{11025} \right\} \\ &+ \frac{n_a}{r} \int_0^{+\infty} d\tau M_{ia}^{(5)} \left\{ \frac{62}{5} \ln \left(\frac{\tau}{2r} \right) + \frac{416}{75} \ln \left(\frac{\tau}{2r_0} \right) + \frac{40318}{1125} \right\} + \mathcal{O} \left(\frac{\ln r}{r^2} \right), \end{aligned} \quad (3.5b)$$

$$\begin{aligned} \text{FP} \square_R^{-1} [M^{-2} I_{(M^2 M_{pq})}^{ij}] &= \frac{\hat{n}_{ijab}}{r} \int_0^{+\infty} d\tau M_{ab}^{(5)} \left\{ -\ln \left(\frac{\tau}{2r} \right) - \frac{176}{105} \right\} \\ &+ \frac{\delta_{ij} n_{ab}}{r} \int_0^{+\infty} d\tau M_{ab}^{(5)} \left\{ -\frac{80}{21} \ln \left(\frac{\tau}{2r} \right) - \frac{32}{21} \ln \left(\frac{\tau}{2r_0} \right) - \frac{3146}{315} \right\} \\ &+ \frac{\hat{n}_{a(i}}{r} \int_0^{+\infty} d\tau M_{j)a}^{(5)} \left\{ \frac{52}{7} \ln \left(\frac{\tau}{2r} \right) + \frac{104}{35} \ln \left(\frac{\tau}{2r_0} \right) + \frac{9472}{525} \right\} \\ &+ \frac{1}{r} \int_0^{+\infty} d\tau M_{ij}^{(5)} \left\{ -\frac{24}{5} \ln \left(\frac{\tau}{2r} \right) + \frac{92}{15} \ln \left(\frac{\tau}{2r_0} \right) + \frac{94}{15} \right\} + \mathcal{O} \left(\frac{\ln r}{r^2} \right), \end{aligned} \quad (3.5c)$$

where the moments are evaluated at time $t - r - \tau$. Recall that in the case of the quadratic nonlinearities, the non-local integrals come only from the source terms having $k = 2$ (see section 3 of paper I). This is not true in the case of the cubic (and higher) nonlinearities: besides the integrals generated by the terms $k = 2$, there are some non-local integrals generated by terms having $k \geq \ell + 3$. These integrals, given by (3.4), depend on the constant length scale r_0 . Thus the particular solution which is picked up by the finite part procedure depends on the constant r_0 used in its definition. This is quite normal, and, of course, not a problem—one needs only to be consistent in using this particular solution, notably when relating the multipole moments which parametrize it to the source variables.

3.2. Integrating the tail terms

The effects which are physically associated with tails of tails come from the retarded integration of the tail part of the cubic source, that is $T_{(M^2 M_{pq})}^{\alpha\beta}$ given by (2.16). The problem is to find the (finite part of the) retarded integral of a source term involving a non-local integral with some Legendre function Q_m , namely

$${}_{k,m}\Psi_L \equiv \text{FP}_{B=0} \square_R^{-1} \left[(r/r_0)^B \hat{n}_L r^{-k} \int_1^{+\infty} dx Q_m(x) F(t - rx) \right]. \quad (3.6)$$

We study only the needed cases, which correspond to $k \geq 1$ (note that with this notation the actual radial dependence of the source when $r \rightarrow \infty$, $t - r = \text{constant}$ is $1/r^{k+1}$). Of course, the problem is more complicated than in section 3.1, because the source term at a given value of $t - r$ is a more complicated function of r .

The detailed computation of ${}_{k,m}\Psi_L$ is relegated to appendix A. Here we report the main results. The case $k = 1$ is the most interesting because it leads to qualitatively new results with respect to the quadratic nonlinear order. When $k = 1$ we find the solution in closed analytic form,

$${}_{1,m}\Psi_L = \hat{n}_L \int_1^{+\infty} dy F^{(-1)}(t - ry) \times \left\{ Q_\ell(y) \int_1^y dx Q_m(x) \frac{dP_\ell}{dx}(x) + P_\ell(y) \int_y^{+\infty} dx Q_m(x) \frac{dQ_\ell}{dx}(x) \right\}, \quad (3.7)$$

where the time anti-derivative is defined by $F^{(-1)}(t) = \int_{-\infty}^t dt' F(t')$ (all the functions involved are zero in the remote past by our assumption of initial stationarity). The solution (3.7) has been obtained thanks, in particular, to the mathematical formula (A.5) in appendix A. To leading order when $r \rightarrow \infty$, $t - r = \text{constant}$, the second integral in the brackets dominates the first one. Inspection of (2.16) shows that when $k = 1$ we need only the case $m = \ell$. Because the second integral in (3.7) can be explicitly worked out in this case, we have a good simplification, since notably the dominant term at infinity follows simply from the expansion (2.9c) of the Legendre function (see appendix A for details). We find a non-local expression with $\ln r$ and $\ln^2 r$ terms:

$${}_{1,\ell}\Psi_L = -\frac{\hat{n}_L}{8r} \int_0^{+\infty} d\tau F^{(-1)}(t - r - \tau) \times \left[\ln^2 \left(\frac{\tau}{2r} \right) + 4 \left(\sum_{j=1}^{\ell} \frac{1}{j} \right) \ln \left(\frac{\tau}{2r} \right) + 4 \left(\sum_{j=1}^{\ell} \frac{1}{j} \right)^2 \right] + o(r^{\ell-2}). \quad (3.8)$$

We use the notation $o(r^{\ell-2})$ with a small o symbol to mean that the product of this remainder with the factor $r^{2-\ell}$, where ε is such that $0 < \varepsilon \ll 1$, tends to zero when $r \rightarrow \infty$. This

notation is simply to account for the presence of powers of logarithms in the expansion at infinity.

In the case $2 \leq k \leq \ell + 2$, the result is simpler (see appendix A) in the sense that it is given by a local expression to order $1/r$,

$${}_{k,m}\Psi_L = - {}_{k,m}\alpha_\ell \frac{\hat{n}_L}{r} F^{(k-3)}(t-r) + o(r^{\varepsilon-2}). \quad (3.9)$$

However, the coefficient is still complicated:

$${}_{k,m}\alpha_\ell = \int_1^{+\infty} dx Q_m(x) \int_x^{+\infty} dz \frac{(z-x)^{k-3}}{(k-3)!} Q_\ell(z). \quad (3.10)$$

The numerical values of the ${}_{k,m}\alpha_\ell$'s for general k , ℓ , and m are computed in (A.17), (A.18). Here we need only some values corresponding to $k=2$ and $k=3$ (see (2.16)). These are presented in tables 1 and 2, respectively.

Table 1. Values of the coefficients ${}_{k,m}\alpha_\ell$ for $k=2$.

m	$\ell=0$	$\ell=1$	$\ell=2$	$\ell=3$	$\ell=4$
0	$\frac{1}{6}\pi^2$	$\frac{1}{2}$	$\frac{1}{4}$	$\frac{11}{72}$	$\frac{5}{48}$
1	$\frac{1}{2}$	$\frac{1}{18}\pi^2 - \frac{1}{3}$	$\frac{1}{8}$	$\frac{1}{12}$	$\frac{13}{216}$
2	$\frac{1}{4}$	$\frac{1}{8}$	$\frac{1}{30}\pi^2 - \frac{1}{4}$	$\frac{1}{18}$	$\frac{1}{24}$
3	$\frac{11}{72}$	$\frac{1}{12}$	$\frac{1}{18}$	$\frac{1}{42}\pi^2 - \frac{7}{36}$	$\frac{1}{32}$
4	$\frac{5}{48}$	$\frac{13}{216}$	$\frac{1}{24}$	$\frac{1}{32}$	$\frac{1}{54}\pi^2 - \frac{205}{1296}$

Table 2. Values of the coefficients ${}_{k,m}\alpha_\ell$ for $k=3$.

m	$\ell=1$	$\ell=2$	$\ell=3$	$\ell=4$
0	$\frac{1}{18}\pi^2 - \frac{1}{12}$	$\frac{5}{72}$	$\frac{1}{48}$	$\frac{23}{2700}$
1	$\frac{1}{8}$	$\frac{1}{90}\pi^2 - \frac{1}{12}$	$\frac{1}{108}$	$\frac{1}{240}$
2	$-\frac{1}{90}\pi^2 + \frac{1}{6}$	$\frac{1}{72}$	$\frac{1}{210}\pi^2 - \frac{1}{24}$	$\frac{11}{4320}$
3	$\frac{7}{216}$	$-\frac{1}{210}\pi^2 + \frac{1}{18}$	$\frac{1}{288}$	$\frac{1}{378}\pi^2 - \frac{79}{3240}$
4	$\frac{1}{48}$	$\frac{5}{864}$	$-\frac{1}{378}\pi^2 + \frac{37}{1296}$	$\frac{1}{800}$

In the last case $k \geq \ell + 3$, we again obtain a non-local integral, but with simply a $1/r$ term without $\ln r$ (instead, there is a $\ln r_0$),

$${}_{k,m}\Psi_L = -\frac{\hat{n}_L}{r} \int_0^{+\infty} d\tau F^{(k-2)}(t-r-\tau) \left[{}_{k,m}\beta_\ell \ln\left(\frac{\tau}{2r_0}\right) + {}_{k,m}\gamma_\ell \right] + o(r^{\varepsilon-2}). \quad (3.11)$$

The coefficients ${}_{k,m}\beta_\ell$ and ${}_{k,m}\gamma_\ell$ are also rather involved,

$${}_{k,m}\beta_\ell = \frac{1}{2} \int_1^{+\infty} dx Q_m(x) \int_{-1}^1 dz \frac{(z-x)^{k-3}}{(k-3)!} P_\ell(z), \quad (3.12a)$$

$${}_{k,m}\gamma_\ell = \frac{1}{2} \int_1^{+\infty} dx Q_m(x) \int_{-1}^1 dz \frac{(z-x)^{k-3}}{(k-3)!} P_\ell(z) \left[-\ln\left(\frac{x-z}{2}\right) + \sum_{j=1}^{k-3} \frac{1}{j} \right]. \quad (3.12b)$$

Fortunately, these coefficients are needed only for the particular set of values $k = 3$, $m = 2$ and $\ell = 0$, corresponding to the only term present in (2.16) in this category, which is the antepenultimate term in (2.16c). From (A.22) we find

$${}_{3,2}\beta_0 = \frac{1}{6}, \quad {}_{3,2}\gamma_0 = \frac{7}{72}. \quad (3.13)$$

With the expression (3.8), the values of coefficients in tables 1 and 2, and the two values (3.13), we have the material for the computation of the retarded integral of $T_{(M^2M_{pq})}^{\alpha\beta}$. The result reads

$$\begin{aligned} \text{FP} \square_R^{-1} [M^{-2} T_{(M^2M_{pq})}^{00}] &= \frac{n_{ab}}{r} \int_0^{+\infty} d\tau M_{ab}^{(5)} \left\{ -4 \ln^2 \left(\frac{\tau}{2r} \right) - 24 \ln \left(\frac{\tau}{2r} \right) - \frac{154}{3} \right\} \\ &+ o(r^{\varepsilon-2}), \end{aligned} \quad (3.14a)$$

$$\begin{aligned} \text{FP} \square_R^{-1} [M^{-2} T_{(M^2M_{pq})}^{0i}] &= \frac{n_a}{r} \int_0^{+\infty} d\tau M_{ai}^{(5)} \left\{ -4 \ln^2 \left(\frac{\tau}{2r} \right) - 16 \ln \left(\frac{\tau}{2r} \right) - \frac{202}{5} \right\} \\ &+ \frac{16}{9} \frac{\hat{n}_{iab}}{r} M_{ab}^{(4)} + o(r^{\varepsilon-2}), \end{aligned} \quad (3.14b)$$

$$\begin{aligned} \text{FP} \square_R^{-1} [M^{-2} T_{(M^2M_{pq})}^{ij}] &= \frac{1}{r} \int_0^{+\infty} d\tau M_{ij}^{(5)} \left\{ -4 \ln^2 \left(\frac{\tau}{2r} \right) - \frac{1}{6} \ln \left(\frac{\tau}{2r_0} \right) - \frac{1217}{120} \right\} \\ &+ \frac{1}{r} \left[\frac{23}{30} \hat{n}_{ijab} M_{ab}^{(4)} + \frac{226}{63} \delta_{ij} n_{ab} M_{ab}^{(4)} - \frac{52}{7} n_{a(i} M_{j)a}^{(4)} \right] + o(r^{\varepsilon-2}). \end{aligned} \quad (3.14c)$$

All the logarithms (and squared logarithms) are computed with $\tau/2r$ except for one, in the last equation (3.14c), which is computed with $\tau/2r_0$ and comes from the formula (3.11). The other logarithms, and squared logarithms, come from (3.8).

4. The monopole–monopole–quadrupole metric

4.1. The metric in the far zone

The retarded integral of the monopole–monopole–quadrupole source, namely

$$u_{(M^2M_{pq})}^{\alpha\beta} = \text{FP} \square_R^{-1} [\Lambda_{(M^2M_{pq})}^{\alpha\beta}], \quad (4.1)$$

is obtained simply as the sum of (3.5) and (3.14). Let us now follow the method proposed in [16] (see also section 2 of paper I), and add to $u_{(M^2M_{pq})}^{\alpha\beta}$ a supplementary term $v_{(M^2M_{pq})}^{\alpha\beta}$ so designed as: (i) to be a solution of the homogeneous wave equation; and (ii) to be such that the sum of $u_{(M^2M_{pq})}^{\alpha\beta}$ and $v_{(M^2M_{pq})}^{\alpha\beta}$ is divergenceless. With (i) and (ii) satisfied, a particular solution of the cubic-order field equations in harmonic coordinates is $u_{(M^2M_{pq})}^{\alpha\beta} + v_{(M^2M_{pq})}^{\alpha\beta}$. Actually we do not follow exactly the construction proposed in [16], but adopt the slightly modified construction of $v_{(M^2M_{pq})}^{\alpha\beta}$ defined by equations (2.11) and (2.12) of paper I. Furthermore, as we control only the dominant behaviour at infinity of the term $u_{(M^2M_{pq})}^{\alpha\beta}$, we must check that this weaker information still permits the construction from the divergence of $u_{(M^2M_{pq})}^{\alpha\beta}$ of the corresponding $1/r$ term in $v_{(M^2M_{pq})}^{\alpha\beta}$. This poses no problem, and the relevant formulae can be found in appendix B. The divergence of $u_{(M^2M_{pq})}^{\alpha\beta}$, computed from (3.5) and (3.14), reads

$$\partial_\beta [M^{-2} u_{(M^2M_{pq})}^{0\beta}] = \frac{176}{105} \frac{n_{ab}}{r} M_{ab}^{(5)} + \mathcal{O}\left(\frac{1}{r^2}\right), \quad (4.2a)$$

$$\begin{aligned} \partial_\beta [M^{-2} u_{(M^2 M_{pq})}^{i\beta}] &= \frac{n_a}{r} \int_0^{+\infty} d\tau M_{ia}^{(6)} \left\{ -\frac{9}{10} \ln \left(\frac{\tau}{2r_0} \right) - \frac{1211}{600} \right\} \\ &+ \frac{36 \hat{n}_{iab}}{35 r} M_{ab}^{(5)} + \mathcal{O} \left(\frac{1}{r^2} \right). \end{aligned} \quad (4.2b)$$

Then the formulae (B.2)–(B.5) in appendix B give the $1/r$ term in $v_{(M^2 M_{pq})}^{\alpha\beta}$ as

$$M^{-2} v_{(M^2 M_{pq})}^{00} = \mathcal{O} \left(\frac{1}{r^2} \right), \quad (4.3a)$$

$$M^{-2} v_{(M^2 M_{pq})}^{0i} = \frac{176 n_a}{105 r} M_{ai}^{(4)} + \mathcal{O} \left(\frac{1}{r^2} \right), \quad (4.3b)$$

$$\begin{aligned} M^{-2} v_{(M^2 M_{pq})}^{ij} &= \frac{1}{r} \int_0^{+\infty} d\tau M_{ij}^{(5)} \left\{ -\frac{9}{10} \ln \left(\frac{\tau}{2r_0} \right) - \frac{499}{280} \right\} \\ &+ \frac{1}{r} \left[-\frac{72}{35} \delta_{ij} n_{ab} M_{ab}^{(4)} + \frac{216}{35} \hat{n}_{a(i} M_{j)a}^{(4)} \right] + \mathcal{O} \left(\frac{1}{r^2} \right). \end{aligned} \quad (4.3c)$$

The complete monopole–monopole–quadrupole metric, defined by

$$h_{(M^2 M_{pq})}^{\alpha\beta} = u_{(M^2 M_{pq})}^{\alpha\beta} + v_{(M^2 M_{pq})}^{\alpha\beta}, \quad (4.4)$$

is therefore obtained by adding up the expressions (3.5), (3.14) and (4.3). We obtain

$$\begin{aligned} M^{-2} h_{(M^2 M_{pq})}^{00} &= \frac{n_{ab}}{r} \int_0^{+\infty} d\tau M_{ab}^{(5)} \left\{ -4 \ln^2 \left(\frac{\tau}{2r} \right) - 4 \ln \left(\frac{\tau}{2r} \right) \right. \\ &\left. + \frac{116}{21} \ln \left(\frac{\tau}{2r_0} \right) - \frac{7136}{2205} \right\} + \mathcal{O}(r^{\varepsilon-2}), \end{aligned} \quad (4.5a)$$

$$\begin{aligned} M^{-2} h_{(M^2 M_{pq})}^{0i} &= \frac{\hat{n}_{iab}}{r} \int_0^{+\infty} d\tau M_{ab}^{(5)} \left\{ -\frac{2}{3} \ln \left(\frac{\tau}{2r} \right) - \frac{4}{105} \ln \left(\frac{\tau}{2r_0} \right) - \frac{716}{1225} \right\} \\ &+ \frac{n_a}{r} \int_0^{+\infty} d\tau M_{ai}^{(5)} \left\{ -4 \ln^2 \left(\frac{\tau}{2r} \right) - \frac{18}{5} \ln \left(\frac{\tau}{2r} \right) \right. \\ &\left. + \frac{416}{75} \ln \left(\frac{\tau}{2r_0} \right) - \frac{22724}{7875} \right\} + \mathcal{O}(r^{\varepsilon-2}), \end{aligned} \quad (4.5b)$$

$$\begin{aligned} M^{-2} h_{(M^2 M_{pq})}^{ij} &= \frac{\hat{n}_{ijab}}{r} \int_0^{+\infty} d\tau M_{ab}^{(5)} \left\{ -\ln \left(\frac{\tau}{2r} \right) - \frac{191}{210} \right\} \\ &+ \frac{\delta_{ij} n_{ab}}{r} \int_0^{+\infty} d\tau M_{ab}^{(5)} \left\{ -\frac{80}{21} \ln \left(\frac{\tau}{2r} \right) - \frac{32}{21} \ln \left(\frac{\tau}{2r_0} \right) - \frac{296}{35} \right\} \\ &+ \frac{\hat{n}_{a(i}}{r} \int_0^{+\infty} d\tau M_{j)a}^{(5)} \left\{ \frac{52}{7} \ln \left(\frac{\tau}{2r} \right) + \frac{104}{35} \ln \left(\frac{\tau}{2r_0} \right) + \frac{8812}{525} \right\} \\ &+ \frac{1}{r} \int_0^{+\infty} d\tau M_{ij}^{(5)} \left\{ -4 \ln^2 \left(\frac{\tau}{2r} \right) - \frac{24}{5} \ln \left(\frac{\tau}{2r} \right) \right. \\ &\left. + \frac{76}{15} \ln \left(\frac{\tau}{2r_0} \right) - \frac{198}{35} \right\} + \mathcal{O}(r^{\varepsilon-2}) \end{aligned} \quad (4.5c)$$

(the moments in the integrands are evaluated at $t - r - \tau$).

4.2. The observable quadrupole moment

The computation we have done so far uses harmonic coordinates, which are convenient for constructing solutions by means of a post-Minkowskian algorithm (essentially because all the components of the field obey some wave equations). However, the harmonic coordinates entail a small disadvantage, namely the associated coordinate cones $t-r$ deviate by a logarithm of r (in first approximation) from the true light cones along which gravity propagates. As a result, the expansion of the metric when $r \rightarrow \infty$, $t-r = \text{constant}$, involves, besides the normal powers of $1/r$, some powers of the logarithm of r [41, 42, 16, 17]. This is clear from the previous result (4.5). The logarithms can be gauged away by going to some radiative coordinates X^μ , which are such that the associated coordinate cones $T-R$ (where $R = |\mathbf{X}|$) become asymptotically tangent to the true light cones at infinity. In this paper it will be sufficient to check that the coordinate transformation

$$X^\mu = x^\mu + G\xi^\mu, \quad (4.6a)$$

where x^μ are the harmonic coordinates and ξ^μ is given by

$$\xi^0 = -2M \ln\left(\frac{r}{r_0}\right), \quad \xi^i = 0, \quad (4.6b)$$

does remove all the logarithms of r in the particular case of the interaction $M^2 \times M_{pq}$ (and at leading order at infinity). Note that we have introduced in the coordinate transformation (4.6) the *same* constant r_0 as used in the definition of the finite-part process in section 3. Actually we could have introduced any constant r_1 . However, the choice $r_1 = r_0$, which is simply a choice of gauge (equivalent to a choice of the origin of time in the far zone), is especially convenient, as it will simplify some formulae below.

Under the coordinate transformation (4.6), the metric is changed to

$$H_{(M^2 M_{pq})}^{\mu\nu}(X) = h_{(M^2 M_{pq})}^{\mu\nu}(X) - \xi^\lambda \partial_\lambda h_{(M M_{pq})}^{\mu\nu} + \frac{1}{2} \xi^\lambda \xi^\sigma \partial_{\lambda\sigma} h_{(M_{pq})}^{\mu\nu} + o(R^{\varepsilon-2}), \quad (4.7)$$

where we keep only the terms corresponding to the interaction $M^2 \times M_{pq}$ and neglect all subdominant terms $o(R^{\varepsilon-2})$. Both sides of (4.7) are expressed with the radiative coordinates X^μ . We substitute in the right-hand side the harmonic-coordinates' linear metric (2.5), quadratic metric (2.8), and cubic metric (4.5), and find that all logarithms disappear to order $1/R$, so that we obtain the 'radiative' metric

$$M^{-2} H_{(M^2 M_{pq})}^{00} = \frac{N_{ab}}{R} \int_0^{+\infty} d\tau M_{ab}^{(5)}(T_R - \tau) \left\{ -4 \ln^2\left(\frac{\tau}{2r_0}\right) + \frac{32}{21} \ln\left(\frac{\tau}{2r_0}\right) - \frac{7136}{2205} \right\} + o(R^{\varepsilon-2}), \quad (4.8a)$$

$$M^{-2} H_{(M^2 M_{pq})}^{0i} = \frac{\hat{N}_{iab}}{R} \int_0^{+\infty} d\tau M_{ab}^{(5)}(T_R - \tau) \left\{ -\frac{74}{105} \ln\left(\frac{\tau}{2r_0}\right) - \frac{716}{1225} \right\} + \frac{N_a}{R} \int_0^{+\infty} d\tau M_{ai}^{(5)}(T_R - \tau) \left\{ -4 \ln^2\left(\frac{\tau}{2r_0}\right) + \frac{146}{75} \ln\left(\frac{\tau}{2r_0}\right) - \frac{22724}{7875} \right\} + o(R^{\varepsilon-2}), \quad (4.8b)$$

$$\begin{aligned}
M^{-2}H_{(M^2M_{pq})}^{ij} &= \frac{\hat{N}_{ijab}}{R} \int_0^{+\infty} d\tau M_{ab}^{(5)}(T_R - \tau) \left\{ -\ln\left(\frac{\tau}{2r_0}\right) - \frac{191}{210} \right\} \\
&+ \frac{\delta_{ij}N_{ab}}{R} \int_0^{+\infty} d\tau M_{ab}^{(5)}(T_R - \tau) \left\{ -\frac{16}{3} \ln\left(\frac{\tau}{2r_0}\right) - \frac{296}{35} \right\} \\
&+ \frac{\hat{N}_{a(i}}{R} \int_0^{+\infty} d\tau M_{j)a}^{(5)}(T_R - \tau) \left\{ \frac{52}{5} \ln\left(\frac{\tau}{2r_0}\right) + \frac{8812}{525} \right\} \\
&+ \frac{1}{R} \int_0^{+\infty} d\tau M_{ij}^{(5)}(T_R - \tau) \left\{ -4 \ln^2\left(\frac{\tau}{2r_0}\right) + \frac{4}{15} \ln\left(\frac{\tau}{2r_0}\right) - \frac{198}{35} \right\} \\
&+ o(R^{\varepsilon-2}). \tag{4.8c}
\end{aligned}$$

The retarded time is denoted by $T_R = T - R$ and the direction to the observer by $N_a = N^a = X^a/R$. Note that

$$K_\nu H_{(M^2M_{pq})}^{\mu\nu} = o(R^{\varepsilon-2}), \tag{4.9}$$

where $K_\nu = (-1, N^i)$ is the Minkowskian null direction to the observer.

From the radiative metric (4.8) we extract the ‘observable’ multipole moments which are the quantities measured by an observer located at infinity. The observable multipole moments U_L and V_L parametrize the algebraic transverse–tracefree (TT) projection of the spatial metric in radiative coordinates,

$$(H^{ij})_{\text{TT}} = -\frac{4}{R} \mathcal{P}_{ijab} \sum_{\ell=2}^{\infty} \frac{1}{\ell!} \left\{ N_{L-2} U_{ijL-2}(T_R) - \frac{2\ell}{\ell+1} N_{aL-2} \varepsilon_{ab(i} V_{j)bL-2}(T_R) \right\} + O\left(\frac{1}{R^2}\right), \tag{4.10}$$

where the TT projection operator is

$$\mathcal{P}_{ijab}(\mathbf{N}) = (\delta_{ia} - N_i N_a)(\delta_{jb} - N_j N_b) - \frac{1}{2}(\delta_{ij} - N_i N_j)(\delta_{ab} - N_a N_b). \tag{4.11}$$

(As we use the field variable H^{ij} instead of the covariant metric, (4.10) differs by a sign from (5.1) in paper I.) The moments U_L and V_L agree at the linearized order with the ℓ th time derivatives of the moments M_L and S_L [14].

Working out the TT projection and comparing the result with (4.10), we readily find that the monopole–monopole–quadrupole metric (4.8) contributes to the *quadrupole* observable moment U_{ij} , and only to this moment, by the expression

$$\delta U_{ij}(T_R) = 2M^2 \int_0^{+\infty} d\tau M_{ij}^{(5)}(T_R - \tau) \left\{ \ln^2\left(\frac{\tau}{2r_0}\right) + \frac{57}{70} \ln\left(\frac{\tau}{2r_0}\right) + \frac{124627}{44100} \right\}. \tag{4.12}$$

We add back the necessary powers of G and $1/c$ (recall that δU_{ij} has the dimension of (mass)(length)²(time)⁻²), and find that the term (4.12) carries in front a factor G^2/c^6 , and therefore represents a small modification of the lowest-order quadrupole radiation at the level of the third post-Newtonian (3PN) order. Let us prove that there is no other contribution to U_{ij} at this level. We know from dimensional arguments that a nonlinear term in U_{ij} involves necessarily a factor $1/c^{3(n-1)+\Sigma\ell_i+s-2}$, where $n \geq 2$ is the order of nonlinearity, $\Sigma\ell_i$ is the sum of multiplicities of the n moments M_L and/or S_L composing the term ($1 \leq i \leq n$), and s is the number of current-type moments S_L (see for instance section V in [35]). Furthermore, $\Sigma\ell_i + s - 2 = 2k$ where k is the number of contractions of indices among all the indices carried by the moments. For a term at the 3PN order we thus have $3(n-1) + 2k = 6$, which has the unique solution $n = 3$ (cubic interaction) and $k = 0$ (no contraction of indices). Then the multiplicities satisfy $\ell_1 + \ell_2 + \ell_3 + s - 2 = 0$, so we

have necessarily $(\ell_1, \ell_2, \ell_3) = (0, 0, 2)$ and $s = 0$, which corresponds indeed to the sole interaction $M^2 \times M_{ij}$.

Equation (5.10) of paper I gives the observable quadrupole moment U_{ij} including all terms up to the 2.5PN order. Therefore, by the previous reasoning, we can simply add to (5.10) of paper I the contribution of tails of tails at 3PN order and obtain the complete U_{ij} to the 3PN order. We have

$$\begin{aligned}
U_{ij}(T_R) = & M_{ij}^{(2)} + 2 \frac{GM}{c^3} \int_0^{+\infty} d\tau M_{ij}^{(4)}(T_R - \tau) \left[\ln \left(\frac{c\tau}{2r_0} \right) + \frac{11}{12} \right] \\
& + \frac{G}{c^5} \left\{ -\frac{2}{7} \int_0^{+\infty} d\tau [M_{a(i}^{(3)} M_{j)a}^{(3)}](T_R - \tau) - \frac{2}{7} M_{a(i}^{(3)} M_{j)a}^{(2)} \right. \\
& \left. - \frac{5}{7} M_{a(i}^{(4)} M_{j)a}^{(1)} + \frac{1}{7} M_{a(i}^{(5)} M_{j)a} + \frac{1}{3} \varepsilon_{ab(i} M_{j)a}^{(4)} S_b \right\} \\
& + 2 \left(\frac{GM}{c^3} \right)^2 \int_0^{+\infty} d\tau M_{ij}^{(5)}(T_R - \tau) \left[\ln^2 \left(\frac{c\tau}{2r_0} \right) + \frac{57}{70} \ln \left(\frac{c\tau}{2r_0} \right) + \frac{124627}{44100} \right] \\
& + \mathcal{O} \left(\frac{1}{c^7} \right). \tag{4.13}
\end{aligned}$$

The various terms are: at 1.5PN order, the dominant tail integral [32]; at 2.5PN order, the quadrupole–quadrupole terms (including, in particular, a non-local (memory) integral) and a quadrupole–dipole term (see paper I and references therein); and, at 3PN order, the tail of tail integral computed in this paper. The formula (4.13) constitutes the main result of this paper, as it gives all the physical effects in the radiation field measured by a far away detector up to the 3PN order.

Note that U_{ij} , when expressed in terms of the intermediate moments M_L and S_L as in (4.13), shows a dependence on the (arbitrary) length scale r_0 . Most of this dependence comes from our definition (4.6) of a radiative coordinate system, and thus can be removed by inserting $T_R = t - r/c - (2GM/c^3) \ln(r/r_0)$ back into (4.13), and expanding the result when $c \rightarrow \infty$, $t - r/c = \text{constant}$, keeping the necessary terms consistently. In doing so, one finds that there remains a r_0 -dependent term at 3PN order, namely

$$U_{ij} = M_{ij}^{(2)} - \frac{214}{105} \ln \left(\frac{r}{r_0} \right) \left(\frac{GM}{c^3} \right)^2 M_{ij}^{(4)} + \text{terms independent of } r_0. \tag{4.14a}$$

This term results simply from our use of the r_0 -dependent formulae (3.4) and (3.11) in constructing the harmonic-coordinates metric. As we see from (4.14a), the dependence of U_{ij} on r_0 (or rather r_0/c) is through the effective quadrupole moment

$$M_{ij}^{\text{eff}} = M_{ij} + \frac{214}{105} \ln \left(\frac{r_0}{c} \right) \left(\frac{GM}{c^3} \right)^2 M_{ij}^{(2)}. \tag{4.14b}$$

This moment is exactly the one which appears in the near-zone expansion of the external metric, when taking into account the appearance of the dominant logarithm of c arising at

the 3PN approximation. See the discussion in the appendix of [29]. The appearance of this $\ln c$ was pointed out by Anderson *et al* [43].

4.3. The energy flux

We now investigate the total energy flux generated by the isolated source (its total gravitational *luminosity*), and especially the non-local contributions therein. From (4.10), the luminosity in terms of the observable moments reads

$$\mathcal{L} = \sum_{\ell=2}^{+\infty} \frac{G}{c^{2\ell+1}} \left\{ \frac{(\ell+1)(\ell+2)}{(\ell-1)\ell!(2\ell+1)!!} U_L^{(1)} U_L^{(1)} + \frac{4\ell(\ell+2)}{(\ell-1)(\ell+1)!(2\ell+1)!!c^2} V_L^{(1)} V_L^{(1)} \right\}. \quad (4.15)$$

The powers of $1/c$ show, notably, that the moments U_L and V_L of higher multipoles ℓ contribute to higher orders in the post-Newtonian expansion of \mathcal{L} .

Let us gather the available information on non-local effects present in the U_L 's and V_L 's. Rather, we consider the time-derivatives of the moments ($U_L^{(1)}$ and $V_L^{(1)}$), because these are the quantities of interest in (4.15). From (4.13) we write the time-derivative of the quadrupole U_{ij} as

$$\begin{aligned} U_{ij}^{(1)} &= M_{ij}^{(3)} + 2 \frac{GM}{c^3} \int_0^{+\infty} d\tau M_{ij}^{(5)} \left[\ln \left(\frac{c\tau}{2r_0} \right) + \frac{11}{12} \right] \\ &\quad + \frac{1}{c^5} \left\{ \text{instantaneous terms } M_{a(i} M_{j)a}, \text{ and } \varepsilon_{ab(i} M_{j)a} S_b \right\} \\ &\quad + 2 \left(\frac{GM}{c^3} \right)^2 \int_0^{+\infty} d\tau M_{ij}^{(6)} \left[\ln^2 \left(\frac{c\tau}{2r_0} \right) + \frac{57}{70} \ln \left(\frac{c\tau}{2r_0} \right) + \frac{124627}{44100} \right] \\ &\quad + \frac{1}{c^7} \left\{ \text{instantaneous terms } M_{ab(i} M_{j)ab}, M_{ab} M_{ijab}, \text{ and } \varepsilon_{ab(i} M_{j)ac} S_{bc} \right\} \\ &\quad + \mathcal{O} \left(\frac{1}{c^8} \right), \end{aligned} \quad (4.16)$$

where we indicate only the index *structure* of the local (instantaneous) terms, but write *in extenso* all the non-local integrals. Note the important fact that the non-local (memory) integral present in U_{ij} at the 2.5PN order is a mere time anti-derivative (see (4.13)), and therefore becomes instantaneous when considering the time derivative. We have added with respect to (4.13) the information that the 3.5PN term is instantaneous, exactly like the 2.5PN term. This follows from the dimensional argument used before. The 3.5PN term is such that $3(n-1) + 2k = 7$, therefore $n = 2$ and $k = 2$. Now we know [32] that the only non-local integrals at the quadratic order $n = 2$ are the tail integral, which is purely of order 1.5PN, and the memory integral, which contributes to the 2.5PN, 3.5PN and higher orders but is in the form of a simple time anti-derivative of an instantaneous functional of the moments M_L and S_L . This proves that the 3.5PN term in (4.16) is indeed instantaneous. Furthermore, its multipole structure follows from $\ell_1 + \ell_2 + s - 2 = 2k = 4$. Note that the instantaneous terms in (4.16) (and other equations below) are instantaneous functionals not only of the moments M_L and S_L but also of the real source variables, i.e. when M_L and S_L are replaced by their explicit expressions as integrals over the source. Indeed the non-local integrals in M_L and S_L are not expected to arise before the 4PN order.

Similarly, we write the relevant higher-order multipole moments, but for them we need less accuracy than for the quadrupole. The results are

$$\begin{aligned}
U_{ijk}^{(1)} &= M_{ijk}^{(4)} + 2 \frac{GM}{c^3} \int_0^{+\infty} d\tau M_{ijk}^{(6)} \left[\ln \left(\frac{c\tau}{2r_0} \right) + \frac{97}{60} \right] \\
&\quad + \frac{1}{c^5} \left\{ \text{instantaneous terms } M_{a(ij} M_{k)a}, \varepsilon_{ab(i} M_{jk)a} S_b, \right. \\
&\quad \left. \text{and } \varepsilon_{ab(i} M_{j\underline{a}} S_{k)b} \right\} + \mathcal{O} \left(\frac{1}{c^6} \right), \tag{4.17a}
\end{aligned}$$

$$\begin{aligned}
V_{ij}^{(1)} &= S_{ij}^{(3)} + 2 \frac{GM}{c^3} \int_0^{+\infty} d\tau S_{ij}^{(5)} \left[\ln \left(\frac{c\tau}{2r_0} \right) + \frac{7}{6} \right] \\
&\quad + \frac{1}{c^5} \left\{ \text{instantaneous terms } M_{a(i} S_{j)a}, M_{aij} S_a, \varepsilon_{ab(i} M_{j)ac} M_{bc}, \right. \\
&\quad \left. \text{and } \varepsilon_{ab(i} S_{j)a} S_b \right\} + \mathcal{O} \left(\frac{1}{c^6} \right), \tag{4.17b}
\end{aligned}$$

$$\begin{aligned}
U_{ijkl}^{(1)} &= M_{ijkl}^{(5)} + \frac{G}{c^3} \left\{ 2M \int_0^{+\infty} d\tau M_{ijkl}^{(7)} \left[\ln \left(\frac{c\tau}{2r_0} \right) + \frac{59}{30} \right] \right. \\
&\quad \left. + \text{instantaneous terms } M_{(ij} M_{kl)} \right\} + \mathcal{O} \left(\frac{1}{c^4} \right), \tag{4.17c}
\end{aligned}$$

$$\begin{aligned}
V_{ijk}^{(1)} &= S_{ijk}^{(4)} + \frac{G}{c^3} \left\{ 2M \int_0^{+\infty} d\tau S_{ijk}^{(6)} \left[\ln \left(\frac{c\tau}{2r_0} \right) + \frac{5}{3} \right] \right. \\
&\quad \left. + \text{instantaneous terms } M_{(ij} S_{k)}, \text{ and } \varepsilon_{ab(i} M_{j\underline{a}} M_{k)b} \right\} + \mathcal{O} \left(\frac{1}{c^4} \right). \tag{4.17d}
\end{aligned}$$

(The tails in the mass octupole and current quadrupole are from (5.8) in [35]. The mass 2^4 -pole and current octupole are computed (including all the instantaneous terms) in (5.11) and (5.12) of paper I.)

The separation made in the moments (4.16), (4.17) between instantaneous and non-local terms yields a similar separation in the energy flux given by (4.15). Furthermore, we introduce in the non-local part of \mathcal{L} a separation between the tail terms, strictly speaking, a term involving the square of the tail, and the tail of tail term. Accordingly, we write

$$\mathcal{L} = \mathcal{L}_{\text{inst}} + \mathcal{L}_{\text{tail}} + \mathcal{L}_{(\text{tail})^2} + \mathcal{L}_{\text{tail}(\text{tail})}. \tag{4.18}$$

Quite evidently, we include in the tail part of the flux, $\mathcal{L}_{\text{tail}}$, all the terms which are made of the cross products of the moments M_L and S_L and of the corresponding tail integrals at

1.5PN order. From (4.15)–(4.17) we obtain, neglecting terms at the 4PN order,

$$\begin{aligned} \mathcal{L}_{\text{tail}} = & \frac{4G^2M}{c^5} \left\{ \frac{1}{5c^3} M_{ij}^{(3)} \int_0^{+\infty} d\tau M_{ij}^{(5)}(T_R - \tau) \left[\ln\left(\frac{c\tau}{2r_0}\right) + \frac{11}{12} \right] \right. \\ & + \frac{1}{189c^5} M_{ijk}^{(4)} \int_0^{+\infty} d\tau M_{ijk}^{(6)}(T_R - \tau) \left[\ln\left(\frac{c\tau}{2r_0}\right) + \frac{97}{60} \right] \\ & + \frac{16}{45c^5} S_{ij}^{(3)} \int_0^{+\infty} d\tau S_{ij}^{(5)}(T_R - \tau) \left[\ln\left(\frac{c\tau}{2r_0}\right) + \frac{7}{6} \right] \\ & + \frac{1}{9072c^7} M_{ijkl}^{(5)} \int_0^{+\infty} d\tau M_{ijkl}^{(7)}(T_R - \tau) \left[\ln\left(\frac{c\tau}{2r_0}\right) + \frac{59}{30} \right] \\ & \left. + \frac{1}{84c^7} S_{ijk}^{(4)} \int_0^{+\infty} d\tau S_{ijk}^{(6)}(T_R - \tau) \left[\ln\left(\frac{c\tau}{2r_0}\right) + \frac{5}{3} \right] + \mathcal{O}\left(\frac{1}{c^8}\right) \right\}. \quad (4.19) \end{aligned}$$

The moments in front of each integral depend on the current time T_R . (For convenience we include in (4.19) the terms associated with the constants $\frac{11}{12}$, $\frac{97}{60}$, etc, though these terms are actually instantaneous. In fact these terms are given by some total time derivatives and thus do not participate in the loss of energy in the source. In the case of binary systems moving on circular orbits, these terms are rigorously zero (see section 5).) The expression (4.19) generalizes to 3.5PN order the expression (5.12) of [35].

Now the ‘tail’² contribution to the flux is given by the square of the tail integral at 1.5PN order, and therefore enters the energy flux at the same 3PN order as the contribution of tails of tails. In fact this contribution could be treated on the same footing as the ‘tail(tail)’ contribution, but it will be clearer in section 5 to investigate it separately. We have

$$\mathcal{L}_{(\text{tail})^2} = \frac{4G^2M}{c^5} \left\{ \frac{GM}{5c^6} \left(\int_0^{+\infty} d\tau M_{ij}^{(5)}(T_R - \tau) \left[\ln\left(\frac{c\tau}{2r_0}\right) + \frac{11}{12} \right] \right)^2 + \mathcal{O}\left(\frac{1}{c^{10}}\right) \right\}. \quad (4.20)$$

Physically, (4.20) represents the energy flux due to the tail part of the wave, in situations where the tail can be separated from the other components of the field. In particular, after the passage of a burst of gravitational radiation (defined by the constancy of the quadrupole moment M_{ij} before and after a certain interval of time), the wave tail will be solely present in the radiation field, and therefore the total energy flux \mathcal{L} will reduce in this case to $\mathcal{L}_{(\text{tail})^2}$.

Next, the ‘tail(tail)’ contribution to the flux involves the cross product of the quadrupole moment and the tail of tail integral at 3PN order. It reads (neglecting 4PN-order terms)

$$\begin{aligned} \mathcal{L}_{\text{tail(tail)}} = & \frac{4G^2M}{c^5} \left\{ \frac{GM}{5c^6} M_{ij}^{(3)} \int_0^{+\infty} d\tau M_{ij}^{(6)}(T_R - \tau) \right. \\ & \left. \times \left[\ln^2\left(\frac{c\tau}{2r_0}\right) + \frac{57}{70} \ln\left(\frac{c\tau}{2r_0}\right) + \frac{124627}{44100} \right] + \mathcal{O}\left(\frac{1}{c^8}\right) \right\}. \quad (4.21) \end{aligned}$$

Note that in contrast to (4.19) the instantaneous terms associated with the constants $\frac{11}{12}$ and $\frac{124627}{44100}$ in (4.20) and (4.21) do contribute to the energy flux, even in the case of binary systems moving on circular orbits (see section 5).

Finally, the instantaneous part of the flux, $\mathcal{L}_{\text{inst}}$, is entirely defined, up to the 3.5PN order included, by the general formula (4.15) together with the previous definitions of $\mathcal{L}_{\text{tail}}$, $\mathcal{L}_{(\text{tail})^2}$ and $\mathcal{L}_{\text{tail(tail)}}$. We do not write the full expression of $\mathcal{L}_{\text{inst}}$ in terms of M_L and S_L because we do not consider it in the application to compact binaries in section 5. It suffices for our purpose to recall that $\mathcal{L}_{\text{inst}}$ is not only instantaneous in terms of the moments M_L and S_L , but also is instantaneous in terms of the real source parameters (neglecting 4PN-order

terms), that is, in the case of compact binaries, of the orbital separation and relative velocity of the two bodies.

5. Application to in-spiralling compact binaries

Last but not least, we specialize the results to binary systems of compact objects (neutron stars or black holes). These systems, when the two objects spiral very rapidly toward each other in the phase just prior to the final coalescence (the orbital motion is highly relativistic in this phase), constitute the most interesting known source of gravitational waves to be observed by VIRGO and LIGO. The rate of in-spiral is fixed by the total energy in the gravitational waves generated by the orbital motion, that is by the binary's total gravitational luminosity \mathcal{L} , which is therefore a crucial quantity to predict. We assume two non-spinning point masses (without internal structure) moving on an orbit which has evolved for a sufficiently long time to have been circularized by the radiation reaction forces. For such (excellent) modelling of in-spiralling compact binaries we compute $\mathcal{L}_{\text{tail}}$, $\mathcal{L}_{\text{tail}}^2$ and $\mathcal{L}_{\text{tail}(\text{tail})}$ to the 3.5PN order included.

To compute the tail part (given by (4.19)), we need the expressions of the multipole moments for circular compact binaries, to 2PN order for the mass quadrupole moment M_{ij} , 1PN order for the mass octupole M_{ijk} and current quadrupole S_{ij} , and Newtonian order for M_{ijkl} and S_{ijk} . These moments have been calculated in [44], and we simply report here their expressions:

$$M_{ij} = \nu m \left\{ x^{(ij)} \left[1 - \frac{\gamma}{42} (1 + 39\nu) - \frac{\gamma^2}{1512} (461 + 18\,395\nu + 241\nu^2) \right] + \frac{r^2}{c^2} v^{(ij)} \left[\frac{11}{21} (1 - 3\nu) + \frac{\gamma}{378} (1607 - 1681\nu + 229\nu^2) \right] + \mathcal{O}\left(\frac{1}{c^5}\right) \right\}, \quad (5.1a)$$

$$M_{ijk} = -\nu \delta m \left\{ x^{(ijk)} (1 - \gamma\nu) + \frac{r^2}{c^2} v^{(ij} x^{k)} (1 - 2\nu) + \mathcal{O}\left(\frac{1}{c^4}\right) \right\}, \quad (5.1b)$$

$$S_{ij} = -\nu \delta m \varepsilon^{ab(i} x^{j)a} v^b \left[1 + \frac{\gamma}{28} (67 - 8\nu) + \mathcal{O}\left(\frac{1}{c^4}\right) \right], \quad (5.1c)$$

$$M_{ijkl} = \nu m x^{(ijkl)} (1 - 3\nu) + \mathcal{O}\left(\frac{1}{c^2}\right), \quad (5.1d)$$

$$S_{ijk} = \nu m \varepsilon^{ab(i} x^{jk)a} v^b (1 - 3\nu) + \mathcal{O}\left(\frac{1}{c^2}\right). \quad (5.1e)$$

The mass parameters are the total mass $m = m_1 + m_2$ ($m = M$ in the notation of the previous sections), the mass difference $\delta m = m_1 - m_2$, and the mass ratio $\nu = m_1 m_2 / m^2$ (satisfying $0 < \nu \leq \frac{1}{4}$). The relative position and velocity of the two point masses are denoted by $x^i = y_1^i - y_2^i$ and $v^i = dx^i/dt$. We use the post-Newtonian parameter

$$\gamma = \frac{Gm}{rc^2}, \quad r = |\mathbf{x}| \quad (5.2)$$

(r is the harmonic-coordinates' distance between the two bodies). The time derivatives of the multipole moments (5.1) are computed using the equations of motion, with maximal 2PN precision needed for the time derivatives of the quadrupole moment. The equations of

motion are

$$\frac{dv^i}{dt} = -\omega_{2\text{PN}}^2 x^i + \mathcal{O}\left(\frac{1}{c^5}\right), \quad (5.3a)$$

$$\omega_{2\text{PN}}^2 = \frac{Gm}{r^3} \left[1 + (-3 + \nu)\gamma + \left(6 + \frac{41}{4}\nu + \nu^2\right)\gamma^2 \right], \quad (5.3b)$$

where the frequency $\omega_{2\text{PN}}$ is the orbital frequency of the exact circular periodic motion at the 2PN order (see [44]). Introducing, instead of γ , the post-Newtonian parameter $x = (Gm\omega_{2\text{PN}}/c^3)^{2/3}$, we have

$$\gamma = x \left\{ 1 + \left(1 - \frac{1}{3}\nu\right)x + \left(1 - \frac{65}{12}\nu\right)x^2 + \mathcal{O}(x^3) \right\}. \quad (5.3c)$$

The computation of $\mathcal{L}_{\text{tail}}$ proceeds as in section VI.B of [35]. Namely, we insert into (4.19) the time derivatives of the moments (5.1), evaluated both at the current time T_R and at all anterior times $T_R - \tau$. Then we work out all contractions of indices and obtain many integrals of logarithms of $c\tau/2r_1$, where r_1 is some constant such as $r_1 = r_0 e^{-11/12}$, multiplied by cosines of some multiples (n) of $\omega_{2\text{PN}}\tau$. All these integrals are computed using the mathematical formula

$$\int_0^{+\infty} dy \ln y e^{-\lambda y} = -\frac{1}{\lambda} (C + \ln \lambda), \quad (5.4)$$

where λ denotes the complex number $\lambda = 2in\omega_{2\text{PN}}r_1/c$, and where $C = 0.577\dots$ is the Euler constant (see, e.g., [45] p 573). See appendix A of [30] for the proof that this formula yields the correct result for in-spiralling compact binaries modulo some error terms of order $\mathcal{O}(c^{-5} \ln c)$, falling in the present case into the uncontrolled remainder of (4.19). In fact we need only the real part of (5.4) for the computation of $\mathcal{L}_{\text{tail}}$, which leads (using $\ln(i\omega) = \ln \omega + i\pi/2$) to a term proportional to π and independent of r_1 . The result (extending the equation (6.18) in [33]) is

$$\begin{aligned} \mathcal{L}_{\text{tail}} = \frac{32c^5}{5G} v^2 \gamma^5 \left\{ 4\pi \gamma^{3/2} - \left(\frac{25\,663}{672} + \frac{109}{8}\nu \right) \pi \gamma^{5/2} \right. \\ \left. + \left(\frac{90\,205}{576} + \frac{3772\,673}{12\,096}\nu + \frac{32\,147}{3024}\nu^2 \right) \pi \gamma^{7/2} + \mathcal{O}(\gamma^4) \right\}, \end{aligned} \quad (5.5a)$$

or, equivalently, in terms of the parameter x ,

$$\begin{aligned} \mathcal{L}_{\text{tail}} = \frac{32c^5}{5G} v^2 x^5 \left\{ 4\pi x^{3/2} - \left(\frac{8191}{672} + \frac{535}{24}\nu \right) \pi x^{5/2} \right. \\ \left. + \left(-\frac{16\,285}{504} + \frac{176\,419}{1512}\nu + \frac{19\,897}{378}\nu^2 \right) \pi x^{7/2} + \mathcal{O}(x^4) \right\}. \end{aligned} \quad (5.5b)$$

The tails contribute only to the *half-integer* post-Newtonian approximations 1.5PN, 2.5PN, and 3.5PN. Now, we know that only the terms given by some *non-local* integrals can contribute to the half-integer post-Newtonian approximations. This follows from an argument presented in section VI.B of [33], which shows that the terms given by instantaneous functionals of the binary's relative position and velocity are zero for half-integer approximations in the energy flux for circular orbits (but only in this case). Thus we conclude that the terms computed in (5.5a) and (5.5b) represent the *complete* 1.5PN, 2.5PN and 3.5PN approximations in \mathcal{L} —no other contributions can come from $\mathcal{L}_{\text{inst}}$ to these orders (and $\mathcal{L}_{(\text{tail})^2}$ and $\mathcal{L}_{\text{tail}(\text{tail})}$ are purely of 3PN order). Being complete, these approximations can thus be compared, in the test-mass limit $\nu \rightarrow 0$ for one body, with the result of black-hole perturbation theory derived up to 4PN order by Tagoshi and Sasaki [48]. For the comparison we must use (5.5b) expressed in a coordinate-independent way by means of the

parameter x . We find that there is perfect agreement, in the limit $\nu \rightarrow 0$, between (5.5b) and the corresponding terms in equation (43) of [48] (see also equation (3.1) of [49]).

Turn now to $\mathcal{L}_{(\text{tail})^2}$ defined by (4.20). The computation is essentially the same as for $\mathcal{L}_{\text{tail}}$, but since we are considering a small 3PN effect, the expression of the quadrupole moment at the Newtonian order is sufficient. Consistently, we use the Newtonian equations of motion (with orbital frequency $\omega^2 = Gm/r^3$). Using the mathematical formula (5.4), in which both the real and imaginary parts are now needed, we readily obtain

$$\begin{aligned} \mathcal{L}_{(\text{tail})^2} = \frac{32c^5}{5G} v^2 \gamma^5 \left\{ \left(16 \left[C + \ln \left(\frac{4\omega r_0}{c} \right) \right] \right)^2 - \frac{88}{3} \left[C + \ln \left(\frac{4\omega r_0}{c} \right) \right] \right. \\ \left. + 4\pi^2 + \frac{121}{9} \right\} \gamma^3 + \mathcal{O}(\gamma^4). \end{aligned} \quad (5.6)$$

Finally, we compute $\mathcal{L}_{\text{tail}(\text{tail})}$ as given by (4.21). Again we need only the Newtonian quadrupole moment and Newtonian equations of motion, however a new ingredient is necessary, which is a mathematical formula analogous to (5.4) but able to deal with the *square* of the logarithm. From [45] p 574, the relevant formula is

$$\int_0^{+\infty} dy \ln^2 y e^{-\lambda y} = \frac{1}{\lambda} \left[\frac{1}{6} \pi^2 + (C + \ln \lambda)^2 \right]. \quad (5.7)$$

Using this formula for in-spiralling compact binaries is justified in the same way as for the earlier formula (5.4). As a result we get

$$\begin{aligned} \mathcal{L}_{\text{tail}(\text{tail})} = \frac{32c^5}{5G} v^2 \gamma^5 \left\{ \left(-16 \left[C + \ln \left(\frac{4\omega r_0}{c} \right) \right] \right)^2 + \frac{456}{35} \left[C + \ln \left(\frac{4\omega r_0}{c} \right) \right] \right. \\ \left. + \frac{4}{3} \pi^2 - \frac{498508}{11025} \right\} \gamma^3 + \mathcal{O}(\gamma^4). \end{aligned} \quad (5.8)$$

Both $\mathcal{L}_{(\text{tail})^2}$ and $\mathcal{L}_{\text{tail}(\text{tail})}$ have the same structure, namely that of a polynomial of the second degree in the combination $C + \ln(4\omega r_0/c)$. However, we see that the coefficients in front of the square of $C + \ln(4\omega r_0/c)$ in (5.6) and (5.8) are exactly opposite, and therefore that the sum of $\mathcal{L}_{(\text{tail})^2}$ and $\mathcal{L}_{\text{tail}(\text{tail})}$ is actually merely linear in the combination $C + \ln(4\omega r_0/c)$. This fact is somewhat surprising because the terms involving the square of $\ln \omega$ are *a priori* allowed. Thus, to the 3PN order, the terms $(\ln \omega)^2$ cancel out and there remains simply a term with $\ln \omega$ (and the associated $\ln c$). The appearance of a $\ln \omega$ at 3PN was first shown in this context by Tagoshi and Nakamura [39]. Recall that the general structure of the post-Newtonian expansion of the near-zone metric involves, besides the regular powers of $1/c$, some arbitrary powers of $\ln c$ [16]. Similarly, one expects that the general structure of \mathcal{L} should involve, when going to higher post-Newtonian approximations, some arbitrary powers of $\ln \omega$. (Note that up to the 5.5PN order \mathcal{L} as computed in the limit $\nu \rightarrow 0$ by Tanaka *et al* [49] is still linear in $\ln \omega$.)

Adding up (5.6) and (5.8), we obtain

$$\mathcal{L}_{(\text{tail})^2 + \text{tail}(\text{tail})} = \frac{32c^5}{5G} v^2 \gamma^5 \left\{ \left(-\frac{1712}{105} \left[C + \ln \left(\frac{4\omega r_0}{c} \right) \right] + \frac{16}{3} \pi^2 - \frac{116761}{3675} \right) \gamma^3 + \mathcal{O}(\gamma^4) \right\}. \quad (5.9)$$

It is not yet possible to compare this result when $\nu \rightarrow 0$ with the one obtained using perturbation theory. Indeed, (5.9) represents only a part of the complete 3PN term in \mathcal{L} , which should also take into account the (hard to get) 3PN contributions in $\mathcal{L}_{\text{inst}}$, which are due to the 3PN relativistic corrections in the quadrupole moment (5.1a) and in the equations

of motion (5.3). (The 3PN contributions in $\mathcal{L}_{\text{inst}}$ are in progress [54, 55].) Nevertheless, we already recognize in (5.9) the same terms with π^2 and the combination $C + \ln(4\omega)$ as in the result of Tagoshi and Sasaki [48]. (To this order one can replace in (5.9) the parameter γ by x .)

The 3PN term obtained in (5.9) depends on the arbitrary length scale r_0 . This is not a problem because the 3PN term in $\mathcal{L}_{\text{inst}}$ is expected also to depend on r_0 through the explicit expression of the intermediate quadrupole moment M_{ij} as a functional of the source's stress-energy tensor. The r_0 -dependent terms in both $\mathcal{L}_{(\text{tail})^2+\text{tail}(\text{tail})}$ and $\mathcal{L}_{\text{inst}}$ should cancel out, so that the physical energy flux \mathcal{L} is indeed independent of r_0 . We leave for future work [54] the check of the latter assertion.

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Appendix A. Formulae required to compute the cubic nonlinearities

In this appendix we compute the finite part of the retarded integral of a source term with multipolarity ℓ , radial dependence with $k \geq 1$, and containing a non-local integral whose kernel is a certain Legendre function $Q_m(x)$ (see (2.9)). Thus,

$${}_{k,m}\Psi_L = \text{FP}_{B=0} \square_R^{-1} \left[(r/r_0)^B \hat{n}_L r^{-k} \int_1^{+\infty} dx Q_m(x) F(t - rx) \right]. \quad (\text{A.1})$$

We tackle first the case $k = 1$. It can be checked in this case that when $r \rightarrow 0$ the source behaves in such a way that the retarded integral (in its usual triple integral form) is convergent, so we can forget about the finite-part procedure. For the present computation it is convenient to use formula (D5) of appendix D in [16], which gives

$$\begin{aligned} {}_{1,m}\Psi_L = & -\frac{\hat{n}_L}{2r} \int_{-\infty}^{t-r} d\xi \int_{(t-r-\xi)/2}^{(t+r-\xi)/2} dw \int_1^{+\infty} dx Q_m(x) F(\xi - (x-1)w) \\ & \times P_\ell \left(1 - \frac{(t-r-\xi)(t+r-\xi-2w)}{2rw} \right), \end{aligned} \quad (\text{A.2})$$

where P_ℓ denotes the Legendre polynomial. Changing the variables (ξ, w) to the new variables (y, z) defined by $\xi - (x-1)w = t - ry$ and $z = 1 - (t-r-\xi)(t+r-\xi-2w)/2rw$, we obtain

$$\begin{aligned} {}_{1,m}\Psi_L = & -\frac{r\hat{n}_L}{2} \int_1^{+\infty} \frac{dx}{x^2-1} Q_m(x) \int_1^{+\infty} dy F(t-ry) \int_{-1}^1 dz P_\ell(z) \\ & \times \left[\frac{xy-z}{\sqrt{(xy-z)^2 - (x^2-1)(y^2-1)}} - 1 \right]. \end{aligned} \quad (\text{A.3})$$

Next we integrate by parts the y -integral, introducing the anti-derivative $F^{(-1)}$ of F . After some manipulation, we get

$$\begin{aligned} {}_{1,m}\Psi_L = & \frac{\hat{n}_L}{2} \int_1^{+\infty} dy F^{(-1)}(t-ry) \int_1^{+\infty} dx Q_m(x) \\ & \times \frac{d}{dx} \left[\int_{-1}^1 \frac{dz P_\ell(z)}{\sqrt{(xy-z)^2 - (x^2-1)(y^2-1)}} \right]. \end{aligned} \quad (\text{A.4})$$

The z -integration can be performed explicitly thanks to the rather interesting mathematical formula

$$\frac{1}{2} \int_{-1}^1 \frac{dz P_\ell(z)}{\sqrt{(xy-z)^2 - (x^2-1)(y^2-1)}} = \begin{cases} P_\ell(x)Q_\ell(y) & (1 < x \leq y), \\ P_\ell(y)Q_\ell(x) & (1 < y \leq x). \end{cases} \quad (\text{A.5})$$

As this formula does not seem to appear in standard textbooks of mathematical formulae such as [45], we present its proof at the end of this appendix.

With (A.5) we obtain

$$\begin{aligned} {}_{1,m}\Psi_L = \hat{n}_L \int_1^{+\infty} dy F^{(-1)}(t-ry) & \left\{ Q_\ell(y) \int_1^y dx Q_m(x) \frac{dP_\ell}{dx}(x) \right. \\ & \left. + P_\ell(y) \int_y^{+\infty} dx Q_m(x) \frac{dQ_\ell}{dx}(x) \right\}. \end{aligned} \quad (\text{A.6})$$

The leading-order term at infinity ($r \rightarrow \infty$, $t-r = \text{constant}$) comes from the second term in the brackets, and we have

$${}_{1,m}\Psi_L = \frac{\hat{n}_L}{r} \int_0^{+\infty} d\tau F^{(-1)}(t-r-\tau) \int_{1+\tau/r}^{+\infty} dx Q_m(x) \frac{dQ_\ell}{dx}(x) + o(r^{\varepsilon-2}) \quad (\text{A.7})$$

(see the notation concerning the remainder in section 3.2). In the case $\ell = m$ (the only one needed in this paper), the x -integral is obtained in closed form, and we obtain

$${}_{1,\ell}\Psi_L = -\frac{\hat{n}_L}{2r} \int_0^{+\infty} d\tau F^{(-1)}(t-r-\tau) \left[Q_\ell \left(1 + \frac{\tau}{r} \right) \right]^2 + o(r^{\varepsilon-2}). \quad (\text{A.8})$$

Using the expansion (2.9c) of the Legendre function, we further obtain

$$\begin{aligned} {}_{1,\ell}\Psi_L = -\frac{\hat{n}_L}{8r} \int_0^{+\infty} d\tau F^{(-1)}(t-r-\tau) \\ \times \left[\ln^2 \left(\frac{\tau}{2r} \right) + 4 \left(\sum_{j=1}^{\ell} \frac{1}{j} \right) \ln \left(\frac{\tau}{2r} \right) + 4 \left(\sum_{j=1}^{\ell} \frac{1}{j} \right)^2 \right] + o(r^{\varepsilon-2}). \end{aligned} \quad (\text{A.9})$$

This is the formula used in section 3.2.

Consider now the case $k \geq 2$, and restrict attention to the leading-order term at infinity. In this case one uses lemma 7.2 in [16], whose hypothesis is satisfied with $N = k - \varepsilon \geq 2 - \varepsilon$. Thus the $1/r$ term in ${}_{k,m}\Psi_L$, which is deduced from (7.2) and (7.6) in [16], is

$${}_{k,m}\Psi_L = \frac{(-1)^\ell \hat{n}_L}{\ell! r} G^{(\ell)}(u) + o(r^{\varepsilon-2}), \quad (\text{A.10})$$

where $u = t - r$ and

$$G(u) = \text{FP}_{B=0} \int_{-\infty}^u ds R_{+\infty}^B \left(\frac{u-s}{2}, s \right), \quad (\text{A.11a})$$

$$\begin{aligned} R_{+\infty}^B(\rho, s) = -\rho^\ell \int_\rho^{+\infty} dy \frac{(\rho-y)^\ell}{\ell!} \left(\frac{2}{y} \right)^{\ell-1} \left(\frac{y}{r_0} \right)^B y^{-k} \\ \times \int_1^{+\infty} dx Q_m(x) F(s - (x-1)y). \end{aligned} \quad (\text{A.11b})$$

By combining together these expressions, and introducing the variable $z = 1 - (u - s)/y$, we get

$$G(u) = -\frac{1}{2^{\ell+1}} \text{FP}_{B=0} \int_1^{+\infty} dx Q_m(x) \int_0^{+\infty} dy \left(\frac{y}{r_0}\right)^B y^{-k+\ell+2} \times \int_{-1}^1 dz (z^2 - 1)^\ell F(u - (x - z)y). \quad (\text{A.12})$$

This expression is still not suitable for practical computations, and we must integrate by parts the y -integral in order to determine the finite part at $B = 0$. The all-integrated terms during the integrations by parts are zero by analytic continuation in B . We must distinguish two cases, $2 \leq k \leq \ell + 2$ and $k \geq \ell + 3$.

When $2 \leq k \leq \ell + 2$ we obtain after $\ell + 2 - k$ integrations by parts the local expression

$$\frac{(-1)^\ell}{\ell!} G^{(\ell)}(u) = - {}_{k,m}\alpha_\ell F^{(k-3)}(u), \quad (\text{A.13})$$

where the coefficient ${}_{k,m}\alpha_\ell$ reads

$${}_{k,m}\alpha_\ell = \frac{(\ell - k + 2)!}{2^{\ell+1}\ell!} \int_1^{+\infty} dx Q_m(x) \int_{-1}^1 dz \frac{(1 - z^2)^\ell}{(x - z)^{\ell-k+3}}. \quad (\text{A.14})$$

A more elegant form of this coefficient is obtained by using the representation of the Legendre function given by

$$Q_\ell(x) = \frac{1}{2^{\ell+1}} \int_{-1}^1 dz \frac{(1 - z^2)^\ell}{(x - z)^{\ell+1}} \quad (\text{A.15})$$

(which is equivalent to the other formulae (2.9), see, e.g., [50]). Then ${}_{k,m}\alpha_\ell$ reads

$${}_{k,m}\alpha_\ell = \int_1^{+\infty} dx Q_m(x) \int_x^{+\infty} dz \frac{(z - x)^{k-3}}{(k - 3)!} Q_\ell(z). \quad (\text{A.16})$$

The numerical values of the coefficient are computed from

$${}_{k,m}\alpha_\ell = \sum_{i=0}^{k-2} \frac{(-1)^i (k - 2)! (2\ell - 2k + 3 + 2i)!!}{i! (k - i - 2)! (2\ell + 1 + 2i)!!} \times (2\ell - 2k + 5 + 4i) \int_1^{+\infty} dx Q_m(x) Q_{\ell-k+2+2i}(x), \quad (\text{A.17})$$

where the remaining integrals are given by (see, e.g., [45])

$$\int_1^{+\infty} dx Q_m(x) Q_p(x) = \begin{cases} \frac{1}{(m - p)(m + p + 1)} \left[\sum_{j=1}^m \frac{1}{j} - \sum_{j=1}^p \frac{1}{j} \right] & (m \neq p), \\ \frac{1}{2p + 1} \left[\frac{\pi^2}{6} - \sum_{j=1}^p \frac{1}{j^2} \right] & (m = p). \end{cases} \quad (\text{A.18})$$

We thus obtain the coefficients reported in tables 1 and 2 of section 3.

In the case $k \geq \ell + 3$, we get after $k - \ell - 2$ integrations by parts the non-local expression

$$\frac{(-1)^\ell}{\ell!} G^{(\ell)}(u) = - \int_0^{+\infty} d\tau F^{(k-2)}(u - \tau) \left[{}_{k,m}\beta_\ell \ln\left(\frac{\tau}{2r_0}\right) + {}_{k,m}\gamma_\ell \right], \quad (\text{A.19})$$

where the coefficients ${}_{k,m}\beta_\ell$ and ${}_{k,m}\gamma_\ell$ are given by

$${}_{k,m}\beta_\ell = \frac{1}{2^{\ell+1}\ell!(k-\ell-3)!} \int_1^{+\infty} dx Q_m(x) \int_{-1}^1 dz (1-z^2)^\ell (z-x)^{k-\ell-3}, \quad (\text{A.20a})$$

$$\begin{aligned} {}_{k,m}\gamma_\ell &= \frac{1}{2^{\ell+1}\ell!(k-\ell-3)!} \int_1^{+\infty} dx Q_m(x) \int_{-1}^1 dz (1-z^2)^\ell (z-x)^{k-\ell-3} \\ &\quad \times \left[-\ln\left(\frac{x-z}{2}\right) + \sum_{j=1}^{k-\ell-3} \frac{1}{j} \right]. \end{aligned} \quad (\text{A.20b})$$

More elegant forms are

$${}_{k,m}\beta_\ell = \frac{1}{2} \int_1^{+\infty} dx Q_m(x) \int_{-1}^1 dz \frac{(z-x)^{k-3}}{(k-3)!} P_\ell(z), \quad (\text{A.21a})$$

$${}_{k,m}\gamma_\ell = \frac{1}{2} \int_1^{+\infty} dx Q_m(x) \int_{-1}^1 dz \frac{(z-x)^{k-3}}{(k-3)!} P_\ell(z) \left[-\ln\left(\frac{x-z}{2}\right) + \sum_{j=1}^{k-3} \frac{1}{j} \right]. \quad (\text{A.21b})$$

The cases $k=3$ and $\ell=0$ needed in this paper yield

$${}_{3,m}\beta_0 = \int_1^{+\infty} dx Q_m(x) = \frac{1}{m(m+1)} \quad (m \geq 1), \quad (\text{A.22a})$$

$${}_{3,m}\gamma_0 = - \int_1^{+\infty} dx Q_m(x) \left[\ln\left(\frac{1}{2}\sqrt{x^2-1}\right) + Q_1(x) \right]. \quad (\text{A.22b})$$

Finally, the formula (7.132.3) in [45] where, however, a factor $2^{2\lambda-\mu}$ in the denominator should be corrected to a factor $2^{2-\mu}$, leads to

$${}_{3,m}\gamma_0 = \begin{cases} -\frac{1}{18}\pi^2 + \frac{7}{12} & (m=1), \\ \frac{1}{m(m+1)(m+2)} \left(\frac{m^2+3m+3}{m+1} - \frac{2}{m-1} \sum_{j=1}^{m-1} \frac{1}{j} \right) & (m \geq 2). \end{cases} \quad (\text{A.22c})$$

When $m=2$ we find the values (3.13).

Proof of the mathematical formula (A.5). The author proved this formula by: (i) verifying that the left-hand side of the formula, namely

$$I_\ell(x, y) \equiv \frac{1}{2} \int_{-1}^1 \frac{dz P_\ell(z)}{\sqrt{(xy-z)^2 - (x^2-1)(y^2-1)}}, \quad (\text{A.23})$$

is a particular solution of the Legendre equation both in variables x and y ; and (ii) showing that this particular solution $I_\ell(x, y)$ is necessarily equal to the solution given by the right-hand side. To prove (ii) one must invoke the behaviour of the Legendre function at infinity, which is $Q_\ell(x) \approx 1/x^{\ell+1}$ when $x \rightarrow \infty$.

A direct and more explicit proof of this formula was then found by H Sivak. This proof consists first of transforming $I_\ell(x, y)$ by means of formula (3.613.1) in Gradshteyn and Ryzhik [45], in order to obtain

$$I_\ell(x, y) = \frac{1}{2\pi} \int_0^\pi dt \int_{-1}^1 \frac{dz P_\ell(z)}{xy - z - \sqrt{(x^2-1)(y^2-1)} \cos t}. \quad (\text{A.24})$$

With the help of Neumann's formula (2.9a) for the Legendre function, we get

$$I_\ell(x, y) = \frac{1}{\pi} \int_0^\pi dt Q_\ell(xy - \sqrt{(x^2 - 1)(y^2 - 1)} \cos t). \quad (\text{A.25})$$

Using now formula (8.795.2) of [45], we find (assuming for instance that $x < y$)

$$I_\ell(x, y) = \frac{1}{\pi} \int_0^\pi dt \left\{ P_\ell(x) Q_\ell(y) + 2 \sum_{k=1}^{+\infty} (-1)^k P_\ell^{-k}(x) Q_\ell^k(y) \cos kt \right\}. \quad (\text{A.26})$$

This yields immediately the desired result

$$I_\ell(x, y) = P_\ell(x) Q_\ell(y). \quad (\text{A.27})$$

Appendix B. The harmonicity algorithm in the far zone

The harmonicity algorithm is a method for adding to the retarded integral of the source,

$$u^{\alpha\beta} = \text{FP}_{B=0} \square_R^{-1} \left[\left(\frac{r}{r_0} \right)^B \Lambda^{\alpha\beta} \right], \quad (\text{B.1})$$

a homogeneous solution $v^{\alpha\beta}$ of the wave equation, which is such that the sum $u^{\alpha\beta} + v^{\alpha\beta}$ is divergence free (and thus satisfies the harmonic-gauge condition). A particular algorithm is defined by (4.12), (4.13) in [16]; a slightly different one by (2.11), (2.12) in paper I. Here we follow paper I.

The harmonicity algorithm to order $1/r$ when $r \rightarrow \infty$, $t - r = \text{constant}$, is as follows. The $1/r$ term of the divergence of $u^{\alpha\beta}$,

$$\partial_\beta u^{\alpha\beta} = \frac{1}{r} U^\alpha(\mathbf{n}, t - r) + \mathcal{O}\left(\frac{1}{r^2}\right), \quad (\text{B.2})$$

is decomposed into multipole moments according to

$$U^0 = \sum_{\ell \geq 0} n_L W_L, \quad (\text{B.3a})$$

$$U^i = \sum_{\ell \geq 0} n_{iL} X_L + \sum_{\ell \geq 1} \{ n_{L-1} Y_{iL-1} + \varepsilon_{iab} n_{aL-1} Z_{bL-1} \}. \quad (\text{B.3b})$$

Comparing this decomposition with the definition (2.11) in paper I, we obtain some relations between the tensors W_L , X_L , etc, and the time-derivatives of the tensors A_L , B_L , etc, in this definition. Next we obtain the $1/r$ term of $v^{\alpha\beta}$,

$$v^{\alpha\beta} = \frac{1}{r} V^{\alpha\beta}(\mathbf{n}, t - r) + \mathcal{O}\left(\frac{1}{r^2}\right), \quad (\text{B.4})$$

by applying formula (2.12) in paper I. Re-expressing the decomposition in terms of the tensors W_L , X_L , etc, we obtain some formulae for the time derivative of $V^{\alpha\beta}$,

$$\frac{dV^{00}}{dt} = -W - n_a [W_a + Y_a - 3X_a], \quad (\text{B.5a})$$

$$\frac{dV^{0i}}{dt} = -Y_i + 3X_i - \varepsilon_{iab} n_a Z_b + \sum_{\ell \geq 2} n_{L-1} W_{iL-1}, \quad (\text{B.5b})$$

$$\begin{aligned} \frac{dV^{ij}}{dt} = & \delta_{ij} X + \sum_{\ell \geq 2} \{ -2\delta_{ij} n_{L-1} X_{L-1} + 6n_{L-2(i} X_{j)L-2} \\ & + n_{L-2} [W_{ijL-2} - 3X_{ijL-2} + Y_{ijL-2}] + 2n_{aL-2} \varepsilon_{ab(i} Z_{j)bL-2} \}. \end{aligned} \quad (\text{B.5c})$$

These are the formulae needed in section 4.1.

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