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Quadrupole–quadrupole gravitational waves

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Abstract. This paper investigates the nonlinear self-interaction of quadrupole gravitational waves generated by an isolated system. The vacuum Einstein field equations are integrated in the region exterior to the system by means of a post-Minkowskian algorithm. Specializing to the quadrupole–quadrupole interaction (at the quadratic nonlinear order), we recover the known results concerning the non-local modification of the ADM mass–energy of the system accounting for the emission of quadrupole waves, and the non-local memory effect due to the re-radiation by the stress–energy distribution of linear waves. We then compute all the local (instantaneous) terms which are associated, in the quadrupole–quadrupole metric, with the latter non-local effects. Expanding the metric at large distances from the system, we obtain the corresponding radiation-field observables, including all non-local and transient contributions. This permits, notably, the completion of the observable quadrupole moment at the $\frac{5}{2}$ post-Newtonian order.

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1. Introduction

In general relativity, the multipole moments of any finite distribution of energy and momentum interact with each other in vacuum, through the nonlinearities in the field equations. In particular, the multipole moments which describe the gravitational waves emitted by an isolated system do not evolve independently, but rather couple together (including to themselves), giving rise to nonlinear physical effects.

The simplest multipole interaction which contributes to the radiation field is that between the (mass-type) quadrupole moment M_{ij} and the mass monopole M. The latter moment is the constant mass–energy of the source as measured at spatial infinity (ADM mass). Associated with the multipole interaction $M_{ij} \times M$ is the nonlinear effect of tails. This effect is due to the backscatter of linear waves (described by M_{ij}) onto the spacetime curvature generated by the mass–energy M. The tails can be computed within the theory of gravitational perturbations of the Schwarzschild background (see, e.g., [1–3]). A consequence of the existence of tails is the non-locality in time, as the tails are in the form of integrals depending on the history of the source from $-\infty$ in the past to the retarded time $t - r/c$ [4–7]. It is known that the tails appear both in the radiation field and in the radiation reaction forces at the $\frac{3}{2}$ post-Newtonian order (1.5PN, or order c^{-3} when $c \to \infty$) relative to the quadrupole radiation [8, 9].

Next in complexity is the interaction of the quadrupole moment with itself, or quadrupole self-interaction $M_{ij} \times M_{kl}$. Two closely related non-local (or hereditary) effects are known for this particular interaction. As shown by Bonnor and collaborators [4, 6, 10, 11], the total mass M is modified by a non-local integral accounting for the energy which is radiated by the quadrupole waves (in agreement with the Einstein quadrupole formula). On the other hand, the radiation field involves a non-local contribution whose physical origin is the re-emission of waves by the linear waves [12–16, 9]. This contribution can be easily computed by using as the source of waves in the right-hand side of the Einstein field equations the effective stress–energy tensor of gravitational waves (averaged over several wavelengths). As shown by Christodoulou [14] and Thorne [15], this implies a permanent change in the wave amplitude from before to after a burst of gravitational waves, which can be interpreted as the contribution of gravitons in the known formulae for the linear memory [17, 18]. The latter nonlinear memory integral appears at the 2.5PN order in the radiation field (1PN order relative to the tail integral).

Now the metric which corresponds to the quadrupole–quadrupole interaction also involves, besides the non-local contributions, many terms which, by contrast, depend on the multipole moments at the sole retarded instant $t - r/c$. In the following we shall often describe these local terms as instantaneous (following the terminology of [8]). The instantaneous terms in the radiation field (to order $1/r$) are transient in the sense that they return to zero after the passage of a burst of gravitational waves. On physical grounds, it can be argued that the instantaneous terms do not play a very important role. However, these terms do exist and form an integral part of the field generated by very relativistic sources like in-spiralling compact binaries (see, e.g., [19]). In fact, the complete quadrupole–quadrupole metric including all the hereditary and instantaneous contributions will be needed in the construction of very accurate theoretical waveforms to be used by the future detectors LIGO and VIRGO.

The present paper is devoted to the computation of the quadrupole–quadrupole metric, using the so-called multipolar-post-Minkowskian method proposed by Blanchet and Damour [20, 21] following previous work by Bonnor [10] (his double-series approximation method) and Thorne [22] (how to start the post-Minkowskian iteration using STF multipole moments). The instantaneous quadrupole–quadrupole terms have never been computed using the multipolar-post-Minkowskian method. However, they have been computed by Hunter and Rotenberg [6] using the double-series method, in the case where the source is axisymmetric. The nonlinear interaction between quadrupoles has also received attention more recently within the double-series method [23].

The main result of this paper (having in mind the application to astrophysical sources to be detected by VIRGO and LIGO) is the completion of the observable quadrupole moment of a general isolated source at the 2.5PN order. To reach this result, we also take into account previous results concerning the tails at the 1.5PN order [9], and the multipole moments given by explicit integrals over the source to 2.5PN order [24, 25]. (Note that the 2.5PN approximation in the observable quadrupole moment gives no contribution to the phase evolution of in-spiralling compact binaries [25], however it will be required when we compute the 2.5PN waveform.)

In a companion paper [26], which we shall refer to as paper II, we investigate the monopole–monopole–quadrupole interaction (at the cubic nonlinear order), which enters the radiation field at the 3PN approximation. The present paper and paper II are part of the programme of computing the field generated by in-spiralling compact binaries to 3PN and even 3.5PN accuracy (see [27–30] for why such a very high accuracy is necessary).

The plan of the paper is as follows. In section 2 we summarize from [20, 21] the method for computing the field nonlinearities. In section 3 we investigate (following [8]) the general structure of the quadratic nonlinearities. Section 4 deals with the explicit computation of the quadrupole–quadrupole metric. The results are presented in the form of tables of numerical coefficients. Finally, in section 5, we expand the metric at infinity and obtain the observable moments in the radiation field (essentially the quadrupole). The technical formulae for integrating the wave equation are relegated to appendix A, and some required results concerning the dipole–quadrupole interaction are derived in appendix B. Henceforth we set $c = 1$, except when we discuss the post-Newtonian order at the end of section 5.

2. Nonlinearities in the external field

We summarize the method set up in [20] for the computation of the quadratic and higher nonlinearities in the field generated by an isolated system. The computation is performed in the exterior (vacuum) weak-field region of the system, where the components of the gravitational field $h^{\mu\nu}$ are numerically small as compared to one. Here $h^{\mu\nu}$ denotes the metric deviation $h^{\mu\nu} = \sqrt{-g}\, g^{\mu\nu} - \eta^{\mu\nu}$, with $g^{\mu\nu}$ the inverse and g the determinant of the metric $g_{\mu\nu}$, and $\eta^{\mu\nu}$ the Minkowski metric diag$(-1, 1, 1, 1)$. In the exterior weak-field region the field $h^{\mu\nu}$ admits a post-Minkowskian expansion,

$$h^{\mu\nu} = G h_1^{\mu\nu} + G^2 h_2^{\mu\nu} + \cdots + G^n h_n^{\mu\nu} + \cdots , \tag{2.1}$$

where G is the Newtonian gravitational constant, which here plays the role of a book-keeping parameter in the nonlinearity expansion. The first term in (2.1) satisfies the vacuum Einstein equations linearized around the Minkowski metric. In harmonic (or De Donder) coordinates this gives two equations,

$$\Box h_1^{\mu\nu} = 0, \tag{2.2a}$$

$$\partial_\nu h_1^{\mu\nu} = 0. \tag{2.2b}$$

In the first equation $\Box$ denotes the flat spacetime wave (d'Alembertian) operator. The second equation (divergenceless of the field) is the harmonic gauge condition. The equations (2.2), supplemented by the condition of retarded potentials, are solved by means of a multipolar expansion. It is known that only two sets of multipole moments, the mass-type moments M_L and current-type moments S_L (both depending on the retarded time $t - r$), are sufficient to parametrize the general multipole expansion (see, e.g., [22]). In terms of these moments the general solution reads [22]

$$h_1^{00} = -4 \sum_{\ell \geqslant 0} \frac{(-1)^\ell}{\ell!} \partial_L \left[r^{-1} M_L(t-r) \right] , \tag{2.3a}$$

$$h_1^{0i} = 4 \sum_{\ell \geqslant 1} \frac{(-1)^\ell}{\ell!} \partial_{L-1} \left[r^{-1} M^{(1)}_{iL-1}(t-r) \right] + 4 \sum_{\ell \geqslant 1} \frac{(-1)^\ell \ell}{(\ell+1)!} \varepsilon_{iab} \partial_{aL-1} \left[r^{-1} S_{bL-1}(t-r) \right] , \tag{2.3b}$$

$$h_1^{ij} = -4 \sum_{\ell \geqslant 2} \frac{(-1)^\ell}{\ell!} \partial_{L-2} \left[r^{-1} M^{(2)}_{ijL-2}(t-r) \right] - 8 \sum_{\ell \geqslant 2} \frac{(-1)^\ell \ell}{(\ell+1)!} \partial_{aL-2} \left[r^{-1} \varepsilon_{ab(i} S^{(1)}_{j)bL-2}(t-r) \right] . \tag{2.3c}$$

Here the superscript (n) denotes n time derivatives. The index L is a shorthand for a multi-index composed of ℓ indices, $L = i_1 i_2 \cdots i_\ell$ (similarly, $L - 1 = i_1 i_2 \cdots i_{\ell-1}$, $aL - 2 = a i_1 \cdots i_{\ell-2}$, and so on), and ∂_L denotes a product of ℓ space derivatives,

$\partial_L = \partial_{i_1}\partial_{i_2}\cdots\partial_{i_\ell}$. The multipole moments M_L and S_L are symmetric and tracefree (STF) with respect to all their indices†.

A priori the multipole moments $M_L(t)$ and $S_L(t)$ are arbitrary functions of time. Potentially they describe the physics of a general isolated source as seen in its exterior field [22, 20]. The only physical restriction is that the mass monopole M (total mass–energy or ADM mass), the mass dipole M_i (position of the centre of mass multiplied by the mass), and the current dipole S_i (total angular momentum) are constant. Technically speaking, this is a consequence of the harmonic gauge condition (2.2b) (see, e.g., [22]). In this paper we shall set the mass dipole M_i to zero by shifting the origin of coordinates to the centre of mass. In order to describe an isolated system, we must implement a condition of no incoming radiation, ensuring that the radiation field is entirely generated by the system. We assume that the field is stationary in the remote past, i.e. that the moments $M_L(t)$ and $S_L(t)$ are constant before some finite instant in the past, say when $t \leqslant -\mathcal{T}$ (hence M_i cannot be a linear function of time and is necessarily constant or zero). This assumption may seem to be somewhat restrictive, but we can check *a posteriori* that the formulae derived in this paper and paper II admit a well defined limit when $-\mathcal{T} \to -\infty$ in more general physical situations, such as the formation of the system by initial gravitational scattering. The multipole moments $M_L(t)$ and $S_L(t)$, subject to the previous restrictions, play the role of 'seed' moments for the construction of the exterior field (2.1). In particular, we shall express the results of this paper in terms of products of M_{ij} with itself. We shall not use the expressions for the multipole moments M_L and S_L as explicit integrals over the source. However, these expressions are known in the post-Newtonian approximation [24, 25], and should be used in applications.

The coefficient of G^2 in (2.1) is the quadratically nonlinear metric, whose precise definition we recall. The field equations for this coefficient read, still using harmonic coordinates,

$$\Box h_2^{\mu\nu} = N_2^{\mu\nu}\,, \tag{2.4a}$$

$$\partial_\nu h_2^{\mu\nu} = 0\,. \tag{2.4b}$$

The d'Alembertian equation involves a quadratic source $N_2^{\mu\nu} = N^{\mu\nu}(h_1, h_1)$ generated by the linearized gravitational field (2.3), where

$$\begin{aligned} N^{\mu\nu}(h,h) = {} & -h^{\rho\sigma}\partial_\rho\partial_\sigma h^{\mu\nu} + \tfrac{1}{2}\partial^\mu h_{\rho\sigma}\partial^\nu h^{\rho\sigma} - \tfrac{1}{4}\partial^\mu h\partial^\nu h \\ & - 2\partial^{(\mu} h_{\rho\sigma}\partial^\rho h^{\nu)\sigma} + \partial_\sigma h^{\mu\rho}(\partial^\sigma h^\nu_\rho + \partial_\rho h^{\nu\sigma}) \\ & + \eta^{\mu\nu}\big[-\tfrac{1}{4}\partial_\lambda h_{\rho\sigma}\partial^\lambda h^{\rho\sigma} + \tfrac{1}{8}\partial_\rho h\partial^\rho h + \tfrac{1}{2}\partial_\rho h_{\sigma\lambda}\partial^\sigma h^{\rho\lambda}\big]\,. \end{aligned} \tag{2.5}$$

From (2.4b) we deduce

$$\partial_\nu N_2^{\mu\nu} = 0\,. \tag{2.6}$$

Then the quadratic metric $h_2^{\mu\nu}$, solving (2.4) and the condition of stationarity in the past, is obtained as the sum of two distinct contributions,

$$h_2^{\mu\nu} = u_2^{\mu\nu} + v_2^{\mu\nu}\,. \tag{2.7}$$

† Our notation is the following: signature $-+++$; Greek indices $= 0, 1, 2, 3$; Latin indices $= 1, 2, 3$; $g = \det(g_{\mu\nu})$; $\eta_{\mu\nu} = \eta^{\mu\nu} =$ flat metric $= \mathrm{diag}(-1, 1, 1, 1)$; $r = |\boldsymbol{x}| = (x_1^2 + x_2^2 + x_3^2)^{1/2}$; $n^i = n_i = x^i/r$; $\partial_i = \partial/\partial x^i$; $n^L = n_L = n_{i_1}n_{i_2}\cdots n_{i_\ell}$ and $\partial_L = \partial_{i_1}\partial_{i_2}\cdots\partial_{i_\ell}$, where $L = i_1 i_2\cdots i_\ell$ is a multi-index with ℓ indices; $n_{L-1} = n_{i_1}\cdots n_{i_{\ell-1}}$, $n_{aL-1} = n_a n_{L-1}$, etc; $\hat{n}_L$ and $\hat{\partial}_L$ are the (symmetric) and trace-free (STF) parts of n_L and ∂_L, also denoted by $n_{\langle L\rangle}$, $\partial_{\langle L\rangle}$; the superscript (n) denotes n time derivatives; $T_{(\alpha\beta)} = \frac{1}{2}(T_{\alpha\beta} + T_{\beta\alpha})$ and $T_{(ij)} = \frac{1}{2}(T_{ij} + T_{ji})$.

Basically, the first contribution $u_2^{\mu\nu}$ is the retarded integral of the source $N_2^{\mu\nu}$. However, in our case the source is in the form of a multipole expansion (valid only in the exterior of the system and singular at $r = 0$), so we cannot directly apply the usual retarded integral operator, whose range of integration intersects the system at retarded time. A way out of this problem, proposed in [20], consists of multiplying the actual source term $N_2^{\mu\nu}$ by a factor $(r/r_0)^B$, where B is a complex number and r_0 denotes a certain constant having the dimension of a length. When the real part of B is large enough, all the power-like singularities of the multipole expansion at the spatial origin of the coordinates $r = 0$ are cancelled. (Actually we consider separately each multipolar piece, with given multipolarity ℓ, so that the maximal power of the singularities is finite, and B can indeed be chosen in such a way.) Applying the retarded integral on each multipolar piece of the product $(r/r_0)^B N_2^{\mu\nu}$ results in a function of B whose definition can be analytically continued to a neighbourhood of $B = 0$, at which point it admits a Laurent expansion. The finite part at $B = 0$ (in short $\mathrm{FP}_{B=0}$) of the latter expansion is our looked-for solution, as it satisfies the correct wave equation ($\Box\, u_2^{\mu\nu} = N_2^{\mu\nu}$), and is like $N_2^{\mu\nu}$ in the form of a multipole expansion. Note that the latter process represents simply a convenient means to find a solution of the wave equation whose source is in the form of a multipole expansion. Other processes could just as well be used, but this one is particularly powerful as it yields many explicit formulae to be used in practical computations (see appendix A and appendix A of paper II). Hence the first contribution in (2.7) reads

$$u_2^{\mu\nu} = \mathrm{FP}_{B=0}\, \Box_R^{-1} \left[\left(\frac{r}{r_0} \right)^B N_2^{\mu\nu} \right], \tag{2.8}$$

where $\Box_R^{-1}$ denotes the usual retarded integral

$$(\Box_R^{-1} f)(\boldsymbol{x}, t) = -\frac{1}{4\pi} \iiint \frac{\mathrm{d}^3\boldsymbol{x}'}{|\boldsymbol{x} - \boldsymbol{x}'|} f(\boldsymbol{x}', t - |\boldsymbol{x} - \boldsymbol{x}'|)\,. \tag{2.9}$$

When dealing with the metric at quadratic order, it can be proved that the B-dependent retarded integral in (2.8) is actually finite when $B \to 0$ (the finite part is not followed by any pole). So $u_2^{\mu\nu}$ is simply given by the value at $B = 0$ of the retarded integral (see [8] and section 3). But this is due to the special structure of the quadratic source $N_2^{\mu\nu}$, and does not remain true for cubic or higher nonlinear approximations (see, for instance, paper II).

The first contribution $u_2^{\mu\nu}$ solves (2.4*a*), but not the harmonic gauge condition (2.4*b*). The divergence of $u_2^{\mu\nu}$, say $w_2^{\mu} = \partial_\nu u_2^{\mu\nu}$, is *a priori* different from zero. Using (2.6) we find

$$w_2^{\mu} = \mathrm{FP}_{B=0}\, \Box_R^{-1} \left[B \left(\frac{r}{r_0} \right)^B \frac{n_i}{r} N_2^{\mu i} \right]. \tag{2.10}$$

The explicit factor B comes from the differentiation of r^B in (2.8) (we use the notation $n_i = \partial_i r = x^i/r$). Owing to this factor, the finite part in (2.10) is in fact a residue at $B = 0$, or coefficient of B^{-1} in the Laurent expansion. (The source term in (2.10) has a structure which is different from $N_2^{\mu\nu}$, and unlike in (2.8) the integral admits in general a (simple) pole at $B = 0$.) The second contribution $v_2^{\mu\nu}$ in (2.7) is then defined in such a way as to compensate exactly the (*a priori*) non-zero divergence w_2^{μ} of $u_2^{\mu\nu}$, while being a homogeneous solution of (2.4*a*). This is possible because w_2^{μ} is a particular retarded solution of $\Box\, w_2^{\mu} = 0$ (in the exterior region). As such, it admits a unique multipolar decomposition in terms of four sets of STF tensors A_L, B_L, C_L, D_L, namely

$$w_2^0 = \sum_{\ell\geqslant 0} \partial_L \left[r^{-1} A_L(t-r)\right] , \tag{2.11a}$$

$$w_2^i = \sum_{\ell\geqslant 0} \partial_{iL} \left[r^{-1} B_L(t-r)\right] + \sum_{\ell\geqslant 1} \left\{\partial_{L-1} \left[r^{-1} C_{iL-1}(t-r)\right] + \varepsilon_{iab}\partial_{aL-1} \left[r^{-1} D_{bL-1}(t-r)\right]\right\} . \tag{2.11b}$$

These tensors can be computed straightforwardly from the known expression (2.10), and the contribution $v_2^{\mu\nu}$ is defined in terms of these tensors. A particular definition was proposed in equation (4.13) of [20], where it was denoted by $q_2^{\mu\nu}$. Here we shall define this second contribution slightly differently, and accordingly we use the different notation $v_2^{\mu\nu}$. The various components of $v_2^{\mu\nu}$ are given by

$$v_2^{00} = -r^{-1} \int A + \partial_a \left[r^{-1} \left(-\int A_a + \iint C_a - 3B_a\right)\right] , \tag{2.12a}$$

$$v_2^{0i} = r^{-1} \left(-\int C_i + 3B_i^{(1)}\right) - \varepsilon_{iab}\partial_a \left[r^{-1} \int D_b\right] - \sum_{\ell\geqslant 2} \partial_{L-1} \left[r^{-1} A_{iL-1}\right] , \tag{2.12b}$$

$$\begin{aligned} v_2^{ij} = -\delta_{ij} r^{-1} B + \sum_{\ell\geqslant 2} \Big\{ & 2\delta_{ij}\partial_{L-1} \left[r^{-1} B_{L-1}\right] - 6\partial_{L-2(i} \left[r^{-1} B_{j)L-2}\right] \\ & + \partial_{L-2}\left[r^{-1}(A^{(1)}_{ijL-2} + 3B^{(2)}_{ijL-2} - C_{ijL-2})\right] \\ & - 2\partial_{aL-2} \left[r^{-1}\varepsilon_{ab(i} D_{j)bL-2}\right]\Big\} . \end{aligned} \tag{2.12c}$$

As in (2.11), all the tensors are evaluated at the retarded time $t-r$. We note that the formulae (2.12) are non-instantaneous, as they depend on the moments M_L and S_L at any time less than $t-r$ through the first and second time anti-derivatives of A, A_a, C_a, denoted, e.g. by $\int A = \int_{-\infty}^{t-r} A(t')\,\mathrm{d}t'$ and $\iint C_a = \int_{-\infty}^{t-r} \int C_a(t')\,\mathrm{d}t'$ (see [20] for a discussion of this). (To quadratic order the tensors $A_L, \ldots, D_L$ are given by some instantaneous functionals of the moments M_L and S_L.) The main property of $v_2^{\mu\nu}$ is $\partial_\nu v_2^{\mu\nu} = -w_2^\mu$, which is easily checked using the expressions (2.12). Furthermore, $\Box\, v_2^{\mu\nu} = 0$, so the quadratic metric (2.7) is, indeed, a solution of both the wave equation (2.4a) and the gauge condition (2.4b). Note that the spatial trace $v_2^{ii} = \delta^{ij} v_2^{ij}$ is especially simple,

$$v_2^{ii} = -3r^{-1} B . \tag{2.12d}$$

The choice of definition (2.12) adopted here, which differs from the choice adopted in [20], is for convenience in future work. Of course, we are free to adopt either definition because such a choice is equivalent to a choice of gauge. However, the definition (2.12) is slightly preferable to the definition proposed in [20] when we want to express the multipole moments M_L and S_L as integrals over the source. Thus we take the present opportunity to redefine the construction of the exterior metric using this new definition. (Actually it can be checked that all intermediate and final results of this paper and paper II are independent of the choice of definition.)

For the third (cubic) and higher nonlinear iterations, the construction of the external metric proceeds along exactly the same lines, namely

$$h_n^{\mu\nu} = u_n^{\mu\nu} + v_n^{\mu\nu} , \tag{2.13}$$

where the first term $u_n^{\mu\nu}$ is the finite part of the retarded integral of the source to the nth post-Minkowskian order, $u_n^{\mu\nu} = \mathrm{FP}_{B=0}\,\Box_R^{-1}[(r/r_0)^B N_n^{\mu\nu}(h_1,\ldots,h_{n-1})]$, and where the second term $v_n^{\mu\nu}$ is defined from the divergence $w_n^{\mu} = \partial_\nu u_n^{\mu\nu}$ by the same formulae (2.11), (2.12). (See [20] for the proof that the construction of the metric can be implemented to any post-Minkowskian order.)

3. Structure of the quadratically nonlinear field

In this section we investigate the structure of the quadratic metric $h_2^{\mu\nu}$ defined by (2.7)–(2.12). For simplicity we omit most of the numerical coefficients and indices, so as to focus our attention on the basic structure of the metric. A precise computation of the numerical coefficients is dealt with in section 4.

The structure of the linearized metric (2.3) is that of a sum of retarded multipolar waves, consisting of p spatial derivatives (say) acting on monopolar waves $r^{-1}X(t-r)$,

$$h_1 \approx \sum \partial_P[r^{-1}X(t-r)]\,. \tag{3.1}$$

Our notation $\approx$ refers to the structure of the expression. By expanding the derivatives ∂_P (which act both on the pre-factor r^{-1} and on the retardation $t-r$) we get

$$h_1 \approx \sum_{j\geqslant 1} \hat{n}_Q r^{-j} Z(t-r)\,, \tag{3.2}$$

where the powers of $1/r$ are $j \geqslant 1$, and where we have expressed the angular dependence of each term using STF products of unit vectors $\hat{n}_Q = n_{\langle i_i} n_{i_2}\ldots n_{i_q\rangle}$ = STF part of n_Q (see our notation defined in the footnote in section 2). In practice, one may compute (3.2) from (3.1) by decomposing ∂_P on the basis of STF spatial derivatives $\hat{\partial}_Q$ and using (A.15) from appendix A. After insertion of (3.2) into the quadratic source term $N_2^{\mu\nu}$, one finds

$$N_2 \approx \sum_{k\geqslant 2} \hat{n}_L r^{-k} F(t-r)\,, \tag{3.3}$$

where the powers of $1/r$ start with $k=2$ (as is clear from the fact that N_2 is quadratic in h_1 which is of order $1/r$). The functions F are composed of sums of quadratic products of derivatives of the functions Z in (3.2).

The main problem is to compute u_2 defined by (2.8). In view of the structure (3.3), it is *a priori* required to compute the finite part $\mathrm{FP}_{B=0}$ of the retarded integral of any term $\hat{n}_L r^{-k} F(t-r)$ with multipolarity ℓ and radial dependence with $k \geqslant 2$. All the required formulae are listed in appendix A (which summarizes results mostly obtained in previous works [20, 8, 9]). Notably, we know from appendix A that the (finite part of the) retarded integral in the case $k=2$ is irreducibly *non-local* or non-instantaneous, see (A.3)–(A.7). When the power k satisfies $3 \leqslant k \leqslant \ell+2$, where ℓ is the multipolarity (this excludes the monopolar case $\ell = 0$), we know that the corresponding retarded integral is instantaneous, and is given by (A.11), (A.12). Finally, when $k \geqslant \ell+3$, we have again a non-local expression, given by (A.13), (A.14), except in special combinations like (A.16) for which the non-local integrals cancel out.

The computation of $u_2 = \mathrm{FP}\,\Box_R^{-1} N_2$ can be implemented by an algebraic computer program, following the successive steps (3.1)–(3.3) and applying to each of the terms composing (3.3) the formulae (A.5), (A.11) and (A.13), (A.14). This is probably the most efficient way to obtain u_2. However, in doing so we would discover that the only non-local integrals left in u_2 come from the source terms having a radial dependence with $k=2$, in other words all the non-local integrals coming from terms having $k \geqslant \ell+3$ actually cancel out. Practically speaking, the source terms having $k \geqslant \ell+3$ turn out to combine into combinations such as (A.16) yielding purely instantaneous contributions. This fact (proved in [8]) is special to the quadratic nonlinearities, and does not remain true at higher orders, e.g. at the cubic order as seen in paper II.

The proof of the latter assertion uses specifically the quadratic structure of the source N_2. Instead of inserting in (2.5) the linearized metric in expanded form (3.2) and then working out all derivatives to arrive at (3.3), we keep the structure of the source as it basically is, i.e. a sum of quadratic products of multipolar waves,

$$N_2 \approx \sum \partial_P \big[r^{-1} X(t-r)\big] \partial_Q \big[r^{-1} Y(t-r)\big] , \tag{3.4}$$

involving spatial multi-derivatives with $P = i_1 \cdots i_p$ and $Q = j_1 \cdots j_q$ and some functions of time X and Y. Then we perform on each term of (3.4) a sequence of operations by parts (i.e. $\partial_i A \partial_j B = \partial_i (A \partial_j B) - A \partial_i \partial_j B$), by which the spatial derivatives acting on the wave on the left (say) are shifted in front and to the right. This leads to

$$N_2 \approx \sum \partial_P \big\{ r^{-1} X(t-r) \partial_R [r^{-1} Y(t-r)] \big\} . \tag{3.5}$$

Only at this stage does one expand the space derivatives ∂_R (inside the curly brackets), while leaving the derivatives ∂_P in front unexpanded. The result reads

$$N_2 \approx \sum_{2 \leqslant k \leqslant \ell+2} \partial_P \left\{ \hat{n}_L r^{-k} H(t-r) \right\} , \tag{3.6}$$

where the functions H are sums of products of X and time-derivatives of Y, and where we have projected the angular dependence of the terms inside the brackets on STF tensors $\hat{n}_L$. The point is that the radial dependence of the terms inside the brackets is related to the multipolarity ℓ by $2 \leqslant k \leqslant \ell + 2$. This can easily be seen from (3.5), as the expansion of the multipolar wave $\partial_R [r^{-1} Y]$ is composed of a sum of terms $\hat{\partial}_L [r^{-1} Y']$ which have $1 \leqslant k \leqslant \ell + 1$ (see (A.1)), yielding $2 \leqslant k \leqslant \ell + 2$ after multiplication by the factor r^{-1} on the left. The next operation is to single out the terms with pure radial dependence $k = 2$. All these terms can be obtained by applying ∂_P on the terms inside the brackets having $k = 2$. In this way one generates, besides all the terms $k = 2$, many other terms with $k \geqslant 3$, but the latter terms can be recombined into terms of the same form as in (3.6) (and thus having $3 \leqslant k \leqslant \ell + 2$). See [8] for the proof. Thus (3.6) can be rewritten as

$$N_2 \approx r^{-2} Q(\boldsymbol{n}, t-r) + \sum_{3 \leqslant k \leqslant \ell+2} \partial_P \left\{ \hat{n}_L r^{-k} F(t-r) \right\} , \tag{3.7}$$

where $Q(\boldsymbol{n}, t-r)$ denotes the coefficient of r^{-2} in the (finite) expansion of the quadratic source when $r \to \infty$ with $t - r =$ constant. The definition of $Q(\boldsymbol{n}, t-r)$ is

$$N_2^{\mu\nu} = \frac{1}{r^2} Q^{\mu\nu}(\boldsymbol{n}, t-r) + \mathrm{O}\left(\frac{1}{r^3}\right) . \tag{3.8}$$

Next, in anticipation of applying the finite part of the retarded integral, we multiply (3.7) by a factor r^B, and introduce r^B inside the brackets using again a series of operations by parts. In this way we get many new terms, but they all involve at least one factor B coming from the differentiation of r^B during the latter operations. Thus

$$r^B N_2 \approx r^{B-2} Q(\boldsymbol{n}, t-r) + \sum_{3 \leqslant k \leqslant \ell+2} \partial_P \left\{ \hat{n}_L r^{B-k} F(t-r) \right\} + \mathrm{O}(B) . \tag{3.9}$$

Applying the retarded integral on both sides of (3.9), commuting $\Box_R^{-1}$ with ∂_P, and taking the finite part, we are left with (the finite part of) retarded integrals of three types of terms: (i) the first term in (3.9) which has radial dependence r^{-2}; (ii) the terms in the brackets of (3.9) having radial dependence such that $3 \leqslant k \leqslant \ell + 2$; and (iii) the terms $\mathrm{O}(B)$ which have the structure (3.3) with any radial dependence k but carry at least one factor B. In case (ii) the retarded integrals are given by the instantaneous expressions (A.11). In case (iii) the retarded integrals are given by (A.18) when the power of B is one, and are zero for higher powers. So in case (iii) the retarded integrals are also instantaneous. Therefore we can state the following result [8]: the only non-local integrals in $u_2 = \mathrm{FP}\,\Box_R^{-1} N_2$ come from case (i), namely from the source terms whose radial dependence is r^{-2}, and which are denoted by $r^{-2}Q(\boldsymbol{n}, t-r)$ in (3.8). These non-local integrals are given by (A.3)–(A.7). This result holds true only in the case of the quadratic nonlinearity. In cubic and higher nonlinearities, some hereditary integrals are generated by source terms with radial dependence such that $k \geqslant \ell + 3$.

Incidentally, note that the decomposition (3.9) of the source shows that the B-dependent retarded integral $\Box_R^{-1}[r^B N_2]$ is *finite* at $B = 0$, i.e. does not involve any pole when $B \to 0$. Thus the finite part at $B = 0$ is simply equal to the value of $\Box_R^{-1}[r^B N_2]$ at $B = 0$. This can be checked from the formulae (A.3), (A.11) and (A.18), which are all finite at $B = 0$. Here again, this is a peculiarity of the quadratic approximation.

We are now in the position to write down the structure of the first contribution u_2. From (3.9) and (A.3), (A.11) and (A.18), we have

$$u_2 \approx t_2 + \sum_{k\geqslant 1} \hat{n}_L r^{-k} G(t-r)\,, \tag{3.10}$$

where the functions $G(t-r)$ depend instantaneously on the multipole moments M_L and S_L (i.e. at the retarded time $t-r$ only), and where the first term is non-local and given by

$$t_2^{\mu\nu} = \Box_R^{-1}[r^{-2}Q^{\mu\nu}(\boldsymbol{n}, t-r)]\,. \tag{3.11}$$

On the other hand, the second contribution v_2, defined by (2.11), (2.12), involves some time anti-derivatives of quadratic products of moments. We denote these time anti-derivatives by s_2. Thus the quadratic-order metric can be written as

$$h_2 \approx t_2 + s_2 + \sum_{k\geqslant 1} \hat{n}_L r^{-k} P(t-r)\,, \tag{3.12}$$

where t_2 is the non-local integral (3.11), where s_2 are some anti-derivatives (given by (4.12) below in the case of the quadrupole–quadrupole interaction), and where we have many instantaneous terms. See (4.13) and table 2 below for the complete expression of h_2 in the case of the quadrupole–quadrupole interaction.

4. The quadrupole–quadrupole metric

We specialize the previous investigation to the case of the interaction between two quadrupole moments M_{ab} and M_{cd}. Thus we keep in the linearized metric (2.3) only the terms corresponding to M_{ab},

$$h_1^{00} = -2\partial_{ab}\left(r^{-1}M_{ab}\right)\,, \tag{4.1a}$$

$$h_1^{0i} = 2\partial_a\left(r^{-1}M_{ai}^{(1)}\right)\,, \tag{4.1b}$$

$$h_1^{ij} = -2r^{-1}M_{ij}^{(2)}\,. \tag{4.1c}$$

(Henceforth we use the same notation for the metric constructed out of the quadrupole moment as for the complete metric involving all multipolar contributions.)

Inserting (4.1) into the quadratic source (2.5), we can work out explicitly all the terms composing the source either in the all-expanded form (3.3) or in the more elaborate form

(3.7). Notably, we find that the terms with radial dependence r^{-2} take the classic form of the stress–energy tensor of a massless field,

$$Q^{\mu\nu}(\boldsymbol{n}, t-r) = k^\mu k^\nu \Pi(\boldsymbol{n}, t-r)\,, \tag{4.2}$$

where k^μ denotes the Minkowskian null vector $k^\mu = (1, n^i)$, and where Π is given by

$$\Pi = n_{abcd} M^{(3)}_{ab} M^{(3)}_{cd} - 4 n_{ab} M^{(3)}_{ac} M^{(3)}_{bc} + 2 M^{(3)}_{ab} M^{(3)}_{ab}\,. \tag{4.3}$$

The quantity Π is proportional to the power (per unit of steradian) carried by the linearized waves. In the general case, $Q^{\mu\nu}$ involves, besides the quadrupole–quadrupole terms, all the interacting terms between multipole moments with $\ell \geqslant 2$ and the mass monopole M. We introduce the STF multipole decomposition of Π,

$$\Pi(\boldsymbol{n}, u) = \sum_{\ell\geqslant 0} n_L\, \Pi_L(u)\,. \tag{4.4}$$

From (4.3), the only non-zero multipolar coefficients are

$$\Pi_0 = \tfrac{4}{5} M^{(3)}_{ab} M^{(3)}_{ab}\,, \tag{4.5a}$$

$$\Pi_{ij} = -\tfrac{24}{7} M^{(3)}_{a\langle i} M^{(3)}_{j\rangle a}\,, \tag{4.5b}$$

$$\Pi_{ijkl} = M^{(3)}_{\langle ij} M^{(3)}_{kl\rangle}\,. \tag{4.5c}$$

(Π_0 denotes the coefficient with multipolarity $\ell = 0$, i.e. the spherical average.) With this definition, and with the help of (A.7), we can write the non-local integral t_2 as

$$t_2^{\mu\nu} = \Box^{-1}_R \left[\frac{k^\mu k^\nu}{r^2} \Pi \right] = -\int_r^{+\infty} \mathrm{d}s \int \frac{\mathrm{d}\Omega'}{4\pi} \frac{k'^\mu k'^\nu}{s - r\boldsymbol{n}\cdot\boldsymbol{n}'} \Pi(\boldsymbol{n}', t-s)\,. \tag{4.6}$$

Some equivalent expressions follow from (A.3)–(A.5). For instance, we can write $t_2^{\mu\nu}$ in the explicit form

$$t_2^{\mu\nu} = \frac{1}{2} \int_r^{+\infty} \mathrm{d}s \left[M^{(3)}_{ij} M^{(3)}_{kl} \partial^{\mu\nu}_{ijkl}\{6\} - 4 M^{(3)}_{ai} M^{(3)}_{aj} \partial^{\mu\nu}_{ij}\{4\} + 2 M^{(3)}_{ab} M^{(3)}_{ab} \partial^{\mu\nu}\{2\} \right], \tag{4.7a}$$

where the moments in the integrand are evaluated at the time $t-s$, where the multi-derivative operators mean, for instance, $\partial^{\mu\nu}_{ij} = \partial^\mu \partial^\nu \partial_i \partial_j$ with $\partial^\mu = (-\partial/\partial s, \partial_i)$, and where we use the special notation

$$\{p\} = \frac{(s-r)^p \ln(s-r) - (s+r)^p \ln(s+r)}{p!r} \tag{4.7b}$$

(see (A.4) in appendix A).

Having the term $t_2^{\mu\nu}$, we undertake the computation of all the instantaneous terms $M_{ab} \times M_{cd}$ in $u_2^{\mu\nu}$ (namely, second terms in (3.10)). The computation is straightforward but tedious. As said above, when doing practical computations (usually by computer), the best method is the somewhat brute force method consisting of obtaining the source in the all-expanded form (3.3), and applying the (finite part of the) retarded integral on each term of (3.3) with $k \geqslant 3$, using the formulae (A.11) and (A.13). This is simpler than working out the source in the more elaborate form (3.9), and using the manifestly instantaneous formulae (A.11) and (A.18). The brute force method also has the advantage that one can *check* that all the non-local integrals except those coming from the r^{-2} term cancel out (as well as the associated logarithms of r). The term u_2 takes the form

$$u_2^{00} = t_2^{00} + \sum_{k=1}^{6} \frac{1}{r^k} \sum_{m=0}^{6-k} \big\{ a_m^k \hat{n}_{abcd} M_{ab}^{(6-k-m)} M_{cd}^{(m)} + b_m^k \hat{n}_{ab} M_{ac}^{(6-k-m)} M_{bc}^{(m)} + c_m^k M_{ab}^{(6-k-m)} M_{ab}^{(m)} \big\}\,, \tag{4.8a}$$

$$u_2^{0i} = t_2^{0i} + \sum_{k=1}^{6} \frac{1}{r^k} \sum_{m=0}^{6-k} \big\{ d_m^k \hat{n}_{iabcd} M_{ab}^{(6-k-m)} M_{cd}^{(m)} + e_m^k \hat{n}_{iab} M_{ac}^{(6-k-m)} M_{bc}^{(m)} + f_m^k n_i M_{ab}^{(6-k-m)} M_{ab}^{(m)} + g_m^k \hat{n}_{abc} M_{ia}^{(6-k-m)} M_{bc}^{(m)} + h_m^k n_a M_{ib}^{(6-k-m)} M_{ab}^{(m)} \big\}\,, \tag{4.8b}$$

$$u_2^{ij} = t_2^{ij} + \sum_{k=1}^{6} \frac{1}{r^k} \sum_{m=0}^{6-k} \big\{ p_m^k \hat{n}_{ijabcd} M_{ab}^{(6-k-m)} M_{cd}^{(m)} + q_m^k \hat{n}_{ijab} M_{ac}^{(6-k-m)} M_{bc}^{(m)} + r_m^k \delta_{ij} \hat{n}_{abcd} M_{ab}^{(6-k-m)} M_{cd}^{(m)} + s_m^k \hat{n}_{ij} M_{ab}^{(6-k-m)} M_{ab}^{(m)} + t_m^k \delta_{ij} \hat{n}_{ab} M_{ac}^{(6-k-m)} M_{bc}^{(m)} + u_m^k \delta_{ij} M_{ab}^{(6-k-m)} M_{ab}^{(m)} + v_m^k \hat{n}_{abc(i} M_{j)a}^{(6-k-m)} M_{bc}^{(m)} + w_m^k \hat{n}_{a(i} M_{j)b}^{(6-k-m)} M_{ab}^{(m)} + x_m^k n_{ab} M_{ij}^{(6-k-m)} M_{ab}^{(m)} + y_m^k \hat{n}_{ab} M_{a(i}^{(6-k-m)} M_{j)b}^{(m)} + z_m^k M_{a(i}^{(6-k-m)} M_{j)a}^{(m)} \big\}\,, \tag{4.8c}$$

where all moments are evaluated at time $t-r$ and where $a_m^k, b_m^k, \ldots, z_m^k$ are purely numerical coefficients. Using the algebraic computer program Mathematica [31], we have obtained the numerical coefficients listed in table 1.

Next we follow the second part of the construction of the metric, and compute the divergence $w_2^\mu = \partial_\nu u_2^{\mu\nu}$ (see (2.10)–(2.12)). To this end, we need the divergence of the integral $t_2^{\mu\nu}$, which is easily evaluated by noticing that $\partial_\nu t_2^{\mu\nu}$ can be written, analogously to (2.10), as some residue at $B = 0$ of a retarded integral (because of the explicit factor B), and, furthermore, that the radial dependence of the integrand is merely r^{-3} (because it comes from the differentiation of the source term r^{-2}). From (A.18), we know that when $k = 3$ the residue is non-zero only when the multipolarity is $\ell = 0$. This immediately yields

$$\partial_\nu t_2^{\mu\nu} = \mathrm{FP}_{B=0}\, \Box_R^{-1} \left[B \left(\frac{r}{r_0} \right)^B \frac{k^\mu}{r^3} \Pi \right] = -\frac{1}{r} \int \frac{\mathrm{d}\Omega}{4\pi} k^\mu \Pi(\boldsymbol{n}, t-r)\,. \tag{4.9}$$

The $\mu = 0$ component of (4.9) is proportional to the angular average of Π, already computed in (4.5*a*). One can check that the $\mu = i$ component is zero. Thus,

$$\partial_\nu t_2^{0\nu} = -\tfrac{4}{5} r^{-1} M_{ab}^{(3)} M_{ab}^{(3)}\,, \tag{4.10a}$$

$$\partial_\nu t_2^{i\nu} = 0\,. \tag{4.10b}$$

Knowing (4.10), we can obtain w_2^μ by direct differentiation of the expressions (4.8), using the coefficients in table 1. Again the computation is quite lengthy, but it provides us with an important check. Indeed, from (2.11) one must find that the divergence w_2^μ is a solution of the source-free d'Alembertian equation, namely $\Box\, w_2^\mu = 0$. This test is very stringent,

Table 1. The numerical values of coefficients entering the expression for $u_2^{\mu\nu}$ as defined in (4.8).

k	m	a_m^k	b_m^k	c_m^k	d_m^k	e_m^k	f_m^k	g_m^k	h_m^k	p_m^k	q_m^k	r_m^k	s_m^k	t_m^k	u_m^k	v_m^k	w_m^k	x_m^k	y_m^k	z_m^k
1	0	$\frac{5}{12}$	$-\frac{4}{7}$	$\frac{1}{50}$	$-\frac{1}{120}$	$-\frac{13}{108}$	$\frac{23}{210}$	$\frac{59}{108}$	$-\frac{8}{35}$	$-\frac{1}{120}$	$\frac{1}{33}$	$-\frac{5}{66}$	$-\frac{5}{756}$	$\frac{103}{378}$	$-\frac{11}{3150}$	$\frac{97}{330}$	$-\frac{325}{189}$	$\frac{29}{189}$	$\frac{125}{378}$	$\frac{2}{525}$
	1	$\frac{29}{24}$	$-\frac{3}{7}$	$-\frac{2}{45}$	$-\frac{1}{15}$	$-\frac{23}{108}$	$\frac{29}{105}$	$\frac{37}{27}$	$\frac{34}{35}$	$-\frac{1}{15}$	$\frac{8}{33}$	$-\frac{61}{264}$	$-\frac{103}{756}$	$\frac{275}{378}$	$\frac{8}{315}$	$\frac{38}{33}$	$-\frac{863}{189}$	$-\frac{2}{189}$	$\frac{415}{378}$	$\frac{4}{21}$
	2	$\frac{37}{24}$	$\frac{11}{7}$	$\frac{11}{30}$	$-\frac{37}{120}$	$\frac{43}{54}$	$-\frac{31}{210}$	$\frac{133}{54}$	$\frac{26}{35}$	$-\frac{37}{120}$	$\frac{37}{33}$	$-\frac{47}{264}$	$-\frac{187}{378}$	$\frac{110}{189}$	$\frac{221}{630}$	$\frac{52}{33}$	$-\frac{94}{27}$	$\frac{236}{189}$	$\frac{139}{189}$	$\frac{166}{105}$
	3	$\frac{37}{24}$	$\frac{11}{7}$	$\frac{11}{30}$	$-\frac{37}{120}$	$\frac{43}{54}$	$-\frac{31}{210}$	$\frac{26}{27}$	$\frac{54}{35}$	$-\frac{37}{120}$	$\frac{37}{33}$	$-\frac{47}{264}$	$-\frac{187}{378}$	$\frac{110}{189}$	$\frac{221}{630}$	$-\frac{47}{33}$	$\frac{314}{189}$	$\frac{155}{189}$	$\frac{139}{189}$	$\frac{166}{105}$
	4	$\frac{29}{24}$	$-\frac{3}{7}$	$-\frac{2}{45}$	$-\frac{1}{15}$	$-\frac{23}{108}$	$\frac{29}{105}$	$\frac{121}{108}$	$-\frac{22}{35}$	$-\frac{1}{15}$	$\frac{8}{33}$	$-\frac{61}{264}$	$-\frac{103}{756}$	$\frac{275}{378}$	$\frac{8}{315}$	$-\frac{23}{66}$	$\frac{1}{189}$	$\frac{52}{189}$	$\frac{415}{378}$	$\frac{4}{21}$
	5	$\frac{5}{12}$	$-\frac{4}{7}$	$\frac{1}{50}$	$-\frac{1}{120}$	$-\frac{13}{108}$	$\frac{23}{210}$	$\frac{8}{27}$	$-\frac{8}{35}$	$-\frac{1}{120}$	$\frac{1}{33}$	$-\frac{5}{66}$	$-\frac{5}{756}$	$\frac{103}{378}$	$-\frac{11}{3150}$	$-\frac{1}{165}$	$-\frac{1}{189}$	$\frac{2}{189}$	$\frac{125}{378}$	$\frac{2}{525}$
2	0	$\frac{25}{6}$	$-\frac{12}{7}$	0	$-\frac{1}{8}$	$-\frac{13}{18}$	$\frac{23}{210}$	$\frac{59}{18}$	$-\frac{8}{35}$	$-\frac{7}{40}$	$\frac{10}{33}$	$-\frac{25}{33}$	$-\frac{5}{252}$	$\frac{103}{126}$	0	$\frac{97}{33}$	$-\frac{325}{63}$	$\frac{29}{63}$	$\frac{125}{126}$	0
	1	$\frac{83}{12}$	$-\frac{1}{7}$	$-\frac{2}{15}$	$-\frac{7}{8}$	$-\frac{5}{9}$	$\frac{1}{6}$	$\frac{53}{18}$	$\frac{2}{5}$	$-\frac{49}{40}$	$\frac{70}{33}$	$-\frac{205}{132}$	$-\frac{7}{18}$	$\frac{86}{63}$	0	$\frac{283}{33}$	$-\frac{538}{63}$	$-\frac{157}{63}$	$\frac{145}{63}$	0
	2	$\frac{25}{4}$	$\frac{4}{7}$	$\frac{1}{3}$	$-\frac{3}{2}$	$\frac{1}{3}$	$\frac{4}{35}$	$\frac{17}{6}$	$\frac{16}{35}$	$-\frac{9}{4}$	$\frac{46}{11}$	$\frac{35}{44}$	$-\frac{6}{7}$	$-\frac{6}{7}$	$\frac{9}{35}$	$-\frac{29}{11}$	0	$\frac{19}{7}$	$-\frac{18}{7}$	$\frac{36}{35}$
	3	$\frac{83}{12}$	$-\frac{1}{7}$	$-\frac{2}{15}$	$-\frac{7}{8}$	$-\frac{5}{9}$	$\frac{1}{6}$	$\frac{89}{18}$	$-\frac{2}{5}$	$-\frac{49}{40}$	$\frac{70}{33}$	$-\frac{205}{132}$	$-\frac{7}{18}$	$\frac{86}{63}$	0	$-\frac{113}{33}$	$\frac{2}{63}$	$\frac{50}{63}$	$\frac{145}{63}$	0
	4	$\frac{25}{6}$	$-\frac{12}{7}$	0	$-\frac{1}{8}$	$-\frac{13}{18}$	$\frac{23}{210}$	$\frac{16}{9}$	$-\frac{8}{35}$	$-\frac{7}{40}$	$\frac{10}{33}$	$-\frac{25}{33}$	$-\frac{5}{252}$	$\frac{103}{126}$	0	$-\frac{2}{33}$	$-\frac{1}{63}$	$\frac{2}{63}$	$\frac{125}{126}$	0
3	0	$\frac{69}{4}$	$-\frac{18}{7}$	$-\frac{1}{5}$	$-\frac{7}{8}$	$-\frac{65}{36}$	0	$\frac{187}{36}$	$-\frac{6}{5}$	$-\frac{7}{4}$	$\frac{15}{11}$	$-\frac{75}{22}$	$-\frac{5}{252}$	$\frac{103}{126}$	0	$\frac{291}{22}$	$-\frac{325}{63}$	$-\frac{160}{63}$	$\frac{125}{126}$	0
	1	$\frac{51}{8}$	$-\frac{20}{7}$	$-\frac{17}{30}$	-3	$-\frac{37}{12}$	$-\frac{19}{70}$	$-\frac{71}{6}$	$-\frac{38}{35}$	$-\frac{27}{4}$	$\frac{72}{11}$	$\frac{45}{88}$	$-\frac{19}{28}$	$-\frac{1}{2}$	$-\frac{9}{70}$	$\frac{45}{11}$	-5	$-\frac{8}{7}$	$-\frac{17}{14}$	$-\frac{18}{35}$
	2	$\frac{51}{8}$	$-\frac{20}{7}$	$-\frac{17}{30}$	-3	$-\frac{37}{12}$	$-\frac{19}{70}$	$\frac{125}{12}$	$-\frac{24}{35}$	$-\frac{27}{4}$	$\frac{72}{11}$	$\frac{45}{88}$	$-\frac{19}{28}$	$-\frac{1}{2}$	$-\frac{9}{70}$	$-\frac{207}{22}$	$\frac{1}{7}$	$\frac{10}{7}$	$-\frac{17}{14}$	$-\frac{18}{35}$
	3	$\frac{69}{4}$	$-\frac{18}{7}$	$-\frac{1}{5}$	$-\frac{7}{8}$	$-\frac{65}{36}$	0	$\frac{40}{9}$	0	$-\frac{7}{4}$	$\frac{15}{11}$	$-\frac{75}{22}$	$-\frac{5}{252}$	$\frac{103}{126}$	0	$-\frac{3}{11}$	$-\frac{1}{63}$	$\frac{2}{63}$	$\frac{125}{126}$	0
4	0	27	$-\frac{39}{7}$	$-\frac{9}{10}$	$-\frac{9}{4}$	$-\frac{13}{4}$	$-\frac{23}{70}$	$-\frac{49}{4}$	$-\frac{102}{35}$	-8	$\frac{45}{11}$	$-\frac{153}{44}$	$\frac{5}{63}$	$\frac{37}{63}$	$-\frac{11}{210}$	$\frac{279}{22}$	$-\frac{320}{63}$	$-\frac{170}{63}$	$\frac{74}{63}$	$\frac{2}{35}$
	1	$-\frac{291}{8}$	$-\frac{44}{7}$	$-\frac{53}{30}$	$-\frac{9}{2}$	$-\frac{13}{2}$	$-\frac{23}{35}$	$-\frac{17}{4}$	$-\frac{78}{35}$	$-\frac{35}{2}$	$\frac{117}{11}$	$\frac{333}{88}$	$-\frac{29}{63}$	$-\frac{16}{9}$	$-\frac{7}{30}$	$-\frac{225}{22}$	$\frac{20}{63}$	$\frac{68}{63}$	$-\frac{116}{63}$	$-\frac{2}{5}$
	2	27	$-\frac{39}{7}$	$-\frac{9}{10}$	$-\frac{9}{4}$	$-\frac{13}{4}$	$-\frac{23}{70}$	8	$\frac{24}{35}$	-8	$\frac{45}{11}$	$-\frac{153}{44}$	$\frac{5}{63}$	$\frac{37}{63}$	$-\frac{11}{210}$	$-\frac{9}{11}$	$\frac{4}{63}$	$-\frac{8}{63}$	$\frac{74}{63}$	$\frac{2}{35}$
5	0	$-\frac{63}{4}$	-9	$-\frac{21}{10}$	$-\frac{9}{4}$	$-\frac{13}{4}$	$-\frac{23}{70}$	$-\frac{49}{4}$	$-\frac{102}{35}$	$-\frac{75}{4}$	$\frac{90}{11}$	$-\frac{9}{44}$	$\frac{25}{84}$	$-\frac{29}{42}$	$-\frac{11}{70}$	$-\frac{18}{11}$	$\frac{5}{21}$	$-\frac{10}{21}$	$\frac{23}{42}$	$\frac{6}{35}$
	1	$-\frac{63}{4}$	-9	$-\frac{21}{10}$	$-\frac{9}{4}$	$-\frac{13}{4}$	$-\frac{23}{70}$	8	$\frac{24}{35}$	$-\frac{75}{4}$	$\frac{90}{11}$	$-\frac{9}{44}$	$\frac{25}{84}$	$-\frac{29}{42}$	$-\frac{11}{70}$	$-\frac{18}{11}$	$\frac{5}{21}$	$-\frac{10}{21}$	$\frac{23}{42}$	$\frac{6}{35}$
6	0	$-\frac{63}{4}$	-9	$-\frac{21}{10}$	0	0	0	0	0	$-\frac{75}{4}$	$\frac{90}{11}$	$-\frac{9}{44}$	$\frac{25}{84}$	$-\frac{29}{42}$	$-\frac{11}{70}$	$-\frac{18}{11}$	$\frac{5}{21}$	$-\frac{10}{21}$	$\frac{23}{42}$	$\frac{6}{35}$

as a single erroneous coefficient in table 1 would almost certainly cause its failure. Thus we determine all the tensors $A_L, \ldots, D_L$ in (2.11). The non-zero ones are given by

$$A = -\tfrac{2}{75} M^{(6)}_{ab} M_{ab} - \tfrac{176}{225} M^{(5)}_{ab} M^{(1)}_{ab} - \tfrac{22}{45} M^{(4)}_{ab} M^{(2)}_{ab} - \tfrac{8}{15} M^{(3)}_{ab} M^{(3)}_{ab} , \qquad (4.11a)$$

$$A_{ij} = -\tfrac{24}{35} \big(M^{(4)}_{a\langle i} M_{j\rangle a} + M^{(3)}_{a\langle i} M^{(1)}_{j\rangle a} \big) , \qquad (4.11b)$$

$$B = -\tfrac{32}{75} M^{(5)}_{ab} M_{ab} - \tfrac{8}{5} M^{(4)}_{ab} M^{(1)}_{ab} + \tfrac{8}{15} M^{(3)}_{ab} M^{(2)}_{ab} , \qquad (4.11c)$$

$$C_{ij} = \tfrac{4}{25} M^{(5)}_{a\langle i} M_{j\rangle a} - \tfrac{4}{5} M^{(4)}_{a\langle i} M^{(1)}_{j\rangle a} + \tfrac{8}{5} M^{(3)}_{a\langle i} M^{(2)}_{j\rangle a} , \qquad (4.11d)$$

$$D_i = \varepsilon_{iab} \big(-\tfrac{2}{5} M^{(5)}_{ac} M_{bc} - 2 M^{(4)}_{ac} M^{(1)}_{bc} - \tfrac{4}{5} M^{(3)}_{ac} M^{(2)}_{bc} \big) . \qquad (4.11e)$$

By inserting the latter values into (2.12), we obtain the components of the second part $v_2^{\mu\nu}$. In particular, we find some time anti-derivatives which define $s_2^{\mu\nu}$ (see (3.12)) as

$$s_2^{00} = \tfrac{4}{5} r^{-1} \int_{-\infty}^{t-r} \mathrm{d}u\, M^{(3)}_{ab} M^{(3)}_{ab}(u) , \qquad (4.12a)$$

$$s_2^{0i} = -\tfrac{4}{5} \varepsilon_{iab} \partial_a \left(r^{-1} \varepsilon_{bcd} \int_{-\infty}^{t-r} \mathrm{d}u\, M^{(3)}_{ce} M^{(2)}_{de}(u) \right) , \qquad (4.12b)$$

$$s_2^{ij} = 0 . \qquad (4.12c)$$

The physical interpretation of these anti-derivatives is clear. Indeed the linearized metric (2.3) depends in particular on the mass monopole M and current dipole S_i of the source (both are constant). Physically, M and S_i represent the total mass–energy and angular momentum of the system before the emission of gravitational radiation. Now the integral s_2^{00} given by (4.12a) represents a small modification, due to the emission of radiation, of the initial mass M. This is clear from the comparison of (4.12a) and (2.3a), showing that there is exact agreement with the energy loss by radiation as given by the standard Einstein quadrupole formula. This result is originally due to Bonnor [10], and Bonnor and Rotenberg [4]. Similarly, the integral s_2^{0i} given by (4.12b) represents a modification of the total angular momentum in agreement with the quadrupole formula for the angular momentum loss. (There is no loss of total linear momentum at the level of the quadrupole–quadrupole interaction (one needs to consider also the mass octupole M_{abc} and/or the current quadrupole S_{ab}).)

The quadratic metric $h_2^{\mu\nu}$ can now be completed. We add up the two contributions $u_2^{\mu\nu}$ (given by (4.8) and table 1) and $v_2^{\mu\nu}$ (given by (4.11) and (2.12)). The local terms are written in the same form as in (4.8). Thus,

$$h_2^{\mu\nu} = t_2^{\mu\nu} + s_2^{\mu\nu} + \sum_{k=1}^{6} \frac{1}{r^k} \sum_{m=0}^{6-k} \Big\{ \text{same expressions as in (4.8}a\text{–}c\text{) but with coefficients } a'^k_m, \ldots, z'^k_m \Big\} , \qquad (4.13)$$

where the non-local integrals $t_2^{\mu\nu}$ and $s_2^{\mu\nu}$ are given by (4.7) and (4.12), and where all the coefficients $a'^k_m, \ldots, z'^k_m$ are listed in table 2.

5. The quadrupole–quadrupole metric in the far zone

We investigate the behaviour of the quadrupole–quadrupole field $h_2^{\mu\nu}$ in the far zone, near future null infinity (i.e. at large distances when we recede from the source at the speed

Table 2. The numerical values of coefficients entering the expression for $h_2^{\mu\nu}$ as defined in (4.13).

k	m	$a_m'^k$	$b_m'^k$	$c_m'^k$	$d_m'^k$	$e_m'^k$	$f_m'^k$	$g_m'^k$	$h_m'^k$	$p_m'^k$	$q_m'^k$	$r_m'^k$	$s_m'^k$	$t_m'^k$	$u_m'^k$	$v_m'^k$	$w_m'^k$	$x_m'^k$	$y_m'^k$	$z_m'^k$
1	0	$\frac{5}{12}$	$-\frac{4}{7}$	$\frac{1}{30}$	$-\frac{1}{120}$	$-\frac{13}{108}$	$\frac{47}{210}$	$\frac{59}{108}$	$-\frac{34}{35}$	$-\frac{1}{120}$	$\frac{1}{33}$	$-\frac{5}{66}$	$-\frac{5}{756}$	$\frac{103}{378}$	$\frac{221}{630}$	$\frac{97}{330}$	$-\frac{325}{189}$	$\frac{29}{189}$	$\frac{125}{378}$	$-\frac{44}{105}$
	1	$\frac{29}{24}$	$-\frac{3}{7}$	$\frac{1}{3}$	$-\frac{1}{15}$	$-\frac{23}{108}$	$\frac{53}{105}$	$\frac{37}{27}$	$-\frac{12}{7}$	$-\frac{1}{15}$	$\frac{8}{33}$	$-\frac{61}{264}$	$-\frac{103}{756}$	$\frac{275}{378}$	$\frac{58}{63}$	$\frac{38}{33}$	$-\frac{863}{189}$	$-\frac{2}{189}$	$\frac{415}{378}$	$-\frac{2}{21}$
	2	$\frac{37}{24}$	$\frac{11}{7}$	$\frac{7}{30}$	$-\frac{37}{120}$	$\frac{43}{54}$	$-\frac{1}{30}$	$\frac{133}{54}$	$-\frac{6}{5}$	$-\frac{37}{120}$	$\frac{37}{33}$	$-\frac{47}{264}$	$-\frac{187}{378}$	$\frac{110}{189}$	$\frac{293}{630}$	$\frac{52}{33}$	$-\frac{94}{27}$	$\frac{236}{189}$	$\frac{139}{189}$	$\frac{46}{105}$
	3	$\frac{37}{24}$	$\frac{11}{7}$	$\frac{7}{30}$	$-\frac{37}{120}$	$\frac{43}{54}$	$-\frac{1}{30}$	$\frac{26}{27}$	$\frac{14}{5}$	$-\frac{37}{120}$	$\frac{37}{33}$	$-\frac{47}{264}$	$-\frac{187}{378}$	$\frac{110}{189}$	$\frac{293}{630}$	$-\frac{47}{33}$	$\frac{314}{189}$	$\frac{155}{189}$	$\frac{139}{189}$	$\frac{46}{105}$
	4	$\frac{29}{24}$	$-\frac{3}{7}$	$\frac{1}{3}$	$-\frac{1}{15}$	$-\frac{23}{108}$	$\frac{53}{105}$	$\frac{121}{108}$	$\frac{24}{35}$	$-\frac{1}{15}$	$\frac{8}{33}$	$-\frac{61}{264}$	$-\frac{103}{756}$	$\frac{275}{378}$	$\frac{58}{63}$	$-\frac{23}{66}$	$\frac{1}{189}$	$\frac{52}{189}$	$\frac{415}{378}$	$-\frac{2}{21}$
	5	$\frac{5}{12}$	$-\frac{4}{7}$	$\frac{1}{30}$	$-\frac{1}{120}$	$-\frac{13}{108}$	$\frac{47}{210}$	$\frac{8}{27}$	$-\frac{6}{35}$	$-\frac{1}{120}$	$\frac{1}{33}$	$-\frac{5}{66}$	$-\frac{5}{756}$	$\frac{103}{378}$	$\frac{221}{630}$	$-\frac{1}{165}$	$-\frac{1}{189}$	$\frac{2}{189}$	$\frac{125}{378}$	$-\frac{44}{105}$
2	0	$\frac{25}{6}$	$-\frac{12}{7}$	0	$-\frac{1}{8}$	$-\frac{13}{18}$	$\frac{47}{210}$	$\frac{59}{18}$	$-\frac{34}{35}$	$-\frac{7}{40}$	$\frac{10}{33}$	$-\frac{25}{33}$	$-\frac{5}{252}$	$\frac{103}{126}$	0	$\frac{97}{33}$	$-\frac{325}{63}$	$\frac{29}{63}$	$\frac{125}{126}$	0
	1	$\frac{83}{12}$	$-\frac{1}{7}$	$-\frac{2}{15}$	$-\frac{7}{8}$	$-\frac{5}{9}$	$\frac{59}{210}$	$\frac{53}{18}$	$-\frac{54}{35}$	$-\frac{49}{40}$	$\frac{70}{33}$	$-\frac{205}{132}$	$-\frac{7}{18}$	$\frac{86}{63}$	0	$\frac{283}{33}$	$-\frac{538}{63}$	$-\frac{157}{63}$	$\frac{145}{63}$	0
	2	$\frac{25}{4}$	$\frac{4}{7}$	$\frac{1}{3}$	$-\frac{3}{2}$	$\frac{1}{3}$	$\frac{4}{35}$	$\frac{17}{6}$	$\frac{16}{35}$	$-\frac{9}{4}$	$\frac{46}{11}$	$\frac{35}{44}$	$-\frac{6}{7}$	$-\frac{6}{7}$	$\frac{9}{35}$	$-\frac{29}{11}$	0	$\frac{19}{7}$	$-\frac{18}{7}$	$\frac{36}{35}$
	3	$\frac{83}{12}$	$-\frac{1}{7}$	$-\frac{2}{15}$	$-\frac{7}{8}$	$-\frac{5}{9}$	$\frac{59}{210}$	$\frac{89}{18}$	$\frac{6}{7}$	$-\frac{49}{40}$	$\frac{70}{33}$	$-\frac{205}{132}$	$-\frac{7}{18}$	$\frac{86}{63}$	0	$-\frac{113}{33}$	$\frac{2}{63}$	$\frac{50}{63}$	$\frac{145}{63}$	0
	4	$\frac{25}{6}$	$-\frac{12}{7}$	0	$-\frac{1}{8}$	$-\frac{13}{18}$	$\frac{47}{210}$	$\frac{16}{9}$	$-\frac{6}{35}$	$-\frac{7}{40}$	$\frac{10}{33}$	$-\frac{25}{33}$	$-\frac{5}{252}$	$\frac{103}{126}$	0	$-\frac{2}{33}$	$-\frac{1}{63}$	$\frac{2}{63}$	$\frac{125}{126}$	0
3	0	$\frac{69}{4}$	$-\frac{18}{7}$	$-\frac{1}{5}$	$-\frac{7}{8}$	$-\frac{65}{36}$	0	$\frac{187}{36}$	$-\frac{6}{5}$	$-\frac{7}{4}$	$\frac{15}{11}$	$-\frac{75}{22}$	$-\frac{5}{252}$	$\frac{103}{126}$	0	$\frac{291}{22}$	$-\frac{325}{63}$	$-\frac{160}{63}$	$\frac{125}{126}$	0
	1	$\frac{51}{8}$	$-\frac{20}{7}$	$-\frac{17}{30}$	-3	$-\frac{37}{12}$	$-\frac{19}{70}$	$-\frac{71}{6}$	$-\frac{38}{35}$	$-\frac{27}{4}$	$\frac{72}{11}$	$\frac{45}{88}$	$-\frac{19}{28}$	$-\frac{1}{2}$	$-\frac{9}{70}$	$\frac{45}{11}$	-5	$-\frac{8}{7}$	$-\frac{17}{14}$	$-\frac{18}{35}$
	2	$\frac{51}{8}$	$-\frac{20}{7}$	$-\frac{17}{30}$	-3	$-\frac{37}{12}$	$-\frac{19}{70}$	$\frac{125}{12}$	$-\frac{24}{35}$	$-\frac{27}{4}$	$\frac{72}{11}$	$\frac{45}{88}$	$-\frac{19}{28}$	$-\frac{1}{2}$	$-\frac{9}{70}$	$-\frac{207}{22}$	$\frac{1}{7}$	$\frac{10}{7}$	$-\frac{17}{14}$	$-\frac{18}{35}$
	3	$\frac{69}{4}$	$-\frac{18}{7}$	$-\frac{1}{5}$	$-\frac{7}{8}$	$-\frac{65}{36}$	0	$\frac{40}{9}$	0	$-\frac{7}{4}$	$\frac{15}{11}$	$-\frac{75}{22}$	$-\frac{5}{252}$	$\frac{103}{126}$	0	$-\frac{3}{11}$	$-\frac{1}{63}$	$\frac{2}{63}$	$\frac{125}{126}$	0
4	0	27	$-\frac{39}{7}$	$-\frac{9}{10}$	$-\frac{9}{4}$	$-\frac{13}{4}$	$-\frac{23}{70}$	$-\frac{49}{4}$	$-\frac{102}{35}$	-8	$\frac{45}{11}$	$-\frac{153}{44}$	$\frac{5}{63}$	$\frac{37}{63}$	$-\frac{11}{210}$	$\frac{279}{22}$	$-\frac{320}{63}$	$-\frac{170}{63}$	$\frac{74}{63}$	$\frac{2}{35}$
	1	$-\frac{291}{8}$	$-\frac{44}{7}$	$-\frac{53}{30}$	$-\frac{9}{2}$	$-\frac{13}{2}$	$-\frac{23}{35}$	$-\frac{17}{4}$	$-\frac{78}{35}$	$-\frac{35}{2}$	$\frac{117}{11}$	$\frac{333}{88}$	$-\frac{29}{63}$	$-\frac{16}{9}$	$-\frac{7}{30}$	$-\frac{225}{22}$	$\frac{20}{63}$	$\frac{68}{63}$	$-\frac{116}{63}$	$-\frac{2}{5}$
	2	27	$-\frac{39}{7}$	$-\frac{9}{10}$	$-\frac{9}{4}$	$-\frac{13}{4}$	$-\frac{23}{70}$	8	$\frac{24}{35}$	-8	$\frac{45}{11}$	$-\frac{153}{44}$	$\frac{5}{63}$	$\frac{37}{63}$	$-\frac{11}{210}$	$-\frac{9}{11}$	$\frac{4}{63}$	$-\frac{8}{63}$	$\frac{74}{63}$	$\frac{2}{35}$
5	0	$-\frac{63}{4}$	-9	$-\frac{21}{10}$	$-\frac{9}{4}$	$-\frac{13}{4}$	$-\frac{23}{70}$	$-\frac{49}{4}$	$-\frac{102}{35}$	$-\frac{75}{4}$	$\frac{90}{11}$	$-\frac{9}{44}$	$\frac{25}{84}$	$-\frac{29}{42}$	$-\frac{11}{70}$	$-\frac{18}{11}$	$\frac{5}{21}$	$-\frac{10}{21}$	$\frac{23}{42}$	$\frac{6}{35}$
	1	$-\frac{63}{4}$	-9	$-\frac{21}{10}$	$-\frac{9}{4}$	$-\frac{13}{4}$	$-\frac{23}{70}$	8	$\frac{24}{35}$	$-\frac{75}{4}$	$\frac{90}{11}$	$-\frac{9}{44}$	$\frac{25}{84}$	$-\frac{29}{42}$	$-\frac{11}{70}$	$-\frac{18}{11}$	$\frac{5}{21}$	$-\frac{10}{21}$	$\frac{23}{42}$	$\frac{6}{35}$
6	0	$-\frac{63}{4}$	-9	$-\frac{21}{10}$	0	0	0	0	0	$-\frac{75}{4}$	$\frac{90}{11}$	$-\frac{9}{44}$	$\frac{25}{84}$	$-\frac{29}{42}$	$-\frac{11}{70}$	$-\frac{18}{11}$	$\frac{5}{21}$	$-\frac{10}{21}$	$\frac{23}{42}$	$\frac{6}{35}$

of light). The degrees of freedom of the radiation field, to leading order in the inverse of the distance, are contained in the transverse and tracefree (TT) projection of the spatial components ij of the metric (denoted by $g_{ij}^{\rm TT}$). These are the so-called observable (or radiative) multipole moments, which are 'measured' in an experiment located far away from the system. The TT projection $g_{ij}^{\rm TT}$, to first order in $1/r$, reads

$$g_{ij}^{\rm TT} = \frac{4}{r}\mathcal{P}_{ijab}\sum_{\ell=2}^{\infty}\frac{1}{\ell!}\left\{n_{L-2}U_{ijL-2} - \frac{2\ell}{\ell+1}n_{aL-2}\varepsilon_{ab(i}V_{j)bL-2}\right\} + \mathrm{O}\left(\frac{1}{r^2}\right) \tag{5.1}$$

(with $G = c = 1$), where U_L and V_L denote the mass-type and current-type observable moments (both are functions of $t - r$), and where the TT projection operator is

$$\mathcal{P}_{ijab}(\boldsymbol{n}) = (\delta_{ia} - n_i n_a)(\delta_{jb} - n_j n_b) - \tfrac{1}{2}(\delta_{ij} - n_i n_j)(\delta_{ab} - n_a n_b)\,. \tag{5.2}$$

The ℓ-dependent coefficients in (5.1) are chosen so that U_L and V_L agree at the linearized order with the ℓth time derivatives of the moments M_L and S_L (compare with (2.3*c*)).

Let us consider first the non-local integral $t_2^{\mu\nu}$ (see (4.6), (4.7)). As we know from (A.8), the asymptotic expansion when $r \to \infty$, $t - r =$ constant of the retarded integral of a source with radial dependence r^{-2} is composed of terms $1/r^n$ and $\ln r/r^n$. As such, $t_2^{\mu\nu}$ behaves like $\ln r/r$ when $r \to \infty$, $t - r =$ constant. The logarithm is due to the deviation of the flat cones $t - r =$ constant in harmonic coordinates from the true spacetime null cones. The metric in harmonic coordinates is not of the normal Bondi-type at future null infinity [32, 33]. Removal of the logarithm is done using radiative coordinates, so defined that the associated flat cones agree, asymptotically, when $r \to +\infty$, with the true null cones (see, e.g., [21]). This method, adopted in [9], permits one to compute the observable moments in the non-local term $t_2^{\mu\nu}$. Here we follow another method, found by Thorne [15] and Wiseman and Will [16], which consists of applying first the TT projection operator (5.2) on $t_2^{\mu\nu}$. Because the TT projection kills any (linear) gauge term in the $1/r$ part of the metric, this method shortcuts the need for a transformation to radiative coordinates. However, one must be cautious in taking the limit $r \to \infty$, $t - r =$ constant using (4.6). It is not allowed, for instance, to work out a leading $1/r$ term from the second expression in (4.6) because this term would involve a divergent integral (in accordance with the fact that the leading term is actually $\ln r/r$). But, as pointed out in [15, 16], the divergent parts of the integral cancel out after application of the TT projection, and at the end one recovers the correct result. Thus, we compute

$$(t_2^{ij})^{\rm TT} = -\mathcal{P}_{ijab}(\boldsymbol{n})\int_r^{+\infty}\mathrm{d}s\int\frac{\mathrm{d}\Omega'}{4\pi}\,\frac{n'^a n'^b}{s - r\boldsymbol{n}\cdot\boldsymbol{n}'}\,\Pi(\boldsymbol{n}', t - s)\,. \tag{5.3}$$

(Note that the TT projection as defined in (5.2) is purely algebraic. Strictly speaking it agrees with the true TT projection only when acting on the leading $1/r$ term.) With the multipole decomposition (4.4) we have

$$(t_2^{ij})^{\rm TT} = -\mathcal{P}_{ijab}(\boldsymbol{n})\sum_{\ell\geqslant 0}\int_{-\infty}^{t-r}\mathrm{d}u\,\Pi_L(u)\int\frac{\mathrm{d}\Omega'}{4\pi}\,\frac{n'_a n'_b n'_L}{t - u - r\boldsymbol{n}\cdot\boldsymbol{n}'}\,. \tag{5.4}$$

We decompose the product of unit vectors $n'_a n'_b n'_L$ on the basis of STF tensors, we drop the terms having zero TT projection, and we express the remaining terms using the Legendre function of the second kind Q_ℓ (see (A.6*c*)). Restoring the traces on the STF tensors, and dropping further terms having zero TT projection, we obtain

$$(t_2^{ij})^{\mathrm{TT}} = -\frac{1}{r}\mathcal{P}_{ijab}\sum_{\ell\geqslant 2}\frac{\ell(\ell-1)}{(2\ell+3)(2\ell+1)(2\ell-1)}n_{L-2}\int_{-\infty}^{t-r}\mathrm{d}u\,\Pi_{abL-2}(u)$$
$$\times\left\{(2\ell-1)Q_{\ell+2}\left(\frac{t-u}{r}\right)-2(2\ell+1)Q_\ell\left(\frac{t-u}{r}\right)\right.$$
$$\left.+(2\ell+3)Q_{\ell-2}\left(\frac{t-u}{r}\right)\right\}. \tag{5.5}$$

The limit at future null infinity can now be applied. Indeed it suffices to insert the expansion of $Q_\ell(x)$ when $x\to 1^+$ as given by (A.8). As expected, the limit is finite, because the terms $\ln(x-1)$ in the expansions of the Q_ℓ's cancel out. This yields [13, 15, 16, 9]

$$(t_2^{ij})^{\mathrm{TT}} = -\frac{2}{r}\mathcal{P}_{ijab}\sum_{\ell\geqslant 2}\frac{n_{L-2}}{(\ell+1)(\ell+2)}\int_{-\infty}^{t-r}\mathrm{d}u\,\Pi_{abL-2}(u)+\mathrm{O}\left(\frac{\ln r}{r^2}\right). \tag{5.6}$$

In the case of the quadrupole–quadrupole interaction we find (using (4.5))

$$(t_2^{ij})^{\mathrm{TT}} = \frac{1}{r}\mathcal{P}_{ijab}\int_{-\infty}^{t-r}\mathrm{d}u\,\left[\tfrac{4}{7}M^{(3)}_{c\langle a}M^{(3)}_{b\rangle c}-\tfrac{1}{15}n_{cd}M^{(3)}_{\langle ab}M^{(3)}_{cd\rangle}\right]+\mathrm{O}\left(\frac{\ln r}{r^2}\right) \tag{5.7}$$

(where $\langle\,\rangle$ denotes the STF projection). The non-local integral (5.7) represents the quadrupole–quadrupole contribution to the nonlinear memory [14–16, 9].

We now turn our attention to the instantaneous part of the metric. All the terms have been obtained in section 4. We need only to apply the TT projection on the $1/r$ part of h_2^{ij} as given by (4.13) and the coefficients listed in table 2. After several transformations using STF techniques, we obtain

$$(h_2^{ij}-t_2^{ij})^{\mathrm{TT}} = \frac{1}{r}\mathcal{P}_{ijab}\big\{-\tfrac{2}{7}M^{(5)}_{c\langle a}M_{b\rangle c}+\tfrac{10}{7}M^{(4)}_{c\langle a}M^{(1)}_{b\rangle c}+\tfrac{4}{7}M^{(3)}_{c\langle a}M^{(2)}_{b\rangle c}$$
$$+n_{cd}\big[\tfrac{7}{10}M^{(5)}_{\langle ab}M_{cd\rangle}+\tfrac{21}{10}M^{(4)}_{\langle ab}M^{(1)}_{cd\rangle}+\tfrac{17}{5}M^{(3)}_{\langle ab}M^{(2)}_{cd\rangle}\big]$$
$$+n_{dg}\varepsilon_{acg}\big(\varepsilon_{ef\langle b}\big[\tfrac{1}{10}M^{(5)}_{c\underline{e}}M_{d\rangle f}-\tfrac{1}{2}M^{(4)}_{c\underline{e}}M^{(1)}_{d\rangle f}\big]\big)\big\}+\mathrm{O}\left(\frac{1}{r^2}\right) \tag{5.8}$$

(where the underbar on $\underline{e}$ means that the index e is to be excluded from the STF projection).

Using (5.7) and (5.8), we deduce the observable moments by comparison with (5.1). The quadrupole–quadrupole interaction contributes only to the observable mass quadrupole moment U_{ij}, mass 2^4-pole moment U_{ijkl}, and current octupole moment V_{ijk}. We find

$$\delta U_{ij} = -\tfrac{2}{7}\int_{-\infty}^{t-r}M^{(3)}_{a\langle i}M^{(3)}_{j\rangle a}+\tfrac{1}{7}M^{(5)}_{a\langle i}M_{j\rangle a}-\tfrac{5}{7}M^{(4)}_{a\langle i}M^{(1)}_{j\rangle a}-\tfrac{2}{7}M^{(3)}_{a\langle i}M^{(2)}_{j\rangle a}\,, \tag{5.9a}$$

$$\delta U_{ijkl} = \tfrac{2}{5}\int_{-\infty}^{t-r}M^{(3)}_{\langle ij}M^{(3)}_{kl\rangle}-\tfrac{21}{5}M^{(5)}_{\langle ij}M_{kl\rangle}-\tfrac{63}{5}M^{(4)}_{\langle ij}M^{(1)}_{kl\rangle}-\tfrac{102}{5}M^{(3)}_{\langle ij}M^{(2)}_{kl\rangle}\,, \tag{5.9b}$$

$$\delta V_{ijk} = \varepsilon_{ab\langle i}\big[\tfrac{1}{10}M^{(5)}_{j\underline{a}}M_{k\rangle b}-\tfrac{1}{2}M^{(4)}_{j\underline{a}}M^{(1)}_{k\rangle b}\big]. \tag{5.9c}$$

Note that the non-local integrals are present in U_{ij} and U_{ijkl}, but not in the current moment V_{ijk}.

Finally, we add back in (5.9) the factor G and the powers of $1/c$ which are required in order to have the correct dimensionality. When this is done we find that δU_{ij} is of order $1/c^5$ in the post-Newtonian expansion (2.5PN order), while both δU_{ijkl} and δV_{ijk} are of order $1/c^3$ or 1.5PN. This permits writing the complete expressions for U_{ij} to 2.5PN order, and U_{ijkl}, V_{ijk} to 1.5PN order. In the case of U_{ij}, the reasoning has been done in [25], which shows that besides the quadrupole–quadrupole terms $M_{ab} \times M_{cd}$, there is an interaction between M_{ab} and the (static) current dipole S_c also at 2.5PN order, and there is the standard contribution of tails (computed in [9]) at 1.5PN order. (In principle there is also an interaction between the mass octupole M_{abc} and the mass dipole M_d at 2.5PN order, but we have chosen $M_d = 0$.) Thus, from (5.9*a*) and equation (5.7) in [25],

$$U_{ij}(t-r/c) = M^{(2)}_{ij} + \frac{2GM}{c^3}\int_{-\infty}^{t-r/c} \mathrm{d}u \left[\ln\left(\frac{t-r/c-u}{2b}\right) + \tfrac{11}{12}\right] M^{(4)}_{ij}(u) + \frac{G}{c^5}\left\{-\tfrac{2}{7}\int_{-\infty}^{t-r/c} \mathrm{d}u\, M^{(3)}_{a\langle i}M^{(3)}_{j\rangle a}(u) + \tfrac{1}{7}M^{(5)}_{a\langle i}M_{j\rangle a} - \tfrac{5}{7}M^{(4)}_{a\langle i}M^{(1)}_{j\rangle a} - \tfrac{2}{7}M^{(3)}_{a\langle i}M^{(2)}_{j\rangle a} + \tfrac{1}{3}\varepsilon_{ab\langle i}M^{(4)}_{j\rangle a}S_b\right\} + \mathrm{O}\left(\frac{1}{c^6}\right). \tag{5.10}$$

The tail integral involves a constant $\frac{11}{12}$ computed in appendix B of [9]. (See also [9] for the definition of the constant b.) The coefficient in front of the term $M_{ab} \times S_c$ is computed in appendix B below. In a similar way, we find that U_{ijkl} and V_{ijk} involve the same types of multipole interactions, but that the terms $M_{ab} \times M_{cd}$ are of the same 1.5PN order as the tail terms. Thus,

$$U_{ijkl}(t-r/c) = M^{(4)}_{ijkl} + \frac{G}{c^3}\left\{2M\int_{-\infty}^{t-r/c} \mathrm{d}u \left[\ln\left(\frac{t-r/c-u}{2b}\right) + \tfrac{59}{30}\right] M^{(6)}_{ijkl}(u) + \tfrac{2}{5}\int_{-\infty}^{t-r/c} \mathrm{d}u\, M^{(3)}_{\langle ij}M^{(3)}_{kl\rangle}(u) - \tfrac{21}{5}M^{(5)}_{\langle ij}M_{kl\rangle} - \tfrac{63}{5}M^{(4)}_{\langle ij}M^{(1)}_{kl\rangle} - \tfrac{102}{5}M^{(3)}_{\langle ij}M^{(2)}_{kl\rangle}\right\} + \mathrm{O}\left(\frac{1}{c^4}\right), \tag{5.11}$$

and

$$V_{ijk}(t-r/c) = S^{(3)}_{ijk} + \frac{G}{c^3}\left\{2M\int_{-\infty}^{t-r/c} \mathrm{d}u \left[\ln\left(\frac{t-r/c-u}{2b}\right) + \tfrac{5}{3}\right] S^{(5)}_{ijk}(u) + \varepsilon_{ab\langle i}\left[\tfrac{1}{10}M^{(5)}_{j\underline{a}}M_{k\rangle b} - \tfrac{1}{2}M^{(4)}_{j\underline{a}}M^{(1)}_{k\rangle b}\right] - 2S_{\langle i}M^{(4)}_{jk\rangle}\right\} + \mathrm{O}\left(\frac{1}{c^4}\right). \tag{5.12}$$

The constants $\frac{59}{30}$ and $\frac{4}{3}$ are obtained from appendix C in [24]. The coefficient of the term $S_{\langle i}M^{(4)}_{jk\rangle}$ is computed in appendix B below.

The observable quadrupole moment U_{ij} up to 2.5PN order obtained in (5.10) yields the total power contained in the radiation field (or energy flux) complete up to the same 2.5PN order. Indeed, it suffices to insert into (5.10) the intermediate quadrupole moment M_{ij} determined in [25] as an explicit integral extending over the source at 2.5PN order (the other moments needed being the octupole M_{ijk} and current quadrupole S_{ij} at 1PN order, and M_{ijkl} and S_{ijk} at Newtonian order, which are all known). Note that in order to obtain the waveform itself (and not only the energy flux it contains) one needs M_{ijk} and S_{ij} at the higher 2PN order.

Appendix A. Formulae required to compute the quadratic nonlinearities

This appendix presents a unified compendium of formulae, many of them taken from previous works [8, 9, 20], which permit the computation of the quadratic nonlinearities (involving any interaction between two multipole moments). The source of the quadratic nonlinearities takes the structure (3.3), so we present the formulae for the finite part (as defined in (2.8)) of the retarded integral of any term $\hat{n}_L r^{-k} F(t-r)$ with multipolarity ℓ and radial dependence r^{-k} where k is an integer $\geqslant 2$. (For practical computations it is convenient to use the source in expanded form (3.3) rather than in the more precise form (3.9).) The problem was solved in [20], where a basic formula for the B-dependent retarded integral was obtained:

$$\Box_R^{-1}\left[\left(\frac{r}{r_0}\right)^B \frac{\hat{n}_L}{r^k} F(t-r)\right] = \frac{2^{k-3}}{(2r_0)^B (B-k+2)(B-k+1)\cdots(B-k-\ell+2)} \times \int_r^{+\infty} \mathrm{d}s\, F(t-s)\hat{\partial}_L \left\{\frac{(s-r)^{B-k+\ell+2}-(s+r)^{B-k+\ell+2}}{r}\right\} . \quad \text{(A.1)}$$

This formula is valid (by analytic continuation) for all values of B in the complex plane, except possibly at integer values of B where there is a simple pole. Note that to the STF product of unit vectors $\hat{n}_L$ in the left-hand side corresponds a STF product of spatial derivatives $\hat{\partial}_L$ in the right-hand side (see the footnote in section 2 for our notation).

Our first case of interest is that of a source having $k = 2$. In this case (A.1) becomes

$$\Box_R^{-1}\left[\left(\frac{r}{r_0}\right)^B \frac{\hat{n}_L}{r^2} F(t-r)\right] = \frac{1}{2(2r_0)^B B(B-1)\cdots(B-\ell)} \times \int_r^{+\infty} \mathrm{d}s\, F(t-s)\hat{\partial}_L \left\{\frac{(s-r)^{B+\ell}-(s+r)^{B+\ell}}{r}\right\} , \quad \text{(A.2)}$$

of which we compute the finite part in the Laurent expansion when $B \to 0$. We repeat briefly the reasoning of [8]: the coefficient in front of the integral admits a simple pole at $B = 0$, but at the same time the integral vanishes at $B = 0$ thanks to the identity (A36) in [20] (see also (4.20a) in [8]). As a result, the right-hand side of (A.2) is finite at $B = 0$, in agreement with the fact that the retarded integral in its usual form (2.9) is convergent, with value

$$\Box_R^{-1}\left[\frac{\hat{n}_L}{r^2} F(t-r)\right] = \frac{(-1)^\ell}{2} \int_r^{+\infty} \mathrm{d}s\, F(t-s) \times \hat{\partial}_L \left\{\frac{(s-r)^\ell \ln(s-r)-(s+r)^\ell \ln(s+r)}{\ell! r}\right\} , \quad \text{(A.3)}$$

where we have removed the reference to taking the finite part at $B = 0$. Note that the length scale r_0 drops out in the result (this is thanks to (4.20a) in [8]). The formula (A.3) can be generalized to the case where the angular dependence is contained in any (non-tracefree) product $k_{\alpha_1}\cdots k_{\alpha_\ell}$ of ℓ Minkowskian null vectors $k_\alpha = (-1, \boldsymbol{n})$,

$$\Box_R^{-1}\left[\frac{k_{\alpha_1}\cdots k_{\alpha_\ell}}{r^2} F(t-r)\right] = \frac{(-1)^\ell}{2} \int_r^{+\infty} \mathrm{d}s\, F(t-s) \times \partial_{\alpha_1}\cdots\partial_{\alpha_\ell} \left\{\frac{(s-r)^\ell \ln(s-r)-(s+r)^\ell \ln(s+r)}{\ell!\, r}\right\} , \quad \text{(A.4)}$$

where the spacetime derivatives in the right-hand side are defined by $\partial_\alpha = (\partial/\partial s, \partial_i)$. A useful alternative form of (A.3), proved, e.g., in appendix A of [9], reads

$$\Box_R^{-1}\left[\frac{\hat{n}_L}{r^2} F(t-r)\right] = -\frac{\hat{n}_L}{r} \int_r^{+\infty} \mathrm{d}s\, F(t-s) Q_\ell\left(\frac{s}{r}\right) , \quad \text{(A.5)}$$

where $Q_\ell(x)$ denotes the ℓth-order Legendre function of the second kind (with branch cut from $-\infty$ to 1, so $x > 1$). The Legendre function is given by

$$Q_\ell(x) = \tfrac{1}{2}\int_{-1}^{1} P_\ell(y)\frac{\mathrm{d}y}{x-y} \tag{A.6a}$$

$$= \tfrac{1}{2}P_\ell(x)\ln\left(\frac{x+1}{x-1}\right) - \sum_{j=1}^{\ell}\frac{1}{j}P_{\ell-j}(x)P_{j-1}(x)\,, \tag{A.6b}$$

where P_ℓ is the Legendre polynomial (see, e.g., [34]). Note that by combining (A.6*a*) and the expansion of $1/(x-\boldsymbol{n}\cdot\boldsymbol{n}')$ in terms of Legendre polynomials (see, e.g., (A26) in [20]), one has

$$\hat{n}_L Q_\ell(x) = \int \frac{\mathrm{d}\Omega'}{4\pi}\frac{\hat{n}'_L}{x-\boldsymbol{n}\cdot\boldsymbol{n}'}\,, \tag{A.6c}$$

where the angular integration $\mathrm{d}\Omega'$ is associated with the unit direction n'^i. Combining (A.5) and (A.6*c*), we obtain another alternative formula,

$$\Box_R^{-1}\left[\frac{1}{r^2}F(\boldsymbol{n}, t-r)\right] = -\int_r^{+\infty}\mathrm{d}s\int\frac{\mathrm{d}\Omega'}{4\pi}\frac{F(\boldsymbol{n}', t-s)}{s-r\boldsymbol{n}\cdot\boldsymbol{n}'}\,. \tag{A.7}$$

This formula is valid for any function $F(\boldsymbol{n}, u)$ (not only for a function having a definite multipolarity ℓ like in (A.5)). It can also be recovered from the retarded integral in its usual form (2.9).

The leading terms in the expansion when $r \to \infty$ (with $t - r =$ constant) follow from the expansion of the Legendre function when $x \to 1^+$. The expression (A.6*b*) yields

$$Q_\ell(x) = -\tfrac{1}{2}\ln\left(\frac{x-1}{2}\right) - \sum_{j=1}^{\ell}\frac{1}{j} + \mathrm{O}[(x-1)\ln(x-1)]\,, \tag{A.8a}$$

and with (A.5) this implies [9]

$$\Box_R^{-1}\left[\frac{\hat{n}_L}{r^2}F(t-r)\right] = \frac{\hat{n}_L}{2r}\int_0^{+\infty}\mathrm{d}\lambda\, F(t-r-\lambda)\left[\ln\left(\frac{\lambda}{2r}\right) + \sum_{j=1}^{\ell}\frac{2}{j}\right] + \mathrm{O}\left(\frac{\ln r}{r^2}\right)\,. \tag{A.8b}$$

This formula gives the leading term $\ln r/r$ and the sub-dominant term $1/r$. If we try to find the leading term using (A.7) instead of (A.5) (i.e. by changing the variable $s = r + \lambda$ in (A.7) and expanding the integrand when $r \to \infty$, $t - r =$ constant), we get formally a $1/r$ term but as a factor of a divergent integral, as expected since the leading term is actually $\ln r/r$.

In the case of a source term corresponding to $k \geqslant 3$, the relevant formula is obtained by performing $k-2$ integrations by parts of (A.1). We get in this way

$$\Box_R^{-1}\left[\left(\frac{r}{r_0}\right)^B\frac{\hat{n}_L}{r^k}F(t-r)\right] = \left(\frac{r}{r_0}\right)^B\sum_{i=0}^{k-3}\alpha_i(B)\hat{n}_L\frac{F^{(i)}(t-r)}{r^{k-i-2}}$$

$$+\,\beta(B)\,\Box_R^{-1}\left[\left(\frac{r}{r_0}\right)^B\frac{\hat{n}_L}{r^2}F^{(k-2)}(t-r)\right], \tag{A.9}$$

where the second term is a retarded integral of the type studied before, and where the coefficients are

$$\alpha_i(B) = \frac{2^i(B-k+2+i)\cdots(B-k+3)}{(B-k+2-\ell+i)\cdots(B-k+2-\ell)(B-k+3+\ell+i)\cdots(B-k+3+\ell)}\,, \tag{A.10a}$$

$$\beta(B) = \frac{2^{k-2}B(B-1)\cdots(B-\ell)}{(B-k+2)\cdots(B-k-\ell+2)(B+\ell)\cdots(B-k+\ell+3)}\,. \tag{A.10b}$$

The factors symbolized by dots decrease by steps of one unit from left to right. Before taking the finite part, one must study the occurrence of poles at $B = 0$ in the coefficients (A.10). Two cases must be distinguished. In the case $3 \leqslant k \leqslant \ell + 2$, none of the denominators in (A.10) vanish at $B = 0$. This implies that the second term in (A.9) is zero at $B = 0$, owing to the explicit factor B in the numerator of $\beta(B)$. Thus we obtain (in the case $3 \leqslant k \leqslant l+2$) a finite result when $B \to 0$, which is local and independent of r_0,

$$\Box_R^{-1}\left[\left(\frac{r}{r_0}\right)^B \frac{\hat{n}_L}{r^k} F(t-r)\right]\Bigg|_{B=0} = -\frac{2^{k-3}(k-3)!(\ell+2-k)!}{(\ell+k-2)!}\,\hat{n}_L \times \sum_{j=0}^{k-3} \frac{(\ell+j)!}{2^j j!(\ell-j)!}\frac{F^{(k-3-j)}(t-r)}{r^{j+1}}\,. \tag{A.11}$$

This formula can be checked by applying the d'Alembertian operator to both sides. When $r \to \infty$, $t - r =$ constant, it gives

$$\Box_R^{-1}\left[\left(\frac{r}{r_0}\right)^B \frac{\hat{n}_L}{r^k} F(t-r)\right]\Bigg|_{B=0} = -\frac{2^{k-3}(k-3)!(\ell+2-k)!}{(\ell+k-2)!}\hat{n}_L \frac{F^{(k-3)}(t-r)}{r} + \mathrm{O}\left(\frac{1}{r^2}\right)\,. \tag{A.12}$$

Next, in the case $k \geqslant \ell+3$, the denominators of both $\alpha_i(B)$ and $\beta(B)$ vanish at $B = 0$, so $\alpha_i(B)$ involves a (simple) pole, while $\beta(B)$ is finite (because the pole is compensated by the factor B in the numerator). In this case ($k \geqslant \ell + 3$) the finite part reads

$$\begin{aligned}\mathrm{FP}_{B=0}\ \Box_R^{-1}&\left[\left(\frac{r}{r_0}\right)^B \frac{\hat{n}_L}{r^k} F(t-r)\right] \\ &= \sum_{i=0}^{k-3} \tilde{\alpha}_i \hat{n}_L \frac{F^{(i)}(t-r)}{r^{k-i-2}} + \frac{(-1)^{k+\ell}2^{k-2}(k-3)!}{(k+\ell-2)!(k-\ell-3)!} \\ &\quad\times\left\{\frac{(-1)^\ell}{2}\ln\left(\frac{r}{r_0}\right)\hat{\partial}_L\left(\frac{F^{(k-\ell-3)}(t-r)}{r}\right) + \Box_R^{-1}\left[\frac{\hat{n}_L}{r^2}F^{(k-2)}(t-r)\right]\right\}\,.\end{aligned} \tag{A.13}$$

The coefficients are given, when $0 \leqslant i \leqslant k - \ell - 4$, by

$$\tilde{\alpha}_i = \frac{(-2)^i(k-3)!}{(k+\ell-2)!(k-\ell-3)!}\frac{(k+\ell-3-i)!(k-\ell-4-i)!}{(k-3-i)!}\,, \tag{A.14a}$$

and, when $k - \ell - 3 \leqslant i \leqslant k - 3$, by

$$\tilde{\alpha}_i = \frac{2^i(-1)^{k+\ell}(k-3)!}{(k+\ell-2)!(k-\ell-3)!}\frac{(k+\ell-3-i)!}{(k-3-i)!(\ell-k+i+3)!} \times\left\{\sum_{j=1}^{k-\ell-3}\frac{1}{j} - \sum_{j=1}^{\ell+3+i-k}\frac{1}{j} - \sum_{j=k-2-i}^{k-3}\frac{1}{j} + \sum_{j=k+\ell-2-i}^{k+\ell-2}\frac{1}{j}\right\}. \tag{A.14b}$$

(One can express (A.14) with the help of the Euler Γ-function and its logarithmic derivative ψ.) Note that (A.13) has a genuine dependence on the length scale r_0 through the logarithm of r/r_0, which is a factor of a homogeneous solution, say

$$\hat{\partial}_L\left(\frac{G(t-r)}{r}\right) = (-1)^\ell \hat{n}_L \sum_{j=0}^{\ell} \frac{(\ell+j)!}{2^j j!(\ell-j)!} \frac{G^{(\ell-j)}(t-r)}{r^{j+1}} . \tag{A.15}$$

The last term in (A.13) involves a non-local integral known from (A.3)–(A.5) or (A.7). There exists a special combination of retarded integrals (A.13) which is purely local, finite at $B = 0$, logarithm free and independent of r_0. This combination, particularly useful in practical computations, reads

$$\Box_R^{-1}\left[\left(\frac{r}{r_0}\right)^B \hat{n}_L \left(2(k-2)\frac{F^{(1)}(t-r)}{r^k} + [(k-1)(k-2)-\ell(\ell+1)]\frac{F(t-r)}{r^{k+1}}\right)\right]\Bigg|_{B=0}$$
$$= \hat{n}_L \frac{F(t-r)}{r^{k-1}} + \gamma\,\hat{\partial}_L\left(\frac{F^{(k-\ell-2)}(t-r)}{r}\right) , \tag{A.16a}$$

where

$$\gamma = \frac{(-1)^k 2^{k-2}(k-3)!}{(k+\ell-1)!(k-\ell-2)!}\big[(k+\ell-1)(k-\ell-2)-(2k-3)(k-2)\big] . \tag{A.16b}$$

The formula is valid when $k \geqslant \ell+2$; when $k = \ell+2$ we recover (A.11). The second term in (A.16a) is a homogeneous solution fixed by our particular way of integrating the wave equation. Thus the first term by itself is a solution of the required equation, but the homogeneous solution must be taken into account when doing practical computations with this method.

To leading order when $r \to \infty$ with $t - r =$ constant, (A.13) reads

$$\mathrm{FP}_{B=0}\,\Box_R^{-1}\left[\left(\frac{r}{r_0}\right)^B \frac{\hat{n}_L}{r^k} F(t-r)\right] = \frac{(-1)^{k+\ell}2^{k-3}(k-3)!}{(k+\ell-2)!(k-\ell-3)!}\frac{\hat{n}_L}{r}$$
$$\times \int_0^{+\infty} \mathrm{d}\lambda\, F^{(k-2)}(t-r-\lambda)\left[\ln\left(\frac{\lambda}{2r_0}\right) + \sum_{j=1}^{k-\ell-3}\frac{1}{j} + \sum_{j=k-2}^{k+\ell-2}\frac{1}{j}\right] + \mathrm{O}\left(\frac{1}{r^2}\right) . \tag{A.17}$$

As in the case $k = 2$ given by (A.8b), the leading $1/r$ term is non-local. However, in contrast with (A.8b), the expansion is free of logarithms of r (this is true to all orders in $1/r$), and depends irreducibly on the length scale r_0.

Finally, we give the expression of the pole part of the retarded integral when $B \to 0$ in the case $k \geqslant \ell+3$. The result, easily obtained from (A.9), (A.10), is

$$\Box_R^{-1}\left[B\left(\frac{r}{r_0}\right)^B \frac{\hat{n}_L}{r^k} F(t-r)\right]\Bigg|_{B=0} = \frac{(-1)^k 2^{k-3}(k-3)!}{(k+\ell-2)!(k-\ell-3)!}\hat{\partial}_L\left(\frac{F^{(k-\ell-3)}(t-r)}{r}\right) . \tag{A.18}$$

As there are only simple poles, the result is zero when the power of B is strictly larger than one.

Appendix B. The quadrupole–(current–)dipole interaction

We start from the linearized metric (2.3) written for the mass quadrupole M_{ab} and the (stationary) current dipole S_a,

$$h_1^{00} = -2\partial_{ab}(r^{-1}M_{ab}), \tag{B.1a}$$

$$h_1^{0i} = -2\varepsilon_{iab}\partial_a(r^{-1})S_b + 2\partial_a(r^{-1}M_{ai}^{(1)}), \tag{B.1b}$$

$$h_1^{ij} = -2r^{-1}M_{ij}^{(2)}, \tag{B.1c}$$

and substitute it into (2.5), keeping only the terms $M_{ab} \times S_c$.

Once the source $N_2^{\mu\nu}$ for this interaction is known in all-expanded form (3.3), we can apply, to each of the terms, the formulae of appendix A. However, as we are only interested in the two coefficients of the terms $M_{ab} \times S_c$ in (5.10) and (5.12), it is sufficient to obtain the $1/r$ part of the metric $h_2^{\mu\nu}$ when $r \to \infty$, $t - r =$ constant. Thus we proceed like in appendix C of [24], where the $1/r$ part of $h_2^{\mu\nu}$ in the cases of quadrupole–monopole interactions was computed. We recall the necessary formulae derived in [24],

$$\mathrm{FP}_{B=0}\,\Box_R^{-1}\left[r^{B-1}\hat{\partial}_L\left(r^{-1}F(t-r)\right)\right] = \frac{(-1)^\ell \hat{n}_L}{2r}\int_0^\infty \mathrm{d}\lambda\, F^{(\ell)}(t-r-\lambda)\left[\ln\left(\frac{\lambda}{2r}\right) + \sum_{j=1}^{\ell}\frac{1}{j}\right] + \mathrm{O}\left(\frac{\ln r}{r^2}\right), \tag{B.2a}$$

$$\mathrm{FP}_{B=0}\,\Box_R^{-1}\left[r^{B}\partial_i(r^{-1})\hat{\partial}_L\left(r^{-1}F(t-r)\right)\right] = \frac{(-1)^\ell}{2(\ell+1)}(n_i\hat{n}_L - \delta_{i\langle a_\ell}n_{L-1\rangle})\frac{F^{(\ell)}(t-r)}{r} + \mathrm{O}\left(\frac{1}{r^2}\right), \tag{B.2b}$$

$$\begin{aligned}\mathrm{FP}_{B=0}\,\Box_R^{-1}\left[r^{B}\partial_{ij}(r^{-1})\hat{\partial}_L\left(r^{-1}F(t-r)\right)\right] &= \frac{(-1)^{\ell+1}}{2(\ell+1)(\ell+2)}\Big\{-\tfrac{4}{3}\delta_{ij}\delta_{\ell 0} + (n_{ij}+\delta_{ij})\hat{n}_L - 2[\delta_{i\langle a_\ell}n_{L-1\rangle}n_j + \delta_{j\langle a_\ell}n_{L-1\rangle}n_i] \\ &\quad + 2\delta_{i\langle a_\ell}\delta_{\underline{j}a_{\ell-1}}n_{L-2\rangle}\Big\}\frac{F^{(\ell+1)}(t-r)}{r} + \mathrm{O}\left(\frac{1}{r^2}\right).\end{aligned} \tag{B.2c}$$

We have added in (B.2c) the term $-\frac{4}{3}\delta_{ij}\delta_{\ell 0}$ (where δ_{ij} and $\delta_{\ell 0}$ are Kronecker symbols) with respect to the formula (C3) given in appendix C of [24]. Indeed this term is mistakenly missing in the formula (C3) of [24] (but it does not change any of the results derived in [24]).

The formulae (B.2) yield straightforwardly the $1/r$ term in the first part $u_2^{\mu\nu}$ of the metric (see (2.8)). Then we compute the ($1/r$ term of the) divergence $w_2^\mu = \partial_\nu u_2^{\mu\nu}$ and deduce from (2.11), (2.12) the second part $v_2^{\mu\nu}$ of the metric. We find that $v_2^{\mu\nu}$ is non-zero in the case $M_{ab} \times S_c$, in contrast to the cases $M \times M_L$ and $M \times S_L$ studied in [24] for which the $v_2^{\mu\nu}$'s are zero. The result of the computation is

$$h_2^{00} = -\tfrac{4}{3}r^{-1}\varepsilon_{abc}n_{ad}M_{bd}^{(4)}S_c + \mathrm{O}(r^{-2}), \tag{B.3a}$$

$$h_2^{0i} = r^{-1}\left[-\tfrac{4}{3}\varepsilon_{abc}n_a M_{ib}^{(4)}S_c - \tfrac{5}{6}\varepsilon_{iab}n_{acd}M_{cd}^{(4)}S_b\right] + \mathrm{O}(r^{-2}), \tag{B.3b}$$

$$h_2^{ij} = r^{-1}\big[\tfrac{1}{3}\varepsilon_{abc}n_{ijad}M^{(4)}_{bd}S_c - \tfrac{8}{3}\varepsilon_{abc}n_{a(i}M^{(4)}_{j)b}S_c + \delta_{ij}\varepsilon_{abc}n_{ad}M^{(4)}_{bd}S_c - 2\varepsilon_{ab(i}n_{ac}M^{(4)}_{j)c}S_b + \tfrac{1}{3}\varepsilon_{ab(i}n_{j)acd}M^{(4)}_{cd}S_b\big] + \mathrm{O}(r^{-2})\,. \tag{B.3c}$$

From (B.3*c*) we obtain the TT projection and then deduce the associated observable moments. Only the mass quadrupole and current octupole moments receive a contribution,

$$\delta U_{ij} = \tfrac{1}{3}\varepsilon_{ab\langle i}M^{(4)}_{j\rangle a}S_b, \tag{B.4a}$$

$$\delta V_{ijk} = -2M^{(4)}_{\langle ij}S_{k\rangle}. \tag{B.4b}$$

We thus obtain the coefficients quoted in (5.10) and (5.12).

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