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1996 Class. Quantum Grav. 13 575

(http://iopscience.iop.org/0264-9381/13/4/002)

Class. Quantum Grav. **13** (1996) 575–584. Printed in the UK

Gravitational waveforms from inspiralling compact binaries to second-post-Newtonian order

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Received 15 January 1996

Abstract. The two independent 'plus' and 'cross' polarization waveforms associated with the gravitational waves emitted by inspiralling, non-spinning, compact binaries are presented, ready for use in the data analysis of signals received by future laser interferometer gravitational-wave detectors such as LIGO and VIRGO. The computation is based on a recently derived expression of the gravitational field at the second-post-Newtonian approximation of general relativity beyond the dominant (Newtonian) quadrupolar field. The use of these theoretical waveforms to make measurements of astrophysical parameters and to test the nature of relativistic gravity is discussed.

PACS numbers: 0430, 0480, 0480N, 9760J, 9760L

1. Introduction

Two large-scale laser interferometer detectors of gravitational waves are now under construction in the US (LIGO experiment [1]) and in Europe (VIRGO experiment [2]). These experiments should detect the gravitational waves generated by inspiralling compact binary systems at cosmological distances [3–7]. These are systems of two compact objects (neutron stars or black holes) spiralling very rapidly around each other in their late stage of evolution immediately preceding the final coalescence. The dynamics of such systems is entirely ruled by the radiation reaction forces due to the emission of the gravitational radiation itself. Estimates of the number of inspiralling/coalescing events each year have been found to be quite promising [8–11].

The LIGO and VIRGO observations of inspiralling compact binaries should provide precise measurements of the masses of the objects, possibly of their spins and possibly, in the case of neutron stars, of their radii [12–19]. The absolute luminosity distance of the binary will be measured independently of any assumption concerning the nature (and masses) of the objects [20]. However, if the intrinsic masses of the objects are known, the cosmological red-shift of the host galaxy where the event took place can be measured. There is hope of deducing in this way a measurement of the Hubble parameter from an assumption concerning the statistical distribution of masses of neutron stars [21, 22]. The puzzle of the origin of gamma-ray bursts could be solved by comparison of the times of

0264-9381/96/040575+10$19.50

arrival of the gravitational waves and of gamma ray bursts [10, 23, 24]. Furthermore, new limits on the validity of alternative theories of gravity, notably scalar–tensor theories [25], and new tests of general relativity in the strong field regime [26] should be possible by monitoring very precisely the inspiralling signal.

To prepare for the analysis of such signals in the LIGO and VIRGO detectors one needs to compute the gravitational radiation field generated by a system of two point-masses moving on a circular orbit (the relevant case because the orbit will have been circularized by radiation reaction forces). Since inspiralling compact binaries are very relativistic, this problem is highly nontrivial and represents a challenge to relativity theorists [5, 7], as its resolution involves carrying out the calculation to a very high order in terms of a post-Newtonian expansion (see Will [27] for a review). The problem can be decomposed into two different problems, which can be referred to respectively as the 'wave generation problem' and the 'radiation reaction problem' (see Damour [28] for a discussion).

The wave generation problem deals with the computation of the gravitational waveforms generated by the binary (at the leading order in $1/r$, where r is the distance of the binary) when the orbital phase and frequency of the binary take some given values ϕ and ω. This problem involves computing the amplitude of each harmonic of the wave corresponding to frequencies which are multiples of the orbital frequency, with the predominant harmonic being at twice the orbital frequency.

The radiation reaction problem consists of determining the evolution of the orbital phase $\phi(t)$ itself as a function of time, from which one deduces the orbital frequency $\omega(t) = \mathrm{d}\phi(t)/\mathrm{d}t$. The actual time variation of $\phi(t)$ is nonlinear because the orbit evolves under the effects of gravitational radiation reaction forces. In principle, it should be determined from the knowledge of the radiation reaction forces acting locally on the orbit. However, these forces are at present not known with sufficient accuracy (only the relative first post-Newtonian corrections are known [29, 30]), so in practice the phase evolution is determined by equating the high-order post-Newtonian energy flux in the waves or energy loss (averaged over one orbit) and the decrease of the correspondingly accurate binding energy of the binary.

Estimates of the precision needed in the resolution of these two problems can be inferred from black-hole perturbation techniques in the special case where the mass of one object is very small as compared with the other mass [31–37]. The required precision is reached when the systematic errors due to the neglect of some higher-order approximation become less than the statistical errors due to noise in the detector. It turns out that the post-Newtonian corrections in the time evolution of the phase (radiation reaction problem) will be measurable in advanced detectors, probably up to three post-Newtonian (3PN) orders beyond the (Newtonian) quadrupole radiation [33, 37]. This corresponds to relativistic corrections in the energy loss as high as order $\sim (v/c_0)^6$, where v is the orbital velocity and where we denote for later convenience the speed of light by c_0. The possibility of measuring such high-order corrections can be understood crudely from the fact that, in order not to suffer a too severe reduction in signal-to-noise ratio, one will have to monitor the phase evolution with an accuracy of one tenth of a cycle over the tens of thousands of cycles during the entire passage through the frequency bandwidth of the detector.

It has been argued [15–17] that most of the accessible information allowing accurate measurements of the binary's intrinsic parameters (such as the two masses) is in fact contained in the phase, because of the accumulation of cycles, and that rather less accurate information is available in the wave amplitude itself. For instance, the relative precision in the determination of the distance r of the source, which affects the wave amplitude, will be less than for the masses, which strongly affect the phase evolution. Consequently, the determination of r does not necessitate as high a post-Newtonian precision as for the masses, and to a good approximation it should be adequate for that purpose to include in the gravitational waveform (as determined by solving the generation problem) post-Newtonian corrections up to an order of less than 3PN order. To determine just what order is required in the waveform would necessitate a measurement-accuracy analysis similar to the ones performed in [17–19], but not restrained to a model waveform taking only the higher-order phase evolution into account (namely the so-called restricted post-Newtonian waveform). To our knowledge, no complete investigation has been performed yet, however we believe that such a study would be desirable. For instance, it is conceivable that conclusions reached regarding the needed accuracy in the determination of the phase in the reaction problem are modified when the full amplitude evolution of the wave is taken into account.

Both the gravitational waveform and associated outgoing flux or energy loss have been obtained recently to the 2PN approximation corresponding to the relative post-Newtonian order $\sim (v/c_0)^4$. Two computations have been performed, one by Blanchet *et al* [38] using a mixed post-Newtonian and post-Minkowskian formalism [39, 40], and one by Will and Wiseman [41] using a purely post-Newtonian formalism due to Epstein and Wagoner [42] and Thorne [43]. The result of these computations for the energy loss has been summarized in [44]. More recently, the precision on the energy loss has been extended to include the next 2.5PN approximation [45].

Even though the 2PN or even 2.5PN precision in the resolution of the reaction problem appears to be still insufficient to make full use of the phase data [33, 37], it is plausible, as previously argued [15–17], that the 2PN waveform amplitude is already close to what will be needed by the LIGO and VIRGO detectors. However, this 2PN waveform is displayed in [38, 41] in a format which is not ready for use in the future analysis of the outputs of LIGO and VIRGO. The aim of this paper is to present the complete, ready-to-use expression of the waveform, including the amplitude of all the wave harmonics present, to 2PN order, for non-spinning bodies. More precisely, we compute the two independent polarization states of the gravitational wave (customarily referred to as the 'plus' and 'cross' polarizations) which define the theoretical 'templates' to be cross correlated with the raw output of the detectors. The templates should be used both 'on line' when searching for the signal and later when the signal is subject to a very precise data analysis involving a more accurate determination of the parameters, and possibly the measurement of other parameters.

In section 2 we present our main result, namely the two plus and cross polarization waveforms to 2PN order as functions of the phase and frequency. For completeness and discussion thereafter, we recall also the results obtained in [38, 41] concerning the time evolution to 2PN order of the phase and frequency. In section 3 we discuss various issues associated with the actual use of these waveforms.

2. The gravitational waveforms to second-post-Newtonian order

The gravitational wave as it propagates through the detector in the source's wave zone is entirely described by the transverse and traceless (TT) asymptotic waveform $h_{ij}^{\mathrm{TT}} = (g_{ij} - \delta_{ij})^{\mathrm{TT}}$, where g_{ij} denotes the spatial covariant metric in a coordinate system adapted to the wave zone, and δ_{ij} is the spatial part of the Minkowski metric (signature $-+++$). The two polarization states h_+ and $h_\times$ are defined by $h_+ = \frac{1}{2}(p_i p_j - q_i q_j) h_{ij}^{\mathrm{TT}}$ and $h_\times = \frac{1}{2}(p_i q_j + p_j q_i) h_{ij}^{\mathrm{TT}}$ where $\boldsymbol{p}$ and $\boldsymbol{q}$ denote two polarization vectors forming, along with the direction $\boldsymbol{n}$ from the source to the detector, an orthonormal right-handed triad. The detector is directly sensitive to that linear combination of polarization waveforms h_+ and

$h_\times$ which is given by

$$h(t) = F_+ h_+(t) + F_\times h_\times(t) \tag{1}$$

where F_+ and $F_\times$ are the so-called beam-pattern functions of the detector depending on two angles giving the direction $-\boldsymbol{n}$ of the source, as seen from the detector, and a polarization angle specifying the orientation of the vectors $\boldsymbol{p}$ and $\boldsymbol{q}$ around that direction. The expressions of F_+ and $F_\times$ in terms of these angles are given explicitly in the case of laser-interferometer detectors by, for example, equation (104) of Thorne [3].

The two polarizations h_+ and $h_\times$ in the case of a binary made of non-spinning point-masses moving on a (quasi-)circular orbit are obtained by a straightforward computation starting from the end results of [38, 41]. We choose the polarization vectors $\boldsymbol{p}$ and $\boldsymbol{q}$ to lie along the major and minor axis (respectively) of the projection onto the plane of the sky of the circular orbit, with $\boldsymbol{p}$ oriented toward the 'ascending node', the point at which body one crosses the plane of the sky moving toward the observer. The TT waveform h_{ij}^{TT} is split as in equation (4.1) of [38] into the sum of an 'instantaneous' contribution $(h_{ij}^{\mathrm{TT}})_{\mathrm{inst}}$ and of a 'tail' contribution $(h_{ij}^{\mathrm{TT}})_{\mathrm{tail}}$ (we comment later on the presence of wave tails in the signal and their possible detection). The instantaneous contribution is given, for example, by equations (4.5) of [38] from which we compute first $(h_+)_{\mathrm{inst}}$ and $(h_\times)_{\mathrm{inst}}$. Then we add to these the tail parts $(h_{+,\times})_{\mathrm{tail}}$ which are already obtained in equations (4.9)–(4.10) of [38]. Identical formulae result from calculating h_+ and $h_\times$ from the circular-orbit limit of equations (6.10) and (6.11) of [41]. Some transformations are necessary in order to express the result in the best form for applications in the LIGO and VIRGO detectors. The final result reads

$$h_{+,\times} = \frac{2Gm\eta}{c_0^2 r}\left(\frac{Gm\omega}{c_0^3}\right)^{2/3}\{H_{+,\times}^{(0)} + x^{1/2}H_{+,\times}^{(1/2)} + xH_{+,\times}^{(1)} + x^{3/2}H_{+,\times}^{(3/2)} + x^2H_{+,\times}^{(2)}\} \tag{2}$$

where the brackets involve a post-Newtonian expansion whose various post-Newtonian terms are given for the plus polarization by

$$H_+^{(0)} = -(1+c^2)\cos 2\psi \tag{3a}$$

$$H_+^{(1/2)} = -\frac{s}{8}\frac{\delta m}{m}[(5+c^2)\cos\psi - 9(1+c^2)\cos 3\psi] \tag{3b}$$

$$H_+^{(1)} = \tfrac{1}{6}[(19+9c^2-2c^4) - \eta(19-11c^2-6c^4)]\cos 2\psi - \tfrac{4}{3}s^2(1+c^2)(1-3\eta)\cos 4\psi \tag{3c}$$

$$\begin{aligned} H_+^{(3/2)} = \frac{s}{192}\frac{\delta m}{m}\{&[(57+60c^2-c^4) - 2\eta(49-12c^2-c^4)]\cos\psi \\ &-\tfrac{27}{2}[(73+40c^2-9c^4) - 2\eta(25-8c^2-9c^4)]\cos 3\psi \\ &+\tfrac{625}{2}(1-2\eta)s^2(1+c^2)\cos 5\psi\} - 2\pi(1+c^2)\cos 2\psi \end{aligned} \tag{3d}$$

$$\begin{aligned} H_+^{(2)} = \tfrac{1}{120}[&(22+396c^2+145c^4-5c^6) + \tfrac{5}{3}\eta(706-216c^2-251c^4+15c^6) \\ &-5\eta^2(98-108c^2+7c^4+5c^6)]\cos 2\psi + \tfrac{2}{15}s^2[(59+35c^2-8c^4) \\ &-\tfrac{5}{3}\eta(131+59c^2-24c^4) + 5\eta^2(21-3c^2-8c^4)]\cos 4\psi \\ &-\tfrac{81}{40}(1-5\eta+5\eta^2)s^4(1+c^2)\cos 6\psi \\ &+\frac{s}{40}\frac{\delta m}{m}\{[11+7c^2+10(5+c^2)\ln 2]\sin\psi - 5\pi(5+c^2)\cos\psi \\ &-27[7-10\ln(3/2)](1+c^2)\sin 3\psi + 135\pi(1+c^2)\cos 3\psi\} \end{aligned} \tag{3e}$$

and for the cross polarization by

$$H_\times^{(0)} = -2c\sin 2\psi \tag{4a}$$

$$H_\times^{(1/2)} = -\frac{3}{4}sc\frac{\delta m}{m}[\sin\psi - 3\sin 3\psi] \tag{4b}$$

$$H_\times^{(1)} = \frac{c}{3}[(17-4c^2) - \eta(13-12c^2)]\sin 2\psi - \tfrac{8}{3}(1-3\eta)cs^2\sin 4\psi \tag{4c}$$

$$\begin{aligned} H_\times^{(3/2)} = \frac{sc}{96}\frac{\delta m}{m}\{&[(63-5c^2) - 2\eta(23-5c^2)]\sin\psi \\ &-\tfrac{27}{2}[(67-15c^2) - 2\eta(19-15c^2)]\sin 3\psi \\ &+\tfrac{625}{2}(1-2\eta)s^2\sin 5\psi\} - 4\pi c\sin 2\psi \end{aligned} \tag{4d}$$

$$\begin{aligned} H_\times^{(2)} = \frac{c}{60}&[(68+226c^2-15c^4) + \tfrac{5}{3}\eta(572-490c^2+45c^4) \\ &-5\eta^2(56-70c^2+15c^4)]\sin 2\psi + \tfrac{4}{15}cs^2[(55-12c^2) \\ &-\tfrac{5}{3}\eta(119-36c^2) + 5\eta^2(17-12c^2)]\sin 4\psi - \tfrac{81}{20}(1-5\eta+5\eta^2)cs^4\sin 6\psi \\ &-\tfrac{3}{20}sc\frac{\delta m}{m}\{[3+10\ln 2]\cos\psi + 5\pi\sin\psi - 9[7-10\ln(3/2)]\cos 3\psi \\ &-45\pi\sin 3\psi\}. \end{aligned} \tag{4e}$$

The notation is as follows. The post-Newtonian expansion in (2) is parametrized by $x \equiv (Gm\omega/c_0^3)^{2/3}$ where ω is the 2PN-accurate orbital frequency of the circular orbit ($\omega = 2\pi/P$ where P is the orbital period) and $m \equiv m_1 + m_2$ is the total mass of the binary. (The expansion (2) is valid up to the neglect of 2.5PN terms of order $\mathrm{O}(x^{5/2})$.) In addition to m we denote $\delta m \equiv m_1 - m_2$ and $\eta \equiv m_1 m_2/m^2$. The vector $\boldsymbol{n}$ along the line of sight from the binary to the detector defines the inclination angle i with respect to the normal to the orbital plane. The normal is chosen to be right-handed with respect to the sense of motion so that we have $0 \leqslant i \leqslant \pi$. The notations c and s are shorthand for the cosine and sine of the inclination angle : $c \equiv \cos i$ and $s \equiv \sin i$. Finally the basic phase variable ψ entering (3) and (4) is defined by

$$\psi = \phi - \frac{2Gm\omega}{c_0^3}\ln\left(\frac{\omega}{\omega_0}\right) \tag{5}$$

where ϕ is the actual orbital phase of the binary, namely the angle oriented in the sense of motion between the ascending node and the direction of body one ($\phi = 0 \bmod 2\pi$ when the two bodies lie along $\boldsymbol{p}$, with body one at the ascending node). The logarithmic term in the definition of ψ involves a constant frequency scale ω_0 which can be chosen arbitrarily (see later). This logarithmic phase modulation was determined in [46, 47] and originates physically from the propagation of tails in the wave zone. Using ψ instead of the actual phase ϕ simplifies somewhat the expression of the waveform, since it permits collecting in a single block all the logarithmic terms (to 2PN order). The variable ψ is also very convenient in black-hole perturbation theory, where it permits the resolution of the Teukolsky equation governing the outgoing radiation up to very high order in the post-Newtonian expansion [35, 36].

The expressions (2)–(5) solve the generation problem for inspiralling compact binaries to 2PN order. Up to now, both the plus and cross polarization waveforms were known for arbitrary masses to 1.5PN order [47, 48], and in the test mass limit $\eta \to 0$ to 1.5PN order [32] and 4PN order [36]. Through 1.5PN order equations (3) and (4) agree with Wiseman [49] (see also [27]) when those formulae are reduced to the circular-orbit limit. Equations (3) and (4) also largely agree with similar formulae given by Poisson [32] and

Wiseman [47], however those formulae contained some typographical errors (see [48]). In any case, the final results given in equations (3) and (4) supersede the previous results. We have checked that our expressions (3) and (4) in the test mass limit $\eta \to 0$ are in perfect agreement with the truncation to 2PN order of the results given in appendix B of Tagoshi and Sasaki [36]. (The comparison shows that the phase variable used by Tagoshi and Sasaki is related to ours by $\psi_{\mathrm{TS}} = \psi + (2Gm\omega/c_0^3)(\ln 2 - \frac{17}{12})$ which clearly corresponds simply to a rescaling of the frequency ω_0.)

In order to obtain the expressions of the waveforms $h_+(t)$ and $h_\times(t)$ as functions of time one needs to replace ψ and ω appearing in equations (2)–(5) by their explicit time evolutions $\psi(t)$ and $\omega(t)$ obtained from the resolution of the radiation reaction problem. This problem has been solved in [38, 41] to 2PN order, and we quote here these results for completeness and later discussion. It is convenient to introduce instead of the local time t flowing in the experimenters' reference frame the dimensionless time variable

$$\Theta = \frac{c_0^3\eta}{5Gm}(t_{\mathrm{c}} - t) \tag{6}$$

where t_{c} is a constant which represents the instant of coalescence of the two point-masses (at which the frequency goes formally to infinity). Then the instantaneous orbital phase $\phi(t)$ is given in terms of the time variable (6) by equation (4.29) in [38] which reads

$$\begin{aligned}\phi(t) = \phi_{\mathrm{c}} - \frac{1}{\eta}\bigg\{&\Theta^{5/8} + \left(\frac{3715}{8064} + \frac{55}{96}\eta\right)\Theta^{3/8} - \frac{3\pi}{4}\Theta^{1/4} \\ &+ \left(\frac{9\,275\,495}{14\,450\,688} + \frac{284\,875}{258\,048}\eta + \frac{1855}{2048}\eta^2\right)\Theta^{1/8}\bigg\}\end{aligned} \tag{7}$$

where ϕ_{c} is another constant representing the value of the phase at t_{c}. (Note that when taking into account higher-order post-Newtonian approximations (starting at 2.5PN) the phase $\phi(t)$ no longer tends to a constant when $t \to t_{\mathrm{c}}$ but instead becomes infinite [45]. In this case the constant ϕ_{c} is simply determined by initial conditions when the frequency of the wave enters the detector's bandwidth.) The orbital frequency is obtained simply by differentiating equation (7) with respect to time ($\omega = \mathrm{d}\phi/\mathrm{d}t$), hence

$$\begin{aligned}\omega(t) = \frac{c_0^3}{8Gm}\bigg\{&\Theta^{-3/8} + \left(\frac{743}{2688} + \frac{11}{32}\eta\right)\Theta^{-5/8} - \frac{3\pi}{10}\Theta^{-3/4} \\ &+ \left(\frac{1\,855\,099}{14\,450\,688} + \frac{56\,975}{258\,048}\eta + \frac{371}{2048}\eta^2\right)\Theta^{-7/8}\bigg\}.\end{aligned} \tag{8}$$

From (7) and (8) one deduces the phase variable ψ using equation (5). Both the expressions (7) and (8) are valid up to the 2PN order which corresponds formally to the same relative precision as for the waveforms (3) and (4). However, it is not *a priori* required for consistency to have the same post-Newtonian precision in both the waveforms and phase. On the contrary, one should use in the waveforms (3) and (4) the best available expression for $\phi(t)$ which will be hopefully determined in the future to a much higher order than 2PN (see [45] for the expression of $\phi(t)$ to 2.5PN order). Related to this, note that the logarithmic term in the phase variable $\psi(t)$ although of formal order 1.5PN, is actually of order 4PN relatively to the dominant term in $\phi(t)$ given by (7) (indeed $\phi(t) - \phi_{\mathrm{c}}$ is of order $\sim c_0^5$ which is the inverse of the order of the radiation reaction effects). Thus the logarithmic term is in fact currently negligible but must be included when the precision on $\phi(t)$ improves in the future to reach 4PN (even without knowing $\phi(t)$ to the 4PN order it may be a good idea to include this term in the templates).

It can be readily checked from the expressions (5)–(8) that any change in the phase variable ψ corresponding to a rescaling of the frequency ω_0 is equivalent, to the considered order, to a shift in the value of the instant of coalescence t_c, namely, the rescaling $\omega_0 \to \lambda\omega_0$ is equivalent to the shift $t_c \to t_c - (2Gm/c_0^3)\ln\lambda$. Thus a particular choice of the frequency scale ω_0 is physically irrelevant since a different choice can always be absorbed into a redefinition of the origin of time in the wave zone. (We have $\omega_0 = (1/4b)\exp(\frac{11}{12} - C)$ where $C = 0.577\ldots$ is Euler's constant and b is a freely specifiable parameter entering the relation between the wave zone coordinate time t and the harmonic coordinates t_H, r_H: $t = t_H - r_H/c_0 - (2Gm/c_0^3)\ln(r_H/c_0 b)$ where r_H is the distance of the source in harmonic coordinates, see [46].) It could be possible to relate ω_0 to the source characteristics, choosing for instance $\omega_0 = c_0^3/Gm$ where m is the total mass of the binary. However, this mass will not be known in practice but it will be used as a parameter in the templates to be varied during the data analysis process, so this choice seems to be somewhat awkward. For practical purposes it is probably more convenient to choose ω_0 in such a way that it is uniform over all templates. For instance, one can relate ω_0 to the detector characteristics by choosing $\omega_0/\pi = 10$ Hz where 10 Hz (say) is the seismic cut-off frequency of the detector [26]. Here we adopt this choice.

3. Discussion

To construct adequate filters for the analysis of inspiralling binary signals, one should proceed as follows. The theoretical waveform $h(t)$ given as a function of time by the above formulae (1)–(8) is discretized and its Fourier transform $\tilde{h}(\Omega)$ is computed numerically and stored. Then the ratio $\tilde{q}(\Omega) = \tilde{h}(\Omega)/S_n(\Omega)$, where $S_n(\Omega)$ is the measured power spectral density of the noise in the detector, defines the Fourier transform of the Wiener filter $q(t)$, which is to be cross-correlated (either in real time or during the more precise analysis later) with the raw output $o(t)$ of the detector composed of the superposition of the actual signal and of the noise (which is supposed here to be Gaussian). Because of the availability of fast Fourier transforms the correlation is computed in the Fourier domain using the (discrete) correlation theorem for the Fourier coefficients $\tilde{q}(\Omega)$ and $\tilde{o}(\Omega)$. This is repeated for each set of parameters in the filter until maximization of the signal-to-noise ratio is obtained yielding the determination of the parameters of the binary (if a signal was really present at this instant).

For two nonrotating test-masses there are four parameters entering the phase of the signal (this is true up to any post-Newtonian order). These can be chosen to be the two mass parameters m and η, and the constants t_c and ϕ_c. Alternatively one can use the chirp mass $\mathcal{M} = \eta^{3/5}m$ and the reduced mass $\mu = \eta m$, and/or the arrival time t_0 and phase ϕ_0 at the seismic frequency ω_0. The mass parameters may also be replaced by two distinct post-Newtonian pieces of the (chirp) time left until coalescence starting from ω_0 [50]. In addition, the amplitude of each harmonic of the signal depends on the distance r, on the inclination angle i, and on the direction of the binary and the polarization angle through the beam-pattern functions F_+ and $F_\times$. Note that r is the cosmological luminosity distance, and that the masses are the red-shifted masses which are equal to the true masses multiplied by $1 + z$ where z is the binary's cosmological redshift. This permits the cosmological measurements proposed in [20–22].

In the case of rotating bodies, there are additional parameters in the signal. Spin–orbit and spin–spin contributions to the waveform and the phase evolution have been obtained by Kidder and co-workers [51, 52]. The overall effect of spins on the accumulated phase during an inspiral was analysed by Blanchet *et al* [44]. This calculation was made with the assumption that the spins of the objects were aligned and perpendicular to the orbital plane. However, including non-aligned spins makes the waveform and accumulated-phase calculations considerably more complicated. In this general case the orbital plane can wobble—thus the inclination angle i in (3) and (4) changes with time—giving rise to an amplitude modulation and a frequency modulation [53]. However, note that for realistic inspiralling neutron star binaries (such as the binary pulsar PSR 1913+16 at the epoch when it finally coalesces) the spin–orbit and spin–spin parameters are expected to be very small, so that the dynamics of these systems will be dominated by the purely gravitational effects investigated here (see the discussion in [44]).

In addition to involving rotating bodies, the binary could move on a slightly eccentric orbit if, being formed late, it reaches the final inspiral stage before the circularization of the orbit by radiation reaction has fully taken place. This would introduce an additional term in the phase (7) involving a new parameter which can conveniently [19] be taken to be $e_0^2\omega_0^{19/9}$, where e_0 is the initial eccentricity at ω_0 (recall that e^2 evolves in time proportionally to $\omega^{-19/9}$ in the quadrupole approximation). The general formulae of [40, 41, 54] can be applied to such non-circular situations.

Another possibility is that general relativity is not the correct theory of gravity because of, for instance, the existence of a scalar spin-0 field besides the spin-2 metric field. In this case there would be still another parameter in the signal describing the coupling of the scalar field with the matter fields. Preliminary analyses of the bounds that could be placed on the coupling parameter ω_{BD} of the Brans–Dicke theory can be found in [19, 25].

More generally, it should be possible to check that the theoretical waveform (1)–(8) is exactly reproduced in the real signal without prejudice of which theory could be the true theory of gravity. A simple method to do this consists of parametrizing the templates by a redundant set of parameters (i.e. by using more parameters than strictly necessary) and measuring them independently by optimal filtering. In this way one can test whether some particular terms involving some specific combinations of parameters are really present in the signal [26]. However, since multiplying the number of independent parameters has the effect of diminishing the accuracy in their measurement, this method necessitates a high signal-to-noise ratio.

An application of this method is the detection in the real signal of effects associated with the tails of waves. Physically the tails come from the backscattering of the linear waves off the static spacetime curvature generated by the total mass m of the binary itself. The tails are characterized by the fact that they are ‘non-instantaneous’, namely they depend on the whole integrated past of the source; however, for binary systems the actual dependence on the past is negligible to 2PN order. Nevertheless, some important signatures of the tail effect remain in the signal, the most important of which is the term with coefficient π in the phase of the wave (7), which is directly due to the tails in the radiation reaction forces. This effect should be easily detectable in the real signal. Other tail contributions enter the binary’s waveform and are due to the propagation of tails in the wave zone. An instance is the logarithmic phase modulation in (5) which can be referred to as a wave tail contribution to the phase (as opposed to the radiation reaction contribution). Even this very small tail-induced phase modulation (of relative 4PN order) has been shown to be detectable with a high, but not exceedingly high, signal-to-noise ratio using the above method [26]. This shows how the observations of inspiralling compact binaries could permit new tests of general relativity in a regime of strong and rapidly-varying gravitational fields.

Acknowledgments

This research is supported in part by the Centre National de la Recherche Scientifique, by the National Science Foundation grants no PHY 92-22902 (Washington University) and AST 94-17371 (Caltech) and by the National Aeronautics and Space Aministration grants no NAGW 3874 (Washington University) and NAGW 4268 (Caltech). One of the authors (AGW) would like to thank Ben Owen for several helpful comments.

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